# CONVERGENCE OF THE MONTE CARLO EM FOR CURVED EXPONENTIAL FAMILIES 

GERSENDE FORT AND ERIC MOULINES

## SUMMARY

The Monte Carlo Expectation Maximization (MCEM) algorithm is a versatile tool for inference in incomplete data models, especially when used in combination with Markov chain Monte Carlo (MCMC) simulation methods.

In this contribution, the almost-sure convergence of the MCEM algorithm is established. It is shown, using uniform versions of ergodic theorems for Markov chains, that MCEM converges under weak conditions on the simulation kernel; practical illustrations are presented, using an hybrid random walk Metropolis Hastings sampler and an independence sampler. The rate of convergence is studied, showing the impact of the simulation schedule on the fluctuation of the parameter estimate at the convergence. A novel averaging procedure is then proposed to reduce the simulation variance and increase the rate of convergence.

Keywords: EM algorithm; Monte-Carlo EM algorithm; Metropolis Hastings algorithms; Averaging procedure

Short Title: Convergence of the MCEM algorithm

AMS-classification: Primary: 65C05, 62-04, Secondary: 60 J 10

Affiliations:<br>G. Fort (Corresponding author), LMC-IMAG, 51, rue des Mathématiques, BP 53, 38041 Grenoble Cedex 9, France, email: Gersende.Fort@imag.fr<br>E. Moulines, ENST, 46, rue Barrault, 75634 Paris Cedex 13, France, email: moulines@tsi.enst.fr

## Introduction

Many problems in computational statistics reduce to the maximization of a criterion

$$
\begin{equation*}
g(\theta):=\int_{\mathcal{X}} h(z ; \theta) \mu(d z) \quad h(\cdot ; \theta)>0, \mu \text {-a.s. } \tag{1}
\end{equation*}
$$

on a feasible set $\Theta$, when $g$ can not computed in closed form. In the terminology of the missing data problem, $g$ is the incomplete data likelihood i.e. the likelihood of the observations for the value of the parameter $\theta, z \in \mathcal{X}$ is the missing data vector and $h$ is the complete data likelihood with respect to (w.r.t.) the reference measure $\mu$, i.e. $h$ is the likelihood of the observations and of the missing data.

The Expectation Maximization (EM) algorithm (Dempster et al. (1977)) is a popular iterative procedure for maximizing $g$. The E step of the algorithm requires the computation of the expectation of the complete $\log$-likelihood w.r.t. the posterior distribution of the missing data. In many situations, this step is intractable; to solve this problem, many approximations of the EM algorithm, which use simulations as an intermediate step, have been proposed (see, e.g. Tanner (1996), Celeux and Diebolt (1992), Delyon et al. (1999)). Perhaps the most popular algorithm for this purpose is the Monte Carlo EM, initially proposed by Wei and Tanner (1991) and later used and studied by many authors (see Sherman et al. (1999) and references therein).

The basic principle behind this algorithm is to replace the expectation step by a blending of Monte Carlo integration procedure with MCMC sampling techniques such as the Gibbs or the Metropolis Hastings algorithms. The MCEM algorithm has been successfully applied in many different settings, including non-linear time-series model (Chan and Ledolter (1995)), generalized linear mixed models with missing data (Chan and Kuk (1997)), full-information item factor models (Meng and Schilling (1996)), genetic models (Guo and Thompson (1991)) and blind deconvolution (Cappé et al. (1999)).

The analysis of the convergence of the MCEM algorithm has been first formally addressed by Biscarat (1994) as a specific example of a random iterative algorithm. The conditions in Biscarat (1994) have been later weakened by Chan and Ledolter (1995). The assumptions in these works are however rather restrictive, because they involve an uniform law of large numbers, i.e., uniform convergence in probability of the Monte Carlo expectation to their corresponding sample average over $\theta$ in a compact subset of the feasible set $\Theta$. This assumption fails to be
verified when Monte Carlo integration is carried out along a single run MCMC algorithm in the simulation step. It can however be verified under reasonable assumptions when Monte Carlo integration is done using independent chains, as shown by (Sherman et al., 1999, Theorem 2) (the difficulty when moving from single run to multiple runs has been overlooked by Chan and Ledolter (1995)). Convergence of random iterative algorithms has also been considered by Shapiro and Wardi (1996), Pierre-Loti-Viaud (1995) and Brandière (1998), also under restrictive assumptions.

Sherman et al. (1999) address a different class of results. These authors focus on the missing data problem, for which $g(\theta)$ is the incomplete data likelihood, depending on the sample size, say $N$ (the dependence on this parameter is implicit in our work, all the results we obtain being conditional to $N$ ). They assume that the Monte Carlo integration is carried out by means of independent chains, and that the number of independent chains, the number of iterations for each chain at each step, and the number of the iterations of the algorithm are functions of $N$. Under these assumptions, the authors derive the rate of convergence of the Monte Carlo estimator obtained as $N \rightarrow \infty$.

The purpose of this paper is to complement the results above, by providing a convergence analysis of the MCEM algorithm which remains valid under assumptions that are verified for a wide class of MCMC simulation techniques, including both single run and multiple runs chains. The proof of convergence is rather different from the schemes used before, avoiding any form of uniform law of large numbers. An averaging technique to improve the rate of convergence is also presented, based on a modification of the averaging techniques (Polyak (1990)).

The paper is organized as follows. In Section 1 we present the MCEM algorithm, and define the stable MCEM algorithm which guarantees the almost sure (a.s.) boundedness of the random recursion. In Section 2, we study the convergence of stable MCEM for curved exponential families when the simulation step is based on MCMC techniques, by assuming an uniform ergodic behavior of the MCMC kernels. In Section 3, the rate of convergence is derived ; it is shown how this rate can be improved, with a very small computational overhead, by using an averaging approach. Section 4 is devoted to an application. The proofs are postponed in Sections 5 to 7.

## 1. The Monte Carlo Expectation Maximization algorithm

In that contribution, we use the terminology of the missing data problem. Let $\Theta \subseteq \mathbb{R}^{l}, \mathcal{X} \subseteq \mathbb{R}^{d}$ endowed with the Borel $\sigma$-field, $\mu$ be a $\sigma$-finite Borel measure on $\mathcal{X}$, and $\{h(z ; \theta), \theta \in \Theta\}$ be a family of positive $\mu$-integrable functions. Any iteration of EM may be formally decomposed into two steps. At iteration $n+1$, the E-step consists in evaluating

$$
\mathcal{Q}\left(\theta, \theta_{n}\right):=\int_{\mathcal{X}} \log h(z ; \theta) \quad p(z ; \theta) \mu(d z)
$$

where

$$
p(z ; \theta):=h(z ; \theta) / g(\theta),
$$

so that

$$
\pi_{\theta}(d z):=p(z ; \theta) \mu(d z)
$$

is a probability distribution which may be interpreted as the posterior distribution of the missing data. In the M-step, the new value of the parameter $\theta_{n+1}$ is set as the maximum over $\Theta$ of $\theta \mapsto \mathcal{Q}\left(\theta, \theta_{n}\right), \theta_{n+1}:=\operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}\left(\phi, \theta_{n}\right)$. It is assumed for simplicity that this maximum exists and is unique (see Wu (1983) for details). The key property of EM is that increasing $\mathcal{Q}\left(\theta, \theta_{n}\right)$ forces an increase of $g$, the function to maximize. It is known that under regularity assumptions, EM instances $\left\{\theta_{n}\right\}$ converge to the set of the stationary points of $g$ ( $\mathrm{Wu}(1983)$ ). In some situations, the E-step is intractable and to deal with these cases, Wei and Tanner (1991) propose to replace the expectation by a Monte Carlo integration, leading to the so-called Monte Carlo EM. The MCMC approach consists in sampling a $\mathcal{X}$-valued Markov chain $\left\{Z_{j}^{n}\right\}_{j}$ from a Markov kernel $P_{\theta_{n}}$, with stationary distribution $\pi_{\theta_{n}}$ and initial distribution $\lambda$ (assumed to be constant over iterations). In the E-step we compute $\mathcal{Q}_{n}\left(\theta, \theta_{n}\right)$

$$
\mathcal{Q}_{n}\left(\theta, \theta_{n}\right):=m_{n}^{-1} \sum_{j=1}^{m_{n}} \log h\left(Z_{j}^{n} ; \theta\right), \quad m_{n} \in \mathbb{Z}_{+}
$$

whereas the M-step remains unchanged. A difficulty when dealing with random sequence $\left\{\theta_{n}\right\}$ is to guarantee the stability (a.s. boundedness). To avoid unnecessary technical conditions, we present a simple modification of the iterative scheme, adapting the algorithm presented by Chen et al. (1988).

The stable MCEM algorithm. A new sequence $\left\{\theta_{n}^{\prime}\right\}$ is obtained by truncating the original recursion: whenever $\operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}_{n}\left(\phi, \theta_{n}^{\prime}\right)$ is outside a specific set, it is re-initialized at a point $\theta_{0}^{\prime}$. In the technique proposed by Chen et al. (1988), the truncation bounds are random functions of the recursion index $n$. The advantage of this approach (compared to projection) is that the truncation does not modify the set of stationary points of the original recursion. More formally, let $\left\{\mathcal{K}_{n}\right\}$ be a sequence of compact subsets such that for any $n \geq 0$,

$$
\begin{equation*}
\mathcal{K}_{n} \subsetneq \mathcal{K}_{n+1}, \quad \Theta=\bigcup_{n \geq 0} \mathcal{K}_{n} \tag{2}
\end{equation*}
$$

Set $p_{0}:=0$ and choose $\theta_{0}^{\prime} \in \mathcal{K}_{0}$. The stable MCEM algorithm is defined as follows

$$
\left\{\begin{array}{ll}
\text { If } \operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}_{n}\left(\phi, \theta_{n}^{\prime}\right) \in \mathcal{K}_{p_{n}}, & \theta_{n+1}^{\prime}:=\operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}_{n}\left(\phi, \theta_{n}^{\prime}\right) \text { and }  \tag{3}\\
p_{n+1}:=p_{n} \\
\text { if } \operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}_{n}\left(\phi, \theta_{n}^{\prime}\right) \notin \mathcal{K}_{p_{n}}, & \theta_{n+1}^{\prime}:=\theta_{0}^{\prime} \text { and }
\end{array} p_{n+1}:=p_{n}+1 .\right.
$$

Note that $p_{n}$ counts the number of re-initializations. It is shown in the sequel that, under appropriate assumptions, $\left\{p_{n}\right\}$ is a.s. finite, meaning that along any trajectory of the algorithm, the number of re-initialization is finite.

## 2. Convergence of the MCEM algorithm for curved exponential family

2.1. Model assumptions. We further restrict our attention to the case where the complete data likelihood $h$ is from the class of the curved exponential densities. We consider the following assumptions which are satisfied in many scenarios.

M1 $\quad \Theta \subseteq \mathbb{R}^{l}, \mathcal{X} \subseteq \mathbb{R}^{d}$, and $\mu$ is a $\sigma$-finite positive Borel measure on $\mathcal{X}$.

Denote by $\langle\cdot ; \cdot\rangle$ the scalar product, by $|\cdot|$ the Euclidean norm and by $\nabla$ the differentiation operator. Let $\phi: \Theta \rightarrow \mathbb{R}, \psi: \Theta \rightarrow \mathbb{R}^{q}$ and $S: \mathcal{X} \rightarrow \mathcal{S} \subseteq \mathbb{R}^{q}$. Define $L: \mathcal{S} \times \Theta \rightarrow \mathbb{R}$ and $h: \mathcal{X} \times \Theta \rightarrow \mathbb{R}^{+} \backslash\{0\}$

$$
L(s ; \theta):=\phi(\theta)+\langle s ; \psi(\theta)\rangle \quad h(z ; \theta):=\exp (L(S(z) ; \theta))
$$

Assume that

M2 (a) $\phi, \psi$ are continuous on $\Theta$ and $S$ is continuous on $\mathcal{X}$.
(b) for all $\theta \in \Theta, \bar{S}(\theta):=\pi_{\theta}(S)$ is finite and continuous on $\Theta$.
(c) there exists a continuous function $\hat{\theta}: \mathcal{S} \rightarrow \Theta$, such that for all $s \in \mathcal{S}, L(s ; \hat{\theta}(s))=$ $\sup _{\theta \in \Theta} L(s ; \theta)$.
(d) $g$ is positive, finite and continuous on $\Theta$, and for any $M>0$, the level set $\{\theta \in$ $\Theta, g(\theta) \geq M\}$ is compact.

Let $\mathcal{L}$ be the set of stationary points of the EM algorithm. With the notations above, $\mathcal{L}$ is given by

$$
\begin{equation*}
\mathcal{L}:=\{\theta \in \Theta, \hat{\theta} \circ \bar{S}(\theta)=\theta\} \tag{4}
\end{equation*}
$$

As shown by $\mathrm{Wu}((\mathrm{Wu}, 1983$, Theorem 2)), under M1-2, if $\Theta$ is open and $\phi$ and $\psi$ are differentiable on $\Theta$, then $g$ is differentiable on $\Theta$ and $\mathcal{L}=\{\theta \in \Theta, \nabla g(\theta)=0\}$. Hence, the set of fixed points of EM coincides with the set of stationary points of $g$. Assume either that

M3 (a) the set $g(\mathcal{L})$ is compact
or
(a') for all compact set $\mathcal{K} \subseteq \Theta, g(\mathcal{L} \cap \mathcal{K})$ is finite.

Note that under M2(d), $g(\mathcal{L})$ is compact iff $\mathcal{L}$ is compact.

Example: Poisson count with random effect. For the purpose of illustration, we consider the estimation of a location parameter in a model of Poisson counts. This model is adapted from Zeger (1988), (see also Chan and Ledolter (1995)). Conditional to the latent variables $Z_{0}, Z_{1}, \ldots, Z_{d}$, the counts $Y_{1}, \ldots, Y_{d}$ are independent and Poisson variables with intensity $\exp (\theta+$ $Z_{k}$ ), where $\theta$ is the unknown translation parameter to estimate in the maximum likelihood sense. $\left\{Z_{k}\right\}$ is a stationary autoregressive process of order $1, Z_{k}=a Z_{k-1}+\sigma \epsilon_{k}$, where $\left\{\epsilon_{k}\right\}$ is an i.i.d standard gaussian noise and the coefficients $|a|<1, \sigma>0$ are known. Set $\mathbf{z}:=\left(z_{0}, \ldots, z_{d}\right)$ a $\mathbb{R}^{d+1}$-valued vector. The complete likelihood may be written as

$$
\begin{equation*}
h(\mathbf{z} ; \theta)=\exp \left(\theta \sum_{k=1}^{d} Y_{k}-\exp (\theta) \sum_{k=1}^{d} \exp \left(z_{k}\right)\right) \tag{5}
\end{equation*}
$$

the dominating measure $\mu$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathcal{X}:=\mathbb{R}^{d+1}$, and the density is given up to irrelevant normalization factor by

$$
\begin{equation*}
\exp \left(\sum_{k=1}^{d} Y_{k} z_{k}-\left(2 \sigma^{2}\right)^{-1}\left(\sum_{k=1}^{d}\left(z_{k}-a z_{k-1}\right)^{2}+(1-a)^{2} z_{0}^{2}\right)\right) \tag{6}
\end{equation*}
$$

Here $\Theta:=\mathbb{R}, \phi(\theta):=\theta \sum_{k=1}^{d} Y_{k}, \psi(\theta):=-e^{\theta}$, and $S(\mathbf{z}):=\sum_{k=1}^{d} e^{z_{k}} \in \mathcal{S}:=\mathbb{R}^{+} \backslash\{0\}$. Assumption M2(a) is trivially verified. Observe that for $y>0, z \in \mathbb{R}, \theta \in \mathbb{R}$, we have $y \theta-e^{\theta+z} \leq$ $-y z+y(\ln (y)-1)$, so that

$$
\begin{equation*}
h(\mathbf{z} ; \theta) \leq \exp \left(\sum_{k=1}^{d} Y_{k}\left(\log \left(Y_{k}\right)-1\right)-\sum_{k=1}^{d} Y_{k} z_{k}\right), \quad \forall \mathbf{z} \in \mathbb{R}^{d+1}, \theta \in \mathbb{R} \tag{7}
\end{equation*}
$$

We easily deduce from (7) that $\sup _{\theta \in \mathbb{R}} g(\theta)<\infty$. (7) also implies that $g$ is uniformly bounded on $\Theta$ and is continuous. Since $\lim _{\theta \rightarrow-\infty} g(\theta)=\lim _{\theta \rightarrow+\infty} g(\theta)=0$, then the level sets are compact, and M2(d) is verified. As $g$ is continuous, M2(b) is trivially checked using similar arguments. M2(c) is verified with

$$
\hat{\theta}(s):=\log \left(\sum_{k=1}^{d} Y_{k}\right)-\log (s)
$$

Finally, $\theta \mapsto g(\theta)$ and its derivatives are analytic on $\Theta$ and analytic functions have only a finite number of zeros in any compact set. As $\mathcal{L}=\{\theta \in \Theta, \nabla g(\theta)=0\}$, then for all compact $\mathcal{K} \subset \Theta$, $\mathcal{L} \cap \mathcal{K}$ is finite and M3(a') is verified.
2.2. Monte Carlo approximation. Let $\left\{\mathcal{K}_{n}\right\}$ be a sequence of compact sets satisfying (2). Given $\theta_{0}^{\prime} \in \mathcal{K}_{0}$ and a probability measure $\lambda$ on $\mathcal{X}$, the stable MCEM sequence $\left\{\theta_{n}^{\prime}\right\}$ is then defined as (see (3))

$$
\left\{\begin{array}{lll}
\text { If } \hat{\theta}\left(\tilde{S}_{n}\right) \in \mathcal{K}_{p_{n}}, & \theta_{n+1}^{\prime}:=\hat{\theta}\left(\tilde{S}_{n}\right) \text { and } & p_{n+1}:=p_{n}  \tag{8}\\
\text { if } \hat{\theta}\left(\tilde{S}_{n}\right) \notin \mathcal{K}_{p_{n}}, & \theta_{n+1}^{\prime}:=\theta_{0}^{\prime} \text { and } & p_{n+1}:=p_{n}+1,
\end{array}\right.
$$

where

$$
\tilde{S}_{n}:=m_{n}^{-1} \sum_{j=1}^{m_{n}} S\left(Z_{j}^{n}\right)
$$

and $\left\{Z_{j}^{n}\right\}$ is sampled from a Markov kernel $P_{\theta_{n}^{\prime}}$ with invariant distribution $\pi_{\theta_{n}^{\prime}}$, and $Z_{0}^{n} \sim \lambda$. To go further, we need to control the $L^{p}$-norm of the fluctuations of the Monte Carlo approximation of $\bar{S}\left(\theta_{n}^{\prime}\right)$ by $\tilde{S}_{n}$.

M4 There exist $p \geq 2$ and $\lambda$, a probability measure on $\mathcal{X}$, such that for any compact set $\mathcal{K} \subseteq \Theta$,

$$
\begin{gathered}
\operatorname{supsup}_{\theta \in \mathcal{K} n \geq 1} n^{-p / 2} \mathbb{E}_{\lambda, \theta}\left[\left|\sum_{k=1}^{n}\left\{S\left(\Phi_{k}\right)-\pi_{\theta}(S)\right\}\right|^{p}\right]<\infty \\
\sup _{\theta \in \mathcal{K}} \sup _{n \geq 1} \sum_{k \geq 1}\left|\lambda P_{\theta}^{k}(S)-\pi_{\theta}(S)\right|<\infty
\end{gathered}
$$

where $\mathbb{E}_{\lambda, \theta}$ is the expectation of the canonical Markov chain $\left\{\Phi_{n}\right\}$ with transition kernel $P_{\theta}$ and initial distribution $\lambda$.

We now state practical conditions upon which M4 is verified. The simplest case is when the kernel $P_{\theta}$ is uniformly ergodic. (See Meyn and Tweedie (1993) for relevant definitions on Markov chains). Let $P$ be a Markov kernel on $\mathcal{X}$.

Proposition 1. Let $P$ be a $\psi$-irreducible aperiodic Markov transition kernel on $\mathcal{X}$. Assume that the whole state space is $\nu_{m}$-small with minorizing constant $\epsilon>0$. Then, $P$ possesses an unique invariant probability measure $\pi$. In addition, for any $p \geq 2$ and any bounded Borel function $g: \mathcal{X} \rightarrow \mathbb{R}^{q}$,

$$
\sum_{k=1}^{\infty}\left|P^{k} g(x)-\pi(g)\right| \leq 2\left(\sup _{\mathcal{X}}|g|\right)\left(1-(1-\epsilon)^{1 / m}\right)^{-1}, \quad \forall x \in \mathcal{X}
$$

and for all $n \geq 1, x \in \mathcal{X}$

$$
\begin{equation*}
\mathbb{E}_{x}\left|\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}\right|^{p} \leq 6^{p} C_{p}\left(\sup _{\mathcal{X}}|g|^{p}\right) \quad\left(1+2\left\{1-(1-\epsilon)^{1 / m}\right\}^{-1}\right)^{p+1} n^{p / 2} \tag{9}
\end{equation*}
$$

where $C_{p}$ is the Rosenthal's constant (see Hall and Heyde (1980), Theorem 2.12).

The proof is in Section 6.

Using this result, assumption M4 is verified provided that $\sup _{\mathcal{X}}|S|<\infty, P_{\theta}$ is for all $\theta \in \Theta$ uniformly ergodic, i.e. $\mathcal{X}$ is $\nu_{m_{\theta}}$-small with minorizing constant $\epsilon_{\theta}$, and for all $\theta$ in a compact subset of $\Theta$, (a) $\epsilon_{\theta}$ is bounded away from zero and (b) $m_{\theta}$ is bounded. This condition is often verified when $\mathcal{X}$ is compact and the kernel depends continuously on $\theta$ (see Section 4 for an illustration). To deal with non-compact state space, the following proposition proved in Section 6 provides convenient sufficient conditions based on the Foster-Lyapunov drift criterion (10).

Proposition 2. Let $P$ be a $\psi$-irreducible aperiodic transition kernel on $\mathcal{X}$ and $C$ be an accessible petite set. Assume that there exist some constants $0<\rho<1, b<\infty$ and a Borel norm-like function $V: \mathcal{X} \rightarrow[1, \infty)$, bounded on $C$ such that

$$
\begin{equation*}
P V \leq \rho V+b \mathbb{1}_{C} \tag{10}
\end{equation*}
$$

Let $p \geq 2$. Choose $M>\sup _{C} V \vee b /\left(1-\rho^{1 / p}\right)^{p}$. Then the set $\{V \leq M\}$ is $\nu_{m}-$ small with minorizing constant $\epsilon>0$ and for any Borel function $g: \mathcal{X} \rightarrow \mathbb{R}^{q},|g| \leq V^{1 / p}$, it holds that for
all $x \in \mathcal{X}, n \geq 1$,

$$
\sum_{k=1}^{\infty}\left|P^{k} g(x)-\pi(g)\right| \leq C \epsilon^{-1}(m+1) M^{1 / p} A^{-1} V^{1 / p}(x)
$$

and

$$
\mathbb{E}_{x}\left|\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}\right|^{p} \leq C \epsilon^{-(p+1)}(m+1)^{p+1} M^{2} A^{-2 p} V(x) \quad n^{p / 2}
$$

where $A:=\left((1-\rho)^{1 / p}-(b / M)^{1 / p}\right)$ and $C$ is a constant which depends only upon $p$.

Hence, if the kernel $P$ depends on a parameter $\theta$, all the quantities appearing in Proposition 2 may depend on $\theta$ and the condition M 4 is verified if, for any compact subset $\mathcal{K} \subset \Theta$, (a) $\sup _{\theta \in \mathcal{K}} \rho_{\theta}<1, \sup _{\theta \in \mathcal{K}} b_{\theta}<\infty, \sup _{\theta \in \mathcal{K}} M_{\theta}<\infty$ and $\sup _{\theta \in \mathcal{K}} m_{\theta}<\infty,(b) \inf _{\theta \in \mathcal{K}} \epsilon_{\theta}>0$ and (c) there exists a measure of probability $\lambda$ on $\mathcal{X}$ such that $\sup _{\theta \in \mathcal{K}} \lambda\left(V_{\theta}\right)<\infty$.

Finally, we need to assume that the number of simulations at each iteration increases at a given rate $\left\{m_{n}\right\}$. The rate of increase depends upon the control of the fluctuation of the Monte Carlo sum. More precisely,

M5 $\quad\left\{m_{n}\right\}$ is a sequence of integers such that $\sum_{n} m_{n}^{-p / 2}<\infty$ where $p$ is given by M4.

Example: Poisson count with random effect (continued). To impute the missing values, we use the hybrid sampler random scan symmetric random walk Metropolis Hastings (henceforth denoted RSM). At each iteration a single component of the missing data vector $z$ drawn at random is updated, using a one-dimensional random walk Metropolis Hastings algorithm, with a proposal distribution having a positive, continuous and symmetric density $q$ w.r.t. the Lebesgue measure on $\mathbb{R}$. This sampler has been studied in Fort et al. (2001). The key findings are summarized here

- for any $\theta \in \Theta$, the RSM kernel $P_{\theta}$ is Lebesgue-irreducible, aperiodic. In addition, for any compact sets $C \subset \mathbb{R}^{d+1}$ and $\mathcal{K} \subset \Theta$, there exist a constant $\epsilon>0$ and a probability measure $\nu$ on $\mathbb{R}^{d+1}$ such that $P_{\theta}^{d+1}(\mathbf{z}, \cdot) \geq \epsilon \nu(\cdot)$ for all $\theta \in \mathcal{K}, \mathbf{z} \in C$.
- Choose $0<s<1$ such that $s(1-s)^{1 / s-1}<(2 d-2)^{-1}$ and set $V_{\theta}(\mathbf{z}):=\pi_{\theta}(\mathbf{z})^{-s}$. Then, for any compact $\mathcal{K} \subset \Theta$,

$$
\limsup _{|z| \rightarrow+\infty} \sup _{\theta \in \mathcal{K}} \frac{P_{\theta} V_{\theta}(\mathbf{z})}{V_{\theta}(\mathbf{z})}<1
$$

Consequently, by applying Proposition 2, it is proved that assumption M4 holds with any real $p \geq 2$ and any probability measure $\lambda$ such that for any compact set $\mathcal{K} \subset \Theta, \sup _{\theta \in \mathcal{K}} \lambda\left(V_{\theta}\right)<\infty$.
2.3. Almost-sure convergence. We now state the main results of our contribution. Under assumptions M1-2, any iteration of the EM algorithm can be written as $\theta_{n+1}=\hat{\theta} \circ \bar{S}\left(\theta_{n}\right)=: T\left(\theta_{n}\right)$, where $T: \Theta \rightarrow \Theta$ is continuous. (Wu, 1983, Theorem 1) proved that (a) $\left\{g\left(\theta_{n}\right)\right\}$ converges to $g\left(\theta^{*}\right)$ for some $\theta^{*}$ in the set $\mathcal{L}$ of the fixed points of $T$, and (b) the limit points of $\left\{\theta_{n}\right\}$ are in $\mathcal{L}$. Under assumptions M1-4, we obtain a similar result for the stable MCEM algorithm. The convergence results hold almost-surely w.r.t. $\overline{\mathbb{P}}$, the probability on the canonical space associated to the trajectories of stable MCEM, started at $\theta_{0}^{\prime}$, given $\lambda$, the initial distribution of the Markov chains, and $\left\{\mathcal{K}_{n}\right\}$, the sequence of compact sets (see Section 5.2 for a precise definition of $\overline{\mathbb{P}}$ ). Denote by $\operatorname{Cl}(A)$ the closure of the set $A$.

Theorem 3. Assume M1-5. Let $\left\{\mathcal{K}_{n}\right\}$ be a sequence of compact sets satisfying (2), $\theta_{0}^{\prime} \in \mathcal{K}_{0}$ and $\lambda$ be given in M4. Consider the stable MCEM random sequence $\left\{\theta_{n}^{\prime}\right\}$ defined by (8). Then,
(i) (a) $\lim _{n} p_{n}<\infty$ w.p. 1 and $\lim \sup _{n}\left|\theta_{n}^{\prime}\right|<\infty$ w.p. 1
(b) $\left\{g\left(\theta_{n}^{\prime}\right)\right\}$ converges w.p. 1 to a connected component of $g(\mathcal{L})$, where $\mathcal{L}$ is given by (4).
(ii) If in addition $g\left(\mathcal{L} \cap \mathrm{Cl}\left(\left\{\theta_{n}^{\prime}\right\}\right)\right)$ has an empty interior, then $\left\{g\left(\theta_{n}^{\prime}\right)\right\}$ converges w.p. 1 to $g^{*}$ and $\left\{\theta_{n}^{\prime}\right\}$ converges to the set $\mathcal{L}_{g^{*}}:=\left\{\theta \in \mathcal{L}, g(\theta)=g^{*}\right\}$.

The proof is given in Section 5.
Remark 4. Using the Sard's Theorem (Bröcker (1975)), it is known that $g(\{\nabla g=0\})$ has an empty interior as soon as the function $g$ is $l$-times differentiable (where $l$ is the dimension of the parameter space). Hence, Theorem $3(i i)$ applies under very weak regularity assumptions.

In many instances, the set $\mathcal{L}$ is made of isolated points and, under suitable conditions, the previous convergence results imply pointwise convergence to some stationary point of $g$. Depending upon the values of the Hessian of $g$, these limiting points are either local maxima, local minima or saddle points. A question of interest is to state conditions upon which the stationary points only coincide with local maxima. To that goal, we formulate some additional regularity assumptions

M6 (a) $\Theta$ is open, (b) for any $s \in \mathcal{S}, \theta \mapsto L(s ; \theta)$ is twice continuously differentiable on $\Theta$, (c) $\theta \mapsto \bar{S}(\theta)$ is twice continuously differentiable on $\Theta$, (d) $\theta \mapsto g(\theta)$ is
continuously differentiable on $\Theta$, (e) $\mathcal{S}$ is open and the convex hull of $S\left(\mathbb{R}^{d}\right)$ is included in $\mathcal{S}$, and (f) $s \mapsto \hat{\theta}(s)$ is twice continuously differentiable on $\mathcal{S}$.

M7
The stationary points of $g$ are isolated. For every stationary point $\theta^{*}$ of $g$, the matrices $-\nabla_{\theta}^{2} L\left(\bar{S}\left(\theta^{*}\right) ; \theta^{*}\right)$ and

$$
\int_{\mathcal{X}} \nabla_{\theta} L\left(S(z) ; \theta^{*}\right)^{t} \nabla_{\theta} L\left(S(z) ; \theta^{*}\right) p\left(z ; \theta^{*}\right) \mu(d z)
$$

are positive definite.

It is shown in Delyon et al. (1999) that under M6-7, the matrix

$$
\nabla T\left(\theta^{*}\right)=\left[\nabla_{\theta}^{2} L\left(s^{*} ; \theta^{*}\right)\right]^{-1}\left(\nabla_{\theta}^{2} L\left(s^{*} ; \theta^{*}\right)-\nabla^{2} \log g\left(\theta^{*}\right)\right), \quad s^{*}:=\bar{S}\left(\theta^{*}\right)
$$

is diagonalizable with positive real eigenvalues. If $\theta^{*}$ is a stable fixed point of $T$, then the modulus off all the eigenvalues of $\nabla T\left(\theta^{*}\right)$ are strictly less than one, and $\theta^{*}$ is a proper maximizer of $g$. If $\theta^{*}$ is hyperbolic (resp. unstable) then it is a saddle-point of $g$ (resp. a local minimum of $g$ ). Recall finally that if the stationary points of $g$ are isolated (that is under M7), convergence to hyperbolic and unstable points, that is convergence to saddle points and local minima of $g$ never occurs w.p. 1 for the MCEM sequence, as shown in Brandière (1998).

Example: Poisson count with random effects (continued) M6 is readily verified. Note that

$$
\nabla \log g(\theta)=\sum_{k=1}^{d} Y_{k}-e^{\theta} \int_{\mathbb{R}^{d+1}} S(\mathbf{z}) p(\mathbf{z} ; \theta) \mu(d \mathbf{z})
$$

and a stationary point $\theta^{*}$ solves the equation:

$$
\sum_{k=1}^{d} Y_{k}=e^{\theta^{*}} \int S(\mathbf{z}) p\left(\mathbf{z} ; \theta^{*}\right) \mu(d \mathbf{z}) \quad \text { i.e. } \quad \sum_{k=1}^{d} Y_{k}=e^{\theta^{*}} \bar{S}\left(\theta^{*}\right)
$$

Since $g$ is analytic (see section 2.1) any compact subset of $\Theta$ contains only a finite number of stationary points of $g$. For a stationary point $\theta^{*}$, note that $-\nabla_{\theta}^{2} L\left(\bar{S}\left(\theta^{*}\right) ; \theta^{*}\right)=e^{\theta^{*}} \bar{S}\left(\theta^{*}\right)$ and

$$
\int \nabla_{\theta} L\left(S(\mathbf{z}) ; \theta^{*}\right)^{t} \nabla_{\theta} L\left(S(\mathbf{z}) ; \theta^{*}\right) p\left(\mathbf{z} ; \theta^{*}\right) \mu(d \mathbf{z})=e^{\theta^{*}} \int\left(S(\mathbf{z})-\bar{S}\left(\theta^{*}\right)\right)^{2} p\left(\mathbf{z} ; \theta^{*}\right) \mu(d \mathbf{z})
$$

so that M7 holds.

## 3. Rate of convergence and averaging

We now study the rate of convergence of $\left\{\theta_{n}^{\prime}\right\}$ (given $\left\{\mathcal{K}_{n}\right\}, \theta_{0}^{\prime} \in \mathcal{K}_{0}$ and $\lambda$ ) to a local maximum $\theta^{*}$ of $g$. Rate of convergence is useful to understand how we should ideally tune the number of simulations $m_{n}$ as a function of the iteration index. It also allows to derive an accelerated version of the algorithm, based on averaging.

Define $G(s):=\bar{S} \circ \hat{\theta}(s)$. The mapping $G$ gives another way to consider an iteration of the EM algorithm, not directly in the parameter space $\Theta$, but in the space of the complete data sufficient statistics $\mathcal{S}$. If $\theta^{*}$ is a fixed point of $T$, i.e. $\theta^{*}=T\left(\theta^{*}\right)=\hat{\theta} \circ \bar{S}\left(\theta^{*}\right)$, then $s^{*}:=\bar{S}\left(\theta^{*}\right)$ is a fixed point of $G$, i.e. $s^{*}=G\left(s^{*}\right)=\bar{S} \circ \hat{\theta}\left(s^{*}\right)$. In addition, $\nabla T\left(\theta^{*}\right)=\nabla \hat{\theta}\left(s^{*}\right) \nabla \bar{S}\left(\theta^{*}\right)$ and $\nabla G\left(s^{*}\right)=\nabla \bar{S}\left(\theta^{*}\right) \nabla \hat{\theta}\left(s^{*}\right)$. Hence $\nabla G\left(s^{*}\right)$ has the same eigenvalues as $\nabla T\left(\theta^{*}\right)$, counting multiplicities together with $(q-l)$ additional eigenvalues equal to zero. The stability properties can thus be directly translated in terms of stability of $s^{*}$; when $\theta^{*}$ is stable, then $s^{*}$ is stable and vice-versa.
3.1. Rate of convergence. We begin by discussing informally the results. Let $\theta^{*}$ be a fixed point of $T$ and let $s^{*}:=\bar{S}\left(\theta^{*}\right)$. There are a priori multiple possible limiting points, so we need to restrict our attention to the set of trajectories that converge to a given limiting point $s^{*}$. For large enough $n$, we may decompose the recursion as follows,

$$
\tilde{S}_{n}-s^{*}=\left(G\left(\tilde{S}_{n-1}\right)-G\left(s^{*}\right)\right)+\tilde{S}_{n}-G\left(\tilde{S}_{n-1}\right)=\Gamma\left(\tilde{S}_{n-1}-s^{*}\right)+\epsilon_{n}+\eta_{n}
$$

where $\Gamma:=\nabla G\left(s^{*}\right)$ and $\left\{\epsilon_{n}\right\}$ is a martingale difference sequence w.r.t. the filtration $\mathcal{F}_{n}:=$ $\sigma\left(\tilde{S}_{0}, \ldots, \tilde{S}_{n}\right)$,

$$
\epsilon_{n}:=\left(\tilde{S}_{n}-\overline{\mathbb{E}}\left[\tilde{S}_{n} \mid \mathcal{F}_{n-1}\right]\right) \mathbb{I}_{\left\{\left|\tilde{S}_{n-1}-s^{*}\right| \leq \delta\right\}}, \quad n \geq 1, \quad \delta>0, \quad \epsilon_{0}:=0
$$

The remainder term $\eta_{n}$ can be expressed as $\eta_{n}:=\eta_{n}^{(1)}+\eta_{n}^{(2)}$, where for $n \geq 1$,

$$
\begin{align*}
& \eta_{n}^{(1)}:=\left(\tilde{S}_{n}-G\left(\tilde{S}_{n-1}\right)\right) \mathbb{I}_{\left\{\left|\tilde{S}_{n-1}-s^{*}\right| \geq \delta\right\}}+\left(\overline{\mathbb{E}}\left[\tilde{S}_{n} \mid \mathcal{F}_{n-1}\right]-G\left(\tilde{S}_{n-1}\right)\right) \mathbb{I}_{\left\{\left|\tilde{S}_{n-1}-s^{*}\right| \leq \delta\right\}},  \tag{11}\\
& \eta_{n}^{(2)}:=\left(G\left(\tilde{S}_{n-1}\right)-G\left(s^{*}\right)-\Gamma\left(\tilde{S}_{n-1}-s^{*}\right)\right)=\sum_{i, j} R_{n-1}(i, j)\left(\tilde{S}_{n-1, i}-s_{i}^{*}\right)\left(\tilde{S}_{n-1, j}-s_{j}^{*}\right), \tag{12}
\end{align*}
$$

and $R_{n}$ is defined componentwise as

$$
R_{n}(i, j):=\int_{0}^{1}(1-t) \frac{\partial^{2} G\left(s^{*}+t\left(\tilde{S}_{n}-s^{*}\right)\right)}{\partial s_{i} \partial s_{j}} d t
$$

It is convenient to decompose the error $\tilde{S}_{n}-s^{*}$ as a sum of a linear term $\mu_{n}$ obeying a linear difference equation driven by the martingale difference $\epsilon_{n}$,

$$
\begin{equation*}
\mu_{n}=\Gamma \mu_{n-1}+\epsilon_{n}, \quad n \geq 1, \quad \text { and } \mu_{0}:=0 \tag{13}
\end{equation*}
$$

and a remainder term $\rho_{n}$

$$
\begin{equation*}
\rho_{n}:=\tilde{S}_{n}-s^{*}-\mu_{n}, \quad n \geq 0 \tag{14}
\end{equation*}
$$

which will be shown to be negligible along the trajectories converging to $s^{*}$. We stress that, because there are possibly several convergence points, the remainder term $\rho_{n}$ as defined above will be small only along trajectories that converge to $s^{*}$.

As shown in the previous section, under the stated assumptions, $\tilde{S}_{n}$ may only converge to stable points of $G$ (hyperbolic points and unstable points are avoided w.p.1), which are associated to a local maximum of the incomplete likelihood $g$. Hence, we may assume that $s^{*}$ is stable, which implies that all the eigenvalues of $\Gamma$ have modulus less than 1 , and thus, that there exist $\gamma<1$ and a constant $C<\infty$ such that for all $k,\left|\Gamma^{k}\right| \leq C \gamma^{k}$, where $|$.$| is any matrix norm. This$ implies that the linear control model (13) above is stable and that,

$$
\mu_{n}=\sum_{k=0}^{n} \Gamma^{k} \epsilon_{n-k}
$$

In many situations, $\gamma$ is very close to one, explaining why the EM algorithm is sometimes slow to converge (see Jamshidian and Jennrich (1997)). Most often, $\gamma$ is unknown. It can however be estimated using e.g. the Louis Information principle (see Delyon et al. (1999)) but this generally involves a significant computational overhead. By construction, the driving error $\left\{\epsilon_{n}\right\}$ is a martingale increment. Observe that if one assumes that for all $n,\left|\tilde{S}_{n-1}-s^{*}\right| \leq \delta$ for some deterministic $s^{*}$ and $\delta$, then there exists a deterministic compact $\mathcal{K} \subseteq \Theta$ such that for all $n$, $\theta_{n}^{\prime} \in \mathcal{K}$. From that remark and M4, it may be asserted that the $L^{p}$-norm of the martingale $\epsilon_{n}$ is inversely proportional to $\sqrt{m_{n}}$, the square root of the number of simulations at step $n$. Hence,

$$
\mu_{n}=O_{\mathrm{L}^{p}}\left(\sum_{k=0}^{n} \gamma^{n-k} m_{k}^{-1 / 2}\right)
$$

we say that $X_{n}=O_{\mathrm{L}^{p}}\left(\alpha_{n}\right)$ where $\alpha_{n} \neq 0$ if $\alpha_{n}^{-1} X_{n}$ is bounded in L ${ }^{p}$. A more explicit expression for the rate of $\mu_{n}$ can be obtained by using the following Lemma, from (Pólya and Szegő, 1976, Result 178 p.39),

Lemma 5. Let $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}, b_{n} \neq 0$, be two sequences such that (i) the power series $f(x):=$ $\sum_{n=1}^{\infty} a_{n} x^{n}$ has a radius of convergence $r$, (ii) $\lim _{n \rightarrow \infty} b_{n} / b_{n+1}=: q$, with $|q|<r$. Define $c_{n}:=\sum_{k=0}^{n} a_{k} b_{n-k}$. Then, $\lim _{n \rightarrow \infty} c_{n} b_{n}^{-1}=f(q)$.

Hence, provided that $\lim _{n} m_{n+1} / m_{n}<\gamma^{-2}$, the linear term $\mu_{n}=O_{\mathrm{L}^{p}}\left(m_{n}^{-1 / 2}\right)$. The constraint $\lim _{n} m_{n+1} / m_{n}<\gamma^{-2}$ is always satisfied when $\left\{m_{n}\right\}$ is subexponential. When $\lim \sup \gamma^{2 n} m_{n}=$ $\infty$, the constraint is no longer satisfied and the rate is strictly lower than $m_{n}^{-1 / 2}$. Of course, this analysis makes sense only if we can prove that $\mu_{n}$ is the leading term of the error $\tilde{S}_{n}-s^{*}$, i.e. $\rho_{n}$ is negligible w.r.t. $\mu_{n}$ along the trajectories of $\tilde{S}_{n}$ that converge to $s^{*}$. More specifically, we have to show that (see Lemma 14, Section 7)

$$
\begin{equation*}
\rho_{n} \mathbb{I}_{\left\{\mathrm{lim}_{n} \tilde{S}_{n}=s^{*}\right\}}=o_{\text {w.p. } 1}\left(m_{n}^{-1 / 2}\right) ; \tag{15}
\end{equation*}
$$

we say that $X_{n}=o_{\text {w.p. } 1}\left(\alpha_{n}\right)$, resp. $X_{n}=O_{\text {w.p. } 1}\left(\alpha_{n}\right)$, where $\alpha_{n} \neq 0$ if $\lim _{n} \alpha_{n}^{-1}\left|X_{n}\right|=0$, w.p.1; resp. $\alpha_{n}^{-1}\left|X_{n}\right|$ is bounded w.p.1.
The discussion above is summarized in the following Theorem.
Theorem 6. Assume M1-7. Let $s^{*}$ be a stable fixed point of the map $G$. Let $\gamma<1$ be the modulus of the largest eigenvalue of $\nabla G\left(s^{*}\right)$. Assume that $1 \leq \lim _{n} m_{n+1} / m_{n}<\gamma^{-2}$. Then, $\mu_{n}=O_{L^{p}}\left(m_{n}^{-1 / 2}\right)$ and $\rho_{n} \mathbb{I}_{\lim _{n} \tilde{S}_{n}=s^{*}}=o_{\mathrm{w} . \mathrm{p} .1}\left(m_{n}^{-1 / 2}\right)$, where $\mu_{n}$ and $\rho_{n}$ are given by (13) and (14).

Theorem 6 shows that, under weak conditions on the sequence $\left\{m_{n}\right\}$, along any trajectory converging to a stable fixed point $s^{*}$, the error $\theta_{n}^{\prime}-\theta^{*}$ (or equivalently $\tilde{S}_{n}-s^{*}$ ), is asymptotically given by $\mu_{n}$. In addition, the $L^{p}$-norm of $\mu_{n}$ decreases as the square root of the number of simulations at step $n$.

To compare the rate of convergence of the MCEM algorithm with other stochastic versions of the EM algorithm, such as the Stochastic Approximation EM (SAEM), it is worthwhile to compute the rate as a function of the number of simulations rather than as a function of the number of iterations. For a generic sequence $\left\{X_{n}\right\}$, define the interpolated sequence $X_{n}^{(i)}=X_{\phi(n)}$ where $\phi$ is defined as the largest integer such that

$$
\sum_{k=0}^{\phi(n)} m_{k}<n \leq \sum_{k=0}^{\phi(n)+1} m_{k} .
$$

The subscript $n$ for the interpolated sequence $\theta_{n}^{\prime(i)}$ refers to the total number of simulations while for the original sequence $\left\{\theta_{n}^{\prime}\right\}$, it coincides with the number of iterations. Assume first
that the number of simulations is increasing at a polynomial rate, i.e. $m_{n}:=n^{\alpha}$ so that $\phi(n) \sim[(1+\alpha) n]^{1 /(1+\alpha)}$. On the simulation time-scale, $\mu_{n}^{(i)}=O_{\mathrm{L}^{p}}\left(n^{-\alpha /(2(1+\alpha))}\right)$ and $\rho_{n}^{(i)}=$ $o_{\text {w.p. } 1}\left(n^{-\alpha /(2(1+\alpha))}\right)$. Hence the rate of convergence is always smaller than $n^{-1 / 2}$, which is the rate of the SAEM algorithm (Delyon et al., 1999, Theorem 7). It is interesting to note that the rate is improved by choosing large values of $\alpha$, whereas small values of $\alpha$ can lead to rather inefficient estimates. In practice, this means that it is better to increase the number of simulations rapidly when the algorithm is approaching convergence, giving thus a theoretical background to well established practice. Assume now that $m_{n}:=m^{n}, m>1$. This choice is advocated in Chan and Ledolter (1995) and in several earlier works on the subject. We get similarly that $\mu_{n}^{(i)}=O_{L^{p}}\left(n^{-1 / 2}\right)$ and $\rho_{n}^{(i)}=o_{\text {w.p. } 1}\left(n^{-1 / 2}\right)$ whenever $1<m<\gamma^{-2}$ : in this case, the rate of convergence is $n^{-1 / 2}$, provided that $m$ is small enough.
3.2. The averaging procedure. This previous discussion evidences that the performance depends critically upon the choice of the schedule which is of course a serious practical drawback. Recently, a data-driven procedure has been proposed by Booth and Hobert (1999). This procedure requires to evaluate the variance of $\tilde{S}_{n}-G\left(\tilde{S}_{n-1}\right)$ which is a challenging problem when MCMC is used to sample the missing data.

We consider here an alternative procedure adapted from a technique developed by Polyak (1990) to improve the rate of convergence for stochastic approximation procedures. To motivate the construction, recall that

$$
\tilde{S}_{n}=s^{*}+\Xi_{n}, \quad \Xi_{n}:=\sum_{k=0}^{n} \Gamma^{n-k} \epsilon_{k}+\rho_{n}
$$

Each value of $\tilde{S}_{n}$ may be seen as an estimator of $s^{*}$ affected by a noise term. The stable MCEM algorithm estimates $s^{*}$ by $\tilde{S}_{n}$ which is an inefficient estimation strategy. By analogy with the regression problem, estimator of $s^{*}$ with reduced variance can be obtained by averaging and weighting the successive estimates $\tilde{S}_{n}$ of $s^{*}$. The regression noise $\Xi_{n}$ being both correlated and heteroscedastic, the best unbiased linear estimator of $s^{*}$ would require to know (or estimate) both the correlation and the variance of $\Xi_{n}$, which is a difficult task. For simplicity, we consider weighted average

$$
\begin{equation*}
\Sigma_{n}:=M_{n}^{-1} \sum_{j=0}^{n} m_{j} \tilde{S}_{j}, \quad \text { and } \quad M_{n}:=\sum_{j=0}^{n} m_{j} \tag{16}
\end{equation*}
$$

where $\tilde{S}_{n}$ is weighted by $m_{n}$, which is a rough estimate of the inverse of the variance of $\Xi_{n}$. $\Sigma_{n}$ may thus be seen as a weighted least-square estimate of $s^{*}$, the weights being (roughly) proportional to the inverse of the noise variance.
Using the decomposition above, $\Sigma_{n}-s^{*}$ may be written as $\Sigma_{n}-s^{*}=\bar{\mu}_{n}+\bar{\rho}_{n}$ where

$$
\begin{equation*}
\bar{\mu}_{n}:=M_{n}^{-1} \sum_{k=0}^{n}\left(\sum_{j=0}^{n-k} m_{j+k} \Gamma^{j}\right) \epsilon_{k}, \quad \bar{\rho}_{n}:=M_{n}^{-1} \sum_{k=0}^{n} m_{k} \rho_{k} . \tag{17}
\end{equation*}
$$

Under M4, $\overline{\mathbb{E}}\left[\left|\epsilon_{n}\right|^{p} \mid \mathcal{F}_{n}\right] \leq 2^{p} \mathrm{Cm}_{n}^{-p / 2}$ where $C$, given by M4, does not depend on the simulation schedule. Then, the martingale form of the Rosenthal's inequality implies that

$$
\left\|\bar{\mu}_{n}\right\|_{\mathrm{L}^{p}} \leq C(p)\left(\left(\sum_{k=0}^{n} m_{k}^{-1}\left(\sum_{j=0}^{n-k} m_{j+k} \gamma^{j}\right)^{2}\right)^{1 / 2}+\left(\sum_{k=0}^{n} m_{k}^{-p / 2}\left(\sum_{j=0}^{n-k} m_{j+k} \gamma^{j}\right)^{p}\right)^{1 / p}\right) M_{n}^{-1}
$$

where $C(p)$ is a constant depending only on $p$. A more explicit expression for the rate of $\bar{\mu}_{n}$ can be obtained from the following Lemma (the proof of which is postponed in Section 7).

Lemma 7. Let $0<\gamma<1$ and $\left\{m_{n}\right\}$ be a positive sequence such that $1 \leq \lim _{n} m_{n+1} / m_{n}=$ : $m<\gamma^{-2}$. Define for some positive integer $r$,

$$
\xi_{n}^{(r)}:=\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1 / r}\left(\sum_{k=0}^{n} m_{k}^{-r / 2}\left(\sum_{j=0}^{n-k} m_{j+k} \gamma^{j}\right)^{r}\right)^{1 / r}
$$

Then, $\lim _{n} \xi_{n}^{(r)}=: \mathcal{B}_{r}(m ; \gamma)$ where
$\mathcal{B}_{r}(m ; \gamma):=\left((1-m \gamma)^{-r}\left[1+\left(m^{r / 2}-1\right) \sum_{l=0}^{r-1}\binom{r}{l}(-1)^{r-l}\left(m^{l-r / 2} \gamma^{l-r}-1\right)^{-1}\right]\right)^{1 / r}, \quad$ if $m \gamma \neq 1$,
$\mathcal{B}_{r}\left(\gamma^{-1} ; \gamma\right):=\left(\left(1-\gamma^{r / 2}\right) \sum_{n}(n+1)^{r} \gamma^{n r / 2}\right)^{1 / r}$.

Hence, provided that $\lim _{n} m_{n+1} / m_{n}=: m<\gamma^{-2}$, this shows that

$$
\begin{equation*}
\lim _{n} M_{n}^{1 / 2}\left\|\bar{\mu}_{n}\right\|_{\mathrm{L}^{p}} \leq C(p) \mathcal{B}_{2}(m, \gamma)+C(p) \mathcal{B}_{p}(m, \gamma) \lim _{n}\left(\sum_{k=0}^{n} m_{k}^{p / 2}\right)^{1 / p} M_{n}^{-1 / 2} . \tag{18}
\end{equation*}
$$

If $m=1$ (this happens for example, for polynomial schedules $m_{n} \propto n^{\alpha}$ or sub-geometrical schedules $m_{n} \propto \exp \left(n^{\alpha}\right), \alpha<1$ ), then $\sum_{k=0}^{n} m_{k}^{p / 2} \sim n m_{n}^{p / 2}$ and $\lim _{n}\left(\sum_{k=0}^{n} m_{k}^{p / 2}\right)^{1 / p} M_{n}^{-1 / 2}=$ 0 . Hence,

$$
\lim _{n} M_{n}^{1 / 2}\left\|\bar{\mu}_{n}\right\|_{\mathrm{L}^{p}} \leq C(p) \mathcal{B}_{2}(1, \gamma) .
$$

If $1<m$, then Lemma 5 implies that $\lim _{n}\left(\sum_{k=0}^{n} m_{k}^{p / 2}\right)^{1 / p} M_{n}^{-1 / 2}=(m-1)^{1 / 2}\left(m^{p / 2}-1\right)^{-1 / p}$. Hence,

$$
\lim _{n} M_{n}^{1 / 2}\left\|\bar{\mu}_{n}\right\|_{\mathrm{L}^{p}} \leq C(p) \mathcal{B}_{2}(m, \gamma)+C(p) \mathcal{B}_{p}(m, \gamma)(m-1)^{1 / 2}\left(m^{p / 2}-1\right)^{-1 / p}
$$

This discussion evidences that the $L^{p}$-norm of the term $\bar{\mu}_{n}$ decreases as $M_{n}^{-1 / 2}$, the inverse of the square root of the total number of simulations up to iteration $n$. In addition, $m \mapsto$ $\mathcal{B}_{2}(m, \gamma)$ increases on $\left[1, \gamma^{-2}\right) \backslash\left\{\gamma^{-1}\right\}$ and the minimum is $\mathcal{B}_{2}(1, \gamma)=(1-\gamma)^{-1}$; when $m=$ $\gamma^{-1}, \mathcal{B}_{2}\left(\gamma^{-1}, \gamma\right)=(1+\gamma)^{1 / 2}(1-\gamma)^{-3 / 2}>\mathcal{B}_{2}(1, \gamma)$. This implies that the upper bound in (18) is minimal for $m=1$ and that the upper bound for the error term is minimum when $\lim _{n} m_{n+1} / m_{n}=1$.
$\bar{\mu}_{n}$ is the leading term in $\Sigma_{n}-s^{*}$ provided that, along any trajectories that converge to $s^{*}, \bar{\rho}_{n}$ is negligible w.r.t. $\bar{\mu}_{n}$, that is $\bar{\rho}_{n} \mathbb{I}_{\lim _{n} \tilde{S}_{n}=s^{*}}=o_{\text {w.p.1 }}\left(M_{n}^{-1 / 2}\right)$. By (37) and (38), $\bar{\rho}_{n} \mathbb{I}_{\lim }^{n} \tilde{S}_{n}=s^{*}=$ $O_{\text {w.p.1 }}(1) O_{L^{p}}\left(n M_{n}^{-1}\right)$. Hence, $\bar{\rho}_{n}$ is negligible compared to $\bar{\mu}_{n}$ whenever the simulation schedule checks the condition $n M_{n}^{-1 / 2}=o(1)$. For example, for geometrical schedules, this condition is always checked whereas for polynomial schedules $m_{n} \propto n^{\alpha}$, one has to choose $\alpha>1$.
The discussion above is summarized in the following Theorem.
Theorem 8. Assume $M 1-M 7$. Let $s^{*}$ be a stable fixed point of the map $G$ and denote $\Gamma:=$ $\nabla G\left(s^{*}\right)$. Let $\gamma<1$ be the modulus of the largest eigenvalue of $\nabla G\left(s^{*}\right)$. Let $M_{n}, \bar{\mu}_{n}$ and $\bar{\rho}_{n}$ be given by (16) and (17). Assume that (i) $1 \leq \lim _{n} m_{n+1} / m_{n}<\gamma^{-2}$, and (ii) $n M_{n}^{-1 / 2}=o(1)$. Then, $\bar{\mu}_{n}=O_{L^{p}}\left(M_{n}^{-1 / 2}\right)$ and $\bar{\rho}_{n} \Pi_{\lim _{n} \tilde{S}_{n}=s^{*}}=o_{\mathrm{w} . \mathrm{p} .1}\left(M_{n}^{-1 / 2}\right)$.

Theorem 8 shows that under weak conditions on the sequence $\left\{m_{n}\right\}$, along any trajectory converging to a stable fixed point $s^{*}$, the error $\Sigma_{n}-s^{*}$ behaves asymptotically as $\bar{\mu}_{n}$; thus, the estimator $\bar{\theta}_{n}:=\hat{\theta}\left(\Sigma_{n}\right)$ (or equivalently $\Sigma_{n}$ ) has a rate proportional to $M_{n}^{-1 / 2}$, that is a rate inversely proportional to the square root of the total number of simulations up to iteration $n$. When expressed on the simulation time-scale, the previous result shows that the $L^{p}$-norm of the leading term $\bar{\mu}_{n}^{(i)}$ is proportional to $n^{-1 / 2}$.
Hence, the averaging procedure improves the rate of convergence. In addition, the discussion above evidences that when averaging is used, it is not recommended to use geometrical schedules. It is better to choose $m_{n}$ in such a way that $\lim _{n} m_{n+1} / m_{n}=1$ and $n M_{n}^{-1 / 2}=o(1)$, which is verified e.g. if $m_{n}$ grows polynomially.

Example: Poisson count with random effects (continued). A plot of $N=100$ observations $Y_{1}, \cdots, Y_{100}$, obtained with $\theta_{\text {true }}=2, a=0.4$ and $\sigma^{2}=1$ is given in Figure 1. To implement stable MCEM, the compact sets $\left\{\mathcal{K}_{n}\right\}$ are chosen as ball of radius $(n+1)$ centered at $\theta_{0}^{\prime}$. The Monte Carlo approximations are computed by use of the hybrid sampler described in Paragraph 2.2. The proposal distribution for each component is a standard Gaussian variable on $\mathbb{R}$, (the mean acceptance rate is $\approx 40 \%$ ). The chains are initialized in a compact ball of radius $r=11$ according to a concatenation rule: if the last sample $Z_{m_{n}}^{n}$ at iteration $n$ is in this ball, then it is the starting point of the following chain i.e. $Z_{0}^{n+1}:=Z_{m_{n}}^{n}$; otherwise, we set $Z_{0}^{n+1}:=r Z_{m_{n}}^{n} /\left|Z_{m_{n}}^{n}\right|$. The simulation schedule increases polynomially $m_{n}:=1000+n^{2}$. In Figure 2, we plot three paths of stable MCEM started respectively at $\theta_{0}^{\prime}=\log \left(N^{-1} \sum Y_{k}\right) \approx 2.41, \theta_{0}^{\prime}=-2$ and $\theta_{0}^{\prime}=4$. After respectively 0,3 and 2 re-initializations, convergence to the point $\theta^{*} \approx 1.88$ may be observed. In Figure 3, we plot a stable MCEM path started from $\theta_{0}^{\prime}=\log \left(N^{-1} \sum Y_{k}\right)$ and its averaged counterpart (i.e. the sequence $\bar{\theta}_{n}$ given by $\bar{\theta}_{n}:=\hat{\theta}\left(\Sigma_{n}\right)$ ). It may be observed that the variation of the averaged path decreases more rapidly than the variation of the stable MCEM path, which illustrates the discussion of Paragraph 3.2.

## 4. An application to product diffusion modeling

We illustrate the previous results by considering the Bass product diffusion model which consists in predicting market penetration of new products and services. Sherman et al. (1999) proved the convergence in the case where the missing data are obtained (at each step) from $m$ independent runs of a Gibbs sampler. These authors assume uniform geometric ergodicity in the total variation distance, and uniform convergence in $L^{2}$ (Assumptions (C5-6)) which seem difficult to directly verify in practice.

The observations $y:=\left\{\left(t_{1}, n_{1}\right), \cdots,\left(t_{d}, n_{d}\right)\right\}$ are the cumulative numbers $n_{j}$ of adopters at a set of increasing instants $t_{j}$. We set $t_{0}=n_{0}:=0$. It is assumed that the $n_{j}$ 's are realizations of a process $N(t)$ at time $t_{j}$, and the $t_{j}$ 's are selected independently of the adoption process. $N(t)$ is a pure birth Markov process with stationary transition probabilities and population adoption rate

$$
\Lambda(t):=(M \pi-N(t))(\varrho+\varsigma N(t))
$$

where $M$ is the population size ( $M$ is known and constant over time), $\pi$ is the proportion of potential adopters, $\varrho \geq 0$ is the innovator coefficient and $\varsigma \geq 0$ is the imitator coefficient. For all
$0 \leq i \leq n_{d}-1, \Lambda\left(t_{i}\right)$ has to be positive. In addition, in order the expected number of adopters not to exceed the number of eventual adopters, we require $\varrho+\varsigma n_{d} \leq 1$. Hence $(\varrho, \varsigma, \pi) \in \Upsilon$ where

$$
\Upsilon:=\left\{(\varrho, \varsigma, \pi) \in(0,1] \times[0,1] \times\left[n_{d} / M, 1\right], \quad 0<\varrho+\varsigma n_{d} \leq 1\right\} .
$$

Our purpose is to compute the maximum likelihood estimator for $\vartheta:=(\varrho, \varsigma, \pi)$, or equivalently the maximum likelihood estimator for $\theta=(\alpha, \beta, \gamma):=\zeta(\vartheta)$ defined as

$$
\zeta(\varrho, \varsigma, \pi):=\left[\begin{array}{c}
-\varsigma \\
\varsigma M \pi-\varrho \\
\varrho M \pi
\end{array}\right] \quad \zeta^{-1}(\alpha, \beta, \gamma):=\left[\begin{array}{c}
1 / 2\left(-\beta+\sqrt{\beta^{2}-4 \alpha \gamma}\right) \\
-\alpha \\
2 \gamma M^{-1}\left(-\beta+\sqrt{\beta^{2}-4 \alpha \gamma}\right)^{-1}
\end{array}\right]
$$

so that $\zeta: \Upsilon \rightarrow \Theta:=\zeta(\Upsilon)$ is continuous. Hence, we want to maximize on $\Theta$ the incomplete data likelihood $g$ given by

$$
g(\theta):=\prod_{j=1}^{d}\left(\prod_{k=n_{j-1}}^{n_{j}-1} \lambda_{k}(\theta)\right) \sum_{i=n_{j-1}}^{n_{j}}\left(\exp \left(-\lambda_{i}(\theta)\left(t_{j}-t_{j-1}\right)\right) \prod_{k=n_{j-1}, k \neq i}^{n_{j}}\left\{\lambda_{k}(\theta)-\lambda_{i}(\theta)\right\}^{-1}\right)
$$

where $\lambda_{i}(\theta):=\alpha i^{2}+\beta i+\gamma$. Computation and maximization of $g$ are not tractable (see Dalal and Weerahandi (1995)). We thus implement the stable MCEM algorithm and solve a missing data problem where missing data are individual adoption times. We write $g(\theta):=\int_{\mathcal{X}} h(\mathbf{z} ; \theta) \mu(d \mathbf{z})$ where (see (Sherman et al., 1999, Eq.(11)))

$$
\mathrm{z}:=\left(z_{1}, \cdots, z_{n_{d}}\right), \quad z_{0}:=0 \quad \mathcal{X}:=\left[0, t_{d}\right]^{n_{d}}
$$

$$
h(\mathbf{z} ; \theta):=\prod_{i=0}^{n_{d}-1} \lambda_{i}(\theta) \exp \left(-\lambda_{i}(\theta)\left(z_{i+1}-z_{i}\right)\right) \quad \exp \left(-\lambda_{n_{d}}(\theta)\left(t_{d}-z_{n_{d}}\right)\right)
$$

and $\mu$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^{n_{d}}$

$$
\mu(d \mathbf{z}):=\mathbb{1}_{0<z_{1}<\cdots<z_{n_{d}}} \prod_{j=1}^{d-1} \mathbb{1}_{z_{n_{j}} \leq t_{j}<z_{n_{j}+1}} \mathbb{I}_{z_{n_{d}} \leq t_{d}} \quad d \mathbf{z}
$$

Define $\psi(\theta):=\theta$ and

$$
\phi(\theta):=-\lambda_{n_{d}}(\theta) t_{d}+\sum_{k=0}^{n_{d}-1} \ln \lambda_{k}(\theta), \quad S(\mathbf{z}):=\left[\sum_{k=1}^{n_{d}}(2 k-1) z_{k} ; \sum_{k=1}^{n_{d}} z_{k} ; 0\right] ;
$$

so that $\log h(\mathbf{z} ; \theta)=\phi(\theta)+\langle S(z) ; \theta\rangle$. M2(a) is readily verified and, as $g$ is continuous on $\Theta, \mathrm{M} 2(\mathrm{~b})$ follows from an application of the Lebesgue theorem. It is trivial to verify that for all $\theta \in \Theta$, $s \in \mathcal{S},-\nabla_{\theta}^{2} L(s ; \theta)$ is positive definite; then, for all $s \in \mathcal{S}$, the function $\theta \mapsto L(s ; \theta)$ is strictly concave on $\Theta$ and $s \mapsto \hat{\theta}(s)$ is well-defined on $\mathcal{S}$. By applying the implicit function theorem, $\hat{\theta}$ is also continuous. M2(c) is thus verified. $\vartheta \mapsto g \circ \zeta(\vartheta)$ is a positive and continuous function on
$\Upsilon$, and $\lim _{\varrho \rightarrow 0} g \circ \zeta(\varrho, \varsigma, \pi)=0$ for any $(\varsigma, \pi)$ showing that the level sets $\{g \circ \zeta \geq M\}, M>0$, are compact subsets of $\Upsilon$. As $\zeta$ is continuous, the level sets $\{g \geq M\}$ are compact subsets of $\Theta$, and M2(d) holds. Finally, $\mathcal{L}$ is a closed subset of the bounded set $\Theta$ which proves M3(a).
To impute the missing values $\mathbf{z}$, we use a Metropolis Hastings Independent Sampler (IS) with proposal distribution $q d \mu$ which is chosen as the product of $d$ distributions of the order statistics of ( $n_{k}-n_{k-1}$ ) independent random variable uniformly distributed on $\left[t_{k-1}, t_{k}\right], 1 \leq k \leq d$, i.e.

$$
q(\mathbf{z}) \mu(d \mathbf{z}):=\left[\prod_{k=1}^{d} \frac{\left(t_{k}-t_{k-1}\right)^{n_{k}-n_{k-1}}}{\left(n_{k}-n_{k-1}\right)!}\right]^{-1} \mathbb{1}_{0<z_{1}<\cdots<z_{n_{d}}} \prod_{j=1}^{d-1} \mathbb{1}_{z_{n_{j}} \leq t_{j}<z_{n_{j}+1}} \quad \mathbb{1}_{z_{n_{d}} \leq t_{d}} d \mathbf{z}
$$

Recall that for an homogeneous Poisson process of rate $\lambda$, the conditional distributions of the arrivals in a given interval given the number of arrival is i.i.d uniform over that interval so that the choice of the proposal is well matched to the target density. With these definitions, the IS kernel, $P_{\theta}$, is Lebesgue-irreducible and aperiodic. It is easily seen that the target density $p(z ; \theta)$ is uniformly bounded for $\theta$ in a compact set $\mathcal{K} \subseteq \Theta$. Thus, there exists some minorizing constant $0<\epsilon<1$ such that $\epsilon p(\mathbf{z} ; \theta) \leq q(\mathbf{z})$ for all $\theta \in \mathcal{K}, \mathbf{z} \in \mathcal{X}$. Hence, for $\mathbf{z} \in \mathcal{X}$, any measurable set A,

$$
P_{\theta}(\mathbf{z}, A) \geq \int_{A} \alpha_{\theta}\left(\mathbf{z}, \mathbf{z}^{\prime}\right) q\left(\mathbf{z}^{\prime}\right) \mu\left(d \mathbf{z}^{\prime}\right) \geq \epsilon \int_{A} p\left(\mathbf{z}^{\prime} ; \theta\right) \mu\left(d \mathbf{z}^{\prime}\right)=\epsilon \pi_{\theta}(A)
$$

where $\alpha_{\theta}\left(\mathbf{z}, \mathbf{z}^{\prime}\right)$ is the acceptation ratio. The condition M4 follows from Proposition 1 , with any $p \geq 2$ and any probability measure $\lambda$ on $\mathcal{X}$.

Simulations (1). We generate $d:=30$ observations at time $t_{j}:=0.25 j$ by choosing $M:=$ $2000,\left(\varrho_{t}, \varsigma_{t}, \pi_{t}\right):=(0.03,0.0004,0.5)$ that is $\left(\alpha_{t}, \beta_{t}, \gamma_{t}\right)=(-0.0004,0.37,30)$. The corresponding cumulative numbers $n_{j}$ appear as stars in Figure 4 (we have $n_{d}=651$ ). The parameter space $\Theta$ is covered by the increasing sequence of compact sets

$$
\mathcal{K}_{n}:=\zeta\left(\left\{(\varrho, \varsigma), \quad 0.0003 / 2^{n} \leq \varrho \leq 1, \quad 0 \leq \varsigma \leq 1, \quad 0 \leq \varrho+\varsigma n_{d} \leq 1\right\} \times\left[n_{d} / M, 1\right]\right), \quad n \geq 0
$$

The initial distribution $\lambda$ of the Markov chains coincides with the proposal distribution of the independent sampler $q d \mu$ described above.
Two paths of stable MCEM started respectively at $\theta_{0}^{\prime}=\left(-510^{-5}, 0.0321,0.3260\right)$ [path 1] and $\theta_{0}^{\prime}=\left(-410^{-5},-0.24,450\right)$ [path 2] are run for 300 iterations. The number of simulations at each iteration increases polynomially $m_{n}=20+n^{1.2}$. After respectively four and zero re-initializations and a small number of iterations, the convergence of both paths to $\theta^{*} \sim(-0.00027,0.2965,37.41)$ may be observed. In Figure 5, we plot the stable MCEM sequences $\left\{\gamma_{n}\right\}$ both converging to $\gamma^{*}=37.41$. In the lower left-hand corner, the first ten values are drawn, showing (a) the four
re-initializations on Path 1 and (b) for both paths, the rapid move towards a neighborhood of the limiting value $\gamma^{*}$. The two paths are drawn in the right subplots (from iteration 9 to 300 ), showing the convergence to the same limiting point $\gamma^{*}$ and a similar variation of the paths.

We then observe the performance of stable MCEM and the averaged counterpart for two polynomial schedules, $m_{n} \sim n^{1.2}$ and $m_{n} \sim n^{2}$. The procedures, run for 300 iterations, start from $\theta_{0}^{\prime}=\left(-510^{-5}, 0.0321,0.3260\right)$. In Figure 6 , we plot the sequences $\left\{\gamma_{n}\right\}$ and $\left\{\bar{\gamma}_{n}\right\}$, respectively obtained by the stable MCEM algorithm and the averaging procedure (the ten first values are discarded). In all cases, convergence to $\gamma^{*}=37.41$ may be observed. COntrary to the variation of the averageds stable MCEM path, the variation of stable MCEM paths depends upon the simulation schedule. Hence, it may be observed that averaging smoothes out the trajectory and improves the rate of convergence.

The same conclusions could be drawn from the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\bar{\alpha}_{n}\right\}$, and $\left\{\bar{\beta}_{n}\right\}$, the plots of which are omitted.

Dalal and Weerahandi Dalal and Weerahandi (1992) derive approximations of mean and variance of the Poisson process $N(t)$. The estimates of the mean functions $\mathbb{E}\left[N\left(t_{j}\right)\right]$ computed from the true value of the parameter $\theta_{t}$ (resp. the stable MCEM estimate $\theta^{*}$ ) appeared as $x$-marks (resp. squares) on Figure 4. The dots curves interpolate points corresponding to $\pm 2$ estimated standard errors from the estimates of the mean $\mathbb{E}\left[N\left(t_{j}\right)\right]$.

Simulations (2). Consider now the prediction of the number of wireless telecommunication services in the United States. Cellular Telecommunications Industry Association performed semi-annual surveys, collected in June and December, from January 1985 to June 2001 (the datas are available on the web site www.wow-com.com/industry/stats/surveys/). In Figure 8, the 34 observations collected at time $1,2, \cdots, 34$, appear as stars. We assume that this count follows a pure birth Markov model (our results suggest it is a good approximation). Since the same person may subscribe to different wireless services, the (true) population size $M$ is unknown. As discussed in Sherman et al., $M$ and $\pi$ enter the model through the product $M \pi$, so any value $M>n_{d}$ is convenient. As $n_{d} \sim 10^{8}$, we set $M=10^{9}$.

Our estimate is computed from the 29 values collected from January 1985 to December 1998 and the last values are used to cross-validate the result. The estimate is computed as the limiting value of a path $\left\{\bar{\theta}_{n}\right\}$ of the averaged procedure run for 200 iterations with a polynomial simulation schedule $m_{n} \sim n^{2}$ and started at $\theta_{0}^{\prime}=\left(-510^{-5}, 0.0321,0.3260\right)$. The paths of $\left\{\bar{\alpha}_{n}\right\}$, $\left\{\bar{\beta}_{n}\right\},\left\{\bar{\gamma}_{n}\right\}$ are plotted in Figure 7. The limiting value is $\theta^{*}=\left(-6.2710^{-11}, 0.16,1.7710^{5}\right)$. The
fitted values (resp. the predicted values) of the mean function $\mathbb{E}\left[N\left(t_{j}\right)\right]$ for $j \in\{1,28\}$ (resp. $j \in\{29,37\}$ ) appear as (down) triangles in Figure 8 (resp. (up) triangles).
Sherman et al.provide an estimate $\theta^{*}$ of $\theta_{t}$ based on the first 23 values collected from January 1985 to December 1995. They obtain $\theta^{*}=\left(-1.0610^{-9}, 0.20097,8.1710^{5}\right)$. Their fitted values (resp. their predicted values) of the mean function $\mathbb{E}\left[N\left(t_{j}\right)\right]$ for $j \in\{1,23\}$ (resp. $j \in\{24,37\}$ ) are represented as diamonds in Figure 8 (resp. squares).
In both cases, the extrapolated values well track the observed datas.

## 5. Proof of Theorem 3

Let $T: \Theta \rightarrow \Theta$ be a point-to-point map. Let $\mathcal{L}$ be a non empty subset of $\Theta$. A positive function $W$ defined on $\Theta$ is said to be a Lyapunov function relatively to $(T, \mathcal{L})$ when, (i) for all $u \in \Theta$, $W \circ T(u)-W(u) \geq 0$ and (ii) for any compact set $\mathcal{K} \subseteq \Theta \backslash \mathcal{L}, \inf _{u \in \mathcal{K}}\{W \circ T(u)-W(u)\}>0$. In the literature, convergence of random iterative maps $\left\{F_{n}\right\}$ that approximate a deterministic iterative map $T$ having a Lyapunov function $W$ is addressed under the assumption that for all compact set $\mathcal{K}$,

$$
\limsup _{n} \sup _{u \in \mathcal{K}}\left|W \circ F_{n}(u)-W \circ T(u)\right|=0 .
$$

When applied to the present problem, this condition is often not checked when MCMC algorithms are used to perform Monte Carlo integration. In this section, we show how this condition can be replaced by the weaker condition

$$
\lim _{n}\left|W \circ F_{n}\left(u_{n}\right)-W \circ T\left(u_{n}\right)\right| \mathbb{I}_{u_{n} \in \mathcal{K}}=0 .
$$

### 5.1. Deterministic results.

Proposition 9. Let $\Theta \subseteq \mathbb{R}^{l}, \mathcal{K}$ be a compact subset of $\Theta$ and $\mathcal{L} \subseteq \Theta$ such that $\mathcal{L} \cap \mathcal{K}$ is compact. Let $W$ be a continuous Lyapunov function relatively to $(T, \mathcal{L})$. Assume that there exists a $\mathcal{K}$ valued sequence $\left\{u_{n}\right\}$ such that $\lim _{n}\left|W\left(u_{n+1}\right)-W \circ T\left(u_{n}\right)\right|=0$. Then $\left\{W\left(u_{n}\right)\right\}$ converges to $a$ connected component of $W(\mathcal{L} \cap \mathcal{K})$. If $W(\mathcal{L} \cap \mathcal{K})$ has an empty interior, $\left\{W\left(u_{n}\right)\right\}$ converges to $w^{\star}$ and $\left\{u_{n}\right\}$ converges to the set $\mathcal{L}_{w^{\star}} \cap \mathcal{K}$ where $\mathcal{L}_{w^{\star}}:=\left\{\theta \in \mathcal{L}, W(\theta)=w^{\star}\right\}$.

Proof. Define the compact set $\mathcal{D}:=W(\mathcal{L} \cap \mathcal{K})$. Let $\mathcal{D}_{\alpha}$ be the $\alpha$-neighborhood of the closed set $\mathcal{D}$ in $\mathbb{R}, \mathcal{D}_{\alpha}:=\{x \in \mathbb{R}, d(x, \mathcal{D})<\alpha\}$. As $\mathcal{D}$ is compact, $\mathcal{D}=\bigcap_{\alpha>0} \mathcal{D}_{\alpha}$. Let $\alpha>0$. Since $\mathcal{D}_{\alpha}$
is a finite union of disjoint bounded open intervals, there exist $n_{\alpha} \geq 0$ and two increasing real valued sequences $\left\{a_{\alpha}(k)\right\}$ and $\left\{b_{\alpha}(k)\right\}, 1 \leq k \leq n_{\alpha}$, such that

$$
\begin{equation*}
\mathcal{D}_{\alpha}=\bigcup_{k \in\left\{1, \cdots, n_{\alpha}\right\}}\left(a_{\alpha}(k), b_{\alpha}(k)\right) \tag{19}
\end{equation*}
$$

$W^{-1}\left(\mathcal{D}_{\alpha / 2}\right)$ is an open neighborhood of $\mathcal{L} \cap \mathcal{K}$, and we define

$$
\begin{equation*}
\epsilon_{\alpha}:=\inf _{\left\{u \in \mathcal{K} \backslash W^{-1}\left(\mathcal{D}_{\alpha / 2}\right)\right\}}\{W \circ T(u)-W(u)\}, \quad \text { and } \quad \rho_{\alpha}:=\epsilon_{\alpha} \wedge \alpha \tag{20}
\end{equation*}
$$

Since $\mathcal{K} \backslash W^{-1}\left(\mathcal{D}_{\alpha / 2}\right)$ is a compact subset of $\mathbb{R}^{d}, \epsilon_{\alpha}$ and $\rho_{\alpha}$ are both positive. We define $\eta_{n+1}:=W\left(u_{n+1}\right)-W \circ T\left(u_{n}\right)$. Then

$$
\begin{equation*}
W\left(u_{n+1}\right)-W\left(u_{n}\right)=W \circ T\left(u_{n}\right)-W\left(u_{n}\right)+\eta_{n+1} \tag{21}
\end{equation*}
$$

and there exists $N_{\alpha} \geq 0$, such that for any $n \geq N_{\alpha}$,

$$
\begin{equation*}
\left|\eta_{n+1}\right| \leq \rho_{\alpha} / 2 \tag{22}
\end{equation*}
$$

By (20), (21),

$$
\begin{equation*}
\left(n \geq N_{\alpha} \text { and } u_{n} \in \mathcal{K} \backslash W^{-1}\left(\mathcal{D}_{\alpha / 2}\right)\right) \Longrightarrow W\left(u_{n+1}\right)-W\left(u_{n}\right) \geq \rho_{\alpha} / 2 \tag{23}
\end{equation*}
$$

Define $k_{\alpha}^{\star}:=\min \left\{1 \leq k \leq n_{\alpha}, \lim \sup _{n} W\left(u_{n}\right)<b_{\alpha}(k)\right\}$ and $I(\alpha):=\left(a_{\alpha}\left(k_{\alpha}^{\star}\right) ; b_{\alpha}\left(k_{\alpha}^{\star}\right)\right)$.
shows that $\left\{W\left(u_{n}\right)\right\}$ is infinitely often (i.o.) in $\mathcal{D}_{\alpha / 2} \subset \mathcal{D}_{\alpha}$, and since $\mathcal{D}_{\alpha}$ is a finite union of intervals, $\left\{W\left(u_{n}\right)\right\}$ is i.o. in an interval of (19); thus, $\lim \sup _{n} W\left(u_{n}\right) \in I(\alpha)$. Let $p \geq N_{\alpha}$ such that $W\left(u_{p}\right) \in I(\alpha)$. We prove by induction that for all $n \geq p, W\left(u_{n}\right) \in I(\alpha)$. By definition, $W\left(u_{p}\right) \in I(\alpha)$. Assume now that for $p \leq k \leq n, W\left(u_{k}\right) \in I(\alpha)$.

- If $W\left(u_{n}\right) \in \mathcal{D}_{\alpha / 2}$, we have $W\left(u_{n}\right) \geq a_{\alpha}\left(k_{\alpha}^{*}\right)+\alpha / 2$. Thus,

$$
W\left(u_{n+1}\right) \geq W\left(u_{n}\right)+\eta_{n+1} \geq a_{\alpha}\left(k_{\alpha}^{*}\right)+\alpha / 2-\rho_{\alpha} / 2 \geq a_{\alpha}\left(k_{\alpha}^{*}\right)
$$

- If $W\left(u_{n}\right) \in \mathcal{D}_{\alpha} \backslash \mathcal{D}_{\alpha / 2}$, then under (20), $W \circ T\left(u_{n}\right)-W\left(u_{n}\right) \geq \rho_{\alpha}$, and (21) and (22) imply that $W\left(u_{n+1}\right) \geq a_{\alpha}\left(k_{\alpha}^{*}\right)+\rho_{\alpha} / 2 \geq a_{\alpha}\left(k_{\alpha}^{*}\right)$.

Hence, the set of the limit points $\mathcal{I}$ of $\left\{W\left(u_{n}\right)\right\}$ is non empty and included in the interval $I(\alpha)$. Let $0<\alpha_{1}<\alpha_{2}$. By definition, $\mathcal{D}_{\alpha_{1}} \subset \mathcal{D}_{\alpha_{2}}$, thus $I\left(\alpha_{1}\right) \subset I\left(\alpha_{2}\right)$ and $\mathcal{I} \subset I\left(\alpha_{1}\right) \cap I\left(\alpha_{2}\right)$. Let $\left\{\alpha_{n}\right\}$ be a decreasing sequence such that $\lim _{n} \alpha_{n}=0$; then $\mathcal{I} \subset \bigcap_{n} I\left(\alpha_{n}\right)$. $\left\{I\left(\alpha_{n}\right)\right\}$ is a decreasing sequence of intervals, $\bigcap_{n} I\left(\alpha_{n}\right)$ is an interval and $\bigcap_{n} I\left(\alpha_{n}\right) \subset W(\mathcal{L} \cap \mathcal{K})$. Hence,
$\left\{W\left(u_{n}\right)\right\}$ converges to this interval which concludes the first part of the proof. The last part is a consequence of (21).

It is proved in Proposition 10 that the compactness assumption of the sequence $\left\{u_{n}\right\}$ can be replaced by a recurrence condition, provided that there exists a Lyapunov function controlling the excursion outside the compact sets of $\Theta$. In Proposition11, we propose a stabilization procedure ensuring this recurrence property for sequences $\left\{u_{n}\right\}$ defined by inhomogeneous maps, $u_{n+1}=F_{n}\left(u_{n}\right)$.

Proposition 10. Let $\Theta \subseteq \mathbb{R}^{l}, T: \Theta \rightarrow \Theta$ and $\mathcal{L} \subset \Theta$. Assume that

A1 there exists a continuous Lyapunov function $W$ for $(T, \mathcal{L})$ such that (a) for all $M>0$, the level set $\{\theta \in \Theta, W(\theta) \geq M\}$ is compact, (b) $\Theta=\bigcup_{n \geq 1}\{\theta \in \Theta, W(\theta) \geq n\}$.
A2 $W(\mathcal{L})$ is compact, or $\mathbf{A 2}{ }^{\prime} W(\mathcal{L} \cap \mathcal{K})$ is finite for all compact set $\mathcal{K} \subseteq \Theta$.
A3 there exists a $\Theta$-valued sequence $\left\{u_{n}\right\}$ such that ( $\boldsymbol{a}$ ) $\left\{u_{n}\right\}$ is infinitely often in a compact subset $\mathcal{G} \subseteq \Theta$ and (b) for any compact set $\mathcal{K} \subseteq \Theta, \lim _{n} \mid W\left(u_{n+1}\right)-W \circ$ $T\left(u_{n}\right) \mid \mathbb{I}_{u_{n} \in \mathcal{K}}=0$.

Then $\left\{u_{n}\right\}$ is in a compact subset of $\Theta$.

Proof. (under the assumption A2) Let $\alpha>0$. Under A1(b) and A2, there exists $M>0$ such that

$$
\mathcal{G} \cup \mathcal{L}_{\alpha} \subset\{\theta \in \Theta, W(\theta) \geq M\}
$$

where $\mathcal{L}_{\alpha}$ is the $\alpha$-neighborhood of $\mathcal{L}$. Define

$$
\begin{equation*}
\epsilon:=\inf _{\{\theta \in \Theta, W(\theta) \geq M-1\} \backslash \mathcal{L}_{\alpha}}\{W \circ T(\theta)-W(\theta)\} \quad \text { and } \quad \rho:=\epsilon \wedge 1 \tag{24}
\end{equation*}
$$

By assumption, $\epsilon>0$ and $\rho>0$. Define $\eta_{n+1}:=W\left(u_{n+1}\right)-W \circ T\left(u_{n}\right)$. Under A3, there exists $N$ such that

$$
\begin{equation*}
\left(n \geq N \text { and } u_{n} \in\{\theta \in \Theta, W(\theta) \geq M-1\}\right) \Rightarrow\left|\eta_{n+1}\right| \leq \rho / 2 \tag{25}
\end{equation*}
$$

Note that

$$
\begin{equation*}
W\left(u_{n+1}\right)-W\left(u_{n}\right)=W \circ T\left(u_{n}\right)-W\left(u_{n}\right)+\eta_{n+1} \tag{26}
\end{equation*}
$$

Since $\left\{u_{n}\right\}$ is infinitely often in the compact set $\mathcal{G}$, there exists $p \geq N$ such that $W\left(u_{p}\right) \geq M-1$. We show by induction that for all $n \geq p, W\left(u_{n}\right) \geq M-1$. The property holds for $n=p$. Assume it holds for $p \leq k \leq n$.

- If $u_{n} \in\{\theta \in \Theta, W(\theta) \geq M\}$, then (24-26) imply that $W\left(u_{n+1}\right) \geq W\left(u_{n}\right)-\rho / 2 \geq$ $M-1 / 2 \geq M-1$.
- If $u_{n} \in\{\theta \in \Theta, W(\theta) \geq M-1\} \backslash \mathcal{L}_{\alpha}$, then (24-26) imply that $W\left(u_{n+1}\right) \geq W\left(u_{n}\right)+\epsilon-$ $\rho / 2 \geq W\left(u_{n}\right) \geq M-1$.

Hence for any $q \geq n, u_{q}$ is in the compact set $\{\theta \in \Theta, W(\theta) \geq M-1\}$.

Proof. (under the assumption $A 2^{\prime}$ ). By assumption, there exists $M$ such that $\mathcal{G} \subset\{\theta \in$ $\Theta, W(\theta) \geq M\}$. As $W(\mathcal{L} \cap\{\theta, W(\theta) \geq M-1\})$ is finite, there exist $\alpha>0$ and $M-1 \leq$ $M^{\prime \prime}<M^{\prime}<M$, such that

$$
\mathcal{L}_{\alpha} \cap\left\{\theta \in \Theta, W(\theta) \geq M^{\prime \prime}\right\} \subset\left\{\theta \in \Theta, W(\theta) \geq M^{\prime}\right\} .
$$

Define

$$
\epsilon:=\inf _{\left\{\theta \in \Theta, W(\theta) \geq M^{\prime \prime}\right\} \backslash \mathcal{L}_{\alpha}}\{W \circ T(\theta)-W(\theta)\} \quad \text { and } \quad \rho:=\epsilon \wedge\left(M^{\prime}-M^{\prime \prime}\right) .
$$

It may be proved that for all large $q, u_{q}$ is in the compact set $\left\{\theta \in \Theta, W(\theta) \geq M^{\prime \prime}\right\}$. The proof is on the same lines as the previous one, and is omitted for brevity.

Let $\left\{F_{n}\right\}: \Theta \rightarrow \Theta$ be a family of point-to-point maps. Choose a sequence of compact subsets $\left\{\mathcal{K}_{n}\right\}$ of $\Theta$ such that for any $n \geq 0$,

$$
\mathcal{K}_{n} \subsetneq \mathcal{K}_{n+1} \quad \Theta=\bigcup_{n \geq 0} \mathcal{K}_{n} .
$$

Let $u_{0} \in \mathcal{K}_{0}$. Set $p_{0}:=0$ and for $n \geq 0$,

$$
\left\{\begin{array}{lll}
\text { If } F_{n}\left(u_{n}\right) \in \mathcal{K}_{p_{n}}, & u_{n+1}:=F_{n}\left(u_{n}\right) \text { and } & p_{n+1}:=p_{n},  \tag{27}\\
\text { if } F_{n}\left(u_{n}\right) \notin \mathcal{K}_{p_{n}} & u_{n+1}:=u_{0} \text { and } & p_{n+1}:=p_{n}+1 .
\end{array}\right.
$$

Proposition 11. Let $\Theta \subseteq \mathbb{R}^{l}, T$ and $\left\{F_{n}\right\}$ be point-to-point maps onto $\Theta$. Let $\left\{u_{n}\right\}$ be the sequence given by (27). Assume (a) A1-2 holds, (b) for all $u \in \mathcal{K}_{0}, \lim _{n}\left|W \circ F_{n}-W \circ T\right|(u)=0$ and (c) for any compact subset $\mathcal{K} \subseteq \Theta, \lim _{n}\left|W \circ F_{n}\left(u_{n}\right)-W \circ T\left(u_{n}\right)\right| \mathbb{1}_{u_{n} \in \mathcal{K}}=0$. Then, $\limsup p_{n} p_{n}<\infty$ and $\left\{u_{n}\right\}$ is a compact sequence.

The proof is along the same lines as Proposition 10 and is omitted for brevity.
5.2. Proof of Theorem 3. Given $\lambda, \theta_{0}^{\prime}$ and the sequence of compact sets $\left\{\mathcal{K}_{n}\right\}$, the process $\left\{\theta_{n}^{\prime}\right\}$ is defined on the canonical space of the inhomogeneous Markov chain $\left\{\left(\tilde{S}_{n}, p_{n}\right)\right\}$. We denote by $\overline{\mathbb{P}}$ (resp. $\overline{\mathbb{E}}$ ) the probability (resp. the expectation) of this canonical Markov chain (the dependence upon $\lambda, \theta_{0}^{\prime}$ and $\left\{\mathcal{K}_{n}\right\}$ is omitted).
We apply Proposition 9 and Proposition 11 with the EM map $T:=\hat{\theta} \circ \bar{S}$ and the random sequence of maps $\left\{F_{n}\right\}, F_{n}(\theta):=\operatorname{argmax}_{\phi \in \Theta} \mathcal{Q}_{n}(\phi, \theta)$.

Proof of (i-a). We check the conditions of Proposition 11. It is well-known that the incomplete data likelihood $g$ is a natural Lyapunov function relatively to the EM map $T$ and to the set $\mathcal{L}$ of the fixed points of $T$. Under M1-3, the conditions A1-2 are verified with $W=g$. Let $\epsilon>0$ and $\mathcal{K} \subseteq \Theta$ be a compact. We prove that $\sum_{n} \mathbb{I}_{\left\{\left|g \circ F_{n}\left(\theta_{n}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{I}_{\left.\theta_{n}^{\prime} \in \mathcal{K} \geq \epsilon\right\}}\right.}$ is finite w.p.1. By the second Borel-Cantelli Lemma, the convergence of the series is implied by the convergence of $\sum_{n} \overline{\mathbb{P}}\left(\left|g \circ F_{n}\left(\theta_{n}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon \mid \mathcal{F}_{n-1}\right)$ w.p. 1 where $\mathcal{F}_{n}:=\sigma\left(\tilde{S}_{k}, k \leq n\right)$. By assumption, $\bar{S}(\mathcal{K})$ is a compact subset of $\mathcal{S}$. For $\delta>0$, define the compact $\bar{S}(\mathcal{K}, \delta):=\left\{s \in \mathbb{R}^{q}, \inf _{t \in \mathcal{K}}|t-s| \leq\right.$ $\delta\}$. Then there exists $\eta(\epsilon, \delta)$ such that for any $x, y \in \bar{S}(\mathcal{K}, \delta)$,

$$
|x-y| \leq \eta(\epsilon, \delta) \Longrightarrow|g \circ \hat{\theta}(x)-g \circ \hat{\theta}(y)| \leq \epsilon
$$

Hence,

$$
\begin{array}{r}
\overline{\mathbb{P}}\left(\left|g \circ F_{n}\left(\theta_{n}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon \mid \mathcal{F}_{n-1}\right)=\overline{\mathbb{P}}\left(\left|g \circ \hat{\theta}\left(\tilde{S}_{n}\right)-g \circ \hat{\theta}\left(\bar{S}\left(\theta_{n}^{\prime}\right)\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon \mid \mathcal{F}_{n-1}\right) \\
=\overline{\mathbb{P}}\left(\left|g \circ \hat{\theta}\left(\tilde{S}_{n}\right)-g \circ \hat{\theta}\left(\bar{S}\left(\theta_{n}^{\prime}\right)\right)\right| \mathbb{I}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon,\left|\tilde{S}_{n}-\bar{S}\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \leq \delta \mid \mathcal{F}_{n-1}\right) \\
+\overline{\mathbb{P}}\left(\left|g \circ \hat{\theta}\left(\tilde{S}_{n}\right)-g \circ \hat{\theta}\left(\bar{S}\left(\theta_{n}^{\prime}\right)\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon,\left|\tilde{S}_{n}-\bar{S}\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}}>\delta \mid \mathcal{F}_{n-1}\right) \\
\leq 2 \overline{\mathbb{P}}\left(\left|\tilde{S}_{n}-\bar{S}\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \alpha \mid \mathcal{F}_{n-1}\right)
\end{array}
$$

with $\alpha:=\delta \wedge \eta(\epsilon, \delta)$. Thus,

$$
\begin{aligned}
\overline{\mathbb{P}}\left(\left|g \circ F_{n}\left(\theta_{n}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon \mid\right. & \left.\mathcal{F}_{n-1}\right) \leq 2 \alpha^{-p} \overline{\mathbb{E}}\left[\left|\tilde{S}_{n}-\bar{S}\left(\theta_{n}^{\prime}\right)\right|^{p} \mid \mathcal{F}_{n-1}\right] \mathbb{I}_{\theta_{n}^{\prime} \in \mathcal{K}} \\
& \leq 2 \alpha^{-p} m_{n}^{-p} \mathbb{E}_{\lambda, \theta_{n}^{\prime}}\left[\left|\sum_{j=1}^{m_{n}}\left\{S\left(\Phi_{j}\right)-\pi_{\theta_{n}^{\prime}}(S)\right\}\right|^{p}\right] \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}}
\end{aligned}
$$

where $p$ is given by M4. Then M4 implies that there exists a finite constant $C:=C(\mathcal{K})$ such that

$$
\mathbb{E}_{\lambda, \theta_{n}^{\prime}}\left[\left|\sum_{j=1}^{m_{n}}\left\{S\left(\Phi_{j}\right)-\pi_{\theta_{n}^{\prime}}(S)\right\}\right|^{p}\right] \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \leq C m_{n}^{p / 2}
$$

and, under M5, the proof is concluded.

Proof of (i-b) and (ii). We check the conditions of Proposition 9. It remains to prove that for any compact set $\mathcal{K} \subseteq \Theta$,

$$
\lim _{n}\left|g\left(\theta_{n+1}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}}=0 \quad \overline{\mathbb{P}} \text {-a.s. }
$$

We proceed as above and consider the a.s. convergence of the random series

$$
\begin{equation*}
\sum_{n} \overline{\mathbb{P}}\left(\left|g\left(\theta_{n+1}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon \mid \mathcal{F}_{n-1}\right) \tag{28}
\end{equation*}
$$

By definition, either $\theta_{n+1}^{\prime}=F_{n}\left(\theta_{n}^{\prime}\right)$ or $\theta_{n+1}^{\prime}=\theta_{0}^{\prime}$ and $p_{n+1}=p_{n}+1$. We have just proved that the number of re-initialization is finite w.p. 1 so that the series

$$
\sum_{n} \overline{\mathbb{P}}\left(\left|g\left(\theta_{n+1}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon, \theta_{n+1}^{\prime}=\theta_{0}^{\prime}, p_{n+1}=p_{n}+1 \mid \mathcal{F}_{n-1}\right)
$$

is finite $\overline{\mathbb{P}}$-a.s. Then (28) is finite iff $\sum_{n} \overline{\mathbb{P}}\left(\left|g\left(\theta_{n+1}^{\prime}\right)-g \circ T\left(\theta_{n}^{\prime}\right)\right| \mathbb{1}_{\theta_{n}^{\prime} \in \mathcal{K}} \geq \epsilon, \theta_{n+1}^{\prime}=F_{n}\left(\theta_{n}^{\prime}\right) \mid \mathcal{F}_{n-1}\right)$ is finite $\overline{\mathbb{P}}$-a.s., which is established above.

## 6. Uniform Rosenthal's inequality

Let $f: \mathcal{X} \rightarrow[1, \infty)$ be a measurable function. For some function $g: \mathcal{X} \rightarrow \mathbb{R}^{q}$, (resp. for some signed measure $\nu$ on $\mathcal{X}$ ), define

$$
\|g\|_{f}:=\sup _{\mathcal{X}} \frac{|g|}{f}, \quad \mathcal{L}_{f}:=\left\{g: \mathcal{X} \rightarrow \mathbb{R}^{q},\|g\|_{f}<\infty\right\} \quad\|\nu\|_{f}:=\sup _{\{g,|g| \leq f\}}|\nu(g)|
$$

Proposition 12. Let $\left(\Omega, \mathcal{A}, \mathcal{F}_{n},\left\{\phi_{n}\right\}, P_{x}\right)$ be a canonical Markov chain with invariant probability measure $\pi$ on $\mathcal{X}$. Assume that there exist $p \geq 2$, some measurable functions $1 \leq f_{0} \leq V_{0} \leq$ $V_{0}^{p} \leq V_{1}<\infty$ and some constants $C_{i}<\infty, i=0,1$, such that for any $x \in \mathcal{X}$

$$
\left\{\begin{array}{l}
\sum_{n}\left\|P^{n}(x, \cdot)-\pi(\cdot)\right\|_{f_{0}} \leq C_{0} V_{0}(x)  \tag{29}\\
\sum_{n}\left\|P^{n}(x, \cdot)-\pi(\cdot)\right\|_{V_{0}^{p}} \leq C_{1} V_{1}(x)
\end{array}\right.
$$

Then, for any Borel function $g: \mathcal{X} \rightarrow \mathbb{R}^{q}, g \in \mathcal{L}_{f_{0}}$,

$$
\mathbb{E}_{x}\left|\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}\right|^{p} \leq\|g\|_{f_{0}}^{p} 6^{p} C_{p} C_{0}^{p}\left(C_{1} V_{1}(x)+\pi\left(V_{0}^{p}\right)\right) n^{p / 2} \quad x \in \mathcal{X}
$$

where $C_{p}$ is the Rosenthal's constant.

Proof. Denote by $\hat{g}(x):=\sum_{k=0}^{\infty}\left\{P^{k} g(x)-\pi(g)\right\}$, the unique solution (up to a constant) of the Poisson equation $\hat{g}-P \hat{g}=g-\pi(g)$. Then $\hat{g} \in \mathcal{L}_{V_{0}}$ and $\|\hat{g}\|_{V_{0}} \leq C_{0}\|g\|_{f_{0}}$. Write

$$
\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}=\sum_{k=1}^{n}\left\{\hat{g}\left(\Phi_{k}\right)-P \hat{g}\left(\Phi_{k-1}\right)\right\}-P \hat{g}\left(\Phi_{n}\right)+P \hat{g}\left(\Phi_{0}\right)
$$

$\left\{\hat{g}\left(\Phi_{k}\right)-P \hat{g}\left(\Phi_{k-1}\right)\right\}$ is a $L^{p}$-martingale increment (w.r.t. the initial distribution $\delta_{x}$ ) and by applying the Minkovsky's inequality and the Rosenthal's inequality (Hall and Heyde, 1980, Theorem 2.12), we get

$$
\begin{aligned}
\mathbb{E}_{x}\left[\left|\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}\right|^{p}\right] & \leq 3^{p-1}\left\{C_{p} \mathbb{E}_{x}\left[\left(\sum_{k=1}^{n} \mathbb{E}_{x}\left[\left|\hat{g}\left(\Phi_{k}\right)-P \hat{g}\left(\Phi_{k-1}\right)\right|^{2} \mid \mathcal{F}_{k-1}\right]\right)^{p / 2}\right]\right. \\
& \left.+C_{p} \mathbb{E}_{x}\left[\sum_{k=1}^{n}\left|\hat{g}\left(\Phi_{k}\right)-P \hat{g}\left(\Phi_{k-1}\right)\right|^{p}\right]+\mathbb{E}_{x}\left[\left|P \hat{g}\left(\Phi_{n}\right)\right|^{p}\right]+|P \hat{g}(x)|^{p}\right\}
\end{aligned}
$$

where $C_{p}$ is the Rosenthal's constant and $\left\{\mathcal{F}_{n}\right\}$ is the natural filtration of the Markov chain $\left\{\Phi_{n}\right\}$. In addition,

$$
\left(\sum_{k=1}^{n} \mathbb{E}_{x}\left[\left|\hat{g}\left(\Phi_{k}\right)-P \hat{g}\left(\Phi_{k-1}\right)\right|^{2} \mid \mathcal{F}_{k-1}\right]\right)^{p / 2} \leq\left(\sum_{k=1}^{n} P|\hat{g}|^{2}\left(\Phi_{k-1}\right)\right)^{p / 2} \leq n^{p / 2-1} \sum_{k=1}^{n} P|\hat{g}|^{p}\left(\Phi_{k-1}\right)
$$

Hence,

$$
\begin{aligned}
& \mathbb{E}_{x}\left[\left|\sum_{k=1}^{n}\left\{g\left(\Phi_{k}\right)-\pi(g)\right\}\right|^{p}\right] \leq 3^{p-1}\left(C_{p}\left(n^{p / 2-1}+2^{p}\right) \sum_{k=1}^{n} P^{k}|\hat{g}|^{p}(x)+P|\hat{g}|^{p}(x)+P^{n+1}|\hat{g}|^{p}(x)\right) \\
& \leq 3^{p-1}\left(C_{p}\left(n^{p / 2-1}+2^{p}\right)+1\right)\left(\left.\sum_{k \geq 1}\left|P^{k}\right| \hat{g}\right|^{p}(x)-\pi\left(|\hat{g}|^{p}\right) \mid+n \pi\left(|\hat{g}|^{p}\right)\right) \\
& \leq 6^{p} C_{p} n^{p / 2}\left(\left.\sum_{k \geq 1}\left|P^{k}\right| \hat{g}\right|^{p}(x)-\pi\left(|\hat{g}|^{p}\right) \mid+\pi\left(|\hat{g}|^{p}\right)\right)
\end{aligned}
$$

Since $\hat{g} \in \mathcal{L}_{V_{0}}, \pi\left(|\hat{g}|^{p}\right) \leq\|\hat{g}\|_{V_{0}}^{p} \pi\left(V_{0}^{p}\right)<\infty$, and by assumption,

$$
\left.\sum_{k=0}^{\infty}\left|P^{k}\right| \hat{g}\right|^{p}(x)-\pi\left(|\hat{g}|^{p}\right) \mid \leq\|\hat{g}\|_{V_{0}}^{p} C_{1} V_{1}(x)
$$

This yields the desired result.

Proof of Proposition 1 . When the state space is $\nu_{m}$-small, it is easily seen that

$$
\sum_{n}\left\|P^{n}(x, \cdot)-\pi(\cdot)\right\|_{T V} \leq 2\left(1-(1-\epsilon)^{1 / m}\right)^{-1}
$$

and the proof of $(9)$ is a trivial application of Proposition 12.
The following proposition gives sufficient conditions, based on nested drift conditions, leading to the explicit bounds (29).

Proposition 13. Let $P$ be a $\psi$-irreducible and aperiodic transition kernel on a general state space $\mathcal{X}$. Let $C \subseteq D$ be some accessible $\nu_{m}$-small sets. Assume there exist some Borel functions $f, V: \mathcal{X} \rightarrow[1, \infty), f \leq V$, some constants $b<\infty$ and $0<a<1$ such that $\sup _{D} V<\infty$ and

$$
\begin{cases}P V(x) \leq V(x)-f(x)+b \mathbb{1}_{C}(x), & \\ f(x) \geq b /(1-a), & x \in D^{c} .\end{cases}
$$

Then, $P$ possesses an invariant probability measure $\pi, \pi(f)<\infty$ and for any probability measure $(\lambda, \mu)$ on $\mathcal{X} \times \mathcal{X}$,

$$
\begin{equation*}
\sum_{n=0}^{\infty}\left|\lambda P^{n} g-\mu P^{n} g\right| \leq\|g\|_{f}\left(\epsilon^{-1} M_{V}+a^{-1}(\lambda(V)+\mu(V))\right) \tag{30}
\end{equation*}
$$

where,

$$
\begin{aligned}
& M_{V}:=\sup _{\left(x, x^{\prime}\right) \in C \times D} \sum_{k=1}^{m-1}\left\{P^{k} f(x)+P^{k} f\left(x^{\prime}\right)\right\} \\
&+ \sup _{\left(x, x^{\prime}\right) \in C \times D}\left(\sum_{k=1}^{m-1}\left\{P^{k} f(x)+P^{k} f\left(x^{\prime}\right)\right\}+a^{-1}\left\{P^{m} V(x)+P^{m} V\left(x^{\prime}\right)\right\}\right) \leq 4 a^{-1}\left(b m+\sup _{D} V\right)
\end{aligned}
$$

with the convention that $\sum_{k=1}^{0} P^{k} f(x)=0$.

Proof. By Theorem 14.0.1 of Meyn and Tweedie (1993), there exists an invariant probability measure $\pi$ such that $\pi(f)<\infty$.
For simplicity, the proof of (30) is restricted to the case $m=1$. The proof of (30) is based on coupling technique which may be summarized as follows. Let $\Delta:=(C \times D) \cup(D \times C)$ and $R$ be the residual kernel defined as

$$
R(x, \cdot):=\left(1-\mathbb{1}_{D}(x) \epsilon\right)^{-1}\left(P(x, \cdot)-\epsilon \mathbb{1}_{D}(x) \nu_{1}(\cdot)\right)
$$

We define a $\mathcal{X} \times \mathcal{X} \times\{0,1\}$-valued process $Z:=\left\{\Omega, \mathcal{A}, Z_{n}=\left(X_{n}, X_{n}^{\prime}, d_{n}\right), P_{x, x^{\prime}, d}\right\}$ such that (a) $P_{x, x^{\prime}, 0}\left(X_{n} \in \cdot\right)=P^{n}(x, \cdot)$ and $P_{x, x^{\prime}, 0}\left(X_{n}^{\prime} \in \cdot\right)=P^{n}\left(x^{\prime}, \cdot\right)$ for all $\left(x, x^{\prime}\right) \in \mathcal{X} \times \mathcal{X},(b)$ there exists a random-time $T$ and $X_{n} \mathbb{I}_{T \leq n}=X_{n}^{\prime} \mathbb{I}_{T \leq n}$. Set $Z_{0}:=\left(x, x^{\prime}, 0\right)$. Each time $\left(X_{k}, X_{k}^{\prime}, d_{k}\right)$ hits the
set $\Delta \times\{0\}$, an $\epsilon$-biased coin is tossed. If the coin comes up head, then the coupling is successful: the next value of $X_{k+1}=X_{k+1}^{\prime}$ is simulated from $\nu_{1}, d_{k+1}=1$, and the two components remain forever coupled. Otherwise, the next values $X_{k+1}$ and $X_{k+1}^{\prime}$ are drawn independently from the residual kernel $R$ and $d_{k+1}=0$. If $\left(X_{k}, X_{k}^{\prime}, d_{k}\right) \in \Delta^{c} \times\{0\}$, then the processes are updated independently from $P$.
Define the coupling time $T:=\inf \left\{n \geq 1, d_{n}=1\right\}$ (with the convention that $\inf \emptyset=\infty$ ), $T_{0}:=\inf \left\{k \geq 0,\left(X_{k}, X_{k}^{\prime}\right) \in \Delta\right\}$ and, for $i \geq 1, T_{i}:=\inf \left\{k>T_{i-1},\left(X_{k}, X_{k}^{\prime}\right) \in \Delta\right\}$ the successive hitting times on $\Delta$. By definition of $T$, we have $X_{n} \mathbb{\Pi}_{T \leq n}=X_{n}^{\prime} \mathbb{I}_{T \leq n}$ and for any Borel function $g \in \mathcal{L}_{f}$,

$$
\begin{equation*}
\sum_{n \geq 0} \int \lambda(d x) \mu(d y)\left|P^{n} g(x)-P^{n} g(y)\right| \leq\|g\|_{f} \mathbb{E}_{\lambda, \mu, 0}\left[\sum_{n=0}^{T-1}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right] . \tag{31}
\end{equation*}
$$

Define

$$
\begin{equation*}
A(f):=(1-\epsilon) \sup _{\left(x, x^{\prime}\right) \in \Delta} \int R(x, d y) R\left(x^{\prime}, d y^{\prime}\right) \mathbb{E}_{y, y^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right] . \tag{32}
\end{equation*}
$$

The first set in the proof consists in showing that

$$
\begin{equation*}
\mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right] \leq \mathbb{I}_{\Delta}\left(x, x^{\prime}\right)\left\{f(x)+f\left(x^{\prime}\right)\right\}+a^{-1} \mathbb{I}_{\Delta^{c}}\left(x, x^{\prime}\right)\left\{V(x)+V\left(x^{\prime}\right)\right\} \tag{33}
\end{equation*}
$$

The case $\left(x, x^{\prime}\right) \in \Delta$ is trivial. For $\left(x, x^{\prime}\right) \in \Delta^{c}$, under the stated assumptions,

$$
\mathbb{E}_{x, x^{\prime}, 0}\left[V\left(X_{1}\right)+V\left(X_{1}^{\prime}\right)\right] \leq V(x)+V\left(x^{\prime}\right)-\left(f(x)+f\left(x^{\prime}\right)\right)+b\left(\mathbb{1}_{C}(x)+\mathbb{1}_{C}\left(x^{\prime}\right)\right) .
$$

Since $\left(x, x^{\prime}\right) \in \Delta^{c}, x \in C$ (resp. $x^{\prime} \in C$ ) implies that $x^{\prime} \in D^{c}$ (resp. $x \in D^{c}$ ), so that

$$
f\left(x^{\prime}\right)-b \mathbb{1}_{C}(x) \geq a f\left(x^{\prime}\right) \quad f(x)-b \mathbb{1}_{C}\left(x^{\prime}\right) \geq a f(x) .
$$

Hence,

$$
\mathbb{E}_{x, x^{\prime}, 0}\left[V\left(X_{1}\right)+V\left(X_{1}^{\prime}\right)\right] \leq V(x)+V\left(x^{\prime}\right)-a\left(f(x)+f\left(x^{\prime}\right)\right), \quad\left(x, x^{\prime}\right) \in \Delta^{c},
$$

and the proof of (33) follows from the so-called Dynkin's formula (Meyn and Tweedie, 1993, Proposition 11.3.2). Note that by (33), $\mathbb{E}_{x, x^{\prime}, 0}\left[T_{0}\right]<\infty$, which implies that $P_{x, x^{\prime}, 0}(T<\infty)=1$ for all $\left(x, x^{\prime}\right) \in \mathcal{X} \times \mathcal{X}$. We now prove that

$$
\begin{equation*}
\mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T-1}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right] \leq \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right]+\epsilon^{-1} A(f) . \tag{34}
\end{equation*}
$$

By the strong Markov property, and by noting that $P_{x, x^{\prime}, 0}\left(d_{T_{j}}=0\right)=(1-\epsilon)^{j}$, for $j \geq 0$,

$$
\begin{gathered}
\mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{j+1}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\} \mathbb{I}_{\{0\}}\left(d_{T_{j}+1}\right)\right]=(1-\epsilon) \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{j}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\} \mathbb{I}_{\{0\}}\left(d_{T_{j-1}+1}\right)\right] \\
+\mathbb{E}_{x, x^{\prime}, 0}\left[\mathbb{I}_{\{0\}}\left(d_{T_{j}+1}\right) \mathbb{E}_{X_{T_{j}+1}, X_{T_{j}+1}^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right]\right] \\
\leq(1-\epsilon) \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{j}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\} \mathbb{I}_{\{0\}}\left(d_{T_{j-1}+1}\right)\right]+A(f)(1-\epsilon)^{j},
\end{gathered}
$$

with the convention $T_{-1}+1=0$. By straightforward recursion,

$$
\begin{align*}
\mathbb{E}_{x, x^{\prime}, 0}\left[\sum _ { n = 0 } ^ { T _ { j + 1 } } \left\{f\left(X_{n}\right)+\right.\right. & \left.\left.f\left(X_{n}^{\prime}\right)\right\} \mathbb{I}_{\{0\}}\left(d_{T_{j}+1}\right)\right] \\
& \leq(1-\epsilon)^{j}\left((1-\epsilon) \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right]+(j+1) A(f)\right) \tag{35}
\end{align*}
$$

Hence,

$$
\begin{aligned}
\mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T-1}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\}\right]= & \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{0}}\left\{f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right\} \mathbb{I}_{d_{T_{0}+1}=1}\right]+ \\
& \sum_{j=0}^{\infty} \mathbb{E}_{x, x^{\prime}, 0}\left[\sum_{n=0}^{T_{j+1}}\left(f\left(X_{n}\right)+f\left(X_{n}^{\prime}\right)\right) \mathbb{I}_{\{0\}}\left(d_{T_{j}+1}\right) \mathbb{I}_{\{1\}}\left(d_{T_{j+1}+1}\right)\right]
\end{aligned}
$$

and (34) follows by noting that $P_{X_{T_{j}}, X_{T_{j}}^{\prime}, 0}\left(d_{T_{j}+1}=1\right)=\epsilon$. The proposition follows from (31) to (35).

The drift condition implies
$\sup _{\left(x, x^{\prime}\right) \in C \times D}\left(\sum_{k=1}^{m-1}\left\{P^{k} f(x)+P^{k} f\left(x^{\prime}\right)\right\}+\left\{P^{m} V(x)+P^{m} V\left(x^{\prime}\right)\right\}\right) \leq 2 b m+\sup _{\left(x, x^{\prime}\right) \in C \times D}\left\{V(x)+V\left(x^{\prime}\right)\right\}$ from which it is easily seen that $M_{V} \leq 4 a^{-1}\left(b m+\sup _{D} V\right)$.

Proof of Proposition 2. The first step is to prove that the level set $D:=\{V \leq M\}$ is small. By assumption, $\sup _{x \in D} \mathbb{E}_{x}\left[\tau_{C}\right]<\infty$, then for any $\eta>0$, there exists $n_{0}$ such that $P_{x}\left(\sigma_{C} \geq n\right) \leq \eta, x \in D$ and $n \geq n_{0}$. Then we can define a distribution $\alpha=\{\alpha(n)\}$ on $\mathbb{Z}_{+}$ such that for $x \in D$ and $0<l<1, \sum_{n} \alpha(n) P^{n}(x, C) \geq \sum_{n \leq n_{0}} \alpha(n) P^{n}(x, C) \geq l(1-\eta)$. As $C$ is petite, there exist some measure $\nu$ on $\mathcal{X}$ and some distribution $\beta=\{\beta(n)\}$ on $\mathbb{Z}_{+}$such that $\sum_{n} \alpha * \beta(n) P^{n}(x, A) \geq l(1-\eta) \nu(A)$ which proves that $D$ is petite. The smallness property of
$D$ deduces from Theorem 5.5.7. Meyn and Tweedie (1993). Note in addition that by definition, $D \supseteq C$. Define

$$
\begin{array}{lll}
f_{0}:=V^{1 / p}, & V_{0}:=V^{1 / p} /\left(1-\rho^{1 / p}\right), & b_{0}:=b^{1 / p} /\left(1-\rho^{1 / p}\right), \\
f_{1}:=V_{0}^{p}, & V_{1}:=V /\left\{(1-\rho)\left(1-\rho^{1 / p}\right)^{p}\right\}, & b_{1}:=b /\left\{(1-\rho)\left(1-\rho^{1 / p}\right)^{p}\right\}, \\
a_{0}:=1-\frac{b^{1 / p}}{\left(1-\rho^{1 / p}\right) M^{1 / p}} & a_{1}:=1-\frac{b}{(1-\rho) M} &
\end{array}
$$

It is easily seen that $P V_{i} \leq V_{i}-f_{i}+b_{i} \mathbb{I}_{C}, i=0,1,1 \leq f_{0} \leq V_{0} \leq V_{0}^{p}=f_{1} \leq V_{1}, 0<a_{i}<1$ and $f_{i} \geq b_{i} /\left(1-a_{i}\right)$ on $D^{c}, i=0,1$. By applying Proposition 13, the inequalities (29) are verified and the constants $C_{i}, i=0,1$, are upper bounded by (this upper bound is not optimal)

$$
\begin{aligned}
C_{0} & \leq 5 \epsilon^{-1}(m+1) M^{1 / p}\left(1-\rho^{1 / p}\right)\left(\left(1-\rho^{1 / p}\right)-(b / M)^{1 / p}\right)^{-1} \\
C_{1} V_{1}(x)+\pi\left(V_{0}^{p}\right) & \leq 5 \epsilon^{-1}(m+1) M\left(1-\rho^{1 / p}\right)^{-p}(1-\rho-b / M)^{-1} V(x)
\end{aligned}
$$

This yields the desired result.

## 7. Proof of Lemmas 7 and 14

### 7.1. Proof of Lemma 14.

Lemma 14. Under the assumptions of theorem 6, we have

$$
\begin{equation*}
\rho_{n} \mathbb{I}_{\left\{\lim _{n} \tilde{S}_{n}=s^{*}\right\}}=o_{\text {w.p. } 1}\left(m_{n}^{-1 / 2}\right) ; \tag{36}
\end{equation*}
$$

Proof. The remainder term $\rho_{n}$ also follows a difference equation of the form

$$
\rho_{n}=\Gamma \rho_{n-1}+\eta_{n}=\left(H_{n-1}+\Gamma\right) \rho_{n-1}+r_{n-1}+\eta_{n}^{(1)}
$$

since $\eta_{n}^{(2)}$ may be decomposed as $\eta_{n}^{(2)}=H_{n-1} \rho_{n-1}+r_{n-1}$ with $H_{n}:=\sum_{1 \leq i \leq q} R_{n}(i, \cdot)\left(2 \mu_{n, i}+\right.$ $\left.\rho_{n, i}\right)$, and $r_{n}:=\sum_{1 \leq i, j \leq q} R_{n}(i, j) \mu_{n, i} \mu_{n, j}$ for $n \geq 0$. Hence we have $\rho_{n}:=\rho_{n}^{(1)}+\rho_{n}^{(2)}$ where

$$
\rho_{n}^{(1)}:=\prod_{k=0}^{n-1}\left(H_{k}+\Gamma\right) \rho_{0}+\sum_{k=1}^{n}\left(\prod_{j=k}^{n-1}\left(H_{j}+\Gamma\right)\right) \eta_{k}^{(1)}, \quad \rho_{n}^{(2)}:=\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1}\left(H_{j}+\Gamma\right)\right) r_{k} .
$$

As $\mu_{n}=O_{L^{p}}\left(m_{n}^{-1 / 2}\right)$ and, by assumption, $\sum_{n} m_{n}^{-p / 2}<\infty$, then $\mu_{n}=o_{\text {w.p.1 }}(1)$, and thus, $\rho_{n} \mathbb{I}_{\lim _{n} \tilde{S}_{n}=s^{*}}=o_{\text {w.p.1 }}(1)$. Hence, $\left|H_{n}\right| \mathbb{I}_{\lim _{n} \tilde{S}_{n}=s^{*}}=o_{\text {w.p.1 }}(1)$, and for any $\gamma<\tilde{\gamma}<1, j \leq n$, $\left|\prod_{k=j}^{n}\left(H_{j}+\Gamma\right)\right| \mathbb{1}_{\lim _{n} \tilde{S}_{n}=s^{*}}=O_{\text {w.p.1 }}\left(\tilde{\gamma}^{n}\right)$. Along trajectories converging to $s^{*}$, the first term in $\rho_{n}^{(1)}$ is $O_{\text {w.p.1 }}(1) O_{L^{p}}\left(\tilde{\gamma}^{n}\right)$ since, by $\mathrm{M} 4, \rho_{0} \in L^{p}$. The first term in $\eta_{n}^{(1)}$ is only finitely-often
non-zero, and by M4, the second term in $\eta_{n}^{(1)}$ is bounded and the bound is inversely proportional to $m_{n}$. Thus, by choosing $\tilde{\gamma}^{-1}>\lim _{n} m_{n+1} / m_{n}$ and by applying Lemma 5 ,

$$
\begin{equation*}
\rho_{n}^{(1)} \mathbb{I}_{\mathrm{lim}_{n} \tilde{S}_{n}=s^{*}}=O_{\text {w.p. } .1}(1) O_{L^{p}}\left(m_{n}^{-1}\right) . \tag{37}
\end{equation*}
$$

Similarly, as $r_{n}=O_{L^{p}}\left(m_{n}^{-1}\right)$,

$$
\begin{equation*}
\rho_{n}^{(2)} \mathbb{I}_{\lim _{n}} \tilde{S}_{n}=s^{*}=O_{\text {w.p. } 1}(1) O_{L^{p}}\left(m_{n}^{-1}\right), \tag{38}
\end{equation*}
$$

and the proof of (36) is completed.

### 7.2. Proof of Lemma 7.

Lemma 15. Let $\left\{a_{n}\right\}$ and $\left\{b_{n}\right\}, b_{n} \neq 0$, be two sequences such that (i) the power series $f(x):=$ $\sum_{n=1}^{\infty} a_{n} x^{n}$ has a radius of convergence $r$, (ii) $\lim _{n \rightarrow \infty} b_{n+1} / b_{n}=: q$, with $|q|<r$. Define $c_{n}:=\sum_{k \geq n} b_{k} a_{k-n}$. Then, $\lim _{n \rightarrow \infty} c_{n} b_{n}^{-1}=f(q)$.

Proof. By assumption, for any $K$ and $\epsilon>0$, there exists $N$ such that for all $n \geq N, \mid b_{n+K} / b_{n}-$ $q^{K} \mid \leq \epsilon$. In addition, there exist some positive constants $A, \epsilon$ such that for all $n, j \geq 0, b_{n+j} / b_{n} \leq$ $A(q+\epsilon)^{j}$.

$$
\left|b_{n}^{-1} \sum_{k \geq n} b_{k} a_{k-n}-\sum_{k \geq 0} q^{k} a_{k}\right| \leq \sum_{k=n}^{n+K}\left|b_{k} / b_{n}-q^{k-n}\right| a_{k-n}+\sum_{k \geq n+K} b_{k} / b_{n} a_{k-n}+\sum_{k \geq K} q^{k} a_{k} .
$$

Let $\epsilon>0$. Then there exists $K$ such that the last two sums are upper bounded by $\epsilon$. Now for those constants $K, \epsilon$, there exists $N$ such that for $n \geq N$, the first sum is lower than $\epsilon$. And the proof is completed.

We now prove Lemma 7 . We shall establish that for $m \gamma \neq 1$,

$$
\begin{equation*}
(1-m \gamma)^{r}\left(\lim _{n} \xi_{n}^{(r)}\right)^{r}=1+m^{r / 2} \sum_{l=0}^{r-1}\binom{r}{l}(-1)^{r-l}\left(m^{l-r / 2} \gamma^{l-r}-1\right)^{-1} \lim _{n} m_{n}^{r / 2}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1} . \tag{39}
\end{equation*}
$$

If $m>1$, then Lemma 5 implies that $\lim _{n} m_{n}^{r / 2}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1}=1-m^{-r / 2}$. If $m=1$, then $\lim _{n} m_{n}^{r / 2}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1}=0$. In both cases, $\lim _{n} m_{n}^{r / 2}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1}=1-m^{-r / 2}$. Thus Lemma 7 holds provided that (39) is established.

First case: $m \gamma<1$. Define $S_{n}:=\sum_{j \geq n} m_{j} \gamma^{j} . \quad S_{n}=\gamma^{n} \sum_{j \geq n} m_{j} \gamma^{j-n}$, and by applying Lemma 15 , since $m<\gamma^{-1}$, it holds

$$
\begin{equation*}
\lim _{n} m_{n}^{-1} \gamma^{-n} S_{n}=(1-m \gamma)^{-1} \tag{40}
\end{equation*}
$$

We write

$$
\begin{aligned}
& \sum_{k=0}^{n} m_{k}^{-r / 2}\left(\sum_{j=0}^{n-k} m_{j+k} \gamma^{j}\right)^{r}=\sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r}\left(S_{k}-S_{n+1}\right)^{r} \\
& \quad=\sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r} S_{k}^{r}+\sum_{l=0}^{r-1}\binom{r}{l}(-1)^{r-l} S_{n+1}^{r-l} \sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r} S_{k}^{l} \\
& =\sum_{k=0}^{n} m_{k}^{r / 2}\left(m_{k}^{-1} \gamma^{-k} S_{k}\right)^{r}+m_{n}^{r / 2} \sum_{l=0}^{r-1}\left({ }_{l}^{r}\right)(-1)^{r-l} \gamma^{r-l}\left(m_{n+1} / m_{n}\right)^{r-l}\left(m_{n+1}^{-1} \gamma^{-(n+1)} S_{n+1}\right)^{r-l} \ldots \\
& \quad \times\left(m_{n}^{r / 2-l} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{(n-k)(r-l)}\right)\left(\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\right)^{-1} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\left(m_{k}^{-1} \gamma^{-k} S_{k}\right)^{l} .
\end{aligned}
$$

By use of the Cesaro Lemma and (40),

$$
\lim _{n}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1} \sum_{k=0}^{n} m_{k}^{r / 2}\left(m_{k}^{-1} \gamma^{-k} S_{k}\right)^{r}=(1-m \gamma)^{-r} .
$$

In addition, for all $l \in\{0, \ldots, r-1\},(m \gamma)^{l-r} m^{r / 2}>1$ showing that $\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}$ diverges to infinity. Then, applying again the Cesaro Lemma and (40),

$$
\lim _{n}\left(\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\right)^{-1} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\left(m_{k}^{-1} \gamma^{-k} S_{k}\right)^{l}=(1-m \gamma)^{-l} .
$$

Finally, as $l<r,(m \gamma)^{r-l} m^{-r / 2}<1$ and Lemma 5 implies that

$$
\lim _{n} m_{n}^{r / 2-l} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{(n-k)(r-l)}=(\gamma m)^{l-r} m^{r / 2}\left(\gamma^{l-r} m^{l-r / 2}-1\right)^{-1} .
$$

Combining these limits gives (39).
Second case: $m \gamma>1$. Define $S_{n}:=\sum_{j=0}^{n} m_{j} \gamma^{j} . S_{n}=\gamma^{n} \sum_{j=0}^{n} m_{j} \gamma^{-(n-j)}$ and by applying Lemma 5 , since $m^{-1}<\gamma$, it holds

$$
\begin{equation*}
\lim _{n} m_{n}^{-1} \gamma^{-n} S_{n}=m \gamma(m \gamma-1)^{-1} \tag{41}
\end{equation*}
$$

We write, with the convention $S_{-1}:=0$,

$$
\begin{aligned}
& \sum_{k=0}^{n} m_{k}^{-r / 2}\left(\sum_{j=0}^{n-k} m_{j+k} \gamma^{j}\right)^{r}=\sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r}\left(S_{n}-S_{k-1}\right)^{r} \\
&=(-1)^{r} \sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r} S_{k-1}^{r}+\sum_{l=0}^{r-1}\binom{r}{l}(-1)^{l} S_{n}^{r-l} \sum_{k=0}^{n} m_{k}^{-r / 2} \gamma^{-k r} S_{k-1}^{l} \\
&=(-1)^{r} \gamma^{-r} \sum_{k=0}^{n} m_{k}^{r / 2}\left(m_{k-1} / m_{k}\right)^{r}\left(m_{k-1}^{-1} \gamma^{-(k-1)} S_{k-1}\right)^{r} \\
&+m_{n}^{r / 2} \sum_{l=0}^{r-1}\binom{r}{l}(-1)^{l} \gamma^{-l}\left(m_{n}^{-1} \gamma^{-n} S_{n}\right)^{r-l} m_{n}^{r / 2-l}\left(\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{(r-l)(n-k)}\right) \ldots \\
& \times\left(\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\right)^{-1} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\left(m_{k} / m_{k-1}\right)^{-l}\left(m_{k-1}^{-1} \gamma^{-(k-1)} S_{k-1}\right)^{l}
\end{aligned}
$$

By use of the Cesaro Lemma and (41),

$$
(-1)^{r} \gamma^{-r} \lim _{n}\left(\sum_{k=0}^{n} m_{k}^{r / 2}\right)^{-1} \sum_{k=0}^{n} m_{k}^{r / 2}\left(m_{k-1} / m_{k}\right)^{r}\left(m_{k-1}^{-1} \gamma^{-(k-1)} S_{k-1}\right)^{r}=(1-m \gamma)^{-r}
$$

In addition, for all $l \in\{0, \ldots, r-1\},(m \gamma)^{l}\left(m \gamma^{2}\right)^{-r / 2}>1$ showing that $\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}$ diverges to infinity. Then, applying again the Cesaro Lemma and (41),
$\lim _{n}\left(\sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\right)^{-1} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{k(l-r)}\left(m_{k} / m_{k-1}\right)^{-l}\left(m_{k-1}^{-1} \gamma^{-(k-1)} S_{k-1}\right)^{l}=\gamma^{l}(m \gamma-1)^{-l}$.
Finally, as $(m \gamma)^{l}\left(m \gamma^{2}\right)^{-r / 2}>1$, Lemma 5 implies that

$$
\lim _{n} m_{n}^{r / 2-l} \sum_{k=0}^{n} m_{k}^{l-r / 2} \gamma^{(n-k)(r-l)}=(\gamma m)^{l-r} m^{r / 2}\left(\gamma^{l-r} m^{l-r / 2}-1\right)^{-1}
$$

Combining these limits gives (39).

## 8. Figures



Figure 1. 100 observations from the Poisson count data model.


Figure 2. Stable MCEM sequences for different initial values, and $m_{n}=\left[n^{2}\right]$. The paths all converge to $\theta^{*}=1.88$ after a finite number of re-initializations.


Figure 3. Stable MCEM sequence with and without averaging both started from $\theta_{0}^{\prime}=2.41$; polynomial schedule $m_{n}=\left[n^{2}\right]$.


Figure 4. Cumulative numbers observed at time $t_{j}=0.25 j, j=1, \cdots, 30$; and estimated means of the count process


Figure 5. Stable MCEM sequences for different initial values and $m_{n}=\left[n^{1.2}\right]$. The paths converge to $\gamma^{*}=37.41$ after a finite number a re-initializations.


Figure 6. MCEM sequence and averaged MCEM sequence for different polynomial schedules. The 10 initial values are omitted.


Figure 7. Stable MCEM sequence with averaging: $\bar{\alpha}_{n}$ [upper left-hand], $\bar{\beta}_{n}$ [upper right-hand] and $\bar{\gamma}_{n}$ [lower left-hand].


Figure 8. Cumulative numbers and estimated means of the count process.

Acknowledgements : The authors would like to thank the anonymous referee for helpful comments.

## References

Biscarat, J. (1994). Almost sure convergence of a class of stochastic approximation algorithms. Stochastic Processes Appl. 50 83-100.

Booth, J. and Hobert, J. (1999). Maximizing generalized linear mixed model likelihoods with an automated Monte Carlo EM algorithm. J. R. Stat. Soc. Ser. B 61 265-285.

Brandière, O. (1998). The dynamic system method and the traps. Adv. in Appl. Probab. 30 137-151.

Bröcker, T. (1975). Differentiable germs and catastrophes. Cambridge University Press, Cambridge. London Mathematical Society Lecture Note Series, No. 17.

Cappé, O., Doucet, A., Lavielle, M. and Moulines, E. (1999). Simulation-based methods for blind maximum-likelihood filter identification. Signal Processing 73 3-25.

Celeux, G. and Diebolt, J. (1992). A stochastic approximation type EM algorithm for the mixture problem. Stochastics Stochastics Rep. 85 119-134.

Chan, J. and Kuk, A. (1997). Maximum likelihood estimation for probit-linear mixed models with correlated random effects. Biometrics 53 86-97.

Chan, K. and Ledolter, J. (1995). Monte Carlo EM estimation for time series models involving counts. J. Amer. Statist. Assoc. 90 242-252.

Chen, H., Guo, L. and Gao, A. (1988). Convergence and robustness of the Robbins- Monro algorithm truncated at randomly varying bounds. Stochastic Processes Appl. 27 217-231.

Dalal, S. and Weerahandi, S. (1992). Some approximations for the moments of a process used in diffusion of new products. Statist. Probab. Lett. 15 181-189.

Dalal, S. and Weerahandi, S. (1995). Estimation of innovation diffusion models with application to a consumer durable. Marketing Letters 6.

Delyon, B., Lavielle, M. and Moulines, E. (1999). Convergence of a stochastic approximation version of the EM algorithm. Ann. Statist. 27 94-128.

Dempster, A. P., Laird, N. M. and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the EM algorithm. J. Roy. Statist. Soc. Ser. B 39 1-38. With discussion.

Fort, G., Moulines, E., Roberts, G. and Rosenthal, J. (2001). On the geometric ergodicity of hybrid samplers. Submitted for publication, http://www-lmc.imag.fr/lmcsms/Gersende.Fort/biblio.html.
Guo, S. and Thompson, A. (1991). Monte-Carlo estimation of variance component models for large complex pedigrees. IMA Journal of Mathematics Applied in Medicine and Biology 8 171-189.

Hall, P. and Heyde, C. C. (1980). Martingale limit theory and its application. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York. Probability and Mathematical Statistics.

Jamshidian, M. and Jennrich, R. (1997). Acceleration of the EM algorithm by using quasiNewton methods. J. R. Stat. Soc. Ser. B 59 569-587.

Meng, X. and Schilling, S. (1996). Fitting full-information item factor models and an empirical investigation of bridge sampling. J. Amer. Statist. Assoc. 91 1254-1267.

Meyn, S. P. and Tweedie, R. L. (1993). Markov chains and stochastic stability. SpringerVerlag London Ltd., London.

Pierre-Loti-Viaud, D. (1995). Random perturbations of recursive sequences with an application to an epidemic model. J. Appl. Probab. 32 559-578.

Pólya, G. and Szegő, G. (1976). Problems and theorems in analysis. Vol. II. Study ed. Springer-Verlag, New York.

Polyak, B. (1990). New stochastic Approximation type procedures. Automation and remote control 51 98-107.

Shapiro, A. and Wardi, Y. (1996). Convergence analysis of stochastic algorithms. Math. Oper. Res. 21 615-628.
Sherman, R., Ho, Y. and Dalal, S. (1999). Conditions for convergence of Monte Carlo EM sequences with an application to product diffusion modeling. Econom. J. 2 248-267.
Tanner, M. A. (1996). Tools for statistical inference. 3rd ed. Springer-Verlag, New York. Methods for the exploration of posterior distributions and likelihood functions.
Wei, G. and Tanner, M. (1991). A Monte-Carlo implementation of the EM algorithm and the poor man's data augmentation algorithms. J. Amer. Statist. Assoc. 85 699-704.

Wu, C. (1983). On the convergence properties of the EM algorithm. Ann. Statist. 11 95-103.
Zeger, S. L. (1988). A regression model for time series of counts. Biometrika 75 621-629.

