Convergence of Perturbed Gradient-based methods for non-smooth convex optimization

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Based on joint works with

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↪ `On Perturbed Proximal-Gradient algorithms (2015, arXiv)`

Problem:

Convergence of an algorithm designed for solving

$$\mathrm{argmin}_{\theta \in \Theta} F(\theta) \qquad \text{with } F(\theta) = f(\theta) + g(\theta)$$

where

- the function g: $\mathbb{R}^d \to [0, \infty]$ is convex, non smooth, not identically equal to $+\infty$, and lower semi-continuous
- the function f:$\mathbb{R}^d \to \mathbb{R}$ is a smooth function

  i.e. f is continuously differentiable and there exists $L > 0$ such that

$$\|\nabla f(\theta) - \nabla f(\theta')\| \leq L \, \|\theta - \theta'\| \qquad \forall \theta, \theta' \in \mathbb{R}^d$$

- $\Theta \subseteq \mathbb{R}^d$ is the domain of g: $\Theta = \{\theta : g(\theta) < \infty\}$.

when f and $\nabla f(\theta)$ are not explicit

Outline

The function g

- Can be evaluated, is convex but is not smooth
- Typically: a constraint in the optimization problem
  - ⋆ optimization restricted to a convex set $\mathcal{K}$

$$g(\theta) \in \{0, +\infty\} = \begin{cases} 0 & \text{if } \theta \in \mathcal{K} \\ +\infty & \text{otherwise} \end{cases}$$

  - ⋆ Sparsity constraints

$$g(\theta) \propto \|\theta\|_1 = \sum_{i=1}^{d} |\theta_i|$$

$$g(\theta) \propto \alpha \sum_{i=1}^{d} |\theta_i| + \frac{(1-\alpha)}{2} \sum_{i=1}^{d} \theta_i^2$$

The function f: Ex. 1, Inference in Latent variable models

- A vector of observations: Y
- A vector of latent variables: U
- A parametric model indexed by $\theta \in \Theta$

Minimize the negative log-likelihood:

$$f(\theta) = -\log \int p(\mathrm{Y}|\mathrm{u};\theta)\, \phi(\mathrm{u}) \mu(\mathsf{d}u)$$

which is (usually) intractable; same thing for the gradient

$$\nabla f(\theta) = -\int \nabla \log p(\mathrm{Y}|\mathrm{u};\theta) \; \frac{p(\mathrm{Y},\mathrm{u};\theta)}{\int p(\mathrm{Y},\mathrm{x};\theta)\mu(\mathsf{d}x)} \mu(\mathsf{d}u)$$

Example 1': (logistic regression, with random effects)

For example, logistic regression with random effects, under sparsity constraints

$$\mathbf{U} \sim \mathcal{N}_q(0, I)$$

$$Y_i | \mathbf{U} \overset{ind.}{\sim} \text{Ber}\left(\frac{\exp\left(x_i' \beta + \sigma\, z_i' \mathbf{U}\right)}{1 + \exp\left(x_i' \beta + \sigma\, z_i' \mathbf{U}\right)} \right)$$

$$\theta = (\beta, \sigma) \in \mathbb{R}^p \times \mathbb{R}_+$$

$$g(\theta) = \lambda \sum_{i=1}^{p} |\beta_i|$$

The function f: Ex. 2, Inference in Markov Random Fields

- Observations: i.i.d. $\mathbb{R}^p$-valued samples $Y_1, \cdots, Y_N$ from the distribution

$$\pi_\theta(y) = \frac{\gamma(y;\theta)}{Z_\theta}$$

  with an intractable normalizing constant Z_θ.
- A parametric model indexed by $\theta \in \mathbb{R}^d$.

Minimize the negative log-likelihood, which is intractable

$$f(\theta) = -\sum_{i=1}^{N} \log \gamma(Y_i;\theta) + N \log Z_\theta$$

and with intractable gradient

$$\nabla f(\theta) = -\sum_{i=1}^{N} \nabla_\theta \log \gamma(Y_i;\theta) + N \int \{\nabla_\theta \log \gamma(\mathrm{u},\theta)\} \, \pi_\theta(\mathrm{du})$$

The function f: Ex.3, Learning on huge data set

- Many component functions (ex. a cost function associated to each observation)

Minimize an additive cost function

$$f(\theta) = \frac{1}{N} \sum_{i=1}^{N} f_i(\theta)$$

intractable since N is large, and same thing for its gradient

$$\nabla f(\theta) = \frac{1}{N} \sum_{i=1}^{N} \nabla f_i(\theta)$$

The function f: Ex.4, Online learning

Minimize a mean value

$$f(\theta) = \int \bar{f}(\theta; \mathrm{u}) \pi(\mathrm{du})$$

when the distribution π is unknown, and only examples/samples from π are available online:

$$\nabla f(\theta) = \int \left\{ \nabla_\theta \bar{f}(\theta; \mathrm{u}) \right\} \; \pi(\mathrm{du})$$

Outline

Examples of problems of the form: $\text{argmin}_\theta\{f(\theta) + g(\theta)\}$

A first order method: the proximal gradient algorithm

Convergence of the Perturbed Proximal Gradient algorithm

Rates of convergence

Acceleration

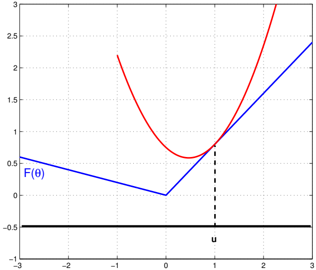
References

The proximal-gradient algorithm (1/2)

$$\mathrm{argmin}_{\theta \in \Theta} \left(\underbrace{f(\theta)}_{C^1 \text{ with Lipschitz gradient}} + \underbrace{g(\theta)}_{\text{not differentiable, convex}} \right)$$

Idea: majorization-minimization iterative method

Nesterov (2004)



We have:

$$F(\theta_{n+1}) \leq F(\theta_n)$$

The proximal-gradient algorithm (2/2)

- Since f is smooth: define a majorizing function $\theta \mapsto Q_\gamma(\theta; \theta_n)$

$$f(\theta) + g(\theta) \leq f(\theta_n) + \langle \nabla f(\theta_n), \theta - \theta_n \rangle + \frac{1}{2\gamma} \|\theta - \theta_n\|^2 + g(\theta)$$

  where $L \leq 1/\gamma$.

- A family of majorizing functions - indexed by γ; all of them are equal to $F(\theta_n)$ at $\theta = \theta_n$
- Update the current solution

$$\theta_{n+1} = \mathrm{argmin}_\theta Q_\gamma(\theta; \theta_n) = \mathrm{argmin}_\theta \left(g(\theta) + \frac{1}{2\gamma} \|\theta - \{\theta_n - \gamma \nabla f(\theta_n)\}\|^2 \right)$$

$$= \mathrm{Prox}_{\gamma, g} \left(\theta_n - \gamma \nabla f(\theta_n)\right)$$

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$$= \mathrm{Prox}_{\gamma_{n+1}, g} \left(\theta_n - \gamma_{n+1} \nabla f(\theta_n)\right)$$

In practice

$$\theta_{n+1} = \mathrm{Prox}_{\gamma_{n+1},g}\left(\theta_n - \gamma_{n+1}\nabla f(\theta_n)\right)$$

- A gradient step w.r.t. the smooth part of $f+g$
- A correction mecanism through the "prox" operator
  - ⋆ when g is the indicator of $\mathcal{K}$

$$\theta_{n+1} = \mathrm{Proj}_{\mathcal{K}}\left(\theta_n - \gamma_{n+1}\nabla f(\theta_n)\right)$$

  - ⋆ when g is the elastic net penalty

$$\theta_{n+1} = (\text{componentwise soft-thresholding of } \theta_n - \gamma_{n+1}\nabla f(\theta_n))$$

- In practice, it may happen that
  - ⋆ the gradient is not explicit but an approximation is available.
  - ⋆ the Prox operator is not explicit.

When the gradient can not computed

$$\theta_{n+1} = \mathrm{Prox}_{\gamma_{n+1},g}\left(\theta_n - \gamma_{n+1} \nabla f(\theta_n)\right)$$

Run a Perturbed Proximal-Gradient Algorithm

$$\begin{aligned}\theta_{n+1} &= \mathrm{Prox}_{\gamma_{n+1},g}\left(\theta_n - \gamma_{n+1}\ H_{n+1}\right) \\ &= \mathrm{Prox}_{\gamma_{n+1},g}\left(\theta_n - \gamma_{n+1}\ \{\nabla f(\theta_n) + \eta_{n+1}\}\right)\end{aligned}$$

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Questions:

- Conditions on η_{n+1}, γ_n for the convergence of the algorithm
- Rates of convergence

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- Conditions on η_{n+1}, γ_n for the convergence of the algorithm
- Rates of convergence
- When H_{n+1} is a Monte Carlo sum
  - ⋆ how many points at each iteration (fixed/increasing batch size; constant/increasing number of draws)
  - ⋆ how to choose the stepsize γ_n : constant or decreasing ?

Outline

A deterministic result for the convergence of $\{\theta_n, n \geq 0\}$

$$\theta_{n+1} = \mathrm{Prox}_{\gamma_{n+1},g}\left(\theta_n - \gamma_{n+1}\ H_{n+1}\right) \qquad \text{with } H_{n+1} \approx \nabla f(\theta_n)$$

Set: $\mathcal{L} = \mathrm{argmin}_\Theta(f+g)$ $\qquad \eta_{n+1} = H_{n+1} - \nabla f(\theta_n)$

Theorem (Atchadé, F., Moulines (2015))

Assume

1. g *convex, lower semi-continuous.*
2. f **convex**, *gradient is lipschitz.*
3. $\sum_n \gamma_n = +\infty$
4. *Convergence of the series*

$$\sum_n \gamma_{n+1}^2 \|\eta_{n+1}\|^2, \qquad \sum_n \gamma_{n+1}\eta_{n+1}, \qquad \sum_n \gamma_{n+1} \left\langle \mathsf{S}_n, \eta_{n+1} \right\rangle$$

where $\mathsf{S}_n = \mathrm{Prox}_{\gamma_{n+1},g}(\theta_n - \gamma_{n+1}\nabla f(\theta_n))$.

Then there exists $\theta_\star \in \mathcal{L}$ *such that* $\lim_n \theta_n = \theta_\star$.

- Result available for both deterministic and stochastic perturbations
- If stochastic perturbations: both biased and unbiased approximation of $\nabla f(\theta_n)$

Sketch of proof:

1. For any minimizer $\theta_\star$ of F

$$\|\theta_{n+1}-\theta_\star\|^2 \leq \|\theta_n-\theta_\star\|^2-\gamma_{n+1}\left(F(\theta_{n+1}) - \min F\right)+\gamma_{n+1}\text{noise}_{n+1} \quad (1)$$

2. Use a (deterministic) Siegmund-Robbins lemma:
   If
$$\sum_n \gamma_n = \infty, \qquad \sum_n \gamma_{n+1}\,\text{noise}_{n+1} < \infty$$
   then the limiting points of $\{\theta_n, n \geq 0\}$ are minimizers of F.

3. Use again (1) to show the convergence of $\{\theta_n\}_n$ to a minimizer of F.

Case of a Monte Carlo approximation

When

$$\nabla f(\theta) = \int H_\theta(x)\, \pi_\theta(\mathrm{d}x)$$

- replace $\nabla f(\theta_n)$ by a Monte Carlo approximation

$$\eta_{n+1} = \frac{1}{m_{n+1}} \sum_{j=1}^{m_{n+1}} H_{\theta_n}(X_{n+1,j}) - \nabla f(\theta_n)$$

  where $\{X_{n+1,j}, j \geq 0\}$ is a Markov chain with inv. dist. π_{θ_n}

- with an increasing number of samples m_{n+1} and a step-size γ_n s.t.

$$\sum_n \gamma_{n+1} = +\infty, \qquad \sum_n \frac{\gamma_{n+1}^2}{m_{n+1}} < \infty, \qquad \sum_n \frac{\gamma_{n+1}}{m_{n+1}} < \infty \text{ when biased approx.}$$

- or with a constant number of samples $m_{n+1} = m$ and a decreasing step-size γ_n s.t.

$$\sum_n \gamma_{n+1} = +\infty \qquad \sum_n \gamma_{n+1}^2 < \infty, \qquad + \text{ ergodicity cond. on the chains}$$

Outline

Examples of problems of the form: $\mathrm{argmin}_{\theta}\{f(\theta)+g(\theta)\}$

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A deterministic result

For non negative weights a_k

$$\sum_{k=1}^{n} a_k \{F(\theta_k) - \min F\} \leq U_n(\theta_\star)$$

Theorem (Atchadé, F., Moulines (2016))

For any $\theta_\star \in \mathcal{L}$,

$$U_n(\theta_\star) = \frac{1}{2} \sum_{k=1}^{n} \left(\frac{a_k}{\gamma_k} - \frac{a_{k-1}}{\gamma_{k-1}} \right) \|\theta_{k-1} - \theta_\star\|^2 + \frac{a_0}{2\gamma_0} \|\theta_0 - \theta_\star\|^2 - \sum_{k=1}^{n} a_k \gamma_k \|\eta_k\|^2 - \sum_{k=1}^{n} a_k \left\langle \mathsf{S}_{k-1} - \theta_\star, \eta_k \right\rangle$$

Case of a Monte Carlo approximation

When

$$\nabla f(\theta_n) = \int H_{\theta_n}(x)\,\pi_{\theta_n}(\mathsf{d}x) \approx \frac{1}{m_{n+1}} \sum_{j=1}^{m_{n+1}} H_{\theta_n}(X_{n+1,j})$$

From the previous result, convergence rates in expectation, in L^q, $\cdots$: e.g.

- with $m_n = m$ and $\gamma_n = O(1/\sqrt{n})$

$$\left\| F\left(\frac{1}{n}\sum_{k=1}^{n}\theta_k\right) - \min F\right\|_{L^q} \leq \left\|\frac{1}{n}\sum_{k=1}^{n} F(\theta_k) - \min F\right\|_{L^q} = O\left(\frac{1}{\sqrt{n}}\right)$$

Case of a Monte Carlo approximation

When

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- with $m_n = m$ and $\gamma_n = O(1/\sqrt{n})$

$$\left\| F\left(\frac{1}{n}\sum_{k=1}^{n} \theta_k\right) - \min F \right\|_{L^q} \leq \left\| \frac{1}{n}\sum_{k=1}^{n} F(\theta_k) - \min F \right\|_{L^q} = O\left(\frac{1}{\sqrt{n}}\right)$$

- with $m_n \sim n$ and $\gamma_n = \gamma$

$$\left\| F\left(\frac{1}{n}\sum_{k=1}^{n} \theta_k\right) - \min F \right\|_{L^q} \leq \left\| \frac{1}{n}\sum_{k=1}^{n} F(\theta_k) - \min F \right\|_{L^q} = O\left(\frac{\ln n}{n}\right)$$

but $\cdots$ with $O(n^2)$ Monte Carlo samples.

Outline

Accelerated Proximal Gradient algorithm

Similarly to the Nesterov acceleration of the gradient algorithm (1983),

$$\theta_{n+1} = \mathrm{Prox}_{\gamma_{n+1},g}\left(\tau_n - \gamma_{n+1} \nabla f(\tau_n)\right)$$
$$\tau_n = \theta_n + \frac{t_{n-1} - 1}{t_n}\left(\theta_n - \theta_{n-1}\right)$$

where t_n is a positive sequence s.t.

$$\gamma_{n+1} t_n (t_n - 1) \leq \gamma_n t_{n-1}^2$$

- The rate of convergence of the exact Proximal-Gradient algorithm:

$$F(\theta_n) - \min F = O\left(\frac{1}{n}\right)$$

- For the Accelerated Proximal-Gradient algorithm Beck and Teboulle, 2009, the rate is

$$F(\theta_n) - \min F = O\left(\frac{1}{n^2}\right)$$

Perturbed Accelerated Proximal Gradient Algorithm

- Sufficient conditions on the stepsizes γ_n, the coefficient t_n and on the perturbation
$$\eta_{n+1} = H_{n+1} - \nabla f(\tau_n)$$
for the convergence of $\{\theta_n, n \geq 0\}$

- Rate of convergence:
  - ⋆ deterministic case: $F(\theta_{n+1}) - \min F = O(t_n^{-2}\gamma_{n+1}^{-1})$
  - ⋆ Monte Carlo case, with $\gamma_n = \gamma$, $t_n = O(n)$, $m_n \sim n^3$:
  $$\mathbb{E}\left[F(\theta_n)\right] - \min F = O\left(\frac{1}{n^2}\right)$$
  but $\cdots$ after n^4 Monte Carlo samples
  - ⋆ (works in progress)

Outline

Convergence of Unbiased Stochastic Proximal-Gradient Algorithm

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