Simultaneous linearization of holomorphic germs in presence of resonances

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Abstract. Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $(\mathrm{d}f_1)_O$ diagonalizable and such that f_1 commutes with f_h for any $h = 2, \ldots, m$. We prove that, under certain arithmetic conditions on the eigenvalues of $(\mathrm{d}f_1)_O$ and some restrictions on their resonances, $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a particular complex manifold invariant under $f_1, \ldots, f_m$.

1. Introduction

One of the main questions in the study of local holomorphic dynamics (see [A] and [B] for general surveys on this topic) is when a given germ of biholomorphism f of $\mathbb{C}^n$ at a fixed point p, which we may place at the origin O, is *holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates, tangent to the identity, conjugating f to its linear part. The answer to this question depends on the set of eigenvalues of $\mathrm{d}f_O$, usually called the *spectrum* of $\mathrm{d}f_O$. In fact, if we denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ the eigenvalues of $\mathrm{d}f_O$, then it may happen that there exists a multi-index $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ with $|k| := k_1 + \cdots + k_n \geq 2$ and such that

$$\lambda^k - \lambda_j := \lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \tag{1}$$

for some $1 \leq j \leq n$; a relation of this kind is called a *resonance* of f, and k is called a *resonant multi-index*. A *resonant monomial* is a monomial $z^k = z_1^{k_1} \cdots z_n^{k_n}$ in the j-th coordinate such that $\lambda^k = \lambda_j$.

One possible generalization of the previous question is to ask when a given set of $m \geq 2$ germs of biholomorphisms $f_1, \ldots, f_m$ of $\mathbb{C}^n$ at the same fixed point, which we may place at the origin, are *simultaneously holomorphically linearizable*, i.e., there exists a local holomorphic change of coordinates conjugating f_h to its linear part for each $h = 1, \ldots, m$.

In [R] we found, under certain arithmetic conditions on the eigenvalues and some restrictions on the resonances, a necessary and sufficient condition for holomorphic linearization. In this article we shall use that result to find a necessary and sufficient condition for holomorphic simultaneous linearization.

Before stating our result we need the following definitions:

Definition 1.1. Let $1 \leq s \leq n$. We say that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_r) \in (\mathbb{C}^*)^n$ *has only level s resonances* if there are only two kinds of resonances:

$$\boldsymbol{\lambda}^k = \lambda_h \iff k \in \widetilde{K}_1, \tag{a}$$

where

$$\widetilde{K}_1 = \left\{ k \in \mathbb{N}^n \;\middle|\; |k| \geq 2, \sum_{p=1}^{s} k_p = 1 \text{ and } \mu_1^{k_{s+1}} \cdots \mu_r^{k_n} = 1 \right\};$$

and

$$(b) \qquad \boldsymbol{\lambda}^k = \mu_j \iff k \in \widetilde{K}_2,$$

where

$$\widetilde{K}_2 = \left\{ k \in \mathbb{N}^n \;\middle|\; |k| \geq 2, k_1 = \cdots = k_s = 0 \text{ and } \exists j \in \{1, \ldots, r\} \text{ s.t. } \mu_1^{k_{s+1}} \cdots \mu_r^{k_n} = \mu_j \right\}.$$

For $s = n$ having only level s resonances means that there are no resonances. When $s < n$, if $(\lambda_1, \ldots, \lambda_s)$ have no resonances, it is easy to verify that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s, 1, \ldots, 1)$ has only level s resonances.

Definition 1.2. Let $n \geq 2$ and let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}^*$ be not necessarily distinct. For any $m \geq 2$ put

$$\widetilde{\omega}(m) = \min_{\substack{2 \leq |k| \leq m \\ k \notin \mathrm{Res}_j(\lambda)}} \min_{1 \leq j \leq n} |\lambda^k - \lambda_j|,$$

where $\mathrm{Res}_j(\lambda)$ is the set of multi-indices $k \in \mathbb{N}^n$, with $|k| \geq 2$, giving a resonance relation for $\lambda = (\lambda_1, \ldots, \lambda_n)$ relative to $1 \leq j \leq n$, i.e., such that $\lambda^k - \lambda_j = 0$. We say that λ *satisfies the reduced Brjuno condition* if there exists a strictly increasing sequence of integers $\{p_\nu\}_{\nu \geq 0}$ with $p_0 = 1$ such that

$$\sum_{\nu \geq 0} p_\nu^{-1} \log \widetilde{\omega}(p_{\nu+1})^{-1} < \infty.$$

Note that the reduced Brjuno condition of order n (i.e., when there are no resonances) is nothing but the usual Brjuno condition introduced in [Br] (see also [M] pp. 25–37 for the one-dimensional case).

Definition 1.3. Let f be a germ of biholomorphism of $\mathbb{C}^n$ fixing the origin O and let $s \in \mathbb{N}$, with $1 \leq s \leq n$. The origin O is called a *quasi-Brjuno fixed point of order* s if $\mathrm{d}f_O$ is diagonalizable and, denoting by $\boldsymbol{\lambda}$ the spectrum of $\mathrm{d}f_O$, we have:

(i) $\boldsymbol{\lambda}$ has only level s resonances;

(ii) $\boldsymbol{\lambda}$ satisfies the reduced Brjuno condition.

We say that f has the origin as a *quasi-Brjuno fixed point* if there exists $1 \leq s \leq n$ such that it is a quasi-Brjuno fixed point of order s.

Definition 1.4. Let $f_1, \ldots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let M be a germ of complex manifold at O of codimension $1 \leq s \leq n$, and f_h-invariant for each $h = 1, \ldots, m$. We say that M is a *simultaneous osculating manifold for* $f_1, \ldots, f_m$ if there exists a holomorphic flat $(1,0)$-connection ∇ of the normal bundle N_M of M in $\mathbb{C}^n$ commuting with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \ldots, m$.

In [R] we saw that the osculating condition was necessary and sufficient to extend a holomorphic linearization from an invariant submanifold to a whole neighbourhood of the origin for a germ f_1 of biholomorphism with a quasi-Brjuno fixed point. Our main theorem shows that the simultaneous osculating condition is also necessary and sufficient to extend a common holomorphic linearization, just assuming that f_1 has a quasi-Brjuno fixed point and commutes with $f_2, \ldots, f_m$:

Theorem 1.1. *Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$, and* *that it commutes with f_h for any $h = 2, \dots, m$. Then $f_1, \dots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \dots, m$, which is a simultaneous osculating manifold for $f_1, \dots, f_m$ and such that $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable.*

A similar topic is studied in [S]. However, his results are not comparable with ours, because his notion of "linearization modulo an ideal" is not suitable for producing a full linearization result, except when there are no resonances at all, whereas in our result we explicitly admit some resonances.

We shall need the following notation: if $g\colon \mathbb{C}^n \to \mathbb{C}$ is a holomorphic function with $g(O) = 0$, and $z = (x, y) \in \mathbb{C}^n$ with $x \in \mathbb{C}^s$ and $y \in \mathbb{C}^{n-s}$, we shall denote by $\mathrm{ord}_x(g)$ the maximum positive integer m such that g belongs to the ideal $\langle x_1, \cdots, x_s \rangle^m$. Furthermore, we shall say that the local coordinates $z = (x, y)$ are *adapted* to the complex submanifold M if in those coordinates M is given by $\{x = 0\}$.

2. Linearization

We first introduced *osculating manifolds* in [R]. A germ f of biholomorphism of $\mathbb{C}^n$ fixing the origin O admits an *osculating manifold M of codimension* $1 \le s \le n$ if there is a germ of f-invariant complex manifold M at O of codimension s such that the normal bundle N_M of M admits a holomorphic flat $(1,0)$-connection that commutes with $\mathrm{d}f|_{N_M}$. Definition 1.4 is the natural extension of this object to the case we are dealing with.

We shall need the following characterization of simultaneous osculating manifolds.

Proposition 2.1. *Let $f_1, \dots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \ge 2$, and let M be a germ of complex manifold at O of codimension $1 \le s \le n$, and f_h-invariant for each $h = 1, \dots, m$. Then M is a simultaneous osculating manifold for $f_1, \dots, f_m$ if and only if there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form*

$$
(2) \qquad \begin{aligned}
x'_i &= \sum_{p=1}^{s} a^{(h)}_{i,p} x_p + \widehat{f}^{(h)}_i(x, y) && \text{for } i = 1, \dots, s,\\
y'_j &= f^{(h)}_j(x, y) && \text{for } j = 1, \dots, r = n - s,
\end{aligned}
$$

with

$$
\mathrm{ord}_x(\widehat{f}^{(h)}_i) \ge 2,
$$

for any $i = 1, \dots, s$ and $h = 1, \dots, m$.

Proof. If there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form (2) with $\mathrm{ord}_x(\widehat{f}^{(h)}_i) \ge 2$ for any $i = 1, \dots, s$ and $h = 1, \dots, m$, then it is obvious to verify that the trivial holomorphic flat $(1,0)$-connection commutes with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \dots, m$.

Conversely, let ∇ be a holomorphic flat $(1,0)$-connection of the normal bundle N_M commuting with $\mathrm{d}f_h|_{N_M}$ for each $h = 1, \dots, m$. It suffices to choose local holomorphic coordinates $z = (x, y)$ adapted to M in which all the connection coefficients Γ^i_{jk} with respect to the local holomorphic frame $\{\pi(\frac{\partial}{\partial x_1}), \dots, \pi(\frac{\partial}{\partial x_s})\}$ of N_M are zero (see [R] Proposition 3.1 and Lemma 3.2), and then the assertion follows immediately from the proof of Theorem 1.3 of [R]. □

Corollary 2.2. *Let $f_1, \dots, f_m$ be m germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin, with $m \geq 2$, and let M be a germ of complex manifold at O of codimension $1 \leq s \leq n$, and f_h-invariant for each $h = 1, \dots, m$. Then M is a simultaneous osculating manifold for $f_1, \dots, f_m$ such that $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable if and only if there exist local holomorphic coordinates $z = (x, y)$ about O adapted to M in which f_h has the form*

$$
(3) \qquad \begin{aligned}
x'_i &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_h + \widehat{f}_i^{(h),1}(x, y) \qquad &&\text{for } i = 1, \dots, s, \\
y'_j &= f_j^{(h)\mathrm{lin}}(x, y) + \widehat{f}_j^{(h),2}(x, y) \qquad &&\text{for } j = 1, \dots, r = n - s,
\end{aligned}
$$

where $f_j^{(h)\mathrm{lin}}(x, y)$ is linear and

$$
(4) \qquad \begin{aligned}
\mathrm{ord}_x(\widehat{f}_i^{(h),1}) &\geq 2, \\
\mathrm{ord}_x(\widehat{f}_j^{(h),2}) &\geq 1,
\end{aligned}
$$

for any $i = 1, \dots, s$, $j = 1, \dots, r$ and $h = 1, \dots, m$.

Proof. One direction is clear.

Conversely, thanks to Proposition 2.1, the fact that M is a simultaneous osculating manifold for $f_1, \dots, f_m$ is equivalent to the existence of local holomorphic coordinates $z = (x, y)$ about O adapted to M, in which f_h has the form (3) with $\mathrm{ord}_x(\widehat{f}_i^{(h),1}) \geq 2$ for any $i = 1, \dots, s$ and $h = 1, \dots, m$. Furthermore, $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable; therefore there exists a local holomorphic change of coordinate, tangent to the identity, and of the form

$$
\begin{aligned}
\tilde{x} &= x, \\
\tilde{y} &= \Phi(y),
\end{aligned}
$$

conjugating f_h to $\tilde{f}_h$ of the form (3) satisfying (4), for each $h = 1, \dots, m$, as we wanted. □

Remark 2.3. It is possible to give the formal analogous of Definition 1.4, and then to prove a formal analogous of Proposition 2.1 and Corollary 2.2, exactly as in [R].

In the proof of Theorem 1.1 we shall use the following result we proved in [R]

Theorem 2.4. (Raissy, 2007) *Let f be a germ of biholomorphism of $\mathbb{C}^n$ having the origin O as a quasi-Brjuno fixed point of order s. Then f is holomorphically linearizable if and only if it admits an osculating manifold M of codimension s such that $f|_M$ is holomorphically linearizable.*

We can now prove our result.

Theorem 2.5. *Let $f_1, \dots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$, and that it commutes with f_h for any $h = 2, \dots, m$. Then $f_1, \dots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \dots, m$, which is a simultaneous osculating manifold for $f_1, \dots, f_m$ and such that $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable.*

Proof. Let M be a germ of complex manifold at O of codimension s, invariant under f_h for each $h = 1, \dots, m$ which is a simultaneous osculating manifold for $f_1, \dots, f_m$ and such that $f_1|_M, \dots, f_m|_M$ are simultaneously holomorphically linearizable. Thanks to the hypotheses we can choose local holomorphic coordinates

$$(x, y) = (x_1, \dots, x_s, y_1, \dots, y_r)$$

such that f_1 is of the form

$$\begin{aligned} x'_i &= \lambda_{1,i} x_i + f_i^{(1),1}(x, y) \quad \text{for } i = 1, \dots, s, \\ y'_j &= \mu_{1,j} y_j + f_j^{(1),2}(x, y) \quad \text{for } j = 1, \dots, r = n - s, \end{aligned}$$

and, for $h = 2, \dots, m$, each f_h is of the form

$$\begin{aligned} x'_i &= \sum_{p=1}^{s} a_{i,p}^{(h)} x_p + f_i^{(h),1}(x, y) \qquad \text{for } i = 1, \dots, s, \\ y'_j &= f_j^{(h)\text{lin}}(x, y) + f_j^{(h),2}(x, y) \quad \text{for } j = 1, \dots, r = n - s, \end{aligned}$$

where $f_j^{(h)\text{lin}}(x, y)$ is linear, and for each $k = 1, \dots, m$

$$\begin{aligned} \text{ord}_x(f_i^{(k),1}) &\geq 2, \\ \text{ord}_x(f_j^{(k),2}) &\geq 1, \end{aligned}$$

that is

$$\begin{aligned} f_i^{(k),1}(x, y) &= \sum_{\substack{|K| \geq 2 \\ |K'| \geq 2}} f_{K,i}^{(k),1} x^{K'} y^{K''} \quad \text{for } i = 1, \dots, s, \\ f_j^{(k),2}(x, y) &= \sum_{\substack{|K| \geq 2 \\ |K'| \geq 1}} f_{K,j}^{(k),2} x^{K'} y^{K''} \quad \text{for } j = 1, \dots, r, \end{aligned}$$

where $K = (K', K'') \in \mathbb{N}^s \times \mathbb{N}^r = \mathbb{N}^n$ and $|K| = \sum_{p=1}^{n} K_p$.

Thanks to Theorem 2.4 and its proof, we know that f_1 is holomorphically linearizable via a linearization ψ of the form

$$\begin{aligned} x_i &= u_i + \psi_i^1(u, v) \quad \text{for } i = 1, \dots, s, \\ y_j &= v_j + \psi_j^2(u, v) \quad \text{for } j = 1, \dots, r, \end{aligned}$$

where $(u, v) = (u_1, \dots, u_s, v_1, \dots, v_r)$ and

$$\begin{aligned} \text{ord}_u(\psi_i^1) &\geq 2, \\ \text{ord}_u(\psi_j^2) &\geq 1, \end{aligned}$$

that is

$$\begin{aligned} \psi_i^1(u, v) &= \sum_{\substack{|K| \geq 2 \\ |K'| \geq 2}} \psi_{K,i}^1 u^{K'} v^{K''} \quad \text{for } i = 1, \dots, s, \\ \psi_j^2(u, v) &= \sum_{\substack{|K| \geq 2 \\ |K'| \geq 1}} \psi_{K,j}^2 u^{K'} v^{K''} \quad \text{for } j = 1, \dots, r. \end{aligned}$$

Since $\psi^{-1}\circ f_1\circ\psi=\mathrm{Diag}(\lambda_{1,1},\ldots,\lambda_{1,s},\mu_{1,1},\ldots,\mu_{1,r})$ commutes with $\tilde f_h=\psi^{-1}\circ f_h\circ\psi$ for each $h=2,\ldots,m$, and $(\lambda_{1,1},\ldots,\lambda_{1,s},\mu_{1,1},\ldots,\mu_{1,r})$ has only level s resonances, it is immediate to verify that $\tilde f_h$ has the form

$$\begin{aligned} u_i' &= \sum_{p=1}^s a_{i,p}^{(h)}u_p+\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,i}}} u_l\tilde f_{l,i}^{(h),1}(v) && \text{for } i=1,\ldots,s,\\ v_j' &= f_j^{(h)\mathrm{lin}}(u,v)+\tilde f_j^{(h),2}(v) && \text{for } j=1,\ldots,r. \end{aligned}$$

Moreover, since $f_h\circ\psi=\psi\circ\tilde f_h$, we have

$$\begin{aligned} &\sum_{p=1}^s a_{i,p}^{(h)}\sum_{\substack{|K|\ge2\\|K'|\ge2}}\psi^1_{K,p}u^{K'}v^{K''}+\sum_{\substack{|K|\ge2\\|K'|\ge2}} f_{K,i}^{(h),1}(u+\psi^1(u,v))^{K'}(v+\psi^2(u,v))^{K''}\\ &\qquad=\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,i}}} u_l\tilde f_{l,i}^{(h),1}(v)\\ &\qquad\quad+\sum_{\substack{|K|\ge2\\|K'|\ge2}}\psi^1_{K,i}\left(\sum_{p=1}^s a_{1,p}^{(h)}u_p+\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,1}}} u_l\tilde f_{l,1}^{(h),1}(v)\right)^{K_1}\cdots\left(\sum_{p=1}^s a_{s,p}^{(h)}u_p+\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,s}}} u_l\tilde f_{l,s}^{(h),1}(v)\right)^{K_s}\\ &\qquad\qquad\times(f^{(h)\mathrm{lin}}(u,v)+\tilde f^{(h),2}(v))^{K''} \end{aligned}\tag{5}$$

for $i=1,\ldots,s$, and

$$\begin{aligned} &\sum_{q=1}^r b_{j,q}^{(h)}\sum_{\substack{|K|\ge2\\|K'|\ge1}}\psi^2_{K,q}u^{K'}v^{K''}+\sum_{p=1}^s c_{j,p}^{(h)}\sum_{\substack{|K|\ge2\\|K'|\ge2}}\psi^1_{K,p}u^{K'}v^{K''}\\ &\qquad\qquad+\sum_{\substack{|K|\ge2\\|K'|\ge1}} f_{K,j}^{(h),2}(u+\psi^1(u,v))^{K'}(v+\psi^2(u,v))^{K''}\\ &\qquad=\tilde f_j^{(h),2}(v)\\ &\qquad\quad+\sum_{\substack{|K|\ge2\\|K'|\ge1}}\psi^2_{K,i}\left(\sum_{p=1}^s a_{1,p}^{(h)}u_p+\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,1}}} u_l\tilde f_{l,1}^{(h),1}(v)\right)^{K_1}\cdots\left(\sum_{p=1}^s a_{s,p}^{(h)}u_p+\sum_{\substack{1\le l\le n\\ \lambda_{1,l}=\lambda_{1,s}}} u_l\tilde f_{l,s}^{(h),1}(v)\right)^{K_s}\\ &\qquad\qquad\times(f_j^{(h)\mathrm{lin}}(u,v)+\tilde f^{(h),2}(v))^{K''} \end{aligned}\tag{6}$$

for $j=1,\ldots,r$.

Now, it is not difficult to verify that there are no terms of the form $u^{K'}v^{K''}$ with $|K'|=1$ in the left-hand side of (5), whereas in the right-hand side terms of this form are given only by the sum of the $u_l\tilde f_{l,i}^{(h),1}(v)$; therefore it must be

$$\tilde f_{l,i}^{(h),1}(v)\equiv0,$$

for all pairs l, i. Similarly, there are no terms of the form $u^{K'}v^{K''}$ with $K'=O$ in the left-hand side of (6), whereas, again, in the right-hand side terms of this form are given by $\tilde f_j^{(h),2}(v)$ only; so

$$\tilde f_j^{(h),2}(v)\equiv0\quad\text{for } j=1,\ldots,r.$$

This proves that $\tilde{f}_h$ is linear for every $h = 2, \ldots, m$, that is ψ is a simultaneous holomorphic linearization for $f_1, \ldots, f_m$.

The other direction is clear. In fact, if f_1 commutes with $f_2, \ldots, f_m$ and $f_1, \ldots, f_m$ are linear, then the eigenspace of f_1 relative to the eigenvalues $\mu_{1,1}, \ldots, \mu_{1,r}$ is a simultaneous osculating manifold for $f_1, \ldots, f_m$ (and $f_1|_M, \ldots, f_m|_M$ are linear), where $(\lambda_{1,1}, \ldots, \lambda_{1,s}, \mu_{1,1}, \ldots, \mu_{1,r})$ is the spectrum of f_1. □

Corollary 2.6. *Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of commuting biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a quasi-Brjuno fixed point of order s, with $1 \leq s \leq n$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable if and only if there exists a germ of complex manifold M at O of codimension s, invariant under f_h for each $h = 1, \ldots, m$ which is a simultaneous osculating manifold for $f_1, \ldots, f_m$ and such that $f_1|_M, \ldots, f_m|_M$ are simultaneously holomorphically linearizable.*

As a final corollary, taking $s = n$ in Theorem 2.5, one gets

Corollary 2.7. *Let $f_1, \ldots, f_m$ be $m \geq 2$ germs of biholomorphisms of $\mathbb{C}^n$, fixing the origin. Assume that f_1 has the origin as a Brjuno fixed point, and that it commutes with f_h for any $h = 2, \ldots, m$. Then $f_1, \ldots, f_m$ are simultaneously holomorphically linearizable.*

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