A PDF method applied to local wind simulation

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Journée Méthodes Particulaires
en Mécanique des Fluides
Météo-France Toulouse

Marsh 16, 2007

Local wind forecasting

Project supported by the French Agency for the Environment and Energy Management (ADEME).

Aim : propose a numerical method to improve the wind forecasting at small scales.

Joint work with

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- P. Drobinski, T. Salameh (Laboratoire de Météorologie Dynamique)
- E. Peirano (ADEME)

Context and Objectives

Development of the wind power resources in France :

in Marsh 2006, the production was around 1 GW.
France plans to produce 10 GW in 2010.

Our objective :

To develop and evaluate a stochastic downscaling method to compute the wind field at small scale, near the surface, from a meteorological field of large scale ($>$ 10-50 km), simulated by a meteorological model.

Geographical framework

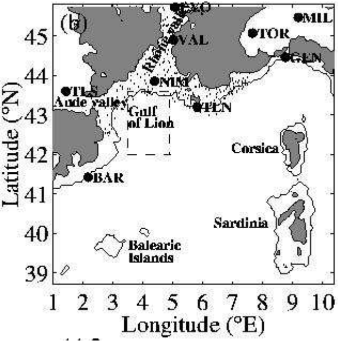
The French part of the Mediterranean basin :
(Languedoc-Roussillon, Provence-Alpes-Côtes d'Azur, Rhône-Alpes) :

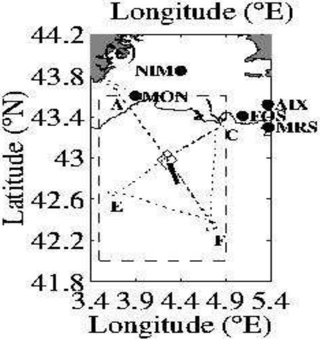
- First region in terms of production, with a high potential to develop.
- Mediterranean climate, mainly forced by the large scale climatic conditions, during the winter (November to Marsh).
- Complex association between large scale and regional scale (10 – 100 km). Important role of orography, the ground and sea contrast.

Meteorological forcing of the Downscaling Stochastic Method (DSM)

We used the numerical model *MM*5 (mesoscale meteorological solver developed at the Pennsylvania State University/National Center for Atmospheric Research, USA)

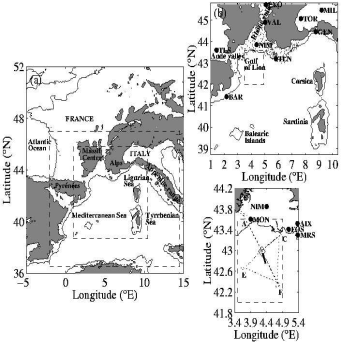
Three computational model domains interactively imbricate:





Meteorological forcing of the (DSM)

MM5 solves the non-hydrostatic equations of motion in a terrain-following sigma coordinates.



Three computational model domains interactively imbricate are used. the horizontal mesh size is 27 km, 9 km and 3 km, respectively.

Coarse domain, medium domain and fine domain are centered at 43.7°N, 4.6°E and cover an area of 1350 km$\times$1350 km, 738 km$\times$738 km and 120 km$\times$174 km, respectively.

Meteorological forcing of (DSM)

The March 24, 1998 Mistral event :

MM5 Initial condition and boundary condition : taken from the ECMWF (European Centre for Medium Range Weather Forecast) reanalyses ERA-40. These reanalysis data are available every six hours on a $1^{\circ} \times 1^{\circ}$ latitude-longitude grid.

Validation of MM5 on this event (radiosoundings launched from Nîmes, on March 24, 1998) see Salameh, Drobinski et all *Atmos. Chem. Phys. Discuss 2006*

Forecasting solvers like $MM5$ have a computational limit in both horizontal and vertical ranges. **Difficult (hopeless) to run $MM5$ with an horizontal resolution finer than $1km$.**

Our down-scaling approach

- Improve the resolution in a given sub-domain (local computations),
- by introducing a local model (compatible with the Navier Stokes equation),
- solved by a particle method which does not require any stability (CFL) condition.

Agenda

- The local model : Lagrangian modeling of turbulent flows.
- The numerical algorithm and related issues.
- Mathematical analysis of the (simplified) generalized (confined) Langevin model.
- Go back to the meteorological context.

Modeling of turbulent flows

Statistical approach of turbulent flows :

The properties of the fluid are all random fields.

The Reynolds averages (or ensemble averages) are expectations:

$$\langle \mathscr{U} \rangle(t,x) := \int_\Omega \mathscr{U}(t,x,\omega) d\mathbb{P}(\omega).$$

The corresponding Reynolds decomposition of the velocity is

$$\mathscr{U}(t,x,\omega) = \langle \mathscr{U} \rangle(t,x) + \mathbf{u}(t,x,\omega).$$

The random field $\mathbf{u}(t,x,\omega)$ is the turbulent part of the velocity.

Incompressible Navier Stokes equation in $\mathbb{R}^3$, for the velocity field $\mathscr{U} = (\mathscr{U}^{(1)}, \mathscr{U}^{(2)}, \mathscr{U}^{(3)})$ and the pressure $\mathscr{P}$:

$$\partial_t \mathscr{U} + (\mathscr{U} \cdot \nabla)\mathscr{U} = \nu \Delta \mathscr{U} - \frac{1}{\rho} \nabla \mathscr{P}, \quad t > 0,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

$$\nabla \cdot \mathscr{U} = \partial_{x_i} \mathscr{U}_i = 0, \quad t \geq 0,\ x \in \mathbb{R}^3, \text{constant mass density } \rho$$

$$\mathscr{U}(0,x) = \mathscr{U}_0(x),\ x \in \mathbb{R}^3.$$

The Reynolds averaged equation for the mean velocity

Assuming Reynolds decomposition, we obtain the unclosed equation

$$\partial_t \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \langle \mathscr{U}^{(j)} \rangle \partial_{x_j} \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \nu \Delta \langle \mathscr{U}^{(i)} \rangle - \frac{1}{\rho} \partial_{x_i} \langle \mathscr{P} \rangle,$$

$\nabla . \langle \mathscr{U} \rangle = 0,\ t \geq 0,\ x \in \mathbb{R}^3,$ constant mass density

$\langle \mathscr{U} \rangle (0, x) = \langle \mathscr{U}_0 \rangle (x),\ x \in \mathbb{R}^3.$ $\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle - \langle \mathscr{U}^{(i)} \rangle \langle \mathscr{U}^{(j)} \rangle.$

One needs to model the equation of the Reynolds stress $(\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle, i, j)$.

Direct modeling of the Reynolds stress :

the so-called k-ε turbulence models,

where

$$\text{kinetic turbulent energy } k(t,x) := \sum_{i=1}^{3} \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle (t,x)$$

and

$$\text{pseudo-dissipation } \varepsilon(t,x) := \nu \sum_{i=1}^{3} \sum_{j=1}^{3} \langle \partial_{x_j} \mathbf{u}^{(i)} \partial_{x_j} \mathbf{u}^{(i)} \rangle (t,x).$$

An alternative approach to compute the Reynolds stress

Let $f_E(V; t, x)$ be the probability density function (PDF) of the random field $\mathscr{U}(t, x)$, then

$$\langle \mathscr{U}^{(i)} \rangle(t, x) = \int_{\mathbb{R}^3} V^{(i)} f_E(V; t, x) dV,$$
$$\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle(t, x) = \int_{\mathbb{R}^3} V^{(i)} V^{(j)} f_E(V; t, x) dV.$$

The closure problem is reported on the PDE satisfied by the probability density function f_E.

In a series of papers (see e.g. Pope 85), Stephen B. Pope propose to model the p.d.f. f_E with a Lagrangian description of the flow.

The Lagrangian approach to compute the Reynolds stresses

Main ideas

- Describe the Lagrangian properties thought a stochastic model with a state vector $(\mathrm{X}, \mathrm{U}, \psi)$ which include
  - particle location,
  - particle velocity
  - scalar variables standing for any particles properties,

  and use a diffusion process to simulate its time rate of change.
- Those models are referred to as Langevin models.
- The associated SDE must be consistent with the macroscopic evolution of the fluid (in particular the averaged Navier-Stokes equation).

The Simplified Langevin model (Pope 94)

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \left(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t) \right) \Big] dt \\ & + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t^{(i)}, \ \forall \ i \in \{1, 2, 3\} \end{cases}$$

+ boundary conditions + wall boundary functions.

where $\varepsilon(t, x)$ and $k(t, x)$ are supposed to be known. $\langle \mathscr{P} \rangle (t, x)$ must be recovered by the Poisson equation

$$\nabla^2 \langle \mathscr{P} \rangle = -\frac{\partial^2 \left\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \right\rangle}{\partial x_i \partial x_j}$$

which guarantees that the averaged Eulerian velocity is divergence free.

Compute the Reynolds averages $\langle \mathscr{U}^{(i)} \rangle (t,x)$ and $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t,x)$

We call $f_L(V,x;t)$ the probability density function of $(\mathrm{U}_t, \mathrm{X}_t)$.

Contrary to f_E, f_F satisfies a closed (nonlinear) PDE : the Fokker-Planck equation a associated to the Langevin SDE.
In the case of incompressible flow with a constant mass density,

$$f_E(V;x,t) = \frac{f_L(V,x;t)}{\int_{\mathbb{R}^3} f_L(V,x;t)dV},$$

and for any bounded measurable function $g(v)$,

$$\langle g(\mathscr{U})\rangle(t,x) = \mathbb{E}\left(g(\mathrm{U}_t)/\mathrm{X}_t = x\right).$$

In particular,

$$\left\langle \mathscr{U}^{(i)} \right\rangle (t,x) = \int_{\mathbb{R}} V^{(i)} \frac{f_L(V,x;t)}{\int_{\mathbb{R}^3} f_L(U,x;t)dU} dV = \mathbb{E}\left(\mathrm{U}_t^{(i)}/\mathrm{X}_t = x\right).$$

The Basic model (Dreeben Pope 98)

Include the instantaneous turbulence frequency ω, satisfying

$$
\begin{cases}
d\mathrm{X}_t = & \mathrm{U}_t dt, \\
d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\
& \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \langle \omega \rangle (t, \mathrm{X}_t) \left(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t) \right) \Big] dt \\
& + \sqrt{C_0 k(t, \mathrm{X}_t) \langle \omega \rangle (t, \mathrm{X}_t)} dW_t^{(i)}, \ \forall\ i \in \{1, 2, 3\} \\
 & \\
d\omega_t = & -C_3 \langle \omega \rangle (t, X_t) \left(\omega_t - \langle \omega \rangle (t, X_t) \right) dt - S_\omega \langle \omega \rangle (t, X_t) \omega_t dt \\
& + \sqrt{2 C_3 C_4 \langle \omega \rangle^2 (t, X_t) \omega_t} dW_t^{(4)}.
\end{cases}
$$

where

$$
S_\omega = C_{\omega 2} + C_{\omega 1} \frac{\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle (t, x)}{\varepsilon(t, x)} \frac{\partial \langle \mathscr{U}^{(i)} \rangle}{\partial x_j}(t, x).
$$

$\varepsilon(t, x)$ is recovered by the closure relation $\langle \omega \rangle (t, x) = \dfrac{\varepsilon(t, x)}{k(t, x)}$.

Application to meteorology

Use the same parametrisation than MM5 :

$$
\begin{cases}
d\mathrm{X}_t = & \mathrm{U}_t dt, \\
d\mathrm{U}_t^{(i)} = & \Big[-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t) \\
& \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \left(\mathrm{U}_t^{(i)} - \left\langle \mathscr{U}^{(i)} \right\rangle (t, \mathrm{X}_t) \right) \Big] dt \\
& + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t^{(i)}, \ \forall \ i \in \{1, 2, 3\}
\end{cases}
$$

- The kinetic turbulent energy $k(t,x) = \frac{1}{2}\langle \mathbf{u}^{(i)}\mathbf{u}^{(i)} \rangle(t,x)$ is computed inside the model.
- $\varepsilon(t,x)$ is recovered by the closure relation $\varepsilon(t,x) = \frac{C}{L} k^{3/2}(t,x)$.

Application to meteorology, with the MM5 forcing

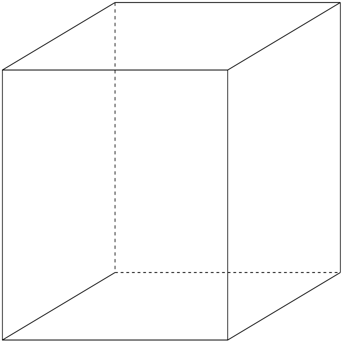
Let $\mathbb{D}$ our local (space) computational domain

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t = & \Big[-\frac{1}{\rho}\nabla \langle \mathscr{P} \rangle (t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t,\mathrm{X}_t)}{k(t,\mathrm{X}_t)} \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t,\mathrm{X}_t)\right)\Big] dt \\ & + \sqrt{C_0 \varepsilon(t,\mathrm{X}_t)} dW_t, \\ & - \sum_{0 \le s \le t} 2 \left(\mathrm{U}_{s^-} \cdot n(\mathrm{X}_s)\right) n(\mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}} \\ & - \sum_{0 \le s \le t} 2 \left(\mathrm{U}_{s^-} \cdot n^{\perp}(\mathrm{X}_s)\right) n^{\perp}(\mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}} \\ & + \sum_{0 \le s \le t} V_{MM5}(s, \mathrm{X}_s) \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}}. \end{cases}$$

The three last terms should guarantee

$$\langle \mathscr{U} \rangle (t,x) := \mathbb{E}\left[\mathrm{U}_t / X_t = x\right] = V_{MM5}(t,x), \forall x \in \partial\mathbb{D}.$$

The numerical framework :

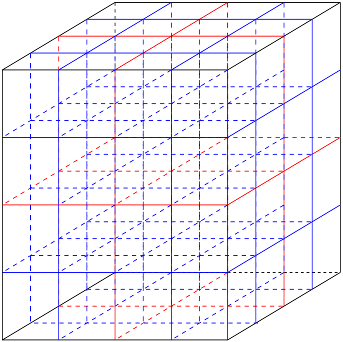


Our computational domain $\mathbb{D}$ (for example, a given cell of the *MM*5 solver.

Boundary condition :

$$\forall x \in \partial\mathbb{D},\ \langle \mathscr{U} \rangle(t,x) = V_{MM5}(t,x)$$

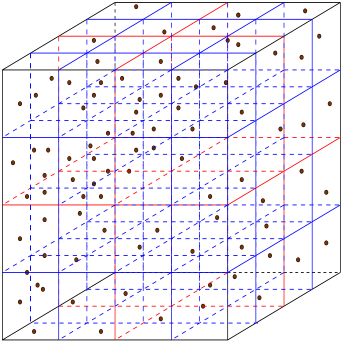
(*MM*5 guideline.)



The computational space is divided in cells of given size.
Particle in cell (P.I.C.) technique to approximate the Eulerian fields like $\left\langle \mathscr{U}^{(i)} \right\rangle (t,x)$.

We compute the Eulerian fields (mean fields) at the center of each sub-cell only.

The numerical framework : particle method



- Introduce N.
- Constant mass density constraint $\Rightarrow$ constant number of particles in each cell.
- Approximation of the conditional expectation :

if $\mathcal{V}_x$ denotes the cell centered in x then the approximation of $\left\langle \mathscr{U}^{(i)} \right\rangle (t,x)$ is given by

$$\left\langle \mathscr{U}^{(i)} \right\rangle (t,x) \simeq \frac{C}{\mathrm{Volume}(\mathcal{V}_x)} \left(\frac{1}{N} \sum_{l=1}^{N} \mathrm{U}_t^{(i),l,N} \, \mathbb{1}_{\{\mathrm{X}_t^{l,N} \in \mathcal{V}_x\}} \right).$$

Particle discretization

$$
\begin{aligned}
\left\langle \mathscr{U}^{(l)} \right\rangle (t,x) &= \mathbb{E}\left(\mathrm{U}_t^{(l)} / \mathrm{X}_t = s\right) \\
&\simeq \frac{\mathbb{E}\left[\mathrm{U}_t^{(l)})\phi_\varepsilon(x,\mathrm{X}_t)\right]}{\mathbb{E}\left[\phi_\varepsilon(x,\mathrm{X}_t)\right]} \\
&\simeq \frac{\frac{1}{N}\sum_{i=1}^{N}\mathrm{U}_t^{(l),i,N}\phi_\varepsilon(x, Xx_t^{i,N})}{\frac{1}{N}\sum_{i=1}^{N}\left[\phi_\varepsilon(x,\mathrm{X}_t^{i,N})\right]}.
\end{aligned}
$$

When $\phi_\varepsilon(x,y) = \mathbb{1}_{\{x \text{ and } y \text{ are in the same sub-cell of size } \varepsilon\}}$,

$$
\left\langle \mathscr{U}^{(i)} \right\rangle (t,x) \simeq \frac{C}{\mathrm{Volume}(\mathcal{V}_x)}\left(\frac{1}{N}\sum_{l=1}^{N}\mathrm{U}_t^{(i),l,N}\,\mathbb{1}_{\{\mathrm{X}_t^{l,N}\in\mathcal{V}_x\}}\right).
$$

The numerical algorithm

The N-Particles dynamic : for $j = 1, \dots, N$

$$\begin{cases} d\mathrm{X}_t^{j,N} = \mathrm{U}_t^{j,N} dt, \\ d\mathrm{U}_t^{(i),j,N} = -\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t^{j,N}) dt \\ \qquad\qquad + D_{\mathrm{U}}(t, X_t^{j,N}) dt + B_{\mathrm{U}}(t, X_t^{j,N}) dW_t^{(i),j,N} \\ \qquad\qquad + MM5 \text{ guideline terms at the boundary, } \forall\ i \in \{1,2,3\} \\ \\ d\omega_t^{j,N} = D_\omega(t, X_t^{j,N}) dt + B_\omega(t, X_t^{j,N}) dW^{(4),j,N}. \end{cases}$$

The coefficients D_{U}, D_ω, B_{U} and B_ω depend on the particles approximations of $\langle \mathscr{U} \rangle$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle$ and its derivatives, $\langle \omega \rangle$.
$-\frac{1}{\rho} \frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \mathrm{X}_t^{j,N})$ ensures that $\nabla \cdot \langle \mathscr{U} \rangle = 0$ and maintains the mass density constant.

A fractional step method : $n\Delta t \longrightarrow (n+1)\Delta t$ (Pope 85)

The N-Particles dynamic : for $j = 1, \ldots, N$, for $n\Delta t \leq t \leq (n+1)\Delta t$,

$$\begin{cases} d\tilde{\mathrm{X}}_t^{j,N} = \tilde{\mathrm{U}}_t^{j,N} dt, \\ d\tilde{\mathrm{U}}_t^{(i),j,N} = -\frac{1}{\rho}\frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \tilde{\mathrm{X}}_t^{j,N})dt \\ \qquad\qquad + D_{\tilde{\mathrm{U}}}(t, X_t^{j,N})dt + B_{\tilde{\mathrm{U}}}(t, X_t^{j,N})dW_t^{(i),j,N} \\ \qquad\qquad + MM5 \text{ guideline terms at the boundary, } \forall\ i \in \{1,2,3\} \\ d\omega_t^{j,N} = D_\omega(t, X_t^{j,N})dt + B_\omega(t, X_t^{j,N})dW^{(4),j,N}. \\ \mathrm{X}_{n\Delta t}^{j,N}, \mathrm{U}_{n\Delta t}^{(i),j,N} \omega_{n\Delta t}^{j,N} \text{ given.} \end{cases}$$

- Correction of the position of the particles $\tilde{\mathrm{X}}_{(n+1)\Delta t}^{j,N} \longrightarrow \mathrm{X}_{(n+1)\Delta t}^{j,N}$, in order to maintain the (discrete) uniform distribution.
- Correction of the particles velocity

$$\tilde{\mathrm{U}}_{(n+1)\Delta t}^{j,N} \longrightarrow \mathrm{U}_{(n+1)\Delta t}^{j,N}$$

such that $\nabla.\langle \mathscr{U}^{(n+1)} \rangle = 0$.

Correction of the positions of the particles

Move the particles, such that the corresponding distribution becomes uniform.
Minimize the global amount of displacement.
The density $\rho(x)$ is an Eulerian quantity approximated thanks to the nearest grid point formula

$$\rho(x_i) = \frac{\#\{\text{particles in } C_i\}}{N_{ppc}}.$$

Can be viewed as a discretization of an optimal continuous transport problem (Brenier) :
Find a transport map $\phi : \mathbb{D} \to \mathbb{D}$, satisfying $\forall A \subset \mathbb{D}$

$$\int_{\phi^{-1}(A)} \rho(x)dx = \int_A \rho_0(x)dx$$

minimizing the L^2-cost

$$K(\phi) = \int_{\mathbb{D}} |x - \phi(x)|^2 dx.$$

Correction of the positions of the particles

Well-known problem, having a well-know solution (see Benamou Brenier 2000 and ref. herein) : ϕ is unique and given by the Monge Ampère equation

$$\phi = \mathbb{1}_{\mathbb{D}} - \nabla\gamma$$

$$\text{with } \gamma \text{ satisfying } \rho(x) = \det \begin{pmatrix} 1 - \frac{\partial^2\gamma}{\partial x_1^2} & -\frac{\partial^2\gamma}{\partial x_1 \partial x_2} & -\frac{\partial^2\gamma}{\partial x_1 \partial x_3} \\ -\frac{\partial^2\gamma}{\partial x_1 \partial x_2} & 1 - \frac{\partial^2\gamma}{\partial x_2^2} & -\frac{\partial^2\gamma}{\partial x_2 \partial x_3} \\ -\frac{\partial^2\gamma}{\partial x \partial x_3} & -\frac{\partial^2\gamma}{\partial x_2 \partial x_3} & 1 - \frac{\partial^2\gamma}{\partial x_3^2} \end{pmatrix},$$

Numerical discretization : difficult

Pope 85 : neglect the nonlinear terms

$$\begin{cases} \Delta\gamma(x) = 1 - \rho(x),\ x \in \mathbb{D}, \\ \left.\dfrac{\partial\gamma}{\partial n}\right|_{\partial\mathbb{D}} = 0. \end{cases}$$

Eventually iterate the process until convergence.

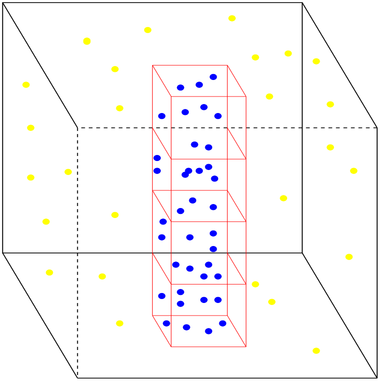
Correction of the positions of the particles : alternative strategy

Suppose $\mathbb{D} = (0, 1)$. The optimal transport is then entirely determined by the transfer condition :

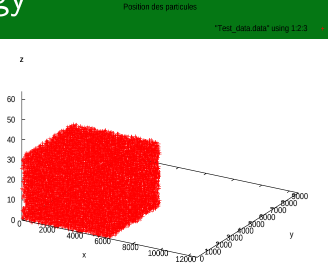
$$\forall\, x \in \mathbb{D}, \quad \phi(x) = \int_0^x \rho(y) dy,$$

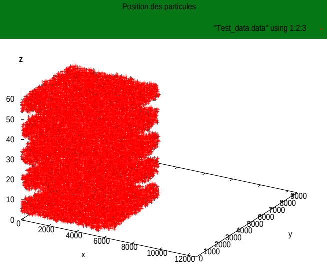
We can solve directly the 1D discrete optimal transport problem.

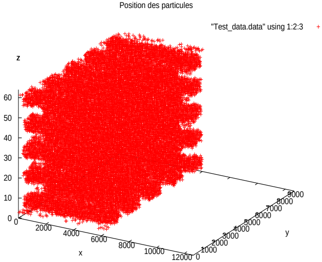
The 3D case is a collection of 1D cases in the three directions.

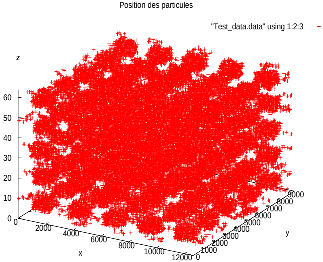


Correction of the positions of the particles : alternative strategy









Sub-optimal transport procedure.

Correction of the positions of the particles : second alternative strategy

Birth-Death procedure in each cell, under the conservation of momentum constraint.

- If $\#\{\text{particles in } C_i\} < N_{ppc}$, add particles with the Eulerian characteristics at x_i .
- If $\#\{\text{particles in } C_i\} > N_{ppc}$, destroy $\#\{\text{particles in } C_i\} - N_{ppc} + 1$ particles and add a new one whose characteristics are their averages.

The field $\tilde{\mathrm{U}}^{(n+1)\Delta t}$ computed from the SDEs does not match the following conservation law:

$$\nabla \cdot \langle \tilde{\mathrm{U}}^{(n+1)\Delta t} \rangle = 0$$

We hence solve the following equation:

$$\begin{cases} \Delta P = -\frac{1}{\Delta t} \nabla \cdot \langle \tilde{\mathrm{U}}^{(n+1)\Delta t} \rangle, \ x \in \mathbb{D}, \\ \frac{\partial P}{\partial n}\Big|_{\partial \mathbb{D}} = 0, \end{cases}$$

and update the velocity field thanks to:

$$\mathrm{U}^{(n+1)\Delta t} = \tilde{\mathrm{U}}^{(n+1)\Delta t} - \Delta t \nabla P.$$

This insures the free divergence of $\langle \mathrm{U}^{(n+1)\Delta t} \rangle$.
Compatibility condition for the Poisson problem :

$$\int_{\partial \mathbb{D}} \langle \tilde{\mathrm{U}}^{(n+1)\Delta t} \rangle ds = \int_{\partial \mathbb{D}} V_{MM5} ds \simeq 0,$$

as $MM5$ is a divergence free solver.

The fractional step method : $n\Delta t \longrightarrow (n+1)\Delta t$

The N-Particles dynamic : for $j = 1, \dots, N$, for $n\Delta t \le t \le (n+1)\Delta t$,

$$\begin{cases} d\tilde{\mathrm{X}}_t^{j,N} = \tilde{\mathrm{U}}_t^{j,N} dt, \\ d\tilde{\mathrm{U}}_t^{(i),j,N} = -\frac{1}{\rho}\frac{\partial \langle \mathscr{P} \rangle}{\partial x_i}(t, \tilde{\mathrm{X}}_t^{j,N}) dt \\ \qquad\qquad +D_{\tilde{\mathrm{U}}}(t, X_t^{j,N}) dt + B_{\tilde{\mathrm{U}}}(t, X_t^{j,N}) dW_t^{(i),j,N} \\ \qquad\qquad +MM5 \text{ guideline terms at the boundary, } \forall\ i \in \{1,2,3\} \\ d\omega_t^{j,N} = D_\omega(t, X_t^{j,N}) dt + B_\omega(t, X_t^{j,N}) dW^{(4),j,N}. \\ \mathrm{X}_{n\Delta t}^{j,N}, \mathrm{U}_{n\Delta t}^{(i),j,N} \omega_{n\Delta t}^{j,N} \text{ given.} \end{cases}$$

- Correction of the particles position : Optimal Transport Problem.
- Correction of the particles velocity : Poisson Equation.

Mathematical study of a very simplified Langevin model

$$\begin{cases} d\mathrm{X}_t = \mathrm{U}_t dt, \\ d\mathrm{U}_t = \mathbb{E}\left[b(\mathrm{U}_t)/X_t\right] dt + dW_t, \;\; t \in [0, T] \end{cases}$$

Nonlinear but smooth drift term $b : \mathbb{R}^d \to \mathbb{R}^d$, bounded continuous.

No divergence free condition.

Related works

Sznitman 86 : PoC for the Burgers' equation :

$$X_t = X_0 + W_t + 2\int_0^t u(s, X_s) ds$$

$u(t, x)dx$ is the law of X_t.

Dermoune 03 : PoC and conditional PoC for pressurless gas equations

$$X_t = X_0 + W_t + \int_0^t \mathbb{E}[v(X_0)/X_s] ds.$$

Propagation of chaos property, formal definition

Definition (Sznitman 89)

Let E be a separable metric space and ν a probability measure on E. A sequence of symmetric probabilities ν^N on E^N is ν-chaotic if for any $\phi_1, \ldots, \phi_k \in C_b(E; \mathbb{R})$, $k \geq 1$,

$$\lim_{N \to \infty} \left\langle \nu^N, \phi_1 \otimes \ldots \otimes \phi_k \otimes 1 \ldots \otimes 1 \right\rangle = \prod_{l=1}^{k} \langle \nu, \phi_l \rangle.$$

Propagation of chaos property, prototypic example :

$$X_t^{i,N} = X_0^i + \int_0^t \frac{1}{N}\sum_{j=1}^N \sigma(X_s^{i,N}, X_s^{j,N})dW_s^i + \int_0^t \frac{1}{N}\sum_{j=1}^N b(X_s^{i,N}, X_s^{j,N})ds, \ t \in [0,T],$$

Theorem

Suppose that $b(\cdot,\cdot)$ and $\sigma(\cdot,\cdot)$ are Lipschitz continuous kernels on $\mathbb{R}^{2d}$. Let P^N be the joint law on $(C([0,T];\mathbb{R}^d))^N$ of the particle system $(X^{1,N},\ldots,X^{N,N})$. The sequence (P^N) is P-chaotic, where P is given by a nonlinear martingale problem.

The P-chaoticity is equivalent to the convergence of the laws of the empirical measures $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_\cdot^{i,N}}$ to δ_P.

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}^d} \sigma(X_s, x)P_s(dx)dW_s + \int_0^t \int_{\mathbb{R}^d} b(X_s, x)P_s(dx)ds, \ t \in [0,T$$

Well posedness of the simplified Langevin model

$$\begin{cases} d\mathrm{X}_t = \mathrm{U}_t dt, \\ d\mathrm{U}_t = \mathbb{E}\left[b(\mathrm{U}_t)/X_t\right] dt + dW_t, \ \ t \in [0, T] \end{cases}$$

Theorem

Let $b : \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function and let (X_0, U_0) be such that $\mathbb{E}_{\mathbb{P}}\left[\|X_0\|_{\mathbb{R}^d} + \|\mathrm{U}_0\|^4_{\mathbb{R}^d}\right] < +\infty$. Then the system admits a unique weak solution.

Well posedness of the simplified Langevin model

$$\begin{cases} d\mathrm{X}_t = \mathrm{U}_t dt, \\ d\mathrm{U}_t = \mathbb{E}\left[b(\mathrm{U}_t)\big/X_t\right] dt + dW_t,\ \ t \in [0, T]. \end{cases}$$

The smoothed system in the space variable x:
for a given regularization ϕ_ε of the Dirac mass in $\mathbb{R}^d$

$$\begin{cases} d\mathrm{X}^\varepsilon_t = \mathrm{U}^\varepsilon_t dt, \\ d\mathrm{U}^\varepsilon_t = \dfrac{\mathbb{E}\left[b(\mathrm{U}^\varepsilon_t)\phi_\varepsilon(x - \mathrm{X}^\varepsilon_t)\right]\Big|_{x=\mathrm{X}^\varepsilon_t}}{\mathbb{E}\left[\phi_\varepsilon(x - \mathrm{X}^\varepsilon_t)\right]\Big|_{x=\mathrm{X}^\varepsilon_t} + \varepsilon} dt + dW_t,\ \ t \in [0, T] \end{cases}$$

$$\mathcal{L}aw(\mathrm{X}^\varepsilon_t, \mathrm{U}^\varepsilon_t) = p^\varepsilon(t, x, u)dxdu.$$

$$\begin{cases} d\mathrm{X}^\varepsilon_t = \mathrm{U}_t dt, \\ d\mathrm{U}^\varepsilon_t = G_\varepsilon[\mathrm{X}^\varepsilon_t, \mathrm{U}^\varepsilon_t, p^\varepsilon_t]dt + dW_t,\ \ t \in [0, T]. \end{cases}$$

Uniqueness results for the laws of (X, U) and $(\mathrm{X}^\varepsilon, \mathrm{U}^\varepsilon)$

Uniqueness result in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ for the mild equations

$$p_t = S_t(p_0) + \int_0^t S'_{t-s}(p_s(.)B[s,.])ds, \text{ where } B[t,x] = \frac{\int_{\mathbb{R}^d} b(u)p_t(x,u)du}{\int_{\mathbb{R}^d} p_t(x,u)du}.$$

$$p_t^\varepsilon = S_t(p_0) + \int_0^t S'_{t-s}(p_s^\varepsilon B_\varepsilon[s,.])ds,$$

$$\text{where } B_\epsilon[s,x] = \frac{\int_{\mathbb{R}^d\times\mathbb{R}^d} b(u')\phi_\epsilon(x-x')p_s^\epsilon(x',u')dx'du'}{\int_{\mathbb{R}^d\times\mathbb{R}^d} \phi_\epsilon(x-x')p_s^\epsilon(x',u')dx'du' + \varepsilon}$$

$$\begin{aligned} \text{with} \quad & S_t(f)(x,u) = \mathbb{E}_{x,u}[f(\mathscr{X}_t, \mathscr{U}_t)] \\ \text{and} \quad & S'_t(f)(x,u) = \mathbb{E}_{x,u}[\nabla_u \cdot f(\mathscr{X}_t, \mathscr{U}_t)]. \end{aligned}$$

$$\mathscr{U}_t = \mathscr{U}_0 + W_t, \ \mathscr{X}_t = \mathscr{X}_0 + \int_0^t \mathscr{U}_s ds.$$

Existence

$$\begin{cases} dX_t^{\varepsilon,i,N} = U_t^{\varepsilon,i,N} dt, \\ dU_t^{\varepsilon,i,N} = \dfrac{\frac{1}{N-1}\sum_{i\neq j} b(U_t^{\varepsilon,i,N})\phi_\varepsilon(X_t^{\varepsilon,j,N} - X_t^{\varepsilon,i,N})}{\frac{1}{N-1}\sum_{i\neq j}\left[\phi_\varepsilon(X_t^{\varepsilon,j,N} - X_t^{\varepsilon,i,N})\right] + \varepsilon} dt + dW_t^i, \;\; t \in [0, T] \end{cases}$$

- Tightness result.
- Propagation of Chaos result.
- Convergence when ε tends to 0.

Existence

$$\begin{cases} d\mathrm{X}_t^{\varepsilon,i,N} = \mathrm{U}_t^{\varepsilon,i,N} dt, \\ d\mathrm{U}_t^{\varepsilon,i,N} = \dfrac{\frac{1}{N-1}\sum_{i\neq j} b(\mathrm{U}_t^{\varepsilon,i,N})\phi_\varepsilon(\mathrm{X}_t^{\varepsilon,j,N} - \mathrm{X}_t^{\varepsilon,i,N})}{\frac{1}{N-1}\sum_{i\neq j}\left[\phi_\varepsilon(\mathrm{X}_t^{\varepsilon,j,N} - \mathrm{X}_t^{\varepsilon,i,N})\right] + \varepsilon} dt + dW_t^i, \;\; t \in [0, T] \end{cases}$$

Theorem

Let $P^{\varepsilon,N}$ the joint law on $(C([0,T];\mathbb{R}^{2d}))^N$ of the particle system $(\mathrm{X}^{\varepsilon,1,N}, \mathrm{U}^{\varepsilon,1,N}, \ldots, \mathrm{X}^{\varepsilon,N,N}, \mathrm{U}^{\varepsilon,N,N})$.

The sequence $(P^{\varepsilon,N})$ is P^ε-chaotic, where P^ε is the law on $C([0,T];\mathbb{R}^{2d})$ of $(\mathrm{X}^\varepsilon, \mathrm{U}^\varepsilon)$.

Equivalently the random measure $\mu^{\varepsilon,N} = \frac{1}{N}\sum_{i=1}^N \delta_{\{\mathrm{X}^{\varepsilon,i,N}, \mathrm{U}^{\varepsilon,i,N}\}}$ converges in law to the deterministic value P^ε.

Spatially Confined Langevin model $\mathbb{D} \in \mathbb{R}^d$

Impact problem with stochastic forcing.

(Deterministic motions, see e.g. Schatzman 98 , Ballard 01).

A Dirichlet condition for the impact problem.

Given a velocity filed V on $[0, T] \times \partial\mathbb{D}$, $\forall t[0, T]$

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t \langle b(\mathrm{X}_s, \mathrm{U}_s)\rangle \, ds + W_t \\ & - \sum_{0 \leq s \leq t} 2 \left(\mathrm{U}_{s-} \cdot n(\mathrm{X}_s)\right) n(\mathrm{X}_s) \, \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}} \\ & - \sum_{0 \leq s \leq t} 2 \left(\mathrm{U}_{s-} \cdot n^{\perp}(\mathrm{X}_s)\right) n^{\perp}(\mathrm{X}_s) \, \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}} \\ & + \sum_{0 \leq s \leq t} V(s, \mathrm{X}_s) \, \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}}. \end{cases}$$

We have to show that, for any $x \in \partial\mathbb{D}$,

$$\langle \mathscr{U} \rangle(t, x) = \mathbb{E}\left[\mathrm{U}_t / X_t = x\right] = V(t, x).$$

Confined Langevin model in $\mathbb{D} = \mathbb{R}^{d-1} \times \mathbb{R}^+$
Averaged no-permeability condition

(X, U) valued on $C([0,T]\,;\mathbb{D}) \times \mathbb{D}([0,T]\,;\mathbb{R}^d)$ s.t.

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t \langle b(\mathscr{U})\rangle (s, \mathrm{X}_s) ds + W_t \\ & - \sum_{0<s\leq t} 2\left(\mathrm{U}_{s-} \cdot n_{\mathbb{D}}(\mathrm{X}_s)\right) n_{\mathbb{D}}(\mathrm{X}_s)\, \mathbb{1}_{\{\mathrm{X}_s \in \partial\mathbb{D}\}} \end{cases}$$

Lemma

The joint law of $(\mathrm{X}_t, \mathrm{U}_t)$ has a density $p(t,x,V)$, satisfying $p(t,x,V) = p(t,x,V - 2(V\cdot n_{\mathbb{D}}(x))n_{\mathbb{D}}(x), \forall x \in \partial\mathbb{D}$. Then we have

$$\mathbb{E}_{\mathbb{P}}\left(\mathrm{U}_t / \mathrm{X}_t = x\right) \cdot n_{\mathbb{D}}(x) = 0,\ \forall x \in \partial\mathbb{D}.$$

Moreover $p(t,x,V)$ is the unique weak solution of the following Vlasov-Fokker-Planck Eq, with a specular boundary condition

The Vlasov-Fokker-Planck Equation

$$\begin{cases} \dfrac{\partial p}{\partial t} = -V \nabla_x p - \nabla_V \cdot \left(\dfrac{\int_{\mathbb{R}^2} b(V) p(t,x,V) dV}{\int_{\mathbb{R}^2} p(t,x,V) dV} \right) + \dfrac{1}{2} \Delta_V p, \\ p_0(x, V) \text{ given,} \\ p(t, 0, V) = p(t, 0, V - 2(V \cdot n_{\mathbb{D}}(x)) n_{\mathbb{D}}(x)). \end{cases}$$

Well posedness of the confined SDE in the half plane

Starting from $(\mathrm{X}_0, \mathrm{U}_0)$ with $\mathrm{X}_0^{(2)} > 0$, and a 2D-Brownian motion (B_t)

$$\mathscr{Y}_t = \mathrm{X}_0 + \int_0^t \mathscr{V}_s ds, \qquad \mathscr{V}_t = \mathrm{U}_0 + B_t.$$

Set $\left(\mathrm{X}_t^{(1)}, \mathrm{U}_t^{(1)}\right) = \left(\mathscr{Y}_t^{(1)}, \mathscr{V}_t^{(1)}\right)$

with $\mathrm{X}_t^{(2)} = |\mathscr{Y}_t^{(2)}|$

$\mathrm{U}_t^{(2)} = \mathscr{V}_t^{(2)} \mathscr{S}_{t^+}$, with $\mathscr{S}_t := sign(\mathscr{Y}_t^{(2)})$.

Lemma

$\mathscr{S}_t$ jump a countable many times and

$$\mathrm{U}_t^{(2)} = \mathrm{U}_0^{(2)} + W_t^{(2)} - 2 \sum_{0<s\leq t} \mathrm{U}_{s-}^{(2)} \mathbb{1}_{\{\mathrm{X}_s^{(2)}=0\}}, \ \mathbb{P}.a.s.$$

where $W_t^{(2)}$ is a Brownian motion.

McKean 63, Lachal 97.

Euler scheme for confined models, $\mathbb{D} = \mathbb{R}^+$

$$\begin{cases} \mathrm{X}_t = & \mathrm{X}_0 + \int_0^t \mathrm{U}_s ds, \\ \mathrm{U}_t = & \mathrm{U}_0 + \int_0^t b(\mathrm{U}_s) ds + W_t - \sum_{0<s\leq t} 2\mathrm{U}_{s-} \mathbb{1}_{\{X_s=0\}} \end{cases}$$

Euler scheme : $\Delta t > 0$ and $K \in \mathbb{N}$ s.t. $T = K\Delta t$; $t_k := k\Delta t$, $1 \leq k \leq K$, $\bar{\mathrm{X}}_0, \bar{\mathrm{U}}_0$ given.

if $\bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k} \geq 0$ then

$$\bar{\mathrm{X}}_{t_{k+1}} = \bar{\mathrm{X}}_{t_k} + \Delta t \bar{\mathrm{U}}_{t_k}$$
$$\bar{\mathrm{U}}_{t_{k+1}} = \bar{\mathrm{U}}_{t_k} + \Delta t b(\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}).$$

else

$$\tau_k = t_k + \bar{\mathrm{X}}_{t_k} / \bar{\mathrm{U}}_{t_k}.$$
$$\bar{\mathrm{X}}_{t_{k+1}} = -(t_{k+1} - \tau_k)\bar{\mathrm{U}}_{t_k}$$
$$\bar{\mathrm{U}}_{t_{k+1}} = \underbrace{-\bar{\mathrm{U}}_{t_k} - (\tau_k - t_k) b(\bar{\mathrm{U}}_{t_k})}_{-\bar{\mathrm{U}}_{\tau_k}} + (t_{k+1} - \tau_k) b(-\bar{\mathrm{U}}_{t_k}) + (W_{t_{k+1}} - W_{t_k}).$$

Weak convergence of the Euler scheme

Lemma

If $b(u) = -cu$ then $h(t,x,u)$ have bounded spatial derivatives up to the order 4 and

$$\left|\mathbb{E}f(\mathrm{X}_T) - \mathbb{E}f(\bar{\mathrm{X}}_T)\right| \leq C\Delta t.$$

$$h(t,x,u) = \mathbb{E}\left(f(X_T^{t,x,u})\right) \text{ solves}$$

$$\begin{cases} \dfrac{\partial h}{\partial t} + u\nabla_x h + b(u)\nabla_u h + \dfrac{1}{2}\Delta_u h = 0, \\ h(t,0,u) = h(t,0,-u), \\ h(T,x,u) = f(x). \end{cases}$$

Go back to MM5 and meteorology

- $\mathbb{D}$ is a MM5 cell of size $12\ 636m \times 9\ 305m \times 64$ (x, y, z), located near the coast or on the sea ($z = \sigma$).

- The wind is about 6 m/s in the dominant direction.
- Time scale : for MM5, the time step is 50 s. The time step for DSM must be smaller : we work with $\Delta t = 1$.
- DSM Cell size near 500 m for the horizontal mesh. (1000 sub-cells, with $N_{ppc} = 100$)
- Parametrisation of the pseudo-dissipation $\varepsilon(t, x) = \frac{C}{L} k^{3/2}(t, x)$.

With a stationary forcing (in time), we observe a stabilization of $k(t, x)$, $\omega(t, x)$ around values compatible with the meteorology.

Conclusion