Construction of good codes from weak Del Pezzo surfaces

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1 Introduction

The geometric construction of error correcting codes goes back to Reed-Solomon and Goppa for curves and to Reed-Muller for affine or projective spaces. In this work, we focus on evaluation codes from algebraic surfaces whose construction works as follows. Given X an algebraic surface over $\mathbf{F}_q$ and D a (Cartier) divisor on X, we denote by $X(\mathbf{F}_q)$ the set of rational points of X and by $H^0(X, D)$ the space of global sections of the divisor D. The associated *evaluation code* is the code whose codewords are the evaluations of the functions of $H^0(X, D)$ at the points of $X(\mathbf{F}_q)$ (see definition 3.1). The length of such a code is thus $\#X(\mathbf{F}_q)$, the dimension is $\dim\big(H^0(X, D)\big)$ (at least if the evaluation map is injective) and the third invariant, the minimum distance, is more difficult to control. It is related to the maximum of rational points that can contain a curve in the linear system $|D|$ (proposition 3.2). The lower this maximum, the better is the minimum distance.

The well known Reed-Muller code of degree d for the projective plane $\mathbf{P}^2$ over $\mathbf{F}_q$ is a nice example. Its codewords are the evaluations of the homogeneous polynomials of degree d at the rational points of $\mathbf{P}^2(\mathbf{F}_q)$. In the geometric setting above, this is the *evaluation code* associated to the algebraic surface $\mathbf{P}^2$, the divisor $d\ell$ where ℓ denotes any line, and the whole set of rational points $\mathbf{P}^2(\mathbf{F}_q)$. Its parameters are well known when $q > d$: the length is the number of rational points $\#\mathbf{P}^2(\mathbf{F}_q) = q^2 + q + 1$, the dimension is the dimension of the space of global section $\dim\big(H^0(\mathbf{P}^2, d\ell)\big) = \binom{d+2}{2}$, and the minimum distance is $q(q - d + 1)$. This minimum distance can be written $(q^2 + q + 1) - (1 + dq)$ and $(1 + dq)$ is nothing else than the maximal number of rational points of a curve lying in the linear system $|d\ell|$ (i.e. the set of plane curves of degree d). In fact, this is the number of points of the union of d lines of $\mathbf{P}^2$ meeting at one point. The existence of this kind of extremely reducible curve over $\mathbf{F}_q$ impacts negatively the minimum distance, since they contain too many rational points. Among other things, this is why evaluation codes associated to more general algebraic surfaces have been considered.

One can distinguish several strategies in the literature to get rid of reducible curves with many components in the linear system. The idea of Couvreur [Cou11] is to work with sublinear systems of $\mathbf{P}^2$ by adding constraints that remove the very reducible sections. In fact by choosing carefully the sublinear system, this kind of sections is no longer defined over the base field, but only over an extension. The number of rational points of irreducible curves that are not absolutely irreducible may fail drastically. In the preceding example of the d intersecting lines, if they are not defined over $\mathbf{F}_q$ but only conjugate over $\mathbf{F}_q$, then one can easily convince ourselves that their union only contains one rational point, their meeting point. Some other examples can be found in Edoukou [Edo08] or Couvreur & Duursma [CD13].

Following Zarzar [Zar07], another fairly repeated strategy is to concentrate on surfaces whose (arithmetic) Cartier class group is free of rank 1. Indeed, this is a natural way to overcome the difficulty of the existence of (very) reducible sections in the linear system $|D|$. Little and Schenck [LS18] have studied anticanonical codes on degree 3 and 4 del Pezzo surfaces having rank 1. In our previous work [BCH+20], we could say that we fill a gap in the study of algebraic geometric codes constructed from del Pezzo surfaces of rank 1. Let us remark that, even if the elements of the linear system $|D|$ are all irreducible, some of them may be absolutely reducible. As in the example of conjugate lines, it is expected that these configurations do not contain too many points but this requires a proof.

In this work, we continue the investigation of codes constructed from del Pezzo surfaces. We do not restrict ourselves to rank one surfaces but above all we consider more general surfaces, that is *non ordinary weak del Pezzo surfaces*. As ordinary del Pezzo surfaces, non ordinary weak del Pezzo surfaces admit a blowing-up description; in the ordinary case, the points that are blown up are in *general position* but in the non ordinary case, they are only in *almost general position* (three points can be colinear and six points can be conconic). The main consequences of these weaker hypotheses on the configuration of points are twofold. First, the surface contains -2-curves (and not only -1-curves). Secondly, the anticanonical divisor is not ample anymore but only big and nef and the anticanonical model is singular with rational double points.

In a concomitant work [BH22], we have computed explicit models for all the *arithmetic types* of weak del Pezzo surfaces of degree at least 3 over a finite field (these types lead to a classification that is coarser than the isomorphism one but that permits to distinguish the main arithmetic properties of the weak del Pezzo surfaces). Taking advantage of this knowledge, we select eight types of (non ordinary) weak del Pezzo that are well suited for coding applications. More precisely, we consider X a (smooth) weak del Pezzo surface of degree d over $\mathbf{F}_q$ and we denote by X_s its (singular) anticanonical model; this is the image of the surface X by the morphism φ associated to $-K_X$ the anticanonical divisor of X. Since $-K_X$ is not ample, the surface X_s is singular with a finite number of rational double points. We study the evaluation code associated to the (singular) surface X_s, the Cartier divisor $-K_{X_s} = \varphi_*(-K_X)$ and the whole set of rational points of X_s (definition 4.1). Except for small values of q, this code has length $n = \#X_s(\mathbf{F}_q)$, dimension $k = d + 1$. The last invariant, the minimum distance $d_{\min}$, is much more subtle to control and requires preparatory calculations.

Before going into details, let us discuss the advantages and disadvantages of considering such weak Del Pezzo surfaces. In the process of construction of a del Pezzo surface, there are blowing-up and blowing-down. The blowing-up may add rational points and thus may increase the length. The blowing-down permits to contract some lines and thus decreases the types of reducible configurations. Since the anticanonical model is no longer smooth, besides the exceptional curves some other curves, in fact the effective roots, can be contracted. If these curves are components of the most reducible sections of the anticanonical divisor on the weak del Pezzo, the parameters of the code could be improved. This is the positive aspect of considering anticanonical model of weak Del Pezzo surfaces. But we should also mention a negative one: because of the singularity of X_s, the notions of Cartier and Weil divisors are not equivalent and this makes it difficult to calculate the minimum distance $d_{\min}$ as we will see below.

The computation of $d_{\min}$ reduces to compute the number:

$$N_q\left(-K_{X_s}\right) = \max\left\{\#C(\mathbf{F}_q) \mid C \in |-K_{X_s}|\right\}.$$

Since every curve of the linear system $|-K_{X_s}|$ is of arithmetic genus 1 (adjunction formula), all its absolutely irreducible curves have a number of rational points which is bounded above by the classic:

$$N_q(1) = \max\left\{\#C(\mathbf{F}_q) \mid C \text{ absolutely irreducible, smooth, genus 1, curves over } \mathbf{F}_q\right\}.$$

By the Weil-Serre bound, we know that $N_q(1) \leq q + 1 + \lfloor 2\sqrt{q} \rfloor$; in fact, except for very special values of q, the Weil-Serre bound turns to be sharp:

$$N_q(1) = \begin{cases} q + \lfloor 2\sqrt{q} \rfloor & \text{if } q = p^e,\ e \geq 5,\ e \text{ odd and } p \mid \lfloor 2\sqrt{q} \rfloor, \\ q + 1 + \lfloor 2\sqrt{q} \rfloor & \text{otherwise} \end{cases}$$

([Ser20, Chap 2, Th 6.3]). Anyway, this bound does not permit to control the number of rational points of reducible or absolutely reducible curves of $|-K_{X_s}|$. Due to the singularities of X_s or more specifically to the difference between the Cartier or Weil divisors or class groups, the expectation that irreducible, but absolutely reducible curves in the linear system do not contain too many points is more difficult to verify. Even if the Cartier class group $\mathrm{CaCl}(X_s)$ is free of rank 1, generated by $-K_{X_s}$, this does not mean that the curves of the linear system $|-K_{X_s}|$ are all irreducible since they can decompose in the Weil class group $\mathrm{Cl}(X_s)$, that is into a sum of Weil irreducible divisors that are not Cartier divisors. To overcome this difficulty, we took fall advantage of the fact that in the context of weak del Pezzo surfaces, explicit models of all the class groups can be computed. This permits us to accurately measure the difference between the Cartier and the Weil divisors. This step uses some basic methods on lattices computations. Then to explicitly compute the maximum $N_q(-K_{X_s})$, we list all the kinds of decompositions into irreducible components that may appear in the linear system $|-K_{X_s}|$. In general, this can be a difficult issue but in our context this task is greatly facilitated by the fact that all the considered surfaces are blowing-up and down of the projective plane: as explained in Hartshorne's classic [Har77, Chap V, beginning of §4 & Remark 4.8.1], we are brought back to the study of some sub-linear systems of plane curves.

We choose examples that illustrate the variety of situations that may occur. In the column $\mathrm{CaCl}(X_s) \hookrightarrow \mathrm{Cl}(X)$ in the tabular below, we see that the Cartier class group $\mathrm{CaCl}(X_s)$ always embeds in the Weil class group $\mathrm{Cl}(X_s)$, and via this embedding $\mathrm{CaCl}(X_s)$ may be equal to $\mathrm{Cl}(X_s)$, or of finite index into $\mathrm{Cl}(X_s)$, or of positive co-rank into $\mathrm{Cl}(X_s)$. Note also that the lattice $\mathrm{CaCl}(X_s)$ is always free, whereas $\mathrm{Cl}(X_s)$ may have a torsion subgroup. In the column $N_q(-K_{X_s})$, one can see that this is not always the absolutely irreducible curves of the linear system that contains the maximum of rational points. Only a case-by-case proof and a carefully study of all the geometric properties permits to estimate the three invariants $[n, k, d_{\min}]$ that are contained in the last column.

	Deg.	Sing.	$\mathrm{CaCl}(X_s) \hookrightarrow \mathrm{Cl}(X)$	$N_q(-K_{X_s})$	$[n, k, d_{\min}]$
§4.2	6	$\mathbf{A}_1$	$2\mathbf{Z} \hookrightarrow \mathbf{Z}$	$2q+1$	$[q^2+1, 7, q^2-2q]$
§4.3	5	$2\mathbf{A}_1$	$\mathbf{Z} \oplus 2\mathbf{Z} \hookrightarrow \mathbf{Z} \oplus \mathbf{Z}$	$2q+2$	$[q^2+q+1, 6, q^2-q-1]$
§4.4	4	$\mathbf{A}_1$	$\mathbf{Z} \simeq \mathbf{Z}$	$\leq N_q(1)$	$[q^2-q+1, 5, \geq q^2-q+1-N_q(1)]$
§4.5	4	$4\mathbf{A}_1$	$\mathbf{Z} \hookrightarrow \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$	$\leq N_q(1)$	$[q^2+1, 5, \geq q^2+1-N_q(1)]$
§4.6	4	$\mathbf{A}_2$	$\mathbf{Z} \simeq \mathbf{Z}$	$\leq N_q(1)$	$[q^2+1, 5, \geq q^2+1-N_q(1)]$
§4.7	4	$\mathbf{D}_5$	$4\mathbf{Z} \hookrightarrow \mathbf{Z}$	$2q+1$	$[q^2+q+1, 5, q^2-q]$
§4.8	3	$\mathbf{A}_1$	$\mathbf{Z} \simeq \mathbf{Z}$	$\leq N_q(1)$	$[q^2+1, 4, \geq q^2+1-N_q(1)]$
§4.9	3	$3\mathbf{A}_2$	$\mathbf{Z} \hookrightarrow \mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z}$	$\leq N_q(1)$	$[q^2+q+1, 4, \geq q^2+q+1-N_q(1)]$

In the tabular above, the inequality $N_q(-K_{X_s}) \leq N_q(1)$ means that the curves of the linear system $|-K_{X_s}|$ that contain the maximum number of points are the absolutely irreducible ones. They are all of arithmetic genus 1, but it may happen that none of these curves is maximal (i.e. has a number of rational points equal to $N_q(1)$). This is why in these cases, one can only give an upper bound for $N_q(-K_{X_s})$ and thus a lower bound for $d_{\min}$. It turns out that we recover two examples of Koshelev [Kos20] (§4.5 and 4.9), where he proved that the linear systems cannot contain a maximal genus one curve for certain finite fields. This permits to increase the lower bound of the minimum distance by one over these fields.

All the presented codes can be easily constructed using a mathematics software system. On the second author's `webpage`, we put a `magma` program that permits to construct all our codes.

2 Generalities on weak del Pezzo surfaces

Let k be a finite field (most of the results remain true on any field), $\bar{k}$ its algebraic closure, $\Gamma = \mathrm{Gal}(\bar{k}/k)$ its absolute Galois group and let σ be the Frobenius automorphism.

In this section, we recall the classical properties of del Pezzo surfaces. In particular, we focus on the specificities of *non ordinary weak* del Pezzo surfaces compared to the *ordinary ones*. The essential references for the content of this section are the book of Manin [Man74] or the more recent one of Dolgachev [Dol12, §8].

2.1 Ordinary versus non ordinary weak del Pezzo surfaces

There are several definitions of a del Pezzo surfaces, even in the Dolgachev's classic [Dol12]; let us start with the definition 8.1.18 of this book.

Definition 2.1. *A smooth projective surface X is a* ***weak del Pezzo*** *surface if its anticanonical divisor $-K_X$ is:*

(i) big, which means that $K_X^{\cdot 2} > 0$,

(ii) and nef, which means that $(-K_X) \cdot D \geq 0$ for any effective divisor D on X.

*The self-intersection $K_X^{\cdot 2}$ is the **degree** of the del Pezzo surface X.*

Thanks to the Nakai-Moishezon criterion [Har77, Chap V, Theorem 1.10], these kinds of surfaces are divided into two cases:

(i) either the inequalities in (*ii*) are all strict ($(-K_X)\cdot D > 0$): the anticanonical divisor is thus **ample** and we say that the del Pezzo surface is an **ordinary** one;

(ii) or there exists an effective divisor D such that $(-K_X)\cdot D = 0$: the anticanonical divisor is **not ample** and we say that the del Pezzo surface is a **non ordinary** one.

These properties have consequences on the **negative curves** on X, those whose self-intersection is negative. Indeed, let C be an absolutely irreducible curve on X of arithmetic genus $\gamma(C)$. By adjunction formula, we know that $C^{\cdot 2} = 2\gamma(C) - 2 + C\cdot(-K_X)$ and since $\gamma(C) \geq 0$ and $C\cdot(-K_X) \geq 0$, we deduce that $C^{\cdot 2} \geq -2$. Thus, negative curves on X have self-intersection -2 or -1. Moreover $C^{\cdot 2} = -2$ if and only if $\gamma(C) = 0$ and $C\cdot(-K_X) = 0$; this means that only non ordinary del Pezzo surfaces can contain (-2)-curves. We also prove the same way that (-1)-curves on weak del Pezzo surfaces must have arithmetic genus equal to 0. This motivates the following definition which deals with negative curves but also divisor classes of curves.

Definition 2.2. *Let X be a weak del Pezzo surface over a field k, let $\overline{X} = X \otimes \overline{k}$ be its extension to the algebraic closure $\overline{k}$ and let $\mathrm{Cl}(\overline{X})$ denote the divisor class group of $\overline{X}$.*

*(i) A divisor class $D \in \mathrm{Cl}(\overline{X})$ is an **exceptional class** if $D^{\cdot 2} = D\cdot K_X = -1$; an absolutely irreducible curve C on X whose class is exceptional is an **exceptional curve**.*

*(ii) A divisor class $D \in \mathrm{Cl}(\overline{X})$ is a **root** if $D^{\cdot 2} = -2$ and $D\cdot K_X = 0$; a curve C on X whose class is a root is an **effective root** and if such a curve is absolutely irreducible then C is called a* (-2)-curve.

It is well known that the geometry of weak del Pezzo surfaces depends to a large extent of these **negative curves**. For example, if X is a weak ordinary del Pezzo surface then all the exceptional classes are the classes of a (unique) exceptional curve and no root is effective. On the contrary, if X is weak non ordinary del Pezzo surface then some exceptional classes may be represented by reducible curves and some roots are effective. These differences of behaviours appear naturally in the blowup description of the generalized del Pezzo surfaces.

2.2 The blow-up model

Over $\overline{k}$, every del Pezzo surface can be obtained by a sequence of blowing ups starting from the projective plane $\mathbf{P}^2$. This description makes most of the invariants of the surface very explicit.

Recall that if $\pi : Y \to X$ is the blowing up of a smooth surface X at a point p, with exceptional divisor E, then $\mathrm{Cl}(Y) = \pi^*\mathrm{Cl}(X) \oplus \mathbf{Z}E$, the intersection pairing on Y satisfying $E^2 = -1$, $\pi^*D\cdot E = 0$ and $\pi^*D\cdot\pi^*D' = D\cdot D'$ for all divisors D and D' of X (the blowing up is an isometry for the intersections pairings). Moreover $K_Y = \pi^*K_X + E$.

We recall that $r \leq 8$ points in $\mathbf{P}^2(\overline{k})$ are said to be

- in *general position* if and only if no three lie on a line, no six lie on a conic, and there is no cubic through seven of them having a singular point at the eighth;
- in *almost general position* if and only if no four lie on a line and no seven lie on a conic.

Del Pezzo surfaces can always be described as follows [Dol12, Th 8.1.15].

Theorem 2.3. *Let X be a generalized del Pezzo surface over k and let $\overline{X} = X \otimes_k \overline{k}$ its extension to $\overline{k}$. If X is of degree d, with $3 \leq d \leq 6$, then $\overline{X}$ is the blowing up of $\mathbf{P}^2$ at $r = 9-d$ points $p_1,\ldots,p_r$ in almost general position; more precisely $\overline{X}$ results in r successive blowups $\pi_1,\ldots,\pi_r$:*

$$\overline{X} \xrightarrow{\pi_r} X_r \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{\pi_1} X_1 := \mathbf{P}^2_{\overline{k}}$$

where $p_i \in X_i$ are in almost general position.

Let E_0 be the class of a line in $\mathbf{P}^2$ and let $E_1,\ldots,E_r$ be the exceptional curves at each stage. Then the divisor class group of $\overline{X}$ with its intersection pairing can be easily described:

$$\mathrm{Cl}(\overline{X}) = \mathbf{Z}E_0 \oplus \mathbf{Z}E_1 \oplus \cdots \oplus \mathbf{Z}E_r \qquad \mathrm{Mat}\left(\_\cdot\_, (E_i)_{0\leq i\leq r}\right) = \mathrm{Diag}(1,-1,\ldots,-1),$$

where Diag denotes the diagonal matrix. This also gives explicitly the canonical class:

$$K_X = -3E_0 + \sum_{i=1}^{r} E_i. \tag{1}$$

The negative classes can be expressed in terms of the basis $E_0, E_1, \ldots, E_r$ [Dol12, §8.2].

Name	Exceptional classes E	Roots R
Conditions	$E^{\cdot 2} = -1$ and $E \cdot (-K_X) = 1$	$R^{\cdot 2} = -2$ and $R \cdot K_X = 0$
Expression in terms of the E_i	E_i, $i \in \{1, \dots, r\}$ $E_{ij} = E_0 - E_i - E_j$, $\{i,j\} \subset \{1, \dots, r\}$ $E_{i_1 \cdots i_5} = 2E_0 - \sum_{j=1}^{5} E_{i_j}$ $\{i_1, i_2, i_3, i_4, i_5\} \subset \{1, \dots, r\}$	$R_{ij} = E_i - E_j$, $\{i,j\} \subset \{1, \dots, r\}$ $R_{ijk} = E_0 - E_i - E_j - E_k$, $-R_{ijk}$ $\{i,j,k\} \subset \{1, \dots, r\}$ $R_{i_1 \cdots i_6} = 2E_0 - \sum_{j=1}^{6} E_{i_j}$, $-R_{i_1 \cdots i_6}$ $\{i_1, i_2, i_3, i_4, i_5, i_6\} \subset \{1, \dots, r\}$

We note that, in the notation E_{ij}, the indices are unordered (which leads to $\binom{r}{2}$ possibilities), whereas they are ordered in the notation R_{ij} since $R_{ji} = -R_{ij}$ (which leads to $2\binom{r}{2}$ possibilities).

Not all these divisor classes are effective and the effectiveness of certain of these classes differentiate some types of Del Pezzo surface.

• In the ordinary case, each exceptional class of divisor is represented by a unique irreducible curve. Either it is one exceptional curve E_i for some $1 \leq i \leq r$ or the strict transform of the line of $\mathbf{P}^2$ passing through p_i and p_j for the class E_{ij} or the strict transform of the (unique) conic of $\mathbf{P}^2$ passing through the $p_{i_1}, \dots, p_{i_5}$ for $E_{i_1 i_2 i_3 i_4 i_5}$. Their intersection graph is an important invariant of the ordinary Del Pezzo surfaces; figures of these graphs for $3 \leq r \leq 5$ can be found in Manin [Man74, §26.9] or in Dolgachev [Dol12, §8.6.3, Figure 8.5]. As for the root classes, no one is effective.

• In the non ordinary cases, where the points are no longer in *general* position but only in *almost general* position, the exceptional divisors are still effective but not necessarily represented by irreducible curves anymore.

For example, if p_1, p_2, p_3 are collinear then the root R_{123} becomes effective since it is the class of the strict transform of the line passing through p_1, p_2, p_3 and the four exceptional classes $E_{12}, E_{13}, E_{23}, E_{12345}$ are represented by reducible curves since

$$E_{12} = R_{123} + E_3, \qquad E_{13} = R_{123} + E_2, \qquad E_{23} = R_{123} + E_1, \qquad E_{12345} = R_{123} + E_{45}.$$

In this case, all other exceptional divisors are still represented by irreducible curves.

Another simple example: if p_2 is chosen to be on E_1, $p_2 \succ p_1$, then the root $E_1 - E_2$ becomes effective since it is the strict transform of E_1. The exceptional classes E_1 and E_{1j} $j \neq 1, 2$ are no longer represented by irreducible curves.

In general, a result of Demazure states that exceptional divisors that are represented by irreducible curves are characterized by the fact that they intersect non negatively (≥ 0) all the irreducible roots [CT88, Proposition 5.5].

Another general result states that the set of irreducible roots (the effective classes represented by an irreducible curve) is necessarily a free family in $\mathrm{Cl}(X \otimes \overline{\mathbb{F}}_q)$ (see loc. cit.). In particular, there are at most r effective irreducible roots. The lattice generated by the effective roots plays a crucial role.

Definition 2.4. *Let X be a generalized del Pezzo surface over k and let $\overline{X} = X \otimes_k \overline{k}$ be its extension to $\overline{k}$. We denote by $\overline{\mathscr{R}}$ the sub-lattice of $\mathrm{Cl}(\overline{X})$ generated by the effective roots and by $\mathscr{R}$ sub-lattice of $\mathrm{Cl}(X)$ defined by $\mathscr{R} = \overline{\mathscr{R}}^{\Gamma}$.*

Following Coray and Tsfasman (see loc. cit.), an important invariant of a weak Del Pezzo surface is the graph of negative curves, which is an analog of the intersection graph of the exceptional divisors/curves introduced above in the ordinary case. To take into account the fact that the surface may be non ordinary, the set of vertices is modified: the vertices corresponding to reducible exceptional divisors are cancelled, while vertices corresponding to effective and irreducible roots are added.

2.3 The anticanonical model X_s

2.3.1 The morphism induced by the anticanonical divisor $-K_X$

Let X be a del Pezzo surface of degree d, with $3 \leq d \leq 6$ and whose canonical divisor is denoted by K_X. In the ordinary case, the anticanonical divisor $-K_X$ is known to be very ample and it induces a projective embedding of X into $\mathbf{P}^d = \mathbf{P}(H^0(X, -K_X))$ (see §3.1 for a review about the space of global sections of a divisor) In the non ordinary case, the anticanonical class $-K_X$ is no longer ample but its linear system remains base point free and gives a morphism from X to a projective space:

Definition 2.5. *Let X be a weak del Pezzo surface of degree d, with $3 \leq d \leq 6$. The image $\varphi(X)$, where $\varphi : X \to \mathbf{P}\left(H^0(X, -K_X)\right) = \mathbf{P}^d$ is the projective morphism associated to the anticanonical divisor $-K_X$ is called the anticanonical model of X and is denoted by X_s. We put $K_{X_s} = \varphi_*(K_X)$.*

This kind of del Pezzo surface corresponds to the definition 8.1.5 in Dolgachev [Dol12]. The fifth talk of Demazure [Dem80, Exposé V] on del Pezzo surfaces contains all the main properties of this anticanonical model.

Proposition 2.6. *The morphism φ satisfies:*

(i) it is not a projective embedding (since $-K_X$ is not ample) but the image X_s is a normal surface whose singularities are rational double points;

(ii) it is the minimal desingularization of X_s, it contracts all the irreducible effective roots on X into the singular points and nothing else;

(iii) the Weil divisor $K_{X_s} = \varphi_*(K_X)$ is a Cartier divisor of X_s which satisfies $\varphi^*(K_{X_s}) = K_X$;

For each singularity, the exceptional divisor of its minimal resolution is a sum of irreducible effective roots (R_i) with $R_i \cdot R_j \in \{0, 1\}$ for any $i \neq j$. As usual in the **ADE** classification of rational double points, we describe the type of a singularity by its dual graph: its vertices correspond to the above roots, and there is an edge between the two vertices when the corresponding roots meet. In the examples below, the types of rational double points that appear correspond to the graphs:

As mentioned in the last item, since the anticanonical model of a non ordinary weak del Pezzo surface is not smooth but only normal, a Weil divisor may not be Cartier and the class groups of Cartier or Weil divisor may differ.

2.3.2 Cartier versus Weil divisors and class groups

Let X be a normal surface; let $k(X)$ be its field of rational functions and $\mathcal{O}_X$ its structural sheaf. We need to review some general facts about divisors in such surfaces (see Liu [Liu02, §7.1 & 7.2] for more details).

• A *prime Weil divisor* X is a prime closed sub-variety of codimension 1 and the group of *Weil divisors* $\mathrm{WDiv}(X)$ is the free abelian group generated by prime Weil divisors. A Weil divisor D can be written $\sum_i n_i C_i$ where the C_i's are irreducible curves on X and where the n_i are integers of which only a finite number are non zero. Such a divisor is said *effective* if $n_i \geq 0$ for all i. Since X is normal, it is regular in codimension 1 and to each rational function $f \in k(X)$, one can associate a Weil divisor (f) which is called *principal*. The set of principal divisors is a sub-group of $\mathrm{WDiv}(X)$.

• A *Cartier divisor*, or a *locally principal divisor* D is a global section of the sheaf $k(X)^\times / \mathcal{O}_X^\times$; it consists in a collection $(U_i, f_i)_{i \in I}$ where $(U_i)_{i \in I}$ is an open covering of X and where the f_i's are rational functions such that the quotients f_i/f_j have neither zeroes nor poles on $U_i \cap U_j$, i.e. such that $f_i/f_j \in \mathcal{O}_X^\times(U_i \cap U_j)$. Two collections $(U_i, f_i)_{i \in I}$ and $(V_j, g_j)_{j \in I}$ represent the same Cartier divisor if on $U_i \cap V_j$, the functions f_i and g_j differ by a multiplicative factor in $\mathcal{O}_X^\times(U_i \cap V_j)$ for every i, j. The set of Cartier divisors can be turned into an abelian group which we denote by $\mathrm{CDiv}(X)$. A Cartier divisor is called *effective* if it can be represented by a collection (U_i, f_i) with $f_i \in \mathcal{O}_X(U_i)$ for every i. A principal Cartier divisor is represented by a collection (X, f), where $f \in k(X)^\times$. The set of principal divisors is also a sub-group of $\mathrm{CDiv}(X)$.

• To each Cartier divisor D one can associate a Weil divisor and this correspondence induces a group homorphism $\mathrm{CDiv}(X) \to \mathrm{WDiv}(X)$; since X is supposed to be normal, this morphism is injective [Liu02, Chap 7, Prop 2.14] and it sends an effective Cartier divisor to an effective Weil one.

• The quotients of the divisor groups $\mathrm{CDiv}(X)$ and $\mathrm{WDiv}(X)$ by the principal divisors are denoted $\mathrm{CaCl}(X)$ and $\mathrm{Cl}(X)$. The previous correspondence induces an injective homorphism $\mathrm{CaCl}(X) \to \mathrm{Cl}(X)$.

These are general facts, but in the context of weak del Pezzo surfaces, we are able to be much more explicit. In particular, one can relate the two groups $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ to the group $\mathrm{Cl}(X) = \mathrm{CaCl}(X)$.

Proposition 2.7. *Let X be a weak del Pezzo surface over k and let X_s be its anticanonical model. Over $\overline{k}$, one has the two exact sequences:*

$$0 \longrightarrow \overline{\mathscr{R}} \longrightarrow \mathrm{Cl}(\overline{X}) \longrightarrow \mathrm{Cl}(\overline{X}_s) \longrightarrow 0 \qquad \Longrightarrow \qquad \mathrm{Cl}(\overline{X}_s) = \mathrm{Cl}(\overline{X})/\overline{\mathscr{R}}, \tag{2}$$

$$0 \longrightarrow \mathrm{CaCl}(\overline{X}_s) \longrightarrow \mathrm{Cl}(\overline{X}) \longrightarrow \mathrm{Hom}\left(\overline{\mathscr{R}}, \mathbf{Z}\right) \qquad \Longrightarrow \qquad \mathrm{CaCl}(\overline{X}_s) = \overline{\mathscr{R}}^{\perp}, \tag{3}$$

where the arrow $\mathrm{Cl}(\overline{X}) \to \mathrm{Hom}\left(\overline{\mathscr{R}}, \mathbf{Z}\right)$ is given by $D \mapsto [R \mapsto D \cdot R]$, and where $\overline{\mathscr{R}}^{\perp} = \{D \in \mathrm{Cl}(\overline{X}) \mid D \cdot R = 0,\ \forall R \in \overline{\mathscr{R}}\}$. Over k, one has $\mathrm{Cl}(X) = \mathrm{CaCl}(X) = \mathrm{CaCl}(\overline{X})^\Gamma = \mathrm{Cl}(\overline{X})^\Gamma$ for X, and for its anticanonical model:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(X_s) \longrightarrow 0 \qquad \Longrightarrow \qquad \mathrm{Cl}(X_s) = \mathrm{Cl}(X)/\mathscr{R}, \tag{4}$$

$$0 \longrightarrow \mathrm{CaCl}(X_s) \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Hom}\left(\overline{\mathscr{R}}, \mathbf{Z}\right)^\Gamma \qquad \Longrightarrow \qquad \mathrm{CaCl}(X_s) = \mathscr{R}^{\perp} \tag{5}$$

Moreover we have an isomorphism $\mathrm{Cl}(X_s) \simeq \mathrm{Cl}(\overline{X}_s)^\Gamma$.

Proof. Let R be the union of effective roots in X and let $U = X \setminus R$ be the open complementary. By a result of Hartshorne [Har77, Chap II, Prop 6.5], we have the exact sequence:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Let U_s be the smooth locus of X_s. This open set is of codimension 2 in X_s and thus $\mathrm{Cl}(X_s) \simeq \mathrm{Cl}(U_s)$ (see loc. cit.). Since the anticanonical map φ_{-K_X} induces an isomorphism from U to U_s one has $\mathrm{Cl}(U_s) \simeq \mathrm{Cl}(U)$ and the sequence (4) follows. Sequence (2) also follows by extending scalars to $\overline{k}$.

On the other hand, we note that the module $\overline{\mathscr{R}}$ being induced [Man74, Chap IV,§29], we know that $H^1(\Gamma, \overline{\mathscr{R}}) = 0$; thus taking the Galois invariants of (2) leads to:

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathrm{Cl}(\overline{X})^\Gamma \longrightarrow \mathrm{Cl}(\overline{X}_s)^\Gamma \longrightarrow 0$$

Now X is smooth and we have $\mathrm{Cl}(X) = \mathrm{Cl}(\overline{X})^\Gamma$ [Sta18, Tag 0CDS]; we deduce the last isomorphism $\mathrm{Cl}(X_s) \simeq \mathrm{Cl}(\overline{X}_s)^\Gamma$.

The exact sequence (3) comes from Bright [Bri13, Prop 1]. We deduce the equality $\mathrm{CaCl}(\overline{X}_s) = \overline{\mathscr{R}}^\perp$, and from [Sta18, Tag 0CDS], we deduce that $\mathrm{CaCl}(X_s) = \mathrm{CaCl}(\overline{X}_s)^\Gamma = (\overline{\mathscr{R}}^\perp)^\Gamma$.

Finally, taking the Galois invariants in the sequence (3) gives the exact sequence in (5). Now since the intersection product is invariant under the Galois action, a divisor in $\mathrm{Cl}(X)$ is orthogonal to $\overline{\mathscr{R}}$ if and only if it is orthogonal to $\mathscr{R}$, and we get the isomorphism in (5). $\square$

2.3.3 Lattice computations

One of the key step of the study of codes from weak del Pezzo surfaces is the explicit computation of the divisor class groups as in (2) and (3). Such computations take place in the group $\mathrm{Cl}(\overline{X})$, which is known to be a free $\mathbf{Z}$-module of finite type endowed with the (non degenerate) intersection bilinear form and involve the root lattices $\overline{\mathscr{R}}$, which is given by some explicit generators and which satisfies $\overline{\mathscr{R}} \cap \overline{\mathscr{R}}^\perp = \{0\}$ (the orthogonal is relative to the intersection pairing).

This is a general issue and let us consider C (for the "class group") a free $\mathbf{Z}$-module of finite rank with a non degenerate symmetric bilinear form $(x, y) \mapsto x \cdot y$ (for the intersection product). Recall that a submodule M of C is a *direct summand* (or *is complemented*) if there exists a submodule N of C such that $C = M \oplus N$; in this case, the submodules M and N are called *complementary submodules* of C [AW92, §3.8,§6.1]. Let $\mathscr{R}$ be a submodule of C such that $\mathscr{R} \cap \mathscr{R}^\perp = \{0\}$ (for the root lattice).

In the context of modules over a principal ideal domain, contrary to what is happening in vector spaces over a field, even if $\mathscr{R} \cap \mathscr{R}^\perp = \{0\}$, the orthogonal submodules $\mathscr{R}$ and $\mathscr{R}^\perp$ may not be complementary submodules. There are at least two different kinds of obstructions for this. Either the submodule $\mathscr{R}$ is not a direct summand or both submodules $\mathscr{R}$ and $\mathscr{R}^\perp$ are direct summands but they are not complementary submodules.

In any case the smallest submodule containing $\mathscr{R}$ which is a direct summand is called the *hull* of $\mathscr{R}$ and is denoted $\mathscr{R}^\sharp$. As for the submodule $\mathscr{R}^\perp$, since it is the kernel of a morphism of free modules, it is always a direct summand. In the same way, even though $\mathscr{R}^\sharp$ and $\mathscr{R}^\perp$ are direct summand, they may or may not be complementary submodules. These phenomenes make the description of the exact sequence:

$$0 \longrightarrow \mathscr{R}^\perp \longrightarrow C/\mathscr{R} \longrightarrow C/\mathscr{R} \oplus \mathscr{R}^\perp \longrightarrow 0$$

a little bit tricky (ie the comparison between the groups of Weil classes and Cartier classes). The main tool for the explicit computation of this sequence is the *Invariant factor theorem for submodules* that will be used twice (see [AW92, Theorem 6.23]).

• First, we apply this result to the submodule $\mathscr{R} \subset C$: there exists a $\mathbf{Z}$-basis $e_1, \ldots, e_n$ of C and $\alpha_1 \mid \cdots \mid \alpha_r$ ($r \leq n$) a sequence of positive integers, called invariant factors, such that $\mathscr{R} = \mathbf{Z}\alpha_1 e_1 \oplus \cdots \oplus \mathbf{Z}\alpha_r e_r$ and $\mathscr{R}^\sharp = \mathbf{Z}e_1 \oplus \cdots \oplus \mathbf{Z}e_r$. The submodule $\mathscr{R}$ is a direct summand of C if and only if $\mathscr{R}^\sharp = \mathscr{R}$, if and only if the invariant factors $\alpha_1, \ldots, \alpha_r$ are all equal to 1.

Put $M = \mathbf{Z}e_{r+1} \oplus \cdots \oplus \mathbf{Z}e_n$ then $\mathscr{R}^\sharp$ and M are complementary submodules, $C = \mathscr{R}^\sharp \oplus M$, and the projections ι_{tors} and ι onto each factors lead to an isomorphism

$$\begin{array}{ccccc} C/\mathscr{R} & \xrightarrow{\simeq} & \mathscr{R}^\sharp/\mathscr{R} & \oplus & M \\ x \bmod \mathscr{R} & \longmapsto & \iota_{\mathrm{tors}}(x) \bmod \mathscr{R} & + & \iota(x) \end{array}$$

In other words, the projection ι_{tors} gives an isomorphism from the torsion submodule of $C/\mathscr{R}$ to the quotient module $\mathscr{R}^\sharp/\mathscr{R}$ which is isomorphic to $\mathbf{Z}/\alpha_1\mathbf{Z} \times \cdots \times \mathbf{Z}/\alpha_r\mathbf{Z}$; the projection ι gives an isomorphism from the torsion-free submodule of $C/\mathscr{R}$ to the submodule M of C.

• Since $\mathscr{R} \cap \mathscr{R}^\perp = \{0\}$, we know that $\mathscr{R}^\perp$ canonically embeds in the quotient $C/\mathscr{R}$; being free of torsion it is a submodule of the torsion-free submodule of $C/\mathscr{R}$. Via ι it thus embeds in M. Using the Invariant factor theorem again, one can choose the basis $e_{r+1}, \ldots, e_n$ of M, in such a way that there exists $\beta_{r+1} \mid \cdots \mid \beta_n$ such that $\iota(\mathscr{R}^\perp) = \mathbf{Z}\beta_{r+1}e_{r+1} \oplus \cdots \oplus \mathbf{Z}\beta_n e_n$. The cokernel $C/\mathscr{R} \oplus \mathscr{R}^\perp$ is then isomorphic to $\mathbf{Z}/\beta_{r+1}\mathbf{Z} \times \cdots \times \mathbf{Z}/\beta_n\mathbf{Z}$. In particular, the canonical embedding of $\mathscr{R}^\perp$ inside M induces an isomorphism if and only if $\beta_{r+1}, \ldots, \beta_n$ are all equal to 1.

In the sequel, we do not give names to the projection morphisms $\iota_{\text{tors}}, \iota$.

3 Codes from surfaces: construction and tools for their study

In this section k is a finite field $\mathbf{F}_q$.

3.1 Evaluation codes from surfaces

Cartier divisors, their spaces of global sections, and the associated complete linear systems are the main ingredients to define and to characterize the parameters of the *evaluation codes* from an algebraic surfaces. Let us recall the definitions and the basic facts concerning these objects. We consider X a (not necessarily smooth, but in fact at least normal here) irreducible surface over k. We denote by $k(X)$ its function field and by $\mathcal{O}_X$ its structural sheaf. Let $D = (U_i, f_i)_{i\in I}$ be a Cartier divisor on this surface X.

A *global section* of D is a function $s \in k(X)$ such that for every $i \in I$, the product sf_i is regular on U_i, that is $sf_i \in \mathcal{O}_X(U_i)$ for all $i \in I$. We denote by $H^0(X, D)$ the set of these sections; this is a vector space which is known to have finite dimension.

By definition, if $s \in H^0(X, D)$ is a global section of D then the Cartier divisor $(U_i, sf_i)_{i\in I}$ is *effective*. It can be shown that two global sections of D lead to the same effective Cartier divisor if and only if they differ by a non zero constant. This means that there is a one-to-one correspondence between the projective space $\mathbf{P}(H^0(X, D))$ and the set of effective Cartier divisors linearly equivalent to D. This last set is called the *complete linear system* associated to the divisor D and is currently denoted by $|D|$, so we have $|D| = \mathbf{P}(H^0(X, D))$. An important invariant of the divisor (or linear system) for our purpose is the maximum of rational points that can contain a curve of $|D|$; we put:

$$N_q(D) = \max\{\#C(\mathbf{F}_q) \mid C \in |D|\}. \tag{6}$$

Definition 3.1. *Let X be a (not necessarily smooth) irreducible surface over k, let D be a Cartier divisor of X and let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of rational points of X. The* ***evaluation code*** *$\mathcal{C}_X(D, \mathcal{P})$ is the image of the evaluation map*

$$\begin{array}{ccc} H^0(X, D) & \longrightarrow & k^n \\ s & \longmapsto & (sf_{i_p})(p) \end{array}$$

where for each point p, the index i_p is chosen in such a way that $p \in U_{i_p}$.

In the preceding definition, the choice of i_p may be not unique but different choices i_p, j_p of these indices lead to homothetic codes since the quotients f_{i_p}/f_{j_p} are non vanishing regular functions on $U_{i_p} \cap U_{j_p}$.

The usual parameters of the evaluation code are related with some invariants of the surface.

Proposition 3.2. *If the evaluation map is injective, a $\mathcal{C}_X(D, X(\mathbf{F}_q))$ has*

(i) ***length*** *equal to $\#X(\mathbf{F}_q)$ the number of rational points of X,*

(ii) ***dimension*** *equal to the dimension of the space $H^0(X, D)$ of global sections,*

(iii) ***minimum distance*** *bounded below by $n - N_q(D)$, where $N_q(D)$ is defined in (6).*

Thanks to this proposition, it is worth noticing that bounding below the minimum distance of an evaluation code $\mathcal{C}_X(D, X(k))$ from a surface X reduces to bounding above $N_q(D)$ the number of points of the curves of the linear system $|D|$ associated to the divisor D. The fewer the number of rational points of the curves in the linear system $|D|$, the higher the minimum distance.

3.2 Blowing up, divisors and (non) complete linear systems

One of the key tools of the construction of the codes from Del Pezzo surfaces are the blowing-up or the blowing down depending on the sense of the arrows.

Let $\pi : Y \to X$ be a sequence of blowing ups where all the surfaces involved are supposed to be smooth, except the last one X which is only supposed to be normal. Such a morphism leads to two natural maps involving different kinds of divisors and divisor class groups.

• First, starting from a Cartier divisor of X, the pullback π^*D is the Cartier divisor on Y defined locally by $(\pi^{-1}U_i, f_i \circ \pi)$. This lead to a morphism $\pi^* : \mathrm{CDiv}(X) \longrightarrow \mathrm{CDiv}(Y)$.

• Secondly, it can be shown that if C irreducible effective Weil divisor of Y, then $\pi(C)$ is either a point or an irreducible effective Weil divisor of X. Then the map $\pi_* : \mathrm{WDiv}(Y) \longrightarrow \mathrm{WDiv}(X)$ defined by $\pi_*(C) = 0$ if $\pi(C)$ is a point and $\pi_*(C) = \pi(C)$ otherwise extends to a group homorphism [Liu02, Chap 9, Lem 2.10].

• Moreover, for every Cartier divisor D of X, one has $\pi_*(\pi^* D) = D$ [Liu02, Chap 9, Prop 2.11] where π_* is applied to the Weil divisor of Y associated to the Cartier divisor $\pi^* D$.

• These two maps induce two homomorphisms $\pi^* : \mathrm{CaCl}(X) \to \mathrm{CaCl}(Y)$ [Liu02, Chap 7, Def 1.34] and $\pi_* : \mathrm{Cl}(Y) \to \mathrm{Cl}(X)$ [Ful98, Chap 1, Th 1.4].

• At the level of global sections and linear systems, the map π^* also induces isomorphisms:

$$\begin{array}{ccc} H^0(X,D) & \stackrel{\simeq}{\longrightarrow} & H^0(Y,\pi^* D) \\ s & \longmapsto & s\circ\pi \end{array} \qquad\qquad \begin{array}{ccc} |D|_X & \stackrel{\simeq}{\longrightarrow} & |\pi^* D|_Y \\ C & \longmapsto & \pi^* C \end{array}$$

where D is a Cartier divisor on X ([Dem80, Exposé V, Cor 2]). Since $\pi_*(\pi^* C) = C$, the inverse of the right isomorphism is nothing else than $\pi_* \, |\pi^* D|_Y \longrightarrow |D|_X$.

We will need to describe the one-to-one correspondence $|D|_X \longrightarrow |\pi^* D|_Y$, when the right divisor is replaced by a divisor of the form $\pi^* D - E$. Some natural sublinear systems appear. Let us go step by step.

• Let $\pi : Y \to X$ be the blowing-up of a smooth surface X at a point $p \in X$ and let E be its exceptional divisor on Y. Since the surfaces are supposed to be smooth, we do not have to distinguish Cartier and Weil divisors. We mainly focus on effective divisors and we call them *curves*. Given C a curve on X, then the pullback $\pi^* C$ is called the *total transform* of C, the closure in Y of $\pi^{-1}(C \setminus \{p\})$, denoted $\widetilde{C}$, is called the *strict transform* of C. These two curves on Y are related by the relation:

$$\pi^* C = \widetilde{C} + m_p(C) E,$$

where $m_p(C)$ denote the multiplicity of C at the point p. In particular, for any $n \geq 0$, the divisor $\pi^* C - nE$ is effective if and only if $m_p(C) \geq n$.

This permits to relate the complete linear system $|\pi^* D - nE|$ on Y to an uncomplete one on X, that is $|D - np|$ the space of curves of $|D|$ which pass through p with multiplicity at least n. In fact this shows that the map $C \mapsto \pi^* C - nE$ leads to a one-to-one correspondence from $|D - np|$ to $|\pi^* D - nE|$ (the other way around, it says that the blowing-up permits to turn uncomplete linear systems into complete ones).

• The same is true if we blow up several points. Let $\pi : Y \to X$ be the blowing-up of a smooth surface X at some points $p_1, \ldots, p_r$, and let $E_1, \ldots, E_r$ be the exceptional divisors. For D a divisor on X. Let $|D - n_1 p_1 - \cdots - n_r p_r|$ denotes the sub-linear system of the complete linear system $|D|$ consisting of curves of $|D|$ which pass through $p_1, \ldots, p_r$ with multiplicities at least $n_1, \ldots, n_r$. The blowing-up permits to turn this incomplete linear system into a complete one: there is a one-to-one correspondence between ([Har77, loc. cit.], [CA00]),

$$\begin{array}{ccc} |D - n_1 p_1 - \cdots - n_r p_r| & \longrightarrow & |\pi^* D - n_1 E_1 - \cdots - n_r E_r| \\ C & \longmapsto & C^\sharp \end{array} \quad \text{where} \quad C^\sharp \stackrel{\text{def.}}{=} \pi^* C - n_1 E_1 - \cdots - n_r E_r. \tag{7}$$

This curve $C^\sharp$ is sometime called the *virtual transform* of C. The total, strict and virtual transforms are thus related by:

$$\pi^* C = \widetilde{C} + \sum_{i=1}^{r} m_{p_i}(C) E_i = C^\sharp + \sum_{i=1}^{r} n_i E_i \qquad \Longrightarrow \qquad C^\sharp = \widetilde{C} + \sum_{i=1}^{r} \left(m_{p_i}(C) - n_i\right) E_i.$$

In particular the virtual and the strict transforms coincide when $m_{p_i}(C) = n_i$ for all i.

• This one-to-one correspondence is still true if some points in the sequence of blowing ups are infinitely near points, that is when some p_j lies on the exceptional divisor of the blow up of another point p_i. In order to describe this, we need to carefully define the sub-linear system associated to a family of infinitely near points. Let us start with only two points: if $p_1 \prec p_2$, that is if p_2 lies on the exceptional curve E_1 above p_1, then for $n_1, n_2 > 0$, the sub-linear system of curves passing through p_1 and p_2 with multiplicities at least n_1 and n_2 is defined by:

$$|D - n_1 p_1 - n_2 p_2| \stackrel{\text{def.}}{=} \{C \in |D - n_1 p_1|, m_{p_2}(\pi^*(C) - n_1 E_1) \geq n_2\} = \left\{C \in |D - n_1 p_1|, m_{p_2}(C^\sharp) \geq n_2\right\}.$$

In particular the sub-system $|D - p_1 - p_2|$ contains all the curves of $|D|$ that pass through p_1 with tangent line at p_1 equal to p_2 union all the curves of $|D|$ singular at p_1; indeed, in the last case $C^\sharp = \pi^*(C) - E_1 = \widetilde{C} + (m_{p_1}(C) - 1) E_1$ has E_1 as a component and thus passes through p_2 (one can check that the conditions $p_1 \in C$ and $p_2 \in \widetilde{C}$ are not linear, which is why we choose $p_2 \in C^\sharp$ instead).

In the same way, if $p_1 \prec p_2 \prec \cdots \prec p_r$, one can define recursively, the sub-linear system $|D - n_1 p_1 - \cdots - n_r p_r|$. With this definition, the one-to-one correspondence (7) is still true.

Let us end by an example: the case $X = \mathbf{P}^2$. If ℓ denotes the class a line, and if E_0 is the pullback of ℓ in Y, then the curves of the complete linear $|dE_0 - n_1 E_1 - \cdots - n_r E_r|_Y$ on Y corresponds bijectively to $|d\ell - n_1 p_1 - \cdots - n_r p_r|$ the (projective) vector space consisting of plane curves of degree d passing through $p_1, \ldots, p_r$ with multiplicities at least $n_1, \ldots, n_r$. For small degrees, it turns out that the irreducible decompositions of such curves can be easily described.

3.3 Blowing up and evaluation codes

Let us return to codes and compare the evaluation codes $\mathcal{C}_X(D, X(k))$ and $\mathcal{C}_Y(\pi^*D, Y(k))$.

Proposition 3.3. *Let X be a normal surface, let p be a point of X and let $\pi : Y \to X$ be the blowing-up of X at p. We denote by $\mathcal{E}$ the divisor sum of the exceptional curves.*

(i) If p is of degree > 1, then the codes $\mathcal{C}_X(D, X(k))$ and $\mathcal{C}_Y(\pi^*D, Y(k))$ are equivalent; moreover the code $\mathcal{C}_Y(\pi^*D - n\mathcal{E}, Y(k))$ can be identified with the sub-code of $\mathcal{C}_X(D, X(k))$ where only the global section having multiplicity at least n at p are evaluated.

(ii) If p is rational, then the code $\mathcal{C}_Y(\pi^*D - n\mathcal{E}, Y(k))$ can be identified with the sub-code of $\mathcal{C}_X(D, X(k) \setminus \{p\})$ where only the global sections having multiplicity at least n at p are evaluated and to which we add the following $(q + 1)$ coordinates: the evaluations at rational points of $\mathbf{P}^1$ of the homogeneous component of degree n of the local equation at p of the section.

Proof. (i) The map $s \mapsto s\circ\pi$ is a one-to-one correspondence from the spaces of functions $H^0(X, D)$ to $H^0(Y, \pi^*D)$. Since the blown points are not rational, the map π induces a one-to-one correspondence from $Y(k)$ to $X(k)$. Thus the codes $\mathcal{C}_X(D, X(k))$ and $\mathcal{C}_Y(\pi^*D, Y(k))$ must be equivalent. By the previous correspondence the global sections of $H^0(Y, \pi^*D - n\mathcal{E})$ are in bijection with the global sections of $H^0(X, D)$ that pass through p with multiplicity at least n and the last statement follows.

(ii) The set $Y(k)$ is in one-to-one correspondence with $(X(k) \setminus \{p\}) \cup \mathcal{E}(k)$ and we only have to compute the evaluations at the points of $E(k)$. We choose an open neighbourhood $U \subset \mathbf{A}^2_{(x,y)}$ of p in which $p = (0, 0)$ and D has local equation $f(x, y) = 0$. Then $\pi^{-1}(U) \subset U \times \mathbf{P}^1_{(u:v)}$ with equation $xv = yu$; there are two affine charts, $\pi^{-1}(U) = V_1 \cup V_2$, with $V_1 \subset \mathbf{A}^2_{(y,u)}$ (resp. $V_2 \subset \mathbf{A}^2_{(x,v)}$) with $\pi(y, u) = (yu, y)$ (resp. $\pi(x, v) = (x, xv)$). On V_1, the divisor $\pi^*D - n\mathcal{E}$ has local equation $\frac{f\circ\pi}{y^n} = \frac{f(yu,y)}{y^n}$. Let $s\circ\pi \in H^0(Y, \pi^*D - n\mathcal{E})$ then $sf \in \mathcal{O}_X(U)$ has multiplicity at least n at p, that is $sf(x, y) = p_n(x, y) + p_{n+1}(x, y) + \cdots$, where p_n is homogeneous of degree n. Thus $\frac{sf\circ\pi}{y^n} = \frac{p_n(yu,y)+p_{n+1}(yu,y)+\cdots}{y^n} = p_n(u, 1) + yq(u, y)$. Evaluating at the point $(0, u) \in \mathcal{E} \cap V_1$, the section- has value $p_n(u, 1)$. The same is true on V_2. □

The examples below provide many examples of this blowing-up operation, especially the one in section 4.7.

4 Anticanonical codes from weak del Pezzo surfaces

In this section we describe some evaluation codes from weak del Pezzo surfaces, we compute their parameters and for some of them a generator matrix. The base field is a finite field $\mathbf{F}_q$ without any other hypothesis excepts sporadically not being too small ($\mathbf{F}_2$ or $\mathbf{F}_3$).

In the first subsection, the general construction is given. We also fix many notations that will be used until the end of the paper.

4.1 General description of the codes and of the main steps of their studies

The evaluation codes (definition 3.1) studied in the sequel are the ones corresponding to the following choices.

Definition 4.1. *Let X be a weak del Pezzo surface over $\mathbf{F}_q$. We call **anticanonical code associated to** X the evaluation code $\mathcal{C}_{X_s}(-K_{X_s}, X_s(\mathbf{F}_q))$, where X_s is the anticanonical model of X, $-K_{X_s}$ is the anticanonical (Cartier) divisor on X_s, and where $X_s(\mathbf{F}_q)$ denotes the set of rational points of X_s.*

Note that we could have considered the evaluation codes $\mathcal{C}_X(-K_X, X(\mathbf{F}_q))$ with the same del Pezzo surfaces, but this leads to worth codes.

In a concomitant work [BH22], we have computed explicit models for all the *arithmetic types* of del Pezzo surfaces over a finite field (these types lead to a classification that is coarser than the isomorphism one but that permit to distinguish the main arithmetic properties of the weak del Pezzo surfaces). Taking advantage of this knowledge, we select eight types of weak del Pezzo that are well suited for coding applications. For each example, our starting point is a blowing-up model of the weak del Pezzo surface, then we study the parameters length, dimension, minimum distance ($[n, k, d_{\min}]_q$) of the associated anticanonical code and last we give a generator matrix (or a program to compute it).

Configuration to blow-up. — The explicit description of the surfaces X and X_s always starts from the projective plane $\mathbf{P}^2$: we first blow up a family of (possibly infinitely near) points $p_1, \ldots, p_r$ to obtain a smooth surface Y; then we may blow down a family of (non intersecting) exceptional curves on Y to obtain the smooth surface X. Last X is mapped to a projective space corresponding to the anticanoncial divisor to lead to the singular surface X_s. To sum up, we have the following diagram:

$$
\begin{array}{ccccl}
Y & & & & \\
 & \searrow^{\chi} & & & \\
\pi \downarrow & & X & & \\
 & & & \searrow^{\varphi} & \\
\mathbf{P}^2 & & \overset{\varepsilon}{\dashrightarrow} & & X_s \subset \mathbf{P}^{\deg(X)}
\end{array}
\qquad
\begin{array}{rl}
\pi & \text{is a sequence of blowing ups at points } p_1, \dots, p_r, \\
\chi & \text{is a sequence of contractions of} \\
 & (-1)\text{-curves } F_1, \dots, F_s, \\
\varphi & \text{is the morphism } \varphi_{-K_X} \text{ associated to} \\
 & \text{the anticanonical divisor } -K_X \text{ of } X, \\
\deg(X) & \text{is the degree of the del Pezzo surface } X, \text{ i.e. } K_X^{\cdot 2}.
\end{array}
\tag{8}
$$

All the surfaces and maps are defined over the base field $\mathbf{F}_q$. The solid arrows π, χ, φ denote maps that are morphisms whereas the dashed arrow ε denotes a map which is a rational one. The need to introduce the auxiliary surface Y is due to the fact that some times, the surface X we want to work with cannot be constructed directly by blowing up the plane at some points. Some contractions may be necessary in order to work with applications that are defined over $\mathbf{F}_q$ (and not only over $\overline{\mathbf{F}}_q$); however this detour is not always useful and in some examples, one has $X = Y$ and the map χ is only the identity.

Two of the parameters $[n, k, d_{\min}]_q$ of the associated anticanonical code are easy to compute.

- The length is nothing else than $\#X_s(\mathbf{F}_q)$. Following the process of blowing ups and down above, it is not difficult to compute this number since blowing up a point adds q rational points or does not change the number of rational points depending on whether the point is rational or not.
- The dimension is nothing else than $d+1$, where d is the degree of the del Pezzo surface X, unless the evaluation map is not injective. This can only occur if $\#X_s(\mathbf{F}_q) \leq N_q(-K_{X_s})$ and we compute last number to estimate the minimum distance. It turns out that the evaluation map is always injective except if the base field is $\mathbf{F}_2$ or $\mathbf{F}_3$ in some cases that are excluded.

As usual, the last parameter, the minimum distance, requires much more preparatory works.

Computation of the divisor class groups. — For these computations, the general ambient space is the geometric divisor class group of Y, which is known to be equal to $\mathrm{Cl}(\overline{Y}) = \mathbf{Z}E_0 \oplus \mathbf{Z}E_1 \oplus \cdots \oplus \mathbf{Z}E_r$, where, as usual, E_i denotes the exceptional curve above p_i in the sequence of blowing ups π. In this lattice, one can easily identify the effective roots in Y, but also in X and we are able to give a basis of the sub-lattice $\overline{\mathscr{R}}$ generated by the effective roots of X over $\overline{\mathbf{F}}_q$. The other (geometric) Cartier and Weil divisor class groups are then given by:

$$\mathrm{Cl}(\overline{X}) = (\mathbf{Z}F_1 \oplus \cdots \oplus \mathbf{Z}F_r)^{\perp}, \qquad \mathrm{CaCl}(\overline{X}_s) = \overline{\mathscr{R}}^{\perp}, \qquad \mathrm{Cl}(\overline{X}_s) = \mathrm{Cl}(\overline{X})/\overline{\mathscr{R}}.$$

(the left orthogonal is computed in the whole $\mathrm{Cl}(\overline{Y})$, the middle one in the sub-lattice $\mathrm{Cl}(\overline{X})$). Using tools of section 2.3.3, explicit bases and canonical embeddings of these geometric divisor class groups can be computed. Taking into account the Galois action, one can also give bases and explicit canonical embedding bases of all the arithmetic divisor class groups. Depending on the examples, the computations are carried out in the geometric groups $\mathrm{Cl}(\overline{X})$ and the Galois invariants are taken in the last step to return in $\mathrm{Cl}(X)$ or we start to compute the Galois invariants and then perform all the computations in $\mathrm{Cl}(X)$. Thanks to Proposition 2.7, these two ways lead to the same results.

Types of decomposition into irreducible components in $|-K_{X_s}|$. — The minimum distance is related to the maximum number of rational points that can contain a (effective) curve in the linear system $|-K_{X_s}|$. To bound above this number of rational points, one way is to study how the curves in this linear system decompose into irreducible components and use the exact number of points if known or the Weil bound if not on each components. Thanks to section 3.2, and since $\varphi^* K_{X_s} = K_X$, we have the following one-to-one correspondences:

$$\begin{array}{ccccc} |-\chi^* K_X|_Y & \xrightarrow{\simeq} & |-K_X|_X & \xrightarrow{\simeq} & |-K_{X_s}|_{X_s} \\ C & \longmapsto & \chi_*(C) & \longmapsto & \varphi_*\left(\chi_*(C)\right) \end{array}$$

The first arrow consists in contracting the family of non-meeting exceptional curves F_i, $1 \leq i \leq s$, the second in contracting the effective roots of X. Thus we are reduced to study the types of decompositions into irreducible components on the smooth surface Y, which is easier. Indeed, we know that $-\chi^* K_X = dE_0 - \sum_{i=1}^r n_i E_i$ for some explicit d and n_i's; in fact, in all examples, $d \in \{3, 4\}$ and $n_i \in \{1, 2\}$. Since Y is the blowing up of $\mathbf{P}^2$ at a family of points, thanks to section 3.2, curves of $|-K_Y|$ are in one-to-one correspondence to the plane curves of a well specified (non complete) linear system of $\mathbf{P}^2$:

$$\begin{array}{ccc} |d\ell - n_1 p_1 - \cdots - n_r p_r| & \xrightarrow{\simeq} & |-\chi^* K_X| \\ C & \longmapsto & C^\sharp \end{array}$$

We are thus reduced to list all the types of decompositions into irreducible components of the plane curves of degree d passing through p_i with multiplicity n_i. Since $d \leq 4$, these absolutely irreducible components must be plane lines, conics, cubics or quartics and an enumeration case by case can be done. More specifically, we follow the steps:

- degree by degree, we list all the possible absolutely irreducible curves that pass through some of the points p_i;
- we compute their Galois-orbits since if an absolutely irreducible component not defined over $\mathbf{F}_q$ appears in the decomposition with multiplicity m, then the same holds for all its conjugates (this permits to get rid of many curves because of their too high degree);
- we combine all these irreducible curves to obtain plane curves in the expected sub-linear system.

In order to make easier this step, we adopt the following notations and conventions. The letters ℓ, q, c, t respectively denote plane lines, quadrics (or conics), cubics and quartics. The indices below these letters are the numbers of the points through which the curve passes. For example, ℓ_1 denotes a line that passes through p_1 (but not through any other point), ℓ_{123} a line that passes through p_1, p_2, p_3 (if it exists), q_{123456} a conic passing through the six points $p_1, \ldots, p_6$ and ℓ or q a line or quadric that do not pass through any p_i. The goal is then to combine all these irreducible plane curves to obtain a curve in the expected linear system.

At the end of this step, we are able to compute the maximum $N_q(-K_{X_s})$ to which the minimum distance is related (proposition 3.2). Comparing with the number $\#X_s(\mathbf{F}_q)$, this also permits us to exclude some too small values of q for which the evaluation map may fail to be injective.

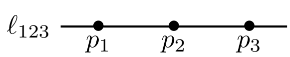
Computation of the global sections from $\mathbf{P}^2$. — Last, if we want to explicitly compute a generator matrix of the code, we need to exhibit a basis of the sub linear system $|d\ell - n_1p_1 - \cdots - n_rp_r|$. Then, by construction we know to which points of $\mathbf{P}^2$ these functions have to be evaluated; in some cases we also need to add some extra evaluation points corresponding to points on some exceptional curves. In any cases, one can compute a generator matrix. This last (concrete) description turns the code into a code close to a Reed-Muller one: the space of polynomials to be evaluated has been restricted, some of the evaluation points have been deleted, some others have been added.

If some readers want to use our code, we put on the second author's `webpage`, a `magma` program that permits to construct all the codes presented below.

4.2 Degree 6, singularity of type $\mathbf{A}_1$

This example corresponds to the type number 3 in degree 6 [BH22].

Configuration to blow-up. — We blow up $\mathbf{P}^2$ at three collinear points that are conjugate over $\mathbf{F}_q$.



$$p_2 = p_1^{\sigma}, \quad p_3 = p_1^{\sigma^2}$$

The resulting surface is a weak del Pezzo surface X whose anticanonical model is denoted X_s. It has a unique singular point of type $\mathbf{A}_1$ which is necessarily rational.

Computation of the divisor class groups. — Over $\overline{\mathbf{F}}_q$, one has

$$\mathrm{Cl}(\overline{X}) = \mathbf{Z}E_0 \oplus \mathbf{Z}E_1 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_3 \qquad \text{and} \qquad -K_X = 3E_0 - E_1 - E_2 - E_3.$$

There is a unique effective root, the strict transform of the line ℓ_{123} passing through the three points p_1, p_2, p_3, and its class is $E_0 - E_1 - E_2 - E_3$. Then

$$\overline{\mathscr{R}} = \mathbf{Z}(E_0 - E_1 - E_2 - E_3) \qquad \overline{\mathscr{R}}^{\perp} = \{a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3 \mid a_0 + a_1 + a_2 + a_3 = 0\}$$
$$= \mathbf{Z}(E_0 - E_1) \oplus \mathbf{Z}(E_0 - E_2) \oplus \mathbf{Z}(E_0 - E_3)$$

Both $\overline{\mathscr{R}}$ and $\overline{\mathscr{R}}^{\perp}$ are direct summand but $\overline{\mathscr{R}}$ and $\overline{\mathscr{R}}^{\perp}$ are not complementary submodules since $\overline{\mathscr{R}} \oplus \overline{\mathscr{R}}^{\perp}$ is of index 2 in $\mathrm{Cl}(\overline{X})$. For a submodule complement to $\overline{\mathscr{R}}$, one can choose:

$$\begin{array}{rcccl} \mathrm{Cl}(\overline{X}) & = & \overline{\mathscr{R}} & \oplus & (\mathbf{Z}E_1 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_3) \\ a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3 & = & a_0(E_0 - E_1 - E_2 - E_3) & + & (a_1 + a_0)E_1 + (a_2 + a_0)E_2 + (a_3 + a_0)E_3 \end{array}$$

This leads to the following isomorphism:

$$\begin{array}{rcl} \mathrm{Cl}(\overline{X})/\overline{\mathscr{R}} & \xrightarrow{\simeq} & \mathbf{Z}E_1 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_3 \\ a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3 \bmod \overline{\mathscr{R}} & \longmapsto & (a_0 + a_1)E_1 + (a_0 + a_2)E_2 + (a_0 + a_3)E_3. \end{array}$$

Via this isomorphism, the submodule $\mathrm{CaCl}(\overline{X}_s) = \overline{\mathscr{R}}^{\perp}$ identifies with $\mathbf{Z}(E_1+E_2)\oplus\mathbf{Z}(E_2+E_3)\oplus\mathbf{Z}(E_1+E_3)$ of invariant factors $1,1,2$ in $\mathbf{Z}E_1\oplus\mathbf{Z}E_2\oplus\mathbf{Z}E_3$.

Over $\mathbf{F}_q$, to recover the class groups $\mathrm{Cl}(X_s)$ and $\mathrm{CaCl}(X_s)$, we only need to take the invariants under the Galois action $(E_0)(E_1E_2E_3)$, what is easy here. One has

$$\begin{aligned}\mathrm{CaCl}(X_s) &= \mathrm{CaCl}(\overline{X}_s)^{\Gamma} = \left(\overline{\mathscr{R}}^{\perp}\right)^{\Gamma} \simeq \mathbf{Z}(3E_0-E_1-E_2-E_3) = \mathbf{Z}(-K_X),\\ \mathrm{Cl}(X_s) &= \mathrm{Cl}(\overline{X}_s)^{\Gamma} = \left(\mathrm{Cl}(\overline{X})/\overline{\mathscr{R}}\right)^{\Gamma} \simeq \mathbf{Z}(E_1+E_2+E_3).\end{aligned}$$

With these identifications, the canonical embedding of $\mathrm{CaCl}(X_s)$ into $\mathrm{Cl}(X_s)$ becomes:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{CaCl}(X_s) & \longrightarrow & \mathrm{Cl}(X_s)\\ & & -K_X & \longmapsto & 2(E_1+E_2+E_3)\end{array}$$

Thus both $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ are free of rank 1, but $\mathrm{CaCl}(X_s)$ is of index 2 into $\mathrm{Cl}(X_s)$. This index has the following consequence: even if $\mathrm{CaCl}(X_s)$ is free of rank one generated by $-K_{X_s}$, a Cartier divisor may decompose into a sum of equivalent Weil irreducible divisors. This explains why we need to investigate how elements of $|-K_X|$ can decompose into irreducible components and how the non ordinary weak del Pezzo surfaces we consider here differ from ordinary ones (compare with [BCH+20]).

Types of decomposition into irreducible components in $|-K_{X_s}|$. — In this example, there is no need to introduce an auxiliary surface Y (one has $Y = X$ and χ is the identity with the notation of the beginning of this section). Since $-K_X = 3E_0 - E_1 - E_2 - E_3$, the virtual transform composed with the push forward lead to a one-to-one correspondence:

$$\begin{array}{ccccc} |3\ell - p_1 - p_2 - p_3| & \longrightarrow & |3E_0 - E_1 - E_2 - E_3| & \longrightarrow & |-K_{X_s}|\\ C & \longmapsto & C^{\sharp} & \longmapsto & \varphi_*(C^{\sharp})\end{array}$$

(the left linear system is on $\mathbf{P}^2$, the middle one on X and the right one on X_s). Then we are reduced to list all the types of decompositions into irreducible components of the curves of $|3\ell - p_1 - p_2 - p_3|$, the sub linear system of cubics passing through the points p_1, p_2, p_3. The orbits of lines, conics, cubics which have degree at most 3 and pass through some of the points p_i's are

$$\ell_1\cup\ell_2\cup\ell_3, \qquad\qquad \ell_{123}, \qquad\qquad c_{123},$$

(of course, implicitly $\ell_2 = \ell_1^{\sigma}$, $\ell_3 = \ell_1^{\sigma^2}$ where σ is a generator of $\mathrm{Gal}(\overline{\mathbf{F}}_q/\mathbf{F}_q)$). Indeed an absolutely irreducible conic q_i or q_{ij} cannot be defined over $\mathbf{F}_q$ and they have at least three conjugates; combining these curves with their conjugates lead to plane curves of degree greater than 6 and thus they cannot appear in our case. A conic q_{123} cannot be absolutely irreducible otherwise it would have three intersection points with the line ℓ_{123}. Let us combine these rational irreducible decompositions in order to construct plane curves in the expected sub-linear system.

First suppose that the decomposition contains a line.

- If this line is ℓ_1, then its conjugates ℓ_2, ℓ_3 must also be geometric components; the only possibility is $\ell_1\cup\ell_2\cup\ell_3$ (line 1 in the tabular below) which is an element of $|3\ell - p_1 - p_2 - p_3|$.

- A component ℓ cannot be completed by a conic passing through the three points and thus if there is a line in the geometric decomposition, ℓ_{123} must be one of them. Since ℓ_{123} already passes through p_1, p_2, p_3 it can be completed by any conic (irreducible or not); this leads to the decompositions of the lines 3 to 7 in the tabular below.

Last, if the decomposition does not contain any line, it must be an irreducible cubic which passes through the three points; this cubic can be smooth or not and we recover the two last lines of the tabular.

	$\lvert 3\ell - p_1 - p_2 - p_3\rvert$ on $\mathbf{P}^2$	$\lvert -K_X\rvert$ on X	$\lvert -K_{X_s}\rvert$ on X_s	Max nb. of pts
1	$\ell_1 \cup \ell_2 \cup \ell_3$	$\widetilde{\ell}_1 \cup \widetilde{\ell}_2 \cup \widetilde{\ell}_3$	$\bigcup_{i=1}^3 \varphi_*(\widetilde{\ell}_i)$	1
2	$\ell_{123} \cup q,\ \ell_{123} \cap q \not\subset \mathbf{P}^2(\mathbf{F}_q)$	$\widetilde{\ell}_{123} \cup \widetilde{q}$	$\varphi_*(\widetilde{q})$	$q+2$
3	$\ell_{123} \cup q,\ \ell_{123} \cap q \subset \mathbf{P}^2(\mathbf{F}_q)$	$\widetilde{\ell}_{123} \cup \widetilde{q}$	$\varphi_*(\widetilde{q})$	q
4	$\ell_{123} \cup \ell \cup \ell',\ \ell_{123} \cap \ell \cap \ell' \neq \emptyset$	$\widetilde{\ell}_{123} \cup \widetilde{\ell} \cup \widetilde{\ell}'$	$\varphi_*(\widetilde{\ell}) \cup \varphi_*(\widetilde{\ell}')$	$2q+1$
5	$\ell_{123} \cup \ell \cup \ell',\ \ell_{123} \cap \ell \cap \ell' = \emptyset$	$\widetilde{\ell}_{123} \cup \widetilde{\ell} \cup \widetilde{\ell}'$	$\varphi_*(\widetilde{\ell}) \cup \varphi_*(\widetilde{\ell}')$	$2q$
6	$2\ell_{123} \cup \ell$	$2\widetilde{\ell}_{123} \cup \widetilde{\ell} \cup \bigcup_{i=1}^3 E_i$	$\varphi_*(\widetilde{\ell}) \cup \bigcup_{i=1}^3 \varphi_*(E_i)$	$q+1$
7	$3\ell_{123}$	$3\widetilde{\ell}_{123} \cup \bigcup_{i=1}^3 2E_i$	$\bigcup_{i=1}^3 2\varphi_*(E_i)$	1
8	c_{123}, singular	$\widetilde{c}_{123}$	$\varphi_*(\widetilde{c}_{123})$	$q+2$
9	c_{123}, smooth	$\widetilde{c}_{123}$	$\varphi_*(\widetilde{c}_{123})$	$\mathrm{N}_q(1)$

Some comments about the three first columns of the previous tabular. The unique irreducible effective root of X is nothing else than the strict transform ℓ_{123} and this explains why this curve disappears in the third column: this line on X is mapped by φ_* to the unique singular point $s \in X_s$. Note also that except in the cases 6 and 7, all the curves have exactly multiplicities 1 at the p_i and thus their strict or virtual transforms are equal. On the contrary, in the remaining cases, the curves on X are the virtual transforms of the ones on $\mathbf{P}^2$. Last, in the decomposition $\varphi_*(\widetilde{\ell}) \cup \varphi_*(\widetilde{\ell}')$, it is worth noticing that irreducible components involves divisors that are not Cartier divisors but only Weil ones on X_s. Indeed the class of $\widetilde{\ell}$ in $\mathrm{Cl}(X)$ is E_0, which is mapped to $E_1 + E_2 + E_3$ in $\mathrm{Cl}(X_s)$, which is not an element of $\mathrm{CaCl}(X_s)$ (equivalently $E_0 \notin \mathscr{R}^\perp$).

Now we make some comments on the numbers of rational points.

Case 1. Since the lines ℓ_1, ℓ_2, ℓ_3 are conjugate a rational point on their union must be at their intersection which contains at most one point. On X the strict transforms $\widetilde{\ell}_1, \widetilde{\ell}_2, \widetilde{\ell}_3$ do not meet the root ℓ_{123} and the contraction does not add any point.

Cases $2, 3, 4$ **&** 5. If the two points of $q \cap \ell_{123}$ are not rational then they are still unrational on $\widetilde{q} \cap \widetilde{\ell}_{123}$ and they are contracted to the singular point s in X_s and thus the image $\varphi_*(\widetilde{q})$ has one more rational point; otherwise, if the two points of $q \cap \ell_{123}$ are rational then they are contracted in X_s and thus the image $\varphi_*(\widetilde{q})$ looses a rational point. The same is true on lines 4 and 5.

Cases 6 **&** 7. The line $\widetilde{\ell}_{123}$ is contracted by φ_* and there are no rational points on the lines E_i.

Cases 8 **&** 9. The starting cubic c_{123} has $(q+1)$ or less than $\mathrm{N}_q(1)$ rational points depending on whether it is singular or smooth. On X_s the number of rational points of $\varphi_*(\widetilde{c}_{123})$ is increased by 1 since the line ℓ_{123} meets the cubic at three conjugate points. The multiplicity of intersection of c_{123} and ℓ_{123} at each point p_i is one (since otherwise, these two curves would have too many intersection points counting with multiplicities). Therefore, the blowing ups at p_1, p_2, p_3 separate the strict transforms $\widetilde{\ell}_{123}$ and $\widetilde{c}_{123}$. Thus $\widetilde{c}_{123}$ and $\varphi_*(\widetilde{c}_{123})$ are isomorphic and have the same number of rational points. Finally, we remark that for every q, one has $q + 1 + \lfloor 2\sqrt{q} \rfloor \leq 2q + 1$ (with equality if and only if $q \in \{2, 3, 4\}$) and thus:

$$N_q(-K_{X_s}) = 2q + 1.$$

Last we note that, except for the cases 1 and 9, all the maximum numbers of points are in fact exact numbers of points. Thus we are not far from having the distribution of weights if the code.

Since p_1, p_2, p_3 are not rational, the three blowing ups do not add any rational point and $\#X(\mathbf{F}_q) = q^2 + q + 1$. Then, the root $\widetilde{\ell}_{123}$ is contracted via the anticanonical morphism and thus $\#X_s(\mathbf{F}_q) = q^2 + 1$. Except if $q = 2$, one has $\#X_s(\mathbf{F}_q) > N_q(-K_{X_s})$ and the evaluation map is injective. With this choice of weak del Pezzo surface, the code of definition 4.1 satisfies the following proposition.

Proposition 4.2. *Let p_1, p_2, p_3 be conjugate collinear point in $\mathbf{P}^2_{\mathbf{F}_q}$, with $q \neq 2$. The anticanonical code associated to the weak del Pezzo surface obtained by blowing up these points has parameters $[q^2 + 1, 7, q^2 - 2q]$.*

Computation of the global sections from $\mathbf{P}^2$. — To construct this kind of codes, one can choose ℓ_{123} to be the line of equation $Y = 0$ in $\mathbf{P}^2$. For any $\zeta \in \mathbf{F}_{q^3} \setminus \mathbf{F}_q$, the point $p_1 = (\zeta : 0 : 1) \in \mathbf{P}^2$ is a degree 3 point whose conjugates $p_2 = (\zeta^\sigma : 0 : 1)$ and $p_3 = (\zeta^{\sigma^2} : 0 : 1)$ are also in ℓ_{123}. Let $X^3 + a_2 X^2 + a_1 X + a_0 \in \mathbf{F}_q[X]$ be the minimal polynomial of ζ over $\mathbf{F}_q$. Then, we easily verify that

$$|3\ell - p_1 - p_2 - p_3| = \left\langle Y^3, Y^2X, Y^2Z, YX^2, YZ^2, YXZ, X^3 + a_2X^2Z + a_1XZ^2 + a_0Z^3 \right\rangle_{\mathbf{F}_q}.$$

Last, the evaluation points are nothing else than the points of $\mathbf{P}^2(\mathbf{F}_q) \setminus \ell_{123}(\mathbf{F}_q)$, plus one point of $\ell_{123}(\mathbf{F}_q)$ since the strict transform of ℓ_{123} is contracted via the anticanonical morphism. Let us denote $(x_i : 1 : z_i)$, $1 \leq i \leq q^2$, the first q^2 points, and let us choose $(0 : 0 : 1) \in \ell_{123}(\mathbf{F}_q)$, then the corresponding generator matrix of the code is:

$$\begin{pmatrix} 1 & \cdots & 1 & 0 \\ x_1 & \cdots & x_{q^2} & 0 \\ z_1 & \cdots & z_{q^2} & 0 \\ x_1^2 & \cdots & x_{q^2}^2 & 0 \\ z_1^2 & \cdots & z_{q^2}^2 & 0 \\ x_1 z_1 & \cdots & x_{q^2} z_{q^2} & 0 \\ P(x_1, 1, z_1) & \cdots & P(x_{q^2}, 1, z_{q^2}) & a_0 \end{pmatrix}, \qquad P(X,Y,Z) = X^3 + a_2 X^2 Z + a_1 X Z^2 + a_0 Z^3.$$

We recover the classical Reed-Muller code on $\mathbf{A}^2$ of degree 2, augmented by one point.

4.3 Degree 5, singularity of type $2\mathbf{A}_1$

This example corresponds to the type number 5 in degree 5 [BH22].

Configuration to blow-up. — We blow up $p_1 \prec p_2$ and $p_3 \prec p_4$, where p_1, p_3 and p_2, p_4 are conjugate points of degree 2.

$$p_3 = p_1^\sigma \qquad p_4 = p_2^\sigma.$$

Since the points p_2, p_4 are infinitely near the points p_1, p_3, they are represented by tangent lines or directions on the picture above. The anticanonical model X_s has two singular points of type $\mathbf{A}_1$ that are conjugate points.

Computation of the divisor class groups. — Over $\overline{\mathbf{F}}_q$, one has:

$$\mathrm{Cl}(\overline{X}) = \mathbf{Z}E_0 \oplus \mathbf{Z}E_1 \oplus \mathbf{Z}E_3 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_4 \qquad \text{and} \qquad -K_X = 3E_0 - E_1 - E_2 - E_3 - E_4.$$

There are two conjugate effective roots, the strict transforms of E_1 and E_3 in the sequence of blowing ups; their classes are $E_1 - E_2$ and $E_3 - E_4$ in such a way that:

$$\overline{\mathscr{R}} = \mathbf{Z}(E_1 - E_2) \oplus \mathbf{Z}(E_3 - E_4), \qquad \overline{\mathscr{R}}^\perp = \{a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 \mid a_1 = a_2,\ a_3 = a_4\} = \mathbf{Z}E_0 \oplus \mathbf{Z}(E_1 + E_2) \oplus \mathbf{Z}(E_3 + E_4).$$

The sub-module $\overline{\mathscr{R}}$ is a direct summand, and as a complementary sub-module one can choose:

$$\begin{array}{rcccl} \mathrm{Cl}(\overline{X}) & = & \overline{\mathscr{R}} & \oplus & \mathbf{Z}E_0 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_4 \\ \sum_{i=0}^4 a_iE_i & = & a_1(E_1 - E_2) + a_3(E_3 - E_4) & + & a_0E_0 + (a_1 + a_2)E_2 + (a_3 + a_4)E_4. \end{array}$$

We deduce the isomorphism:

$$\begin{array}{rcl} \mathrm{Cl}(\overline{X}_s) \simeq \mathrm{Cl}(\overline{X})/\overline{\mathscr{R}} & \xrightarrow{\simeq} & \mathbf{Z}E_0 \oplus \mathbf{Z}E_2 \oplus \mathbf{Z}E_4 \\ \sum_{i=0}^4 a_iE_i \bmod \overline{\mathscr{R}} & \longmapsto & a_0E_0 + (a_1 + a_2)E_2 + (a_3 + a_4)E_4 \end{array}.$$

Since $\mathrm{CaCl}(\overline{X}_s) \simeq \overline{\mathscr{R}}^\perp$, this class group is a rank 3 free sub-group of $\mathrm{Cl}(\overline{X})$. Via the previous isomorphism it is mapped to the sub-group $\mathbf{Z}E_0 \oplus \mathbf{Z}2E_2 \oplus \mathbf{Z}2E_4$, of invariant factors $1, 2, 2$.

The arithmetic groups $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ can be computed by taking the invariants under the Galois action which is $(E_0)(E_1E_3)(E_2E_4)$. Via the previous isomorphism, if we set $\mathcal{E} := E_2 + E_4$, the canonical embedding $0 \to \mathrm{CaCl}(X_s) \to \mathrm{Cl}(X_s)$ is only:

$$\underbrace{\mathbf{Z}E_0 \oplus \mathbf{Z}2\mathcal{E}}_{\simeq \mathrm{CaCl}(X_s)} \subset \underbrace{\mathbf{Z}E_0 \oplus \mathbf{Z}\mathcal{E}}_{\simeq \mathrm{Cl}(X_s)}.$$

In other terms, $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ are both free of rank 2 and via the canonical embedding, the first one has invariant factors $1, 2$ inside the second one.

Types of decomposition into irreducible components in $|-K_{X_s}|$. — Since the two class groups are not rank one, one expects to find a wide variety of possible decompositions into irreducible components for the curves in the linear system $|-K_{X_s}|$. In order to list all these types, we start form $\mathbf{P}^2$ and use the one-to-one correspondences:

$$\begin{array}{ccccc} |3\ell - p_1 - p_3 - p_2 - p_4| & \longrightarrow & |3E_0 - E_1 - E_3 - E_2 - E_4| & \longrightarrow & |-K_{X_s}| \\ C & \longmapsto & C^\sharp & \longmapsto & \varphi_*\left(C^\sharp\right). \end{array}$$

The curves of the left linear system are nothing else than the plane cubics over $\mathbf{F}_q$ passing through p_1, p_3 that are either smooth at p_1, p_3 with tangent lines p_2, p_4 respectively or singular at these points.

Our notations are the same: ℓ_{13} is the line (p_1p_3) which is rational, ℓ_{12} and ℓ_{34} are the lines (p_1p_2) respectively (p_3p_4) (that is the lines of $\mathbf{P}^2$ passing through p_1, respectively p_3, whose strict transform pass through p_2, respectively p_4); these last two lines are conjugate. The orbits of lines, conics, cubics having degree less than 3 and passing through some of the points p_i's are

$$\ell_1 \cup \ell_3, \qquad \ell_{13}, \qquad \ell_{12} \cup \ell_{34}, \qquad q_{13}, \qquad q_{1234}, \qquad c_{13}, \qquad c_{1234}$$

(of course, implicitly $\ell_3 = \ell_1^\sigma$, $\ell_{34} = \ell_{12}^\sigma$). We have just to combine these rational irreducible decompositions in order to construct plane curves in the expected sub-linear system.

Suppose that there is at least one line in the absolute irreducible decomposition.

- If this line is ℓ_{12}, then by rationality, ℓ_{34} is also an absolute irreducible component. Since $\ell_{12} \cup \ell_{34}$ already passes through p_1, p_2, p_3, p_4, one can complete by any rational line ℓ or by the line ℓ_{13} (see cases 1 and 2 in the tabular below).
- If this line is ℓ_{13}, then the two incidence conditions at p_1 and p_3 are satisfied. The complement component must be a (maybe reducible) conic whose strict transform passes through p_2 and p_4; this conic must necessarily pass through p_1, p_3. This conditions suffice since the union of ℓ_{13} with any conic passing through p_1, p_3 is singular. The complement can be the union $\ell_{12} \cup \ell_{34}$ (same as case 2), or $\ell_1 \cup \ell_3 = \ell_1^\sigma$, or ℓ_{13} itself union any other line, or twice ℓ_{13}, or q_{13}, or q_{1234}.
- If this line is ℓ a line that does not pass through the p_i's, then the complement conic must be either $\ell_{12} \cup \ell_{34}$ as in first case, or a conic passing through the four points.

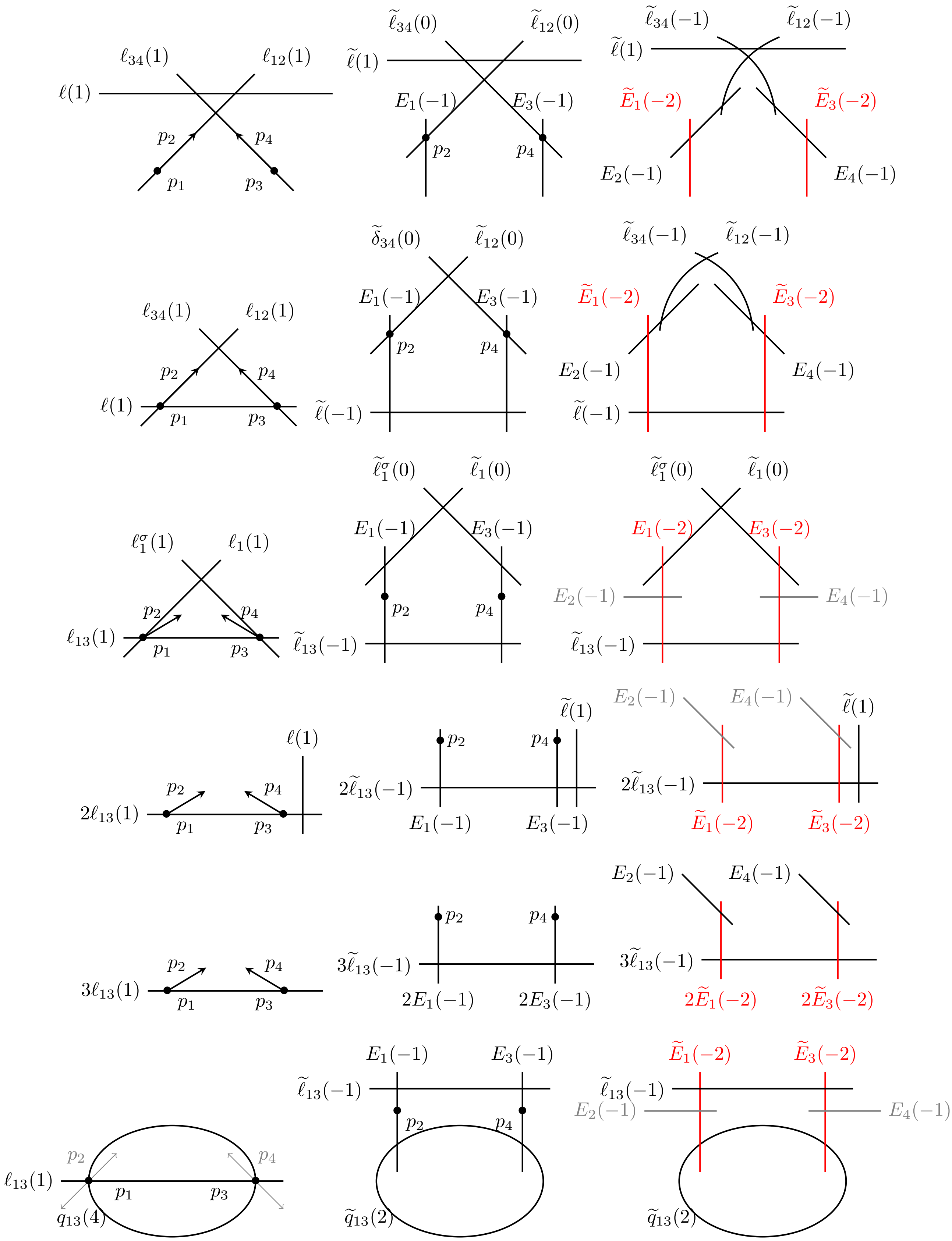
Last, if there is not any line in the absolute irreducible decomposition, then the cubic must be absolutely irreducible and it has to pass through the four points.

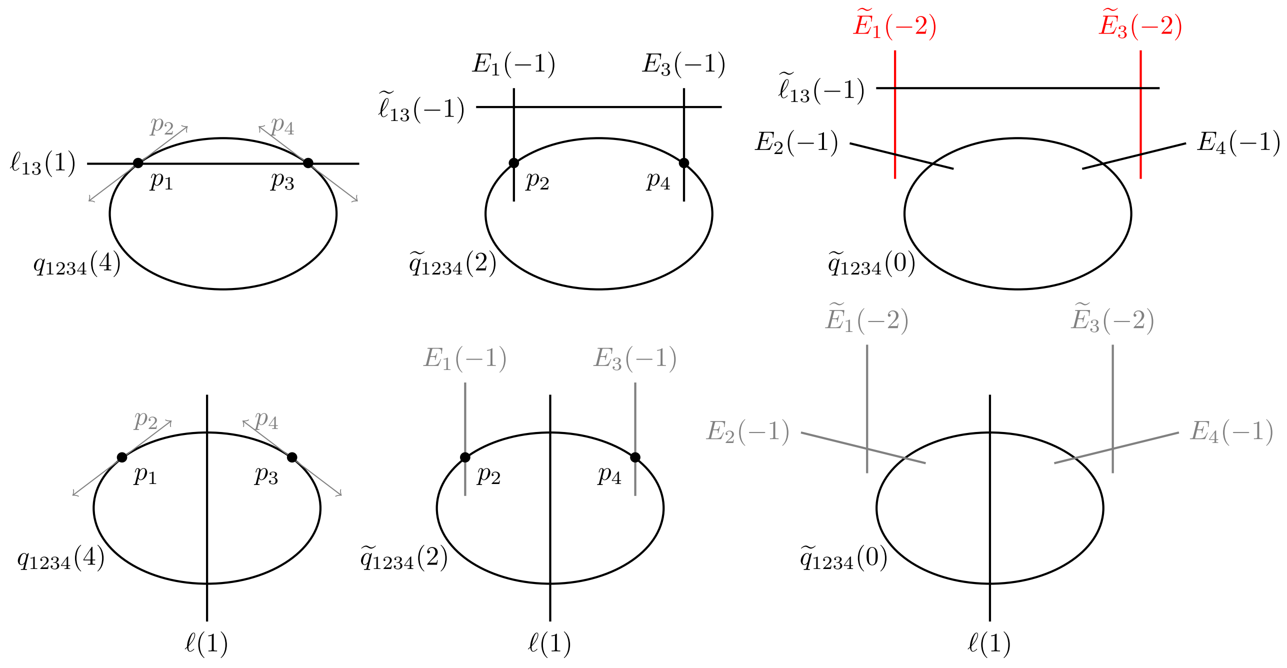
That being, the possible cubics are listed below. The irreducible effective roots of X are the (conjugate) strict transforms $\widetilde{E}_1$ and $\widetilde{E}_3$; since they do not meet, their contraction lead to two (conjugate) singular points s and s^σ on X_s.

	$\left\|3\ell - \sum_{i_1}^4 p_i\right\|$ on $\mathbf{P}^2$	$\|-K_X\|$ on X	$\|-K_{X_s}\|$ on X_s	Max nb. of pts
1	$\ell_{12} \cup \ell_{34} \cup \ell$	$\widetilde{\ell_{12}} \cup \widetilde{\ell_{34}} \cup \widetilde{\ell}$	$\varphi_*(\widetilde{\ell_{12}}) \cup \varphi_*(\widetilde{\ell_{34}}) \cup \varphi_*(\widetilde{\ell})$	$q+2$
2	$\ell_{12} \cup \ell_{34} \cup \ell_{13}$	$\widetilde{\ell_{12}} \cup \widetilde{\ell_{34}} \cup \widetilde{\ell}_{13} \cup \widetilde{E}_1 \cup \widetilde{E}_3 \cup E_2 \cup E_4$	$\varphi_*(\widetilde{\ell_{12}}) \cup \varphi_*(\widetilde{\ell_{34}}) \cup \varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(E_2) \cup \varphi_*(E_4)$	$q+2$
3	$\ell_{13} \cup \ell_1 \cup \ell_3$	$\widetilde{\ell}_{13} \cup \widetilde{\ell}_1 \cup \widetilde{\ell}_3 \cup \widetilde{E}_1 \cup \widetilde{E}_3$	$\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(\widetilde{\ell}_1) \cup \varphi_*(\widetilde{\ell}_3)$	$q+2$
4	$2\ell_{13} \cup \ell$	$2\widetilde{\ell}_{13} \cup \widetilde{\ell} \cup \widetilde{E}_1 \cup \widetilde{E}_3$	$2\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(\widetilde{\ell})$	$2q+1$
5	$3\ell_{13}$	$3\widetilde{\ell}_{13} \cup 2\widetilde{E}_1 \cup 2\widetilde{E}_3 \cup E_2 \cup E_4$	$3\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(E_2) \cup \varphi_*(\widetilde{E}_2)$	$q+1$
6	$\ell_{13} \cup q_{13}$	$\widetilde{\ell}_{13} \cup \widetilde{q}_{13} \cup \widetilde{E}_1 \cup \widetilde{E}_3$	$\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(\widetilde{q}_{13})$	$2q+2$
7	$\ell_{13} \cup q_{1234}$	$\widetilde{\ell}_{13} \cup \widetilde{q}_{1234} \cup \widetilde{E}_1 \cup \widetilde{E}_3 \cup E_2 \cup E_4$	$\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(\widetilde{q}_{1234}) \cup \varphi_*(E_2) \cup \varphi_*(E_4)$	$2q+2$
8	$\ell \cup q_{1234}$	$\widetilde{\ell} \cup \widetilde{q}_{1234}$	$\varphi_*(\widetilde{\ell}) \cup \varphi_*(\widetilde{q}_{1234})$	$2q+2$
9	c_{1234}	$\widetilde{c}_{1234}$	$\varphi_*(\widetilde{c}_{1234})$	$\mathrm{N}_q(1)$

We draw all the preceding decompositions in order to illustrate what is going on. The blowing up $\pi : X \to \mathbf{P}^2$ is decomposed into two blowing ups $\pi = \pi_2 \circ \pi_1$, where $\pi_1 : X_1 \to \mathbf{P}^2$ is the blowing up at p_1 and p_3, and where $\pi_2 : X \to X_1$ is the blowing up at p_2 and p_4. The left column is the drawing of the starting configuration in $\mathbf{P}^2$, the middle one the configuration after having blowing up p_1 and p_3, the right one the configuration in X. The operation from a column to the next one is the virtual transform. Curves drawn in gray are not part of virtual transform, curves drawn in red are the effective roots; these curves are contracted in X_s (but we do not draw this step). In brackets, to the right of the name of a curve, we put its self-intersection. We draw all the cases of the preceding tabular, except the cubic case.

In any case, one can verify that the union of the black curves passes through p_1, p_3, p_2, p_4 and that the divisor class is equal to $-K_X = 3E_0 - E_1 - E_3 - E_2 - E_4$.





Some comments about the number of points.

Cases 1, 2, & 3. The unions of lines $\ell_{12} \cup \ell_{34}$, or $\ell_1 \cup \ell_3$ (recall that $\ell_3 = \ell_1^\sigma$), contain a unique rational point, the intersection point of the two lines. Except ℓ (cases 1 and 2) or ℓ_{13} (case 3), all the lines in the decomposition are not defined over $\mathbf{F}_q$ and di not contain any rational points. Thus to the previous single point we have to add the $(q+1)$ rational points of the line ℓ or ℓ_{13}.

Case 4. The two (black) components have $(q+1)$ rational points but they meet at a rational point, thus their union contains $2q+1$ rational points.

Case 5. The only component that contains rational points is $\varphi_*(\widetilde{\ell}_{13})$.

Cases 6, 7, & 8. In these cases, they are two disjoint components that contain $(q+1)$ rational points.

Finally:

$$N_q\left(-K_{X_s}\right) = 2q+2.$$

Since none of the points p_i is rational, the surfaces X and X_s still have q^2+q+1 points. For every q, one has $\#X_s(\mathbf{F}_q) > N_q\left(-K_{X_s}\right)$ and the evaluation map is always injective. The parameters of the code are thus given by:

Proposition 4.3. *Let $p_1 \prec p_2$ and $p_3 \prec p_4$ be such that p_1, p_3 and p_2, p_4 are conjugate points of degree 2. The anticanonical code of the weak del Pezzo surface obtained by blowing up these points has parameters $[q^2+q+1, 6, q^2-q-1]$.*

Computation of the global sections from $\mathbf{P}^2$. — Let $d \in \mathbf{F}_q$ be a non-square and put $\zeta = \sqrt{d} \in \mathbf{F}_{q^2}$. We choose $p_1 = (\zeta : 0 : 1)$ and $p_2 = (\zeta : 1)$. Then $p_3 = (-\zeta : 0 : 1)$, the line $\ell_{13} = (p_1 p_3)$ has equation $Y = 0$, the line $\ell_{12} = (p_1 p_2)$ corresponds to the zeros of the linear form $L = X - \zeta(Y+Z)$ and the line $\ell_{34} = (p_3 p_4)$ corresponds to the zeros of the linear form $L^\sigma = X + \zeta(Y+Z) = 0$. The linear forms Y, L, L^σ generate the global sections of ℓ and we easily prove that

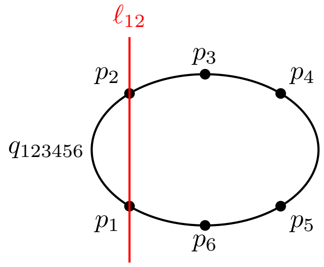
$$\begin{aligned}
|3\ell - p_1 - p_3 - p_2 - p_4| &= \left\langle Y^3, Y^2 L, Y^2 L^\sigma, Y L L^\sigma, L^2 L^\sigma, L(L^\sigma)^2 \right\rangle_{\overline{\mathbf{F}}_q} \\
&= \left\langle Y^3, Y^2(L+L^\sigma), Y^2\zeta(L-L^\sigma), Y L L^\sigma, L L^\sigma(L+L^\sigma), L L^\sigma \zeta(L-L^\sigma) \right\rangle_{\mathbf{F}_q} \\
&= \left\langle Y^3, Y^2 X, Y^2(Y+Z), Y\Pi, X\Pi, (Y+Z)\Pi \right\rangle_{\mathbf{F}_q},
\end{aligned}$$

where $\Pi = L L^\sigma = X^2 - d(Y+Z)^2$. The evaluation points are nothing else than all the points of $\mathbf{P}^2(\mathbf{F}_q)$.

4.4 Degree 4, singularity of type $\mathbf{A}_1$

This example corresponds to the type number 8 in degree 4 [BH22].

Configuration to blow-up and down. — We blow up six points on a conic, the first two p_1 and p_2 being conjugate (or rational), the last four p_3, p_4, p_5, p_6 being conjugate. This leads to a degree 3 weak Del Pezzo surface $Y \xrightarrow{\pi} \mathbf{P}^2$ as in (8). In this surface, the strict transform of the line ℓ_{12} passing through p_1 and p_2 is a rational (-1)-curve that can be contracted: the codomain of the contraction $Y \xrightarrow{\chi} X$ is the degree 4 weak Del Pezzo surface we want to work with in this section.



$$\begin{array}{c} p_2 = p_1^{\sigma} \\ \hline p_4 = p_3^{\sigma} \\ p_5 = p_3^{\sigma^2} \\ p_6 = p_3^{\sigma^3} \end{array}$$

The anticanonical model of this weak del Pezzo surface X has a unique singular point.

Computation of the divisor class groups. — On $\overline{Y}$, one has $\mathrm{Cl}(\overline{Y}) = \bigoplus_{i=0}^{6} \mathbf{Z}E_i$ and 27 exceptional classes of divisor, namely the 6 exceptional lines E_i, $1 \leq i \leq 6$, the 15 strict transforms of the lines passing through two of the six points, $E_{ij} = E_0 - E_i - E_j$, $1 \leq i < j \leq 6$, and the 6 strict transforms of the quadrics passing through five of the six points, $Q_i = 2E_0 - \sum_{j \neq i} E_j$, $1 \leq 6$. Due to weaknesses, among these classes, the quadric ones are not represented by irreducible curves. Indeed $2E_0 - \sum_{j \neq i} E_j = E_i + \left(2E_0 - \sum_{j=1}^{6} E_j\right)$ and the last class is nothing else than the class of the unique effective root, the strict transform of the quadric q_{123456} passing through all the six points.

The group $\mathrm{Cl}(\overline{X})$ can be identified with $\mathbf{Z}(E_0 - E_1 - E_2)^{\perp}$ via the orthogonal projection onto this space. This projection is given by:

$$\begin{array}{rcrcl} \mathrm{Cl}(\overline{Y}) & = & \mathbf{Z}(E_0 - E_1 - E_2) & \oplus & (\mathbf{Z}(E_0 - E_1) \oplus \mathbf{Z}(E_0 - E_2) \oplus \mathbf{Z}E_3 \oplus \cdots \oplus \mathbf{Z}E_6) \\ \sum_{i=0}^{6} a_i E_i & = & (-a_0 - a_1 - a_2)(E_0 - E_1 - E_2) & + & \left[(a_0 + a_2)(E_0 - E_1) + (a_0 + a_1)(E_0 - E_2) + \sum_{i=3}^{6} a_i E_i\right] \end{array},$$

and thus

$$\mathrm{Cl}(\overline{X}) = \mathbf{Z}L_1 \oplus \mathbf{Z}L_2 \oplus \mathbf{Z}E_3 \oplus \cdots \oplus \mathbf{Z}E_6, \qquad \text{where} \qquad L_i = E_0 - E_i, \quad i = 1, 2.$$

In particular, the anticanonical divisors are related by

$$-K_Y = 3E_0 - \sum_{i=1}^{6} E_i = -(E_0 - E_1 - E_2) + \underbrace{2L_1 + 2L_2 - \sum_{i=3}^{6} E_i}_{-K_X}$$

Only the exceptional classes of $\overline{Y}$ that do not meet $E_0 - E_1 - E_2$ are mapped to exceptional classes on $\overline{X}$; for $3 \leq i \leq 6$, this leaves the classes:

$$E_i \longmapsto E_i, \qquad E_{1i} \longmapsto L_1 - E_j, \qquad E_{2i} \longmapsto L_2 - E_j, \qquad Q_i \longmapsto L_1 + L_2 - \sum_{j \in \{3,\ldots,6\} \setminus \{i\}} E_j.$$

As for the unique effective root of $\overline{Y}$, it is mapped to the root $L_1 + L_2 - E_3 - E_4 - E_5 - E_6$ and the last four exceptional classes are not represented by irreducible curves. Thus one has

$$\overline{\mathscr{R}} = \mathbf{Z}(L_1 + L_2 - E_3 - \cdots - E_6), \qquad \overline{\mathscr{R}}^{\perp} = \left\{ a_1 L_1 + a_2 L_2 + \sum_{i=3}^{6} E_i \mid a_1 + a_2 + \sum_{i=3}^{6} a_i = 0 \right\}.$$

In order to take into account the Galois action, which acts via $(L_1 L_2)(E_3 E_4 E_5 E_6)$, we put $\mathcal{L} = L_1 + L_2$ and $\mathcal{E} = \sum_{i=3}^{6} E_i$. We easily verify that

$$\mathrm{Cl}(X) = \mathrm{CaCl}(X) = \mathbf{Z}\mathcal{L} \oplus \mathbf{Z}\mathcal{E} = \mathbf{Z}(\mathcal{L} - \mathcal{E}) \oplus \mathbf{Z}\mathcal{E}, \qquad \mathscr{R} = \mathbf{Z}(\mathcal{L} - \mathcal{E}), \qquad \mathscr{R}^{\perp} = \mathbf{Z}(2\mathcal{L} - \mathcal{E}) = \mathbf{Z}K_X.$$

This leads to the following isomorphism:

$$\begin{array}{rcc} \mathrm{Cl}(X_s) \simeq \mathrm{Cl}(X)/\mathscr{R} & \xrightarrow{\simeq} & \mathbf{Z}\mathcal{E} \\ a\mathcal{L} + b\mathcal{E} \bmod \mathscr{R} & \longmapsto & (a+b)\mathcal{E} \end{array}$$

Via this isomorphism, the sub-module $\mathrm{CaCl}(X_s) = \mathscr{R}^{\perp} = \mathbf{Z}(2\mathcal{L} - \mathcal{E}) = \mathbf{Z}K_X$ is mapped to $\mathbf{Z}\mathcal{E}$ itself. In conclusion both $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ are free $\mathbf{Z}$-module of rank 1 and the canonical embedding turns to be an isomorphism.

Types of decomposition into irreducible components in $|-K_{X_s}|$. — Recall that

$$-K_X = 2L_1 + 2L_2 - \sum_{i=3}^{6} E_i = 4E_0 - 2E_1 - 2E_2 - \sum_{i=3}^{6} E_i.$$

Global sections of $-K_X$ are thus related to quartics of $\mathbf{P}^2$. More precisely, one has the following one-to-one correspondences:

$$\begin{array}{ccccc} \left|4\ell - 2p_1 - 2p_2 - \sum_{i=3}^{6} p_i\right|_Y & \longrightarrow & \left|4E_0 - 2E_1 - 2E_2 - \sum_{i=3}^{6} E_i\right|_X & \longrightarrow & |-K_{X_s}|_{X_s} \\ C & \longmapsto & \chi_*\left(C^\sharp\right) & \longmapsto & \varphi_*\left(\chi_*\left(C^\sharp\right)\right), \end{array}$$

and we need to list all the quadrics of $\mathbf{P}^2$ having multiplicity at least 2 at p_1 and p_2 and passing through the p_i for $3 \le i \le 6$. Note that in the correspondences above, we skip the surface Y. Recall that, as in (8), we have $\mathbf{P}^2 \xleftarrow{\pi} Y \xrightarrow{\chi} X$ and the morphism χ here is the contraction of the strict transform of the line passing through p_1 and p_2.

The orbits of lines, respectively conics, having degree less than 4 and passing through some of the points p_i's are

$$\ell_1 \cup \ell_2, \qquad \ell_3 \cup \ell_4 \cup \ell_5 \cup \ell_6, \qquad \ell_{12}, \qquad \ell_{35} \cup \ell_{46}, \qquad \ell_{13} \cup \ell_{24} \cup \ell_{15} \cup \ell_{26}, \qquad \ell_{14} \cup \ell_{25} \cup \ell_{16} \cup \ell_{23},$$

respectively:

$$q_1 \cup q_2, \qquad q_{12}, \qquad q_{35} \cup q_{46}, \qquad q_{3456}, \qquad q_{1235} \cup q_{1246}, \qquad q_{123456}.$$

The only orbits of cubics or quartics having degree less than 4 that pass through the points p_i are c_{123456} and t_{123456}. We now combine the rational irreducible decompositions in order to construct plane curves in the expected sub-linear system.

First, suppose that the decomposition into absolute irreducible components contains a line

- If this line joins one of the first two points to one of the last four points, i.e. a line ℓ_{ij} with $i \in \{1, 2\}$ and $j \in \{3, 4, 5, 6\}$, then this line has degree 4 and it turns out that its orbit under the Galois action lies in the linear system; (cases 1 and 2 in the tabular below).

- If this line is ℓ_{12}, which is rational, then this line appears with multiplicity at most 2. If $2\ell_{12}$ is a part of the decomposition then the complementary conic must be rational and pass through the last four points: the conic must be $\ell_{35} \cup \ell_{46}$ or q_{3456} or q_{123456} (cases 3, 4, 5). If ℓ_{12} has multiplicity 1, then the complementary cubic passes through the six points. Since, except ℓ_{12}, all the lines passing through some p_i have even degree, this cubic cannot be a union of three lines. The remaining cases are thus $q_{123456} \cup \ell$ or c_{123456} (cases 6 and 7).

- If the line is ℓ_i for $i \ge 3$, then it has degree (at least) 4 and its orbit under Galois has degree 4 (or greater) without passing through p_1 and p_2. It does not work.

- If the line is ℓ_1 then its conjugate ℓ_1^σ passes through p_2; the complement is a conic passing through the six points, and it must be q_{123456} (case 8).

- Last if this line is ℓ a line passing through none of the six points then the complementary cubic passes through the six points with multiplicity 2 at p_1 and p_2. Since an irreducible plane cubic has at most one singular point, this cubic must be reducible and it is the union of a line and a conic, whose meeting points are the singular points, that is p_1 and p_2. Thus the line must be ℓ_{12}, and the conic is q_{123456} and we recover case 6.

Secondly, suppose that there are only two absolutely irreducible conics in the decomposition. For the union of these two conics to be singular at p_1 and p_2, they must pass through p_1 and p_2. Taking into account the rationality, there are only three possibilities, cases 9, 10, 11.

Last if the quartic is absolutely irreducible, then it must pass through the six points with multiplicity 2 at p_1 and p_2.

In the tabular below, we summarize all the possibilities. As noted below, the strict transform $\widetilde{\ell}_{12}$ in Y is contracted in X via the morphism $Y \xrightarrow{\chi} X$; this explains why the curve disappears in the middle column. Then from X to X_s, it is the irreducible effective root $\widetilde{q}_{123456}$ that is contracted by the morphism $X \xrightarrow{\varphi} X_s$. Thus on X_s, there are two specific rational points, p the image of the contraction of $\widetilde{\ell}_{12}$ and s the image of the contraction of $\widetilde{q}_{123456}$.

	$\left\|4\ell - 2p_1 - 2p_2 - \sum_{i=3}^{6} p_i\right\|$ on $\mathbf{P}^2$	$\|-K_X\|$ on X	$\|-K_{X_s}\|$ on X_s	Max nb. of pts
1	$\ell_{13} \cup \ell_{24} \cup \ell_{15} \cup \ell_{26}$	$\widetilde{\ell}_{13} \cup \widetilde{\ell}_{24} \cup \widetilde{\ell}_{15} \cup \widetilde{\ell}_{26}$	$\varphi_*(\widetilde{\ell}_{13}) \cup \varphi_*(\widetilde{\ell}_{24}) \cup \varphi_*(\widetilde{\ell}_{15}) \cup \varphi_*(\widetilde{\ell}_{26})$	0
2	$\ell_{14} \cup \ell_{25} \cup \ell_{16} \cup \ell_{23}$	$\widetilde{\ell}_{14} \cup \widetilde{\ell}_{25} \cup \widetilde{\ell}_{16} \cup \widetilde{\ell}_{23}$	$\varphi_*(\widetilde{\ell}_{14}) \cup \varphi_*(\widetilde{\ell}_{25}) \cup \varphi_*(\widetilde{\ell}_{16}) \cup \varphi_*(\widetilde{\ell}_{23})$	0
3	$2\ell_{12} \cup \ell_{35} \cup \ell_{46}$	$\widetilde{\ell}_{35} \cup \widetilde{\ell}_{46}$	$\varphi_*(\widetilde{\ell}_{35}) \cup \varphi_*(\widetilde{\ell}_{46})$	2
4	$2\ell_{12} \cup q_{3456}$	$\widetilde{q}_{3456}$	$\varphi_*(\widetilde{q}_{3456})$	$q+2$
5	$2\ell_{12} \cup q_{123456}$	$\widetilde{q}_{123456} \cup E_1 \cup E_2$	$\{p, s\} \cup \varphi_*(E_1) \cup \varphi_*(E_2)$	2
6	$\ell_{12} \cup q_{123456} \cup \ell$	$\widetilde{q}_{123456} \cup \widetilde{\ell}$	$\varphi_*(\widetilde{\ell})$	$q+2$
7	$\ell_{12} \cup c_{123456}$	$\widetilde{c}_{123456}$	$\varphi_*(\widetilde{c}_{123456})$	$\mathrm{N}_q\,(1)$
8	$\ell_1 \cup \ell_1^\sigma \cup q_{123456}$	$\widetilde{\ell}_1 \cup \widetilde{\ell}_1^\sigma \cup \widetilde{q}_{123456}$	$\varphi_*(\widetilde{\ell}_1) \cup \varphi_*(\widetilde{\ell}_1^\sigma)$	2
9	$q_{12} \cup q_{123456}$	$\widetilde{q}_{12} \cup \widetilde{q}_{123456}$	$\varphi_*(\widetilde{q}_{12})$	$q+2$
10	$q_{1235} \cup q_{1246}$	$\widetilde{q}_{1235} \cup \widetilde{q}_{1246}$	$\varphi_*(\widetilde{q}_{1235}) \cup \varphi_*(\widetilde{q}_{1246})$	2
11	$2q_{123456}$	$\widetilde{q}_{123456} \cup \bigcup_{i=3}^{6} E_i$	$\{s\} \cup \bigcup_{i=3}^{6} \varphi_*(E_i)$	1
12	t_{123456} singular at p_1, p_2	$\widetilde{t}_{123456}$	$\varphi_*(\widetilde{t}_{123456})$	$\mathrm{N}_q\,(1)$

Some comments about the numbers of points are in order.

Cases 1 & 2. The four lines $\ell_{13}, \ell_{24}, \ell_{15}, \ell_{26}$ are conjugate, they do not meet and thus their union in $\mathbf{P}^2$ does not contain any rational point. In the blowing-up of $\mathbf{P}^2$ at the six points, the strict transforms of the lines $\ell_{13}, \ell_{24}, \ell_{15}, \ell_{26}$ no longer meet the strict transform of ℓ_{12}. Thus the contraction of this line does not add any rational point. Since none of the lines $\ell_{13}, \ell_{24}, \ell_{15}, \ell_{26}$ can be a tangent line to q_{123456} at some p_i (otherwise the line and the quadric would have too many intersection points by Bezout), blowing-up the p_i, $1 \leq i \leq 6$, separates the strict transforms of the lines and the strict transform of the quadric. The contraction of this quadric neither add rational points. This proves that these configurations do not contain any rational point.

Case 3. The lines ℓ_{35} and ℓ_{46} are conjugate to each other. Their intersection point is the unique rational point of their union in $\mathbf{P}^2$. This point is still on $\varphi_*(\widetilde{\ell}_{35}) \cup \varphi_*(\widetilde{\ell}_{46})$. The added point is p which comes from the contraction of $\widetilde{\ell}_{12}$. Note that the contraction of $\widetilde{q}_{123456}$ does not add any point since the lines ℓ_{35} and ℓ_{46} are separated from $\widetilde{q}_{123456}$ in the blow-ups. This is because none of these lines can be a tangent to q_{123456} at one of the six points (otherwise the line and the quadric would have too many intersection points by Bezout).

Case 4. The conics q_{123456} and q_{3456} are separated by the blowing up of the last four points and thus the point s does not belong to the final section. The strict transforms of ℓ_{12} and of q_{3456} meet at two points that are mapped to p by the contraction of $\widetilde{\ell}_{12}$. If these two points are not rational, p is an added rational point of the final section.

Case 5 & 11. The resulting sections contain some exceptional curves E_i in their supports since the multiplicities at some points p_i are strictly greater than the ones expected. Since the points p_i are not rational, none of the E_i contain rational points and the only rational points are $\{p, s\}$ or $\{s\}$.

Case 6. The point p lies on $\varphi_*(\widetilde{\ell})$ but it comes from the intersection point between ℓ and ℓ_{12} (which cannot be one of the p_i since it is a rational point) so no points are added in the contraction of ℓ_{12}. As for the point s, it lies also on $\varphi_*(\widetilde{\ell})$ and it could add one more point if ℓ meet q_{123456} at two conjugate points.

Case 7. The cubic must be smooth at p_1 and p_2; indeed if it would be singular at one of these points, by Galois conjugation, it would be singular at both of them and it would have too many singular points. Thus, to make the multiplicity greater than 2 at p_1, p_2, the complementary line must be ℓ_{12}. This line meets the cubic at p_1 and p_2 and a third point which must be rational and not on q_{123456}. Therefore, the contraction of $\widetilde{\ell}_{12}$ pass through p but does not add any rational points to $\varphi_*(\widetilde{\ell}_{12})$. As for the contraction of the strict transform of q_{123456}, it does not add points either since blowing up the six points separates the cubic and q_{123456}.

Case 8. The meeting point of the curves ℓ_1 and ℓ_1^σ is necessarily rational and it is the unique rational point of their union in $\mathbf{P}^2$. The strict transforms $\widetilde{\ell}_1$ and $\widetilde{\ell}_1^\sigma$ do not meet $\widetilde{\ell}_{12}$ and thus the point p does belong to the final section. The contraction of the root $\widetilde{q}_{123456}$ add the point s except if the meeting of the curves ℓ_1 and ℓ_1^σ already belongs to q_{123456}.

Case 9. The conics q_{12} and q_{123456} are separated by the blowing ups. If the conic q_{12} is chosen in such a way that the tangent line at p_1 equals ℓ_{12}, then $\widetilde{q}_{12}$ and $\widetilde{\ell}_{12}$ meet at two unrational points in Y and the contraction of $\widetilde{\ell}_{12}$ adds the rational point p to $\widetilde{q}_{12}$ in X. This explains why the final number of points is $(q+1)+1$.

Case 10. The two conics are conjugate. Besides p_1 and p_2, they meet at two other points (Bezout) that can be rational. If so these points are the only points of $\mathbf{P}^2(\mathbf{F}_q)$ that belong to the union of the two conics. Blowing up the six points disconnect the two conics from the strict transforms of ℓ_{12} and q_{123456}. So no points are added.

Case 12. Since the quartic has at least two singular points, it geometric genus must be at most 1.

As predicted by the class group computations, all the curves on X_s in the linear system are irreducible; they are not necessarily absolutely irreducible but it turns out that curves that are not absolutely irreducible never contain too many rational points.

In any case, one has

$$N_q\left(-K_{X_s}\right) \leq N_q(1).$$

Since none of the points p_i is rational, the surface Y has q^2+q+1 rational points. Since X is obtained by contracting $\widetilde{\ell}_{12}$, it contains q^2+1 rational points. In the same way, after contracting $\widetilde{q}_{123456}$, the surface X_s has q^2-q+1 rational points. Since $q^2-q+1 \leq N_q(1)$ for $q \in \{2,3\}$, the evaluation map may be non injective and we do not consider the codes with these two values.

Proposition 4.4. *Suppose that $q \neq 2,3$. Let $p_1, \dots, p_6 \in \mathbf{P}^2_{\mathbf{F}_q}$ be six conconic points, such that p_1, p_2 and p_3, p_4, p_5, p_6 are conjugate. The anticanonical code of the weak del Pezzo surface obtained by blowing up these six points and then blowing down the strict transform of the line (p_1p_2) has parameters $[q^2-q+1, 5, \geq q^2-q+1-\mathrm{N}_q(1)]$.*

Computation of the global sections from $\mathbf{P}^2$. — Let Q be a quadratic polynomial that defines q_{123456} and L_{ij} a linear form that defines the line ℓ_{ij}. Then

$$\left| 4\ell - 2p_1 - 2p_2 - \sum_{i=3}^{6} p_i \right| = \langle QL_{12}X, QL_{12}Y, QL_{12}Z, L_{12}^2L_{35}L_{46}, L_{13}L_{24}L_{15}L_{26}\rangle_{\mathbf{F}_q}$$

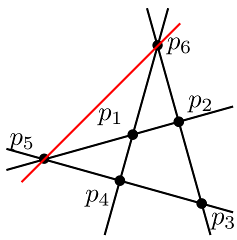
The three first sections are clearly linearly independent. The fourth one cannot be a linear combination of the three first ones since otherwise Q would be reducible. Last, the fifth one cannot be a linear combination of the four first ones since otherwise L_{12} would divide $L_{13}L_{24}L_{15}L_{26}$.

4.5 Degree 4, singularity of type $4\mathbf{A}_1$

This example corresponds to the type number 48 in degree 4 [BH22]. We recover an example already studied by Koshelev [Kos20, §1.2]. Our point of view slightly differs from Koshelev's one, so even if this example appears in the literature, we choose to go into details.

Configuration to blow-up and down. — The context is still the one described in (8) with a non trivial map $Y \xrightarrow{\chi} X$.

Let $p_1, p_2 = p_1^{\sigma}, p_3 = p_1^{\sigma^2}, p_4 = p_1^{\sigma^3} \in \mathbf{P}^2$ be four conjugate points in general position (no three of them are collinear) and, as usual, let $\ell_{12}, \ell_{23}, \ell_{34}, \ell_{14}$ denote the lines $(p_1p_2), (p_2p_3), (p_3p_4), (p_1p_4)$; they are conjugate. Let p_5 be the intersection point of ℓ_{12} and ℓ_{34} and p_6 be the intersection point of ℓ_{23} and ℓ_{14}. They are also conjugate and we denote by ℓ_{56} the rational line passing through p_5, p_6.



$$p_2 = p_1^{\sigma} \quad p_3 = p_1^{\sigma^2} \quad p_4 = p_1^{\sigma^3} \quad p_6 = p_5^{\sigma}$$

We blow up these six points to obtain a degree 3 weak del Pezzo surface Y. The strict transform of the line ℓ_{56}, of class $E_0 - E_5 - E_6$, is an exceptional curve that can be contracted to obtain the degree four weak del Pezzo surface X we consider here. The anticanonical model of this surface has four singular points of type $\mathbf{A}_1$ (since the four irreducible effective roots do not intersect, see below).

Computation of the divisor class groups. — Over $\overline{\mathbf{F}}_q$, we know that $\mathrm{Cl}(\overline{Y}) = \bigoplus_{i=0}^{6} \mathbf{Z}E_i$ and that $-K_Y = 3E_0 - \sum_{i=1}^{6} E_i$. Moreover the surface Y has four irreducible effective roots, the strict transforms of the lines $\ell_{125}, \ell_{236}, \ell_{345}, \ell_{146}$ whose conjugate classes in $\mathrm{Cl}(\overline{Y})$ are:

$$E_0 - E_1 - E_2 - E_5, \qquad E_0 - E_2 - E_3 - E_6, \qquad E_0 - E_3 - E_4 - E_5, \qquad E_0 - E_1 - E_4 - E_6.$$

The group $\mathrm{Cl}(\overline{X})$ identifies with $\mathbf{Z}(E_0-E_5-E_6)^{\perp}$ inside $\mathrm{Cl}(\overline{Y})$. Since

$$\begin{array}{rclcl}\mathrm{Cl}(\overline{Y}) & = & \mathbf{Z}(E_0-E_5-E_6) & \overset{\perp}{\oplus} & \left[\mathbf{Z}(E_0-E_5)\oplus\mathbf{Z}(E_0-E_6)\oplus\bigoplus_{i=1}^{4}\mathbf{Z}E_i\right] \\ \sum_{i=0}^{6}a_iE_i & = & (-a_0-a_5-a_6)(E_0-E_5-E_6) & + & (a_0+a_6)(E_0-E_5)+(a_0+a_5)(E_0-E_6)+\sum_{i=1}^{4}a_iE_i\end{array} \tag{9}$$

one has

$$\mathrm{Cl}(\overline{X})=\mathbf{Z}(E_0-E_5)\oplus\mathbf{Z}(E_0-E_6)\oplus\mathbf{Z}E_1\oplus\mathbf{Z}E_2\oplus\mathbf{Z}E_3\oplus\mathbf{Z}E_4.$$

In particular,

$$-K_X=2(E_0-E_5)+2(E_0-E_6)-E_1-E_2-E_3-E_4=4E_0-2E_5-2E_6-E_1-E_2-E_3-E_4.$$

On $\overline{X}$, there are still four effective roots, the image by the contraction of the effective roots on $\overline{Y}$:

$$\overline{\mathscr{R}}=\mathbf{Z}(E_0-E_1-E_2-E_5)\oplus\mathbf{Z}(E_0-E_2-E_3-E_6)\oplus\mathbf{Z}(E_0-E_3-E_4-E_5)\oplus\mathbf{Z}(E_0-E_1-E_4-E_6).$$

On $\overline{X}$ there are 16 exceptional classes,

$$(E_0-E_i)-E_j,\ i\in\{5,6\},\ j\in\{1,2,3,4\},\quad (E_0-E_5)+(E_0-E_6)-E_{i_1}-E_{i_2}-E_{i_3},\ \{i_1,i_2,i_3\}\subset\{1,2,3,4\},$$

and E_1,E_2,E_3,E_4. Only these last four classes are represented by irreducible exceptional curves. We recover the graph number 9 of the Proposition 6.1 of Coray & Tsfasman [CT88].

The Galois group acts as as a 4-cycle on the roots and as $(E_1E_2E_3E_4)$ on the (-1)-curves. Let us put $\mathcal{F}=(E_0-E_5)+(E_0-E_6)$ and $\mathcal{E}=E_1+E_2+E_3+E_4$, in such a way that $\mathcal{F}^{\cdot 2}=2$, $\mathcal{E}^{\cdot 2}=-4$ and $\mathcal{F}\cdot\mathcal{E}=0$. Then one has:

$$\begin{cases}\mathscr{R}=\mathbf{Z}2(\mathcal{F}-\mathcal{E})\\ \mathrm{Cl}(X)=\mathbf{Z}(\mathcal{F}-\mathcal{E})\oplus\mathbf{Z}\mathcal{E}\end{cases}\quad\Longrightarrow\quad\begin{array}{ccccc}\mathrm{Cl}(X_s)=\mathrm{Cl}(X)/\mathscr{R} & \xrightarrow{\simeq} & \mathbf{Z}/2\mathbf{Z}(\mathcal{F}-\mathcal{E}) & \oplus & \mathbf{Z}\mathcal{E}\\ a\mathcal{F}+b\mathcal{E} & \longmapsto & a(\mathcal{F}-\mathcal{E}) \bmod \mathscr{R} & + & (a+b)\mathcal{E}\end{array}$$

As for the Cartier class group, we find $\mathrm{CaCl}(X_s)=\mathscr{R}^{\perp}=\mathbf{Z}(2\mathcal{F}-\mathcal{E})=\mathbf{Z}(-K_X)$ which embeds in $\mathrm{Cl}(X_s)$ via $-K_X\mapsto\mathcal{E}$. Thus, via the canonical embedding, $\mathrm{CaCl}(X_s)$ and the free part of $\mathrm{Cl}(X_s)$ are isomorphic.

Types of decomposition into irreducible components in $|-K_{X_s}|$. — The situation looks like the preceding one except that the multiplicity is at points p_5,p_6 here. One has:

$$\begin{array}{ccccc}\left|4\ell-\sum_{i=1}^{4}p_i-2p_5-2p_6\right|_Y & \longrightarrow & \left|4E_0-2E_5-2E_6-\sum_{i=1}^{4}E_i\right|_X & \longrightarrow & |-K_{X_s}|_{X_s}\\ C & \longmapsto & \chi_*\left(C^{\sharp}\right) & \longmapsto & \varphi_*\left(\chi_*\left(C^{\sharp}\right)\right).\end{array}$$

Thus we are reduced to list all the types of irreducible decompositions of quadrics passing through the six points, the last two with multiplicities at least 2.

The orbits of lines of degree less than 4 that involve the six points are

$$\ell_5\cup\ell_6,\qquad \ell_1\cup\ell_2\cup\ell_3\cup\ell_4,\qquad \ell_{56},\qquad \ell_{13}\cup\ell_{24},\qquad\text{and}\qquad \ell_{125}\cup\ell_{236}\cup\ell_{345}\cup\ell_{146}.$$

There are only two ways (cases 1 and 2 below) to combine these configurations in order to obtain a curve in the expected linear system.

The orbits of conics of degree less than 4 that involve the six points are q_{1234},q_{56} and $q_{1356}\cup q_{2456}$. There are only two ways (cases 3 and 4 below) to combine the configurations of lines and conics in order to obtain a curve in the expected linear system.

If the decomposition contains a cubic, it must be smooth at p_5 and p_6 and the complement must be ℓ_{56}; this is case 5. This leads to the list below. Let us note that on X the curve $\widetilde{\ell}_{56}$ in Y is contracted by χ. On X_s, this contraction is mapped to a smooth rational point p. This surface contains also four singular points s_i, $1\leq i\leq 4$, coming from the contraction of the four roots; they are conjugate and of degree 4.

	$\left\lvert 4\ell-\sum_{i=1}^{4}p_i-2p_5-2p_6\right\rvert$ on $\mathbf{P}^2$	$\lvert -K_X\rvert$ on X	$\lvert -K_{X_s}\rvert$ on X_s	Max nb. of pts
1	$2\ell_{56}\cup\ell_{13}\cup\ell_{24}$	$\widetilde{\ell}_{13}\cup\widetilde{\ell}_{24}$	$\varphi_*(\widetilde{\ell}_{13})\cup\varphi_*(\widetilde{\ell}_{24})$	2
2	$\ell_{125}\cup\ell_{236}\cup\ell_{345}\cup\ell_{146}$	$\widetilde{\ell}_{125}\cup\widetilde{\ell}_{236}\cup\widetilde{\ell}_{345}\cup\widetilde{\ell}_{146}\cup\bigcup_{i=1}^{4}E_i$	$\{s_i\}\cup\bigcup_{i=1}^{4}\varphi_*(E_i)$	0
3	$2\ell_{56}\cup q_{1234}$	$\widetilde{q}_{1234}$	$\varphi_*(\widetilde{q}_{1234})$	$q+2$
4	$q_{1356}\cup q_{2456}$	$\widetilde{q}_{1356}\cup\widetilde{q}_{2456}$	$\varphi_*(\widetilde{q}_{1356})\cup\varphi_*(\widetilde{q}_{2456})$	2
5	$c_{123456}\cup\ell_{56}$	$\widetilde{c}_{123456}$	$\varphi_*(\widetilde{c}_{123456})$	$\mathrm{N}_q(1)$
6	t_{123456} singular at p_5,p_6	$\widetilde{t}_{123456}$	$\varphi_*(\widetilde{t}_{123456})$	$\mathrm{N}_q(1)$

Some comments on the number of rational points.

Case 1. The lines ℓ_{13} and ℓ_{24} are conjugate, their meeting point is the unique rational point of their union. After the contraction of $\widetilde{\ell}_{56}$, these two lines have one more rational point in common, the point p.

Case 3. If q_{1234} meets ℓ_{56} at two conjugate points, the contraction of $\widetilde{\ell}_{56}$ adds the point p to the other rational points.

Case 4. Blowing up p_5 and p_6 separates the strict transforms $\widetilde{q}_{1356}$ and $\widetilde{q}_{2456}$ from ℓ_{56}. So the contraction of this curve do not add any point. On $\mathbf{P}^2$ the union of conics $\widetilde{q}_{1356}$ and $\widetilde{q}_{2456}$ has at most two rational points: their meeting points that differ from p_5, p_6 if they are rational.

Case 5. Necessarily the cubic is smooth at p_5, p_6 and the tangent lines at these points cannot be equal to ℓ_{56}. Therefore blowing up p_5, p_6 separates the curves ℓ_{56} and $\widetilde{c}_{123456}$ above p_5 and p_6. Besides p_5, p_6 the curves ℓ_{56} and c_{123456} meet at a third point (Bezout) which is necessarily rational. Via the contraction of $\widetilde{\ell}_{56}$, this line concentrates at this third point and no points are added on $\widetilde{c}_{123456}$.

Case 6. Necessarily blowing up p_5 and p_6 disconnects the strict transforms $\widetilde{t}_{123456}$ and $\widetilde{\ell}_{56}$. Moreover, it desingularizes the quartic t since the singularities at p_5 and p_6 must be ordinary. Blowing up $p_1, \ldots, p_4$ also disconnects $\widetilde{t}$ from all the effective roots. Finally $\varphi_*(t_{123456})$ turns to be an elliptic curve.

Finally $N_q(-K_{X_s}) \leq N_q(1)$.

Since none of the points p_i is rational, $\#X(\mathbf{F}_q) = \#\mathbf{P}^2(\mathbf{F}_q) = q^2+q+1$. Then contracting the rational line $\widetilde{\ell}_{56}$ decreases the number by q and $\#X_s(\mathbf{F}_q) = q^2+1$. Except for $q = 2$, this number is stricly greater that $N_q(1)$ and the evaluation map is injective.

Proposition 4.5. *Suppose $q \neq 2$. Let $p_1, p_2 = p_1^\sigma, p_3 = p_1^{\sigma^2}, p_4 = p_1^{\sigma^3} \in \mathbf{P}^2_{\mathbf{F}_q}$ be four conjugate points in general position (no three of them are collinear) and let p_5 (resp. p_6) be the point of intersection of the lines (p_1p_2) and (p_3p_4) (resp. (p_2p_3) and (p_1p_4)). The anticanonical code of the weak del Pezzo surface obtained by blowing up these six points and then blowing down the strict transform of the line (p_5p_6) has parameters $[q^2+1, 5, \geq q^2+1-\mathrm{N}_q(1)]$.*

As proved by Koshelev [Kos20, §1.2], for some values of q, the minimum distance can be improved by one. Since the argument is very nice, we choose to briefly sketch it below. The idea is to prove that all the elliptic curves in our linear system must have a rational 2-torsion point and thus an even number of points. Since for some q, the maximum $N_q(1)$ is odd, this means that $N_q(-K_{X_s}) < N_q(1)$ and our bound for the minimum distance can be improved by 1. Let c be a cubic passing through the six points. Then, for any choice of the origin, the alignments of points permit to show that:

$$\begin{array}{rl} & p_1+p_2+p_5 \\ = & p_1+p_4+p_6 \\ = & p_2+p_3+p_6 \\ = & p_3+p_4+p_5 \end{array} \quad \Longrightarrow \quad \begin{array}{rl} & p_1-p_3 \\ = & p_2-p_4 \\ = & p_4-p_2 \\ = & p_6-p_5 \end{array} \quad \Longrightarrow \quad 2(p_2-p_4)=0,$$

and these points must be rational since $p_2 - p_4 = p_6 - p_5$ with p_5, p_6 conjugate. The case of the quartics in the linear system works in the same way but it is a little bit more technical since we need to know the group law on this kind of curve.

Computation of the global sections from $\mathbf{P}^2$. — In this example, we do not find a nice explicit basis for the global sections. Instead, we choose to present a `magma` code that permits to construct the generator matrix.

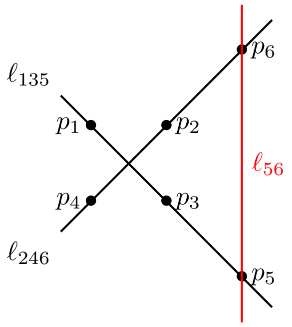
```
clear ;

q := 7 ;
Fq<xi> := FiniteField(q) ;
P2<X,Y,Z> := ProjectiveSpace(Fq, 2) ; XYZ<X,Y,Z> := CoordinateRing(P2) ;
phi := map < P2 -> P2 | [X^q,Y^q,Z^q] > ;

// Some basic routines such as verifying if some points are in general position
load "Utilities.magma" ;

// Choose randomly a degree 4 point in general position.
repeat
  p1 := Random(P2(ext< Fq | 4>)) ;
  p2 := phi(p1) ; p3 := phi(p2) ; p4 := phi(p3) ;
  Cl1234 := Cluster(p1) ;
until EnPositionGenerale([p1,p2,p3,p4]) ;

//// To compute the points p5, p6 we need the extend the scalars to Fq4
P2_Fq4 := BaseChange(P2, ext < Fq | 4 >) ;

p1_Fq4 := P2_Fq4!ElementToSequence(p1) ; p2_Fq4 := P2_Fq4!ElementToSequence(p2) ;
p3_Fq4 := P2_Fq4!ElementToSequence(p3) ; p4_Fq4 := P2_Fq4!ElementToSequence(p4) ;

L12 := Scheme(P2_Fq4, Sections(LinearSystem(LinearSystem(P2_Fq4, 1), [p1_Fq4,p2_Fq4]))[1]) ;
L34 := Scheme(P2_Fq4, Sections(LinearSystem(LinearSystem(P2_Fq4, 1), [p3_Fq4,p4_Fq4]))[1]) ;

p5_Fq4 := Points(L12 meet L34)[1] ;
p5 := P2(ext < Fq | 2 >)!ElementToSequence(p5_Fq4) ;
//// End of the computation over Fq4

p6 := phi(p5) ;
Cl56 := Cluster(p5) ;
Cl56_square := Cluster(P2, Ideal(Cl56)^2) ;
```

```
L := LinearSystem(LinearSystem(P2, 4), Cl1234);
L := LinearSystem(L, Cl56_square);
TheSections := Sections(L);

L56 := Scheme(P2, Sections(LinearSystem(LinearSystem(P2, 1), Cl56))[1]);
S := (Points(P2) diff Points(L56)) join {@ Points(L56)[1]@};

G := Matrix(Fq,5,1+q^2, &cat [[Evaluate(f,ElementToSequence(p)) : p in S] : f in TheSections]);
TheCode := LinearCode(G);

printf "Generator matrix G =\n%o\n", G;
lg := Length(TheCode); dim := Dimension(TheCode); d_min := MinimumWeight(TheCode);
printf "q = %o,\n[n,k,d] = [%o, %o, %o]\n", q, lg, dim, d_min;
printf "What was provided (not taking into account Kashelev remark) = [%o, 5, %o]",
                                              q^2+1, q^2 - q - Floor(2*Sqrt(q));
```

4.6 Degree 4, singularity of type $\mathbf{A}_2$

This example corresponds to the type number 30 in degree 4 [BH22]. This type works almost as the one described in section 4.5.

Configuration to blow-up and down. — Let $p_1, p_2 = p_1^\sigma, p_3 = p_1^{\sigma^2}, p_4 = p_1^{\sigma^3} \in \mathbf{P}^2$ be four conjugate points in general position (no three of them are collinear). The two lines (p_1p_3) and (p_2p_4) are conjugate. We choose a degree 2 point p_5 on (p_1p_3) and we let $p_6 = p_5^\sigma$ which lies on (p_2p_4).



$$p_2 = p_1^\sigma \quad p_3 = p_1^{\sigma^2} \quad p_4 = p_1^{\sigma^3} \quad p_6 = p_5^\sigma$$

The surfaces of the diagram (8) are the following: we blow up the six points to obtain the degree 3 weak del Pezzo surface Y. On this surface, the strict transform of the line ℓ_{56}, of class $E_0 - E_5 - E_6$, is an exceptional curve that can be contracted to obtain the weak degree four weak del Pezzo surface X defined over $\mathbf{F}_q$. The anticanonical model has a unique singular point of type $\mathbf{A}_2$ (since there are only two irreducible effective root that meet, see below).

Computation of the divisor class groups. — Over $\overline{\mathbf{F}}_q$, we know that $\mathrm{Cl}(\overline{Y}) = \bigoplus_{i=0}^{6} \mathbf{Z}E_i$ and that $-K_Y = 3E_0 - \sum_{i=1}^{6} E_i$. There are only two irreducible effective roots on $\overline{Y}$, the strict transforms of the lines ℓ_{135} and ℓ_{246} whose conjugate classes in $\mathrm{Cl}(\overline{Y})$ are $E_0 - E_1 - E_3 - E_5$ and $E_0 - E_2 - E_4 - E_6$.

The group $\mathrm{Cl}(\overline{X})$ identifies with $\mathbf{Z}(E_0 - E_5 - E_6)^\perp$ inside $\mathrm{Cl}(\overline{Y})$. We recover the same orthogonal decomposition as in (9). In particular, we still have $-K_X = 4E_0 - \sum_{i=1}^{4} E_i - 2E_5 - 2E_6$. On $\overline{X}$ there are still two (conjugate) irreducible effective roots, of classes $E_0 - E_1 - E_3 - E_5$ and $E_0 - E_2 - E_4 - E_6$.

We follow the same computation as in section 4.5 and we still put $\mathcal{E} = (E_0 - E_5) + (E_0 - E_6)$ and $\mathcal{F} = E_1 + E_2 + E_3 + E_4$. Then, for X we have:

$$-K_X = 2\mathcal{F} - \mathcal{E}, \qquad \mathscr{R} = \mathbf{Z}(\mathcal{F} - \mathcal{E}), \qquad \text{and} \qquad \mathrm{Cl}(X) = \mathbf{Z}(\mathcal{F} - \mathcal{E}) \oplus \mathbf{Z}\mathcal{E},$$

and for X_s we deduce that:

$$\mathrm{CaCl}(X_s) = \mathbf{Z}(2\mathcal{F} - \mathcal{E}) = \mathbf{Z}(-K_X) \qquad \begin{array}{ccc} \mathrm{Cl}(X_s) & \xrightarrow{\simeq} & \mathbf{Z}\mathcal{E} \\ a\mathcal{F} + b\mathcal{E} & \longmapsto & (a+b)\mathcal{E} \end{array} \qquad \begin{array}{ccc} \mathrm{CaCl}(X_s) & \xrightarrow{\simeq} & \mathrm{Cl}(X_s) \\ -K_X & \mapsto & \mathcal{E} \end{array}$$

The canonical embedding induces an isomorphism between the two class groups.

Types of decomposition into irreducible components in $|-K_X|$. — Since the two class groups are isomorphic and free of rank one, all the sections are necessarily irreducible. However, they can be absolutely reducible and we need to review all the possibilities.

As in section 4.5, we are reduced to list all the types of irreducible decompositions of quartics passing through the six points, the last two with multiplicities at least 2. We follow the same line.

The orbits of lines of degree less than 4 that involve the six points are

$$\ell_5 \cup \ell_6, \quad \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell_4, \quad \ell_{56}, \quad \ell_{12} \cup \ell_{23} \cup \ell_{34} \cup \ell_{14}, \quad \ell_{16} \cup \ell_{25} \cup \ell_{36} \cup \ell_{45}, \quad \text{and} \quad \ell_{135} \cup \ell_{246}.$$

There are five ways (cases 1 to 5 below) to combine these configurations in order to obtain a curve in the expected linear system.

The orbits of conics of degree less than 4 that involve the six points are q_{1234} and q_{56}. Note that compared to the example of section 4.5, the orbit $q_{1356} \cup q_{2456}$ does not appear since a conic q_{1356} cannot be irreducible otherwise it would have three intersection points with the line ℓ_{135}. There are only two ways (cases 6 and 7 below) to combine the configurations of lines and conics in order to obtain a curve in the expected linear system.

If the decomposition contains a cubic, it must be smooth at p_5 and p_6 and the complement must be ℓ_{56}; this is case 8.

This leads to the list below. In Y, the strict transforms $\widetilde{\ell}_{135}$ and $\widetilde{\ell}_{246}$ are separated from the strict transform $\widetilde{\ell}_{56}$. In X this last curve is contracted to a smooth rational point p. Last, another difference from the example of section 4.5: the two effective roots of X meet and they are thus contracted by the anticanonical morphism φ_* onto the same point s. This point is the unique singular point of X_s and it is necessarily a rational point.

	$\left\|4\ell - \sum_{i=1}^{4} p_i - 2p_5 - 2p_6\right\|$ on $\mathbf{P}^2$	$\|-K_X\|$ on X	$\|-K_{X_s}\|$ on X_s	Max nb. of pts
1	$2\ell_{56} \cup \ell_{135} \cup \ell_{246}$	$\widetilde{\ell}_{56} \cup \widetilde{\ell}_{135} \cup \widetilde{\ell}_{246} \cup E_5 \cup E_6$	$\varphi_*(E_5) \cup \varphi_*(E_6) \cup \{s, p\}$	2
2	$\ell_{56} \cup \ell_{135} \cup \ell_{246} \cup \ell$	$\widetilde{\ell}_{135} \cup \widetilde{\ell}_{246} \cup \widetilde{\ell}$	$\varphi_*(\widetilde{\ell})$	$q+2$
3	$\ell_{16} \cup \ell_{25} \cup \ell_{36} \cup \ell_{45}$	$\widetilde{\ell}_{16} \cup \widetilde{\ell}_{25} \cup \widetilde{\ell}_{36} \cup \widetilde{\ell}_{45}$	$\varphi_*(\widetilde{\ell}_{16}) \cup \varphi_*(\widetilde{\ell}_{25}) \cup \varphi_*(\widetilde{\ell}_{36}) \cup \varphi_*(\widetilde{\ell}_{45})$	0
4	$\ell_5 \cup \ell_6 \cup \ell_{135} \cup \ell_{246}$	$\widetilde{\ell}_5 \cup \widetilde{\ell}_6 \cup \widetilde{\ell}_{135} \cup \widetilde{\ell}_{246}$	$\varphi_*(\widetilde{\ell}_5) \cup \varphi_*(\widetilde{\ell}_6) \cup \{s\}$	2
5	$2\ell_{135} \cup 2\ell_{246}$	$\widetilde{\ell}_{135} \cup \widetilde{\ell}_{246} \cup E_1 \cup E_2 \cup E_3 \cup E_4$	$\varphi_*(E_1) \cup \varphi_*(E_2) \cup \varphi_*(E_3) \cup \varphi_*(E_4)$	1
6	$\ell_{135} \cup \ell_{246} \cup q_{56}$	$\widetilde{\ell}_{135} \cup \widetilde{\ell}_{246} \cup \widetilde{q}_{56}$	$\varphi_*(\widetilde{q}_{56}) \cup \{s\}$	$q+2$
7	$2\ell_{56} \cup q_{1234}$	$\widetilde{q}_{1234} \cup \{p\}$	$\varphi_*(\widetilde{q}_{1234}) \cup \varphi_*(\{p\})$	$q+2$
8	$c_{123456} \cup \ell_{56}$	$\widetilde{c}_{123456}$	$\varphi_*(\widetilde{c}_{123456})$	$\mathrm{N}_q(1)$
9	t_{123456} singular at p_5, p_6	$\widetilde{t}_{123456}$	$\varphi_*(\widetilde{t}_{123456})$	$\mathrm{N}_q(1)$

Some comments about the numbers of points.

Case 1 & 5. The exceptional curves E_5 and E_6 do not contain any rational points. The other components are all contracted to the points p or s. The same is true in case 5, without the point p.

Case 2. The ending curve passes through p but the contraction of $\widetilde{\ell}_{56}$ does not add any rational point since ℓ_{56} and ℓ meet at a rational point. The ending curve passes through s and the contraction of the roots add a point if ℓ meet ℓ_{135} and ℓ_{246} outside the meeting point of these two curves.

Case 3. All theses lines and ℓ_{135}, ℓ_{246} and ℓ_{56} are separated by the blowing ups. Since the four lines cannot contain any rational point, neither does their image in X_s.

Case 4. The conjugate lines ℓ_5 and ℓ_6 contain a unique rational point, their intersection point, to which is added the point s.

Case 6. The curve $\widetilde{q}_{56}$ no longer meets $\widetilde{\ell}_{135}$, $\widetilde{\ell}_{246}$ and $\widetilde{\ell}_{56}$. The ending curve contains the rational points of q_{56} plus the point s.

Case 7. If q_{1234} meets ℓ_{56} at two conjugate points then in X, after $\widetilde{\ell}_{56}$ being contracted, the strict transform $\widetilde{q}_{1234}$ passes through p which is an additional rational point. Necessarily the blowing ups of $p_1, \ldots, p_4$ separate the strict transforms $\widetilde{q}_{1234}$, $\widetilde{\ell}_{135}$ and $\widetilde{\ell}_{246}$ and the roots contraction does not add any point.

Cases 8 & 9. Same as §4.5.

Finally $N_q(-K_{X_s}) \leq N_q(1)$.

As in the previous example, one has $\#X_s(\mathbf{F}_q) = q^2+1$, and for $q = 2$, the evaluation map may not be injective.

Proposition 4.6. *Suppose $q \neq 2$. Let $p_1, p_2 = p_1^\sigma, p_3 = p_1^{\sigma^2}, p_4 = p_1^{\sigma^3} \in \mathbf{P}^2$ be four conjugate points in general position (no three of them are collinear), let p_5 be a point of the line (p_1p_3) inside $\mathbf{P}^2(\mathbf{F}_{q^2})$ and let $p_6 = p_5^\sigma$ in such a way that p_6 lies on (p_2p_4). The anticanonical code of the weak del Pezzo surface obtained by blowing up these six points and then blowing down the strict transform of the line (p_5p_6) has parameters $[q^2+1, 5, \geq q^2+1-\mathrm{N}_q(1)]$.*

Computation of the global sections from $\mathbf{P}^2$. — This example looks like the previous one and we do not find a nice explicit basis for the global sections. A slightly modification of the code given for the previous example leads to a program which permits to compute a generator matrix.

4.7 Degree 4, singularity of type $\mathbf{D}_5$

This example corresponds to the type number 58 in degree 4 [BH22].

Configuration to blow-up. — In this example, the surfaces Y and X of diagram (8) are equal and we obtain directly the surface X by blowing up $\mathbf{P}^2$ at five rational points $p_1 \prec p_2 \prec \cdots \prec p_5$, with p_1, p_2, p_3 collinear. Let us denote by $\pi_1, \ldots, \pi_5$ these five blowups at $p_1, \ldots, p_5$ respectively:

$$\mathbf{P}^2 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} X_3 \xleftarrow{\pi_4} X_4 \xleftarrow{\pi_5} X, \qquad \pi : X \to \mathbf{P}^2$$

The fact that p_1, p_2, p_3 are collinear means that there is a line ℓ_{123} of $\mathbf{P}^2$ whose strict transform by π_1 passes through p_2 and whose strict transform by $\pi_2 \circ \pi_1$ passes through p_3. The anticanonical model of this weak del Pezzo surface has a unique singular point of type $\mathbf{D}_5$ (since there are five irreducible effective roots whose intersection graph is $\mathbf{D}_5$, see the picture at the end of this example).

Computation of the divisor class groups. — Since all the blown-up points are rational, there is no need to work with the base change $\overline{X}$. The irreducible effective classes of roots are the strict transform of ℓ_{123} and of E_1, E_2, E_3, E_4, whose classes in $\mathrm{Cl}(X)$ are:

$$E_0 - E_1 - E_2 - E_3, \qquad E_1 - E_2, \qquad E_2 - E_3, \qquad E_3 - E_4, \qquad \text{and} \qquad E_4 - E_5.$$

The submodule $\mathscr{R}$, generated by these classes, is a direct summand and for example $\mathrm{Cl}(X) = \mathscr{R} \oplus \mathbf{Z}E_5$; the projection onto the factor $\mathbf{Z}E_5$ leads to an isomorphism $\mathrm{Cl}(X)/\mathscr{R} \to \mathbf{Z}E_5$. As for the submodule $\mathscr{R}^\perp$, it is defined by the equations $a_1 = a_2 = \cdots = a_5$ and $a_0 + a_1 + a_2 + a_3 = 0$ and thus $\mathscr{R}^\perp = \mathbf{Z}K_X$. Since

$$-K_X = 3\,(E_0 - E_1 - E_2 - E_3) + 2\,(E_1 - E_2) + 4\,(E_2 - E_3) + 6\,(E_3 - E_4) + 5\,(E_4 - E_5) + 4E_5,$$

via the preceding isomorphism, the module $\mathscr{R}^\perp$ embeds via $-K_X \mapsto 4E_5$.

In brief, both divisor class groups $\mathrm{CaCl}(X_s)$ and $\mathrm{Cl}(X_s)$ are free rank one $\mathbf{Z}$-modules, the first one being of index 4 in the latter via the canonical embedding.

For this example, it makes sense to reverse the order of the paragraphs and we start to compute a basis of the global sections.

Computation of the global sections from $\mathbf{P}^2$. — We need to compute a basis of the sublinear system on $\mathbf{P}^2$

$$|3\ell - p_1 - \cdots - p_5|\,.$$

So we consider a cubic of $\mathbf{P}^2_{X,Y,Z}$ whose restriction to the affine space $\mathbf{A}^2_{x_1,y_1}$ ($x_1 = \frac{X}{Z}$, $y_1 = \frac{Y}{Z}$ and $Z \neq 0$) is defined by the equation:

$$C_1(x_1, y_1) = a_{30}x_1^3 + a_{21}x_1^2y_1 + a_{20}x_1^2 + a_{12}x_1y_1^2 + a_{11}x_1y_1 + a_{10}x_1 + a_{03}y_1^3 + a_{02}y_1^2 + a_{01}y_1 + a_{00} = 0$$

We choose $p_1 = (0,0) \in \mathbf{A}^2_{x_1,y_1}$. The cubic passes through the point p_1 if and only if $a_{00} = 0$.

Let x_2, y_2 be the coordinates of the affine chart of the blowing up of $\mathbf{A}^2_{x_1,y_1}$ at p_1 defined by $x_1 = x_2$ and $y_1 = x_2y_2$. In this chart, the exceptional divisor E_1 has equation $x_2 = 0$ and the strict transform of C_1 is defined by:

$$C_2(x_2, y_2) = \frac{1}{x_2}C_1(x_2, x_2y_2) = a_{30}x_2^2 + a_{21}x_2^2y_2 + a_{20}x_2 + a_{12}x_2^2y_2^2 + a_{11}x_2y_2 + a_{10} + a_{03}x_2^2y_2^3 + a_{02}x_2y_2^2 + a_{01}y_2 = 0.$$

We choose $p_2 = (0,0) \in \mathbf{A}^2_{x_2,y_2}$ which corresponds to the line ℓ_{123} with affine equation $y_1 = 0$ in $\mathbf{A}^2_{x_1,y_1}$ or $Y. = 0$ in $\mathbf{P}^2_{X,Y,Z}$. The cubic passes through the point p_2 if and only if $a_{10} = 0$.

Let x_3, y_3 be the coordinates of the affine chart of the blowing up of $\mathbf{A}^2_{x_2,y_2}$ at p_2 defined by $x_2 = x_3$ and $y_2 = x_3y_3$. In this chart, the exceptional divisor E_2 has equation $x_3 = 0$ and the strict transform of C_2 is defined by:

$$C_3(x_3, y_3) = \frac{1}{x_3}C_2(x_3, x_3y_3) = a_{30}x_3 + a_{21}x_3^2y_3 + a_{20} + a_{12}x_3^3y_3^2 + a_{11}x_3y_3 + a_{03}x_3^4y_3^3 + a_{02}x_3^2y_3^2 + a_{01}y_3 = 0.$$

Since p_1, p_2, p_3 are colinear, we have to choose $p_3 = (0,0) \in \mathbf{A}^2_{x_3,y_3}$. The cubic passes through the point p_3 if and only if $a_{20} = 0$.

Let x_4, y_4 be the coordinates of the affine chart of the blowing up of $\mathbf{A}^2_{x_3,y_3}$ at p_3 defined by $x_3 = x_4$ and $y_3 = x_4y_4$. In this chart, the exceptional divisor E_3 has equation $x_4 = 0$ and the strict transform of C_3 is defined by:

$$C_4(x_4, y_4) = \frac{1}{x_4}C_2(x_4, x_4y_4) = a_{30} + a_{21}x_4^2y_4 + a_{12}x_4^4y_4^2 + a_{11}x_4y_4 + a_{03}x_4^6y_4^3 + a_{02}x_4^3y_4^2 + a_{01}y_4 = 0.$$

Since $p_4 \in E_3$, we have to choose $p_4 = (0, \alpha) \in \mathbf{A}^2_{x_4,y_4}$ and since p_1, p_2, p_3, p_4 are **not** colinear, necessarily $\alpha \neq 0$. The cubic passes through the point p_4 if and only if $a_{30} + \alpha a_{01} = 0$.

Last, let x_5, y_5 be the coordinates of the affine chart of the blowing up of $\mathbf{A}^2_{x_4,y_4}$ at p_4 defined by $x_4 = x_5$ and $y_4 = \alpha + x_5 y_5$. In this chart, the exceptional divisor E_4 has equation $x_5 = 0$ and the strict transform of C_4 is defined by:

$$\begin{aligned} C_5(x_5, y_5) &= \frac{1}{x_5} C_2(x_5, x_5 y_5) \\ &= \frac{1}{x_5}\left[a_{30} + a_{21} x_5^2(\alpha + x_5 y_5) + a_{12} x_5^4 (\alpha + x_5 y_5)^2 + a_{11} x_5 (\alpha + x_5 y_5) + a_{03} x_5^6 (\alpha + x_5 y_5)^3 \right. \\ &\qquad \left. + a_{02} x_5^3 (\alpha + x_5 y_5)^2 + a_{01}(\alpha + x_5 y_5)\right] \\ &\equiv \alpha a_{11} + \alpha a_{21} x_5 + a_{01} y_5 + a_{11} x_5 y_5 \bmod x_5^2 \mathbf{F}_q[x_5, y_5]. \end{aligned} \tag{10}$$

Since $p_5 \in E_4$, one can choose $p_5 = (0, \beta) \in \mathbf{A}^2_{x_5,y_5}$. The cubic passes through the point p_5 if and only if $\alpha a_{11} + \beta a_{01} = 0$.

To sum up, the global sections are defined by

$$a_{00} = a_{10} = a_{20} = 0, \qquad a_{30} = -\alpha a_{01}, \qquad \text{and} \qquad a_{11} = -\frac{\beta}{\alpha} a_{01}.$$

The fact that $\alpha \neq 0$ is important here. In the projective setting, this leads to the basis

$$|3\ell - p_1 - \cdots - p_5| = \left\langle \alpha Y Z^2 - \beta XYZ - \alpha^2 X^3, Y^3, X^2 Y, XY^2, Y^2 Z \right\rangle_{\mathbf{F}_q}. \tag{11}$$

Types of decomposition into irreducible components in $|-K_X|$. — Since $\mathrm{CaCl}(X_s)$ if of index 4 inside $\mathrm{Cl}(X_s)$, even if these two groups are free of rank 1, an irreducible Cartier divisor may decompose into Weil irreducible components. In order to lower bound the minimum distance, we need to review all these kinds of decompositions into irreducible components for the curves of the anticanonical linear system on X_s. As usual, we start form $\mathbf{P}^2$ and use the one-to-one correspondences:

$$\begin{array}{ccccc} |3\ell - p_1 - \cdots - p_5|_{\mathbf{P}^2} & \longrightarrow & |-K_X|_X & \longrightarrow & |-K_{X_s}|_{X_s} \\ C & \longmapsto & C^\sharp & \longmapsto & \varphi\left(C^\sharp\right) \end{array}$$

where $C^\sharp$ denotes the virtual transform of C in the composition of the five blowups.

Thanks to the preceding computation, for every curve in $|3\ell - p_1 - \cdots - p_5|_{\mathbf{P}^2}$ there exists $\alpha_1, \ldots, \alpha_5 \in \mathbf{F}_q$ such this curves is defined by

$$\alpha_1 \left(\alpha Y Z^2 - \beta XYZ - \alpha^2 X^3\right) + \alpha_2 Y^3 + \alpha_3 X^2 Y + \alpha_4 XY^2 + \alpha_5 Y^2 Z = 0.$$

We deduce that such a curve can decompose in six different ways, as listed in the tabular below:

- either a cubic c_{12345} for which p_1 is a smooth flex point with tangent line equal to ℓ_{123}, if $\alpha_1 \neq 0$ (case 1);
- or a cubic singular at p_1 which contains ℓ_{123} as a component, if $\alpha_1 = 0$, the complementary component, of discriminant $\alpha_3 \alpha_5^2$ (up to a constant), being either
  - a quadric q_{12} smooth at p_1 with tangent line ℓ_{123}, if $\alpha_3 \neq 0$ and $\alpha_5 \neq 0$ (case 2),
  - or the union of two lines, if $\alpha_3 \neq 0$ and $\alpha_5 = 0$, $\alpha_3 = 0$ and $\alpha_5 \neq 0$, $\alpha_3 = \alpha_5 = 0$ and $\alpha_4 \neq 0$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$ (cases 3, 4, 5, 6 respectively).

	$\left\|3\ell - \sum_{i=1}^5 p_i\right\|$ on $\mathbf{P}^2$	$\|-K_X\|$ on X	$\|-K_{X_s}\|$ on X_s	Max nb. of pts
1	c_{12345}	$\widetilde{c}_{12345}$	$\varphi_*(\widetilde{c}_{12345})$	$\mathrm{N}_q(1)$
2	$\ell_{123} \cup q_{12}$	$\widetilde{\ell}_{123} \cup \widetilde{q}_{12} \cup \widetilde{E}_1 \cup 2\widetilde{E}_2 \cup 2\widetilde{E}_3 \cup \widetilde{E}_4$	$\varphi_*(\widetilde{q}_{12})$	$q+1$
3	$\ell_{123} \cup \ell_1 \cup \ell'_1$	$\widetilde{\ell}_{123} \cup 2\widetilde{E}_1 \cup 2\widetilde{E}_2 \cup 2\widetilde{E}_3 \cup \widetilde{E}_4 \cup \widetilde{\ell}_1 \cup \widetilde{\ell}'_1$	$\varphi_*(\widetilde{\ell}_1) \cup \varphi_*(\widetilde{\ell}'_1)$	$2q+1$
4	$2\ell_{123} \cup \ell$	$2\widetilde{\ell}_{123} \cup \widetilde{E}_1 \cup 2\widetilde{E}_2 \cup 3\widetilde{E}_3 \cup 2\widetilde{E}_4 \cup E_5 \cup \widetilde{\ell}$	$\varphi_*(E_5) \cup \varphi_*(\widetilde{\ell})$	$2q+1$
5	$2\ell_{123} \cup \ell_1$	$2\widetilde{\ell}_{123} \cup 2\widetilde{E}_1 \cup 3\widetilde{E}_2 \cup 4\widetilde{E}_3 \cup 3\widetilde{E}_4 \cup 2E_5 \cup \widetilde{\ell}_1$	$2\varphi_*(E_5) \cup \varphi_*(\widetilde{\ell}_1)$	$2q+1$
6	$3\ell_{123}$	$3\widetilde{\ell}_{123} \cup 2\widetilde{E}_1 \cup 4\widetilde{E}_2 \cup 6\widetilde{E}_3 \cup 5\widetilde{E}_4 \cup 4E_5$	$4\varphi_*(E_5)$	$q+1$

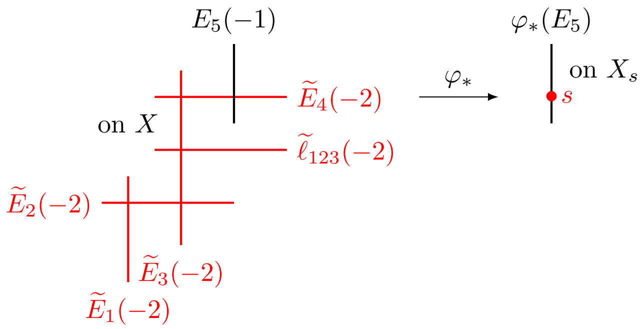
Let us give details for the computation of the virtual transform $\pi^\sharp(C)$ if, for example, $C = \ell_{123} \cup \ell_1 \cup \ell_1'$:

$$\begin{aligned}
C_1 &= \pi_1^\sharp(C) = \widetilde{\ell}_{123} + \widetilde{\ell}_1 + \widetilde{\ell}_1' + 2E_1 && [p_1 \in \ell_{123} \cap \ell_1 \cap \ell_1' \ \Rightarrow\ m_{p_1}(C) = 3] \\
C_2 &= \pi_2^\sharp(C_1) = \widetilde{\ell}_{123} + \widetilde{\ell}_1 + \widetilde{\ell}_1' + 2\widetilde{E}_1 + 2E_2 && \left[p_2 \in \widetilde{\ell}_{123} \cap E_1 \ \Rightarrow\ m_{p_2}(C_1) = 3\right] \\
C_3 &= \pi_3^\sharp(C_2) = \widetilde{\ell}_{123} + \widetilde{\ell}_1 + \widetilde{\ell}_1' + 2\widetilde{E}_1 + 2\widetilde{E}_2 + 2E_3 && \left[p_3 \in \widetilde{\ell}_{123} \cap E_2 \ \Rightarrow\ m_{p_3}(C_2) = 3\right] \\
C_4 &= \pi_4^\sharp(C_3) = \widetilde{\ell}_{123} + \widetilde{\ell}_1 + \widetilde{\ell}_1' + 2\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + E_4 && [p_4 \in E_3 \ \Rightarrow\ m_{p_4}(C_3) = 2] \\
C^\sharp &= \pi_5^\sharp(C_4) = \widetilde{\ell}_{123} + \widetilde{\ell}_1 + \widetilde{\ell}_1' + 2\widetilde{E}_1 + 2\widetilde{E}_2 + 2\widetilde{E}_3 + \widetilde{E}_4 && [p_5 \in E_4 \ \Rightarrow\ m_{p_5}(C_4) = 1]\,.
\end{aligned}$$

This leads to the following decomposition of the canonical class into a sum of effective classes

$$-K_X = (E_0 - E_1 - E_2 - E_3) + (E_0 - E_1) + (E_0 - E_1) + 2(E_1 - E_2) + 2(E_2 - E_3) + 2(E_3 - E_4) + (E_4 - E_5)$$

The intersection graph of the irreducible effective roots in X is connected (see figure below) and all these curves are contracted by the morphism φ to a single rational singular pointy s (of singularity type D_5).



We comment on the numbers of points.

Cases 2 & 6. All the components in X are roots that are contracted, except $\widetilde{q}_{12}$ and E_5 respectively. These two strict transforms meet the tree of roots at only one point and by φ_* they are mapped to isomorphic curves that pass through s.

Case 3. Except $\widetilde{\ell}_1$ and $\widetilde{\ell}_1'$, all the components on X are irreducible effective roots and they are mapped to the point s by the morphism φ_*. After the contraction, the curves $\varphi_*(\widetilde{\ell}_1)$ and $\varphi_*(\widetilde{\ell}_1')$ meet at this singular point, thus their union contains $2q+1$ rational points.

Cases 4 & 5. The line E_5 does not intersect the lines $\widetilde{\ell}$ or $\widetilde{\ell}_1$ in X. However, since ℓ and ℓ_{123} meet at some point of $\mathbf{P}^2$ (not equal to p_1), the lines $\widetilde{\ell}$ and $\widetilde{\ell}_{123}$ intersect in X; in the same way since ℓ_1 passes through p_1, the lines $\widetilde{\ell}_1$ and $\widetilde{E}_1$ intersect in X. Therefore, $\varphi_*(E_5)$ and $\varphi_*(\widetilde{\ell})$ or $\varphi_*(E_5)$ and $\varphi_*(\widetilde{\ell}_1)$ both intersect at s. Thus the two unions has $2q+1$ rational points.

Finally $N_q\left(-K_{X_s}\right) = 2q+1$.

The surface X_s has a unique singular point s. All the irreducible effective roots of X, that is $\widetilde{\ell}_{123}, \widetilde{E}_1, \ldots, \widetilde{E}_4$ are contracted to this single point. The last exceptional curve E_5 meets E_4 and thus $\varphi(E_5)$ passes through s. In conclusion the rational points of $X_s(\mathbf{F}_q)$ are in one-to-one correspondence with $\left(\mathbf{P}^2(\mathbf{F}_q) \setminus \ell_{123}(\mathbf{F}_q)\right) \cup E_5(\mathbf{F}_q)$, which counts q^2+q+1 elements. This number is always strictly greater that $2q+1$ and the evaluation map is always injective.

Proposition 4.7. *Let $p_1 \prec p_2 \prec p_3 \prec p_4 \prec p_5$ be infinitely near rational points. Suppose that the first three ones are collinear. The anticanonical code of the weak del Pezzo surface obtained by blowing up these points has parameters $[q^2+q+1, 5, q^2-q]$.*

The construction of a generator matrix of this code is a nice application of proposition 3.3. It consists of two blocks, the left one, of size $5 \times q^2$, contains the evaluations of the five global sections of (11) at every point of $\mathbf{P}^2(\mathbf{F}_q) \setminus \ell_{123}(\mathbf{F}_q)$, the right one, of size $5 \times (q+1)$ contains the evaluations of the homogeneous parts of degree 1 of the five global sections of (11) at every point of $\mathbf{P}^1(\mathbf{F}_q)$. Letting $y_5 = \beta + z_5$ in (10), this homogeneous part of degree 1 equals $(\alpha a_{21} + \beta a_{11})x_5 + a_{01}z_5$. Finally, we get the explicit matrix:

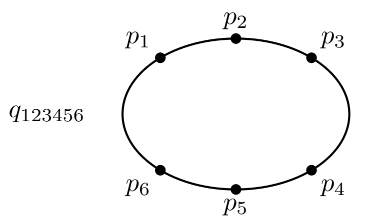
$$\left(\begin{array}{cc|cc}
\alpha y z^2 - \beta x y z - \alpha^2 x^3 & \vdots & -\beta^2 u + \alpha v & \vdots \\
y^3 & \vdots & 0 & \vdots \\
x^2 y & (x:y:z) \in \mathbf{P}^2(\mathbf{F}_q) \mid y \neq 0 & \alpha u & (u:v) \in \mathbf{P}^1(\mathbf{F}_q) \\
x y^2 & \vdots & 0 & \vdots \\
y^2 z & \vdots & 0 & \vdots
\end{array}\right),$$

where $\alpha \in \mathbf{F}_q^*$ and $\beta \in \mathbf{F}_q$.

4.8 Degree 3, singularity of type $\mathbf{A}_1$

This example corresponds to the type number 11 in degree 3 [BH22].

Configuration to blow-up and down. — We blow up six conjugate points $p_1, \ldots, p_6 \in \mathbf{P}^2$ on a smooth conic q_{123456}.



$$\begin{aligned} p_2 &= p_1^{\sigma} \\ p_3 &= p_1^{\sigma^2} \\ p_4 &= p_1^{\sigma^3} \\ p_5 &= p_1^{\sigma^4} \\ p_6 &= p_1^{\sigma^5} \end{aligned}$$

The resulting surface X is a weak del Pezzo of degree 3, whose anticanonical model X_s has a unique singular point of type $\mathbf{A}_1$.

Computation of the divisor class groups. — We have:

$$\mathrm{Cl}(\overline{X}) = \sum_{i=0}^{6} \mathbf{Z}E_i \qquad \text{and} \qquad \mathrm{Cl}(X) = \mathbf{Z}E_0 \oplus \mathbf{Z}\mathcal{E}, \qquad \text{where } \mathcal{E} = \sum_{i=1}^{6} E_i.$$

There is a unique irreducible effective root, the strict transform of the conic q_{123456}, whose class is $2E_0 - \mathcal{E}$. The root module $\mathscr{R}$, generated this class, is a direct summand, $\mathrm{Cl}(X) = \mathscr{R} \oplus \mathbf{Z}E_0$. The projection onto the second factor leads to an isomorphism

$$\begin{array}{ccc} \mathrm{Cl}(X)/\mathscr{R} & \longrightarrow & \mathbf{Z}E_0 \\ E_0 \bmod \mathscr{R} & \longmapsto & E_0 \\ \mathcal{E} \bmod \mathscr{R} & \longmapsto & 2E_0 \end{array}.$$

As for the module $\mathscr{R}^{\perp}$, inside $\mathrm{Cl}(\overline{X})$ it is defined by the single equation $2a_0 + a_1 + \cdots + a_6 = 0$; after taking the Galois invariants, we obtain $\mathrm{CaCl}(X_s) = \mathbf{Z}K_X$, whose image by the previous isomorphism is also $\mathbf{Z}E_0$. Therefore $\mathrm{CaCl}(X_s) \simeq \mathrm{Cl}(X_s)$ and both of them are free of rank one.

Types of decomposition into irreducible components in $|-K_X|$. — This proves that all the sections of the anticanonical divisor are irreducible. As in our previous work [BCH+20], we expect that the curves of the associated linear system can contain at most $\mathrm{N}_q(1)$ rational points. However we need to investigate the types of irreducible decompositions. Here this is easy since one can check that the only Galois orbits of lines or conics or cubics that pass through at least one point p_i are $\ell_{14} \cup \ell_{25} \cup \ell_{36}$ or q_{123456} or c_{123456} (all the others lead to $\mathbf{F}_q$-curves of degree strictly greater than 3). Combining them in order to construct a curve in the expected sub-linear system leads to very few decompositions:

	$\left\lvert 3\ell - \sum_{i=1}^{6} p_i \right\rvert$ on $\mathbf{P}^2$	$\lvert -K_X \rvert$ on X	$\lvert -K_{X_s} \rvert$ on X_s	Max nb. of pts
1	$\ell_{14} \cup \ell_{25} \cup \ell_{36}$	$\widetilde{\ell}_{14} \cup \widetilde{\ell}_{25} \cup \widetilde{\ell}_{36}$	$\varphi_*(\widetilde{\ell}_{14}) \cup \varphi_*(\widetilde{\ell}_{25}) \cup \varphi_*(\widetilde{\ell}_{36})$	1
2	$\ell \cup q_{123456}$	$\widetilde{\ell} \cup \widetilde{q}_{123456}$	$\varphi_*(\widetilde{\ell}) \ni s$	$q+2$
3	c_{123456}	$\widetilde{c}_{123456}$	$\varphi_*(\widetilde{c}_{123456})$	$\mathrm{N}_q(1)$

The number of rational point in case 1 is at most 1 if the three lines meet. In case 2, during the process, if the two meeting points of ℓ and q_{123456} are not rational, then the singular point s is an additional rational point on $\varphi_*(\widetilde{\ell})$. We deduce that $N_q(-K_{X_s}) \leq N_q(1)$.

Since the blown up points are not rational, the blowing ups do not add point on the surface and $\#X(\mathbf{F}_q) = q^2 + q + 1$. Then, the irreducible effective root is contracted and thus $\#X_s(\mathbf{F}_q) = q^2 + 1$. If $q = 2$, the evaluation map may fail to be injective.

Proposition 4.8. *Suppose $q \neq 2$. Let $p_1, \ldots, p_6 \in \mathbf{P}^2$ be six conjugate points lying on a smooth conic. The anticanonical code of the weak del Pezzo surface obtained by blowing up these points has parameters $[q^2 + 1, 4, \geq q^2 + 1 - \mathrm{N}_q(1)]$.*

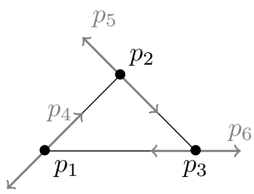
Computation of the global sections from $\mathbf{P}^2$. — Let Q denote the conic passing through $p_1, \dots, p_6$ and let L_{14}, L_{25}, L_{36} be the linear forms whose zeros are the lines ℓ_{14}, ℓ_{25}, ℓ_{36}. Then

$$H^0\left(\mathbf{P}^2, 3\ell - \sum_{i=1}^{6} p_i\right) = \langle XQ, YQ, ZQ, L_{14}L_{25}L_{36}\rangle_{\mathbf{F}_q}$$

4.9 Degree 3, singularity of type $3\mathbf{A}_2$

This example corresponds to the type number 76 in degree 6 [BH22]. As in section 4.5, this example appears in Koshelev's work [Kos20, §1.1] but with another point of view.

Configuration to blow-up. — First we blow up three non-collinear conjugate points p_1, p_2, p_3. This leads to a degree 6 del Pezzo surface with three exceptional conjugate curves E_1, E_2, E_3, the other exceptional curves being the strict transforms $\ell_{12}, \ell_{13}, \ell_{23}$ of the lines joining two of the three points. Then we blow up three other points p_4, p_5, p_6 with $p_{i+3} \succ p_i$, and more precisely p_4 is the intersection point of E_1 and $\widetilde{\ell}_{12}$, p_5 is the intersection point of E_2 and $\widetilde{\ell}_{23}$ and p_6 is the intersection point of E_3 and $\widetilde{\ell}_{13}$. These points are also conjugate and the resulting surface X is a weak degree three del Pezzo surface, with three new exceptional curves E_4, E_5, E_6. The anticanonical model X_s has three conjugate singular points of type $\mathbf{A}_2$.



$$p_2 = p_1^{\sigma} \quad p_5 = p_4^{\sigma}$$
$$p_3 = p_1^{\sigma^2} \quad p_6 = p_4^{\sigma^2}.$$

The point p_4 lies on the strict transform of the line (p_1p_2), which we denote by ℓ_{124}. In the same way we introduce the lines ℓ_{235} and ℓ_{136}.

Computation of the divisor class groups. — There are six irreducible effective roots, the strict transforms of E_1, E_2, E_3 and the strict transforms of $\ell_{124}, \ell_{235}, \ell_{136}$; their classes are:

$$R_1 = E_1 - E_4, \qquad R_2 = E_2 - E_5, \qquad R_3 = E_3 - E_6,$$
$$R_1' = E_0 - E_1 - E_2 - E_4, \qquad R_2' = E_0 - E_2 - E_3 - E_5, \qquad R_3' = E_0 - E_1 - E_3 - E_6.$$

The absolute Galois group acts on this six root classes as $(R_1R_2R_3)(R_1'R_2'R_3')$ and also on the exceptional curves as $(E_1E_2E_3)(E_4E_5E_6)$ (the first three exceptional curves are the total transforms of the exceptional curves on the degree 6 del Pezzo surface, they are no longer irreducible).

We have

$$\mathrm{Cl}(\overline{X}) = \bigoplus_{i=0}^{6} \mathbf{Z}E_i \qquad \overline{\mathscr{R}} = \bigoplus_{i=1}^{3} \mathbf{Z}R_i \oplus \mathbf{Z}R_i' \qquad \text{and} \qquad \overline{\mathscr{R}}^{\perp} = \mathbf{Z}K_X.$$

Let us put:

$$\mathcal{E} = E_1 + E_2 + E_3, \quad \mathcal{E}' = E_4 + E_5 + E_6, \quad \mathcal{R} = R_1 + R_2 + R_3 = \mathcal{E} - \mathcal{E}', \quad \mathcal{R}' = R_1' + R_2' + R_3' = 3E_0 - 2\mathcal{E} - \mathcal{E}'.$$

One easily verify that

$$\mathrm{Cl}(\overline{X})^{\Gamma} = \mathbf{Z}E_0 \oplus \mathbf{Z}\mathcal{E} \oplus \mathbf{Z}\mathcal{E}', \quad \mathscr{R} = \overline{\mathscr{R}}^{\Gamma} = \mathbf{Z}\mathcal{R} \oplus \mathbf{Z}\mathcal{R}' = \mathbf{Z}(\mathcal{E} - \mathcal{E}') \oplus \mathbf{Z}(3E_0 - 2\mathcal{E} - \mathcal{E}'), \quad \text{and} \quad \mathscr{R}^{\perp} = \mathbf{Z}K_X.$$

It turns out that the submodule $\mathscr{R}$ is not a direct summand in $\mathrm{Cl}(X)$; indeed

$$\mathscr{R} = \mathbf{Z}(\mathcal{E} - \mathcal{E}') \oplus \mathbf{Z}3(E_0 - \mathcal{E}) \subset \mathbf{Z}(\mathcal{E} - \mathcal{E}') \oplus \mathbf{Z}(E_0 - \mathcal{E}) \oplus \mathbf{Z}\mathcal{E}' = \mathrm{Cl}(X)$$

(we have just replaced $3E_0 - 2\mathcal{E} - \mathcal{E}'$ by $(3E_0 - 2\mathcal{E} - \mathcal{E}') - (\mathcal{E} - \mathcal{E}')$ in the initial basis). Therefore the projection onto the two last factors leads to an isomorphism:

$$\begin{array}{rcccc} \mathrm{Cl}(X_s) \simeq \mathrm{Cl}(X)/\mathscr{R} & \longrightarrow & \mathbf{Z}/3\mathbf{Z}(E_0 - \mathcal{E}) & \oplus & \mathbf{Z}\mathcal{E}' \\ a_0E_0 + a\mathcal{E} + a'\mathcal{E}' \bmod \mathscr{R} & \longmapsto & (a_0 \bmod 3)\,(E_0 - \mathcal{E}) & + & (a_0 + a + a')\mathcal{E}' \end{array}$$

Via this isomorphism the group $\mathrm{CaCl}(X_s) = \mathscr{R}^{\perp} = \mathbf{Z}K_X$ embeds via $-K_X \mapsto \mathcal{E}'$; this means that $\mathrm{CaCl}(X_s)$ is isomorphic to the free part of $\mathrm{Cl}(X_s)$ and these two groups are free of rank one.

Types of decomposition into irreducible components in $|-K_X|$. — As in the previous case, the global sections of the divisor $|-K_{X_s}|$ are irreducible but not necessarily absolutely irreducible. As usual, we list the Galois orbits of lines or conics or cubics of degree less than 3 that pass through at least one of the six points. The only possibilities are

$$\ell_1 \cup \ell_2 \cup \ell_3, \qquad \ell_{124} \cup \ell_{235} \cup \ell_{136}, \qquad q_{123}, \qquad c_{123}, \qquad c_{123456}.$$

(it is important to keep in mind that a curve which passes through p_4 necessarily passes through p_1). There are only two combinations that lead to a cubic which passes through the six points:

	$\left\|3\ell - \sum_{i=1}^{6} p_i\right\|$ on $\mathbf{P}^2$	$\|-K_X\|$ on X	$\|-K_{X_s}\|$ on X_s	Max nb. of pts
1	$\ell_{124} \cup \ell_{235} \cup \ell_{136}$	$\widetilde{\ell}_{124} \cup \widetilde{\ell}_{235} \cup \widetilde{\ell}_{136} \cup \widetilde{E}_1 \cup \widetilde{E}_2 \cup \widetilde{E}_3 \cup E_4 \cup E_5 \cup E_6$	$\varphi_*(E_4) \cup \varphi_*(E_5) \cup \varphi_*(E_6)$	0
2	c_{123456}	$\widetilde{c}_{123456}$	$\varphi_*(\widetilde{c}_{123456})$	$\mathrm{N}_q\,(1)$

The roots of X are $\widetilde{\ell}_{124}$, $\widetilde{E}_2$ (mapped to a singular point $s \in X_s$), $\widetilde{\ell}_{235}$, $\widetilde{E}_3$ (mapped to a singular point $s^\sigma \in X_s$), and $\widetilde{\ell}_{136}$, $\widetilde{E}_1$ (mapped to a singular point $s^{\sigma^2} \in X_s$). The curves E_i, $i = 4, 5, 6$, are not defined over $\mathbf{F}_q$ and do not contain any rational point. In conclusion $N_q\,(-K_{X_s}) \leq N_q(1)$.

Since the points $p_1, \ldots, p_6$ are not rational the blowing ups do not add any rational point, and since the singular points are not rational the contractions do not add any rational point also. Thus $\#X_s(\mathbf{F}_q) = q^2 + q + 1$, this number is always strictly greater than $N_q(1)$ and we deduce the parameters given below.

Proposition 4.9. *The weak del Pezzo surface of degree* 3 *associated to the configuration specified at the beginning of this section has parameters* $[q^2 + q + 1, 4, \geq q^2 + q + 1 - \mathrm{N}_q\,(1)]$.

Koshelev [Kos20, §1.1] proves that the minimum distance can be improved by 1 for some q since he shows that cubics of the considered linear system must have a 3-torsion point.

Computation of the global sections from $\mathbf{P}^2$. — Let L_{12}, L_{23}, L_{13} be the three conjugate linear forms that respectively define the lines $\ell_{124}, \ell_{235}, \ell_{136}$ in $\mathbf{P}^2$. The family L_{12}, L_{23}, L_{13} is a $\overline{\mathbf{F}}_q$-basis of $H^0(\mathbf{P}^2, \ell)$, while the family of degree 3 monomials in L_{12}, L_{23}, L_{13} is a $\overline{\mathbf{F}}_q$-basis of $H^0(\mathbf{P}^2, 3\ell)$. A cubic in this space can be written:

$$a_1 L_{12}^3 + a_2 L_{23}^3 + a_3 L_{13}^3 + b_1 L_{12} L_{23}^2 + c_1 L_{13} L_{23}^2 + b_2 L_{12} L_{13}^2 + c_2 L_{23} L_{13}^2 + b_3 L_{13} L_{12}^2 + c_3 L_{23} L_{12}^2 + d L_{12} L_{23} L_{13}$$

Such a cubic pass through p_1 if and only if $a_2 = 0$ (since p_1 is a common zero of L_{12} and L_{13}). In the same way it passes through p_2 and p_3 if and only if $a_3 = 0$ and $a_1 = 0$. Now passing through p_4 means that if this cubic is not singular at p_1 then its tangent line at this point must be ℓ_{12}. After deshomogenizing by putting $L_{23} = 1$ (this is possible since L_{23} does not vanish at p_1) this means that the linear component $b_1 L_{12} + c_1 L_{13}$ should be proportional to L_{12}; necessarily $c_1 = 0$. In the same way passing through p_5 (resp. p_6) means that $b_2 = 0$ (resp. $c_3 = 0$). Finally, one has

$$H^0\left(\mathbf{P}^2, 3\ell - \textstyle\sum_{i=1}^{6} p_i\right) = \langle L_{12} L_{23}^2, L_{23} L_{13}^2, L_{13} L_{12}^2, L_{12} L_{23} L_{13} \rangle_{\overline{\mathbf{F}}_q}$$

In order to deduce a $\mathbf{F}_q$-base, we consider θ any primitive element of $\mathbf{F}_{q^3}$ over $\mathbf{F}_q$. The linear independence of homomorphisms permits to prove that the matrix $(\sigma^i(\theta^j))_{1 \leq i,j \leq 3}$ is invertible. Let us put:

$$\begin{aligned} C_1 &= L_{12} L_{23}^2 + L_{23} L_{13}^2 + L_{13} L_{12}^2 \\ C_\theta &= \theta L_{12} L_{23}^2 + \sigma(\theta) L_{23} L_{13}^2 + \sigma^2(\theta) L_{13} L_{12}^2 \\ C_{\theta^2} &= \theta^2 L_{12} L_{23}^2 + \sigma(\theta^2) L_{23} L_{13}^2 + \sigma^2(\theta^2) L_{13} L_{12}^2 \end{aligned}$$

then $C, C_\theta, C_{\theta^2}$ are defined over $\mathbf{F}_q$, as the product $L_{12} L_{23} L_{13}$ and one has:

$$H^0\left(\mathbf{P}^2, 3\ell - \textstyle\sum_{i=1}^{6} p_i\right) = \langle C_1, C_\theta, C_{\theta^2}, L_{12} L_{23} L_{13} \rangle_{\mathbf{F}_q}$$

The birational morphism

$$\begin{array}{ccc} \mathbf{P}^2 & \dashrightarrow & \mathbf{P}^4 \\ (X : Y : Z) & \longmapsto & (C_1 : C_\theta : C_{\theta^2} : L_{12} L_{23} L_{13}) \end{array}$$

has X_s as image in $\mathbf{P}^4$. Thus, if $r_1, \ldots, r_{q^2+q+1}$ denote the rational points of $\mathbf{P}^2$, one of the generating matrix of this code is nothing else than:

$$\begin{pmatrix} C_1(r_1) & \cdots & C_1(r_{q^2+q+1}) \\ C_\theta(r_1) & \cdots & C_\theta(r_{q^2+q+1}) \\ C_{\theta^2}(r_1) & \cdots & C_{\theta^2}(r_{q^2+q+1}) \\ L_{12} L_{23} L_{13}(r_1) & \cdots & L_{12} L_{23} L_{13}(r_{q^2+q+1}) \end{pmatrix}$$

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