Noname manuscript No.
(will be inserted by the editor)

Vanishing cycles and Cartan eigenvectors

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Received: date / Accepted: date

March 15, 2016

Abstract Using the ideas coming from the singularity theory, we study the eigenvectors of the Cartan matrices of finite root systems, and of q-deformations of these matrices.

1 Introduction

Let $A(R)$ be the Cartan matrix of a finite root system R. The coordinates of its eigenvectors have an important meaning in the physics of integrable systems; we will say more on this below.

The aim of this note is a study of these numbers, and of their q-deformations, using some results coming from the singularity theory.

We discuss three ideas:
(a) Cartan/Coxeter correspondence;
(b) Sebastiani - Thom product;
(c) Givental's q-deformations.
Let us explain what we are talking about.

Let us suppose that R is simply laced, i.e. of type A, D, or E. These root systems are in one-to-one correspondence with (classes of) simple singularities
$f : \mathbb{C}^N \to \mathbb{C}$, cf. [1]. Under this correspondence, the root lattice $Q(R)$ is identified with the lattice of vanishing cycles, and the Cartan matrix $A(R)$ is the intersection matrix with respect

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to a *distinguished base*. The action of the Weyl group on $Q(R)$ is realized by Gauss - Manin monodromies - this is the Picard - Lefschetz theory (for some details see §2 below).

Remarkably, this geometric picture provides a finer structure: namely, the symmetric matrix $A = A(R)$ comes equipped with a decomposition

$$A = L + L^t \tag{1}$$

where L is a nondegenerate triangular ”Seifert form”, or ”variation matrix”. The matrix

$$C = -L^{-1}L^t \tag{2}$$

represents a Coxeter element of R; geometrically it is the operator of ”classical monodromy”.

We call the relation (1) - (2) between the Cartan matrix and the Coxeter element the *Cartan/Coxeter correspondence*. It works more generally for non-symmetric A (in this case (1) should be replaced by

$$A = L + U \tag{3}$$

where L is lower triangular and U is upper triangular), and is due to Coxeter, cf. [5], no. 1, p. 767, see §3 below.

In a particular case (corresponding to a bipartition of the Dynkin graph) this relation is equivalent to an observation by R.Steinberg, cf. [18], cf. §3.3 below.

This corresppondence allows one to relate the eigenvectors of A and C, cf. Theorem 1.

A decomposition (1) will be called *a polarization* of the Cartan matrix A. In 4.1 below we introduce an operation of *Sebastiani - Thom*, or *joint* product $A * B$ of Cartan matrices (or of polarized lattices) A and B. The root lattice of $A * B$ is the tensor product of the root lattice of A and the root lattice of B. With respect to this operation the Coxeter eigenvectors factorize very simply.

For example, the lattices E_6 and E_8 decompose into three ”quarks”:

$$E_6 = A_3 * A_2 * A_1 \tag{4}$$

$$E_8 = A_4 * A_2 * A_1 \tag{5}$$

These decompositions are the main message from the singularity theory, and we discuss them in detail in this note.

We use (4), (5), and the Cartan/Coxeter correspondence to get expressions for all Cartan eigenvectors of E_6 and E_8; this is the first main result of this note, see 4.9, 4.11 below.

(An elegant expression for all the Cartan eigenvectors of all finite root systems was given by P.Dorey, cf. [6] (a), Table 2 on p. 659.)

In the paper [9], A.Givental has proposed a q-twisted version of the Picard - Lefschetz theory, which gave rise to a q-deformation of A,

$$A(q) = L + qL^t. \tag{6}$$

Again, as Givental remarked, the decomposition (3) allows us to drop the assumption of symmetry in the definition above. In the last section, §5, we calculate the eigenvalues and eigenvectors of $A(q)$ in terms of the eigenvalues and eigenvectors of A. This is the second main result of this note.

It turns out that if λ is an eigenvalue of A then

$$\lambda(q) = 1 + (\lambda - 2)\sqrt{q} + q \tag{7}$$

will be an eigenvalue of $A(q)$. The coordinates of the corresponding eigenvector $v(q)$ are obtained from the coordinates of $v = v(1)$ by multiplication by appropriate powers of q; this is related to the fact that the Dynkin graph of A is a tree, cf. 5.2. For an example of E_8, see (25).

In physics the coordinates of the Perron - Frobenius Cartan eigenvectors appear as the particle masses (or, dually, as the soliton energies) in affine Toda field theories, cf. [6, 7].

Historically, E_8 made its first appearance in the pioneering papers by A. B. Zamolodchikov [19] on the two-dimensional critical Ising model in a magnetic field.

The Appendix outlines some of the results of a neutron scattering experiment [20] where one has observed the two lowest-mass E_8 particles of the Zamolodchikov theory [19].

2 Recollections from the singularity theory

Here we recall some classical constructions and statements, cf. [1].

2.1 Lattice of vanishing cycles

Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point at 0, with $f(0) = 0$. We will be interested only in polynomial functions (from the list below, cf. §2.4), so $f \in \mathbb{C}[x_1, \ldots, x_N]$. The *Milnor ring* of f is defined by

$$\mathrm{Miln}(f, 0) = \mathbb{C}[[x_1, \ldots, x_N]]/(\partial_1 f, \ldots, \partial_N f)$$

where $\partial_i := \partial/\partial x_i$; it is a finite-dimensional commutative $\mathbb{C}$-algebra. (In fact, it is a Frobenius, or, equivalently, a Gorenstein algebra.) The number

$$\mu := \dim_{\mathbb{C}} \mathrm{Miln}(f, 0)$$

is called the multiplicity or Milnor number of $(f, 0)$.

A *Milnor fiber* is

$$V_z = f^{-1}(z) \cap \bar{B}_\rho$$

where

$$\bar{B}_\rho = \{(x_1, \ldots, x_N)| \sum |x_i|^2 \leq \rho\}$$

for $1 \gg \rho \gg |z| > 0$.

For z belonging to a small disc $D_\epsilon = \{z \in \mathbb{C}| \; |z| < \epsilon\}$, the space V_z is a complex manifold with boundary, homotopically equivalent to a bouquet $\vee S^{N-1}$ of μ spheres, [15].

The family of free abelian groups

$$Q(f; z) := \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu, \;\; z \in \overset{\bullet}{D}_\epsilon := D_\epsilon \setminus \{0\}, \tag{8}$$

($\tilde{H}$ means that we take the reduced homology for $N = 1$), carries a flat Gauss - Manin conection.

Take $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$; the lattice $Q(f;t)$ does not depend, up to a canonical isomorphism, on the choice of t. Let us call this lattice $Q(f)$. The linear operator

$$T(f) : Q(f) \overset{\sim}{\longrightarrow} Q(f) \tag{9}$$

induced by the path $p(\theta) = e^{i\theta}t,\ 0 \leq \theta \leq 2\pi$, is called the classical monodromy of the germ $(f, 0)$.

In all the examples below $T(f)$ has finite order h. The eigenvalues of $T(f)$ have the form $e^{2\pi ik/h},\ k \in \mathbb{Z}$. The set of suitably chosen k's for each eigenvalue are called the *spectrum* of our singularity.

2.2 Morse deformations

The $\mathbb{C}$-vector space $\mathrm{Miln}(f, 0)$ may be identified with the tangent space to the base B of the miniversal defomation of f. For

$$\lambda \in B^0 = B \setminus \Delta$$

where $\Delta \subset B$ is an analytic subset of codimension 1, the corresponding function $f_\lambda : \mathbb{C}^N \to \mathbb{C}$ has μ nondegenerate Morse critical points with distinct critical values, and the algebra $\mathrm{Miln}(f_\lambda)$ is semisimple, isomorphic to $\mathbb{C}^\mu$.

Let $0 \in B$ denote the point corresponding to f itself, so that $f = f_0$, and pick $t \in \mathbb{R}_{>0} \cap \overset{\bullet}{D}_\epsilon$ as in §2.1.

Afterwards pick $\lambda \in B^0$ close to 0 in such a way that the critical values $z_1, \ldots z_\mu$ of f_λ have absolute values $\ll t$.

As in §2.1, for each

$$z \in \tilde{D}_\epsilon := D_\epsilon \setminus \{z_1, \ldots z_\mu\}$$

the Milnor fiber V_z has the homotopy type of a bouquet $\vee S^{N-1}$ of μ spheres, and we will be interested in the middle homology

$$Q(f_\lambda; z) = \tilde{H}_{N-1}(V_z; \mathbb{Z}) \overset{\sim}{=} \mathbb{Z}^\mu$$

The lattices $Q(f_\lambda; z)$ carry a natural bilinear product induced by the cup product in the homology which is symmetric (resp. skew-symmetric) when N is odd (resp. even).

The collection of these lattices, when $z \in \tilde{D}_\epsilon$ varies, carries a flat Gauss - Manin connection.

Consider an "octopus"

$$Oct(t) \subset \mathbb{C}$$

with the head at t: a collection of non-intersecting paths p_i ("tentacles") connecting t with z_i and not meeting the critical values z_j otherwise. It gives rise to a base

$$\{b_1, \ldots, b_\mu\} \subset Q(f_\lambda) := Q(f_\lambda; t)$$

(called "distinguished") where b_i is the cycle vanishing when being transferred from t to z_i along the tentacle p_i, cf. [8], [1].

The Picard - Lefschetz formula describes the action of the fundamental group $\pi_1(\tilde{D}_\epsilon; t)$ on $Q(f_\lambda)$ with respect to this basis. Namely, consider a loop γ_i which turns around z_i along the tentacle p_i, then the corresponding transformation of $Q(f_\lambda)$ is the reflection (or transvection) $s_i := s_{b_i}$, cf. [14], Théorème fondamental, Ch. II, p. 23.

The loops γ_i generate the fundamental group $\pi_1(\tilde{D}_\epsilon)$. Let

$$\rho: \ \pi_1(\tilde{D}_\epsilon; t) \to GL(Q(f_\lambda))$$

denote the monodromy representation. The image of ρ, denoted by $G(f_\lambda)$ and called the *monodromy group of* f_λ, lies inside the subgroup
$O(Q(f_\lambda)) \subset GL(Q(f_\lambda))$ of linear transformations respecting the above mentioned bilinear form on $Q(f_\lambda)$.

The subgroup $G(f_\lambda)$ is generated by $s_i, 1 \leq i \leq \mu$.

As in §2.1, we have the monodromy operator

$$T(f_\lambda) \in G(f_\lambda),$$

the image by ρ of the path $p \subset \tilde{D}_\epsilon$ starting at t and going around all points $z_1, \ldots, z_\mu$.

This operator $T(f_\lambda)$ is now a product of μ simple reflections

$$T(f_\lambda) = s_1 s_2 \ldots s_\mu,$$

- this is because the only critical value 0 of f became μ critical values $z_1, \ldots, z_\mu$ of f_λ.

One can identify the relative (reduced) homology $\tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z})$ with the dual group $\tilde{H}_{N-1}(V_t; \mathbb{Z})^*$, and one defines a map

$$\text{var} : \tilde{H}_{N-1}(V_t, \partial V_t; \mathbb{Z}) \to \tilde{H}_{N-1}(V_t; \mathbb{Z}),$$

called a *variation operator*, which translates to a map

$$L : Q(f_\lambda)^* \xrightarrow{\sim} Q(f_\lambda)$$

("Seifert form") such that the matrix $A(f_\lambda)$ of the bilinear form in the distinguished basis is

$$A(f_\lambda) = L + (-1)^{N-1} L^t,$$

and

$$T(f_\lambda) = (-1)^{N-1} L L^{-t}.$$

A choice of a path q in B connecting 0 with λ, enables one to identify $Q(f)$ with $Q(f_\lambda)$, and $T(f)$ will be identified with $T(f_\lambda)$.

The image $G(f)$ of the monodromy group $G(f_\lambda)$ in $GL(Q(f)) \cong GL(Q(f_\lambda))$ is called the monodromy group of f; it does not depend on a choice of a path q.

2.3 Sebastiani - Thom factorization

If $g \in \mathbb{C}[y_1, \ldots, y_M]$ is another function, the sum, or **join** of two singularities $f \oplus g : \ \mathbb{C}^{N+M} \to \mathbb{C}$ is defined by

$$(f \oplus g)(x, y) = f(x) + g(y)$$

Obviously we can identify

$$\text{Miln}(f \oplus g) \cong \text{Miln}(f) \otimes \text{Miln}(g)$$

Note that the function $g(y) = y^2$ is a unit for this operation.

It follows that the singularities $f(x_1, \ldots, x_N)$ and

$$f(x_1, \ldots, x_N) + x_{M+1}^2 + \ldots + x_{N+M}^2$$

are "almost the same". In order to have good signs (and for other purposes) it is convenient to add some squares to a given f to get $N \equiv 3 \mod (4)$.

The fundamental Sebastiani - Thom theorem, [16], says that there exists a natural isomorphism of lattices

$$Q(f \oplus g) \cong Q(f) \otimes_{\mathbb{Z}} Q(g),$$

and under this identification the full monodromy decomposes as

$$T_{f \oplus g} = T_f \otimes T_g$$

Thus, if

$$\mathrm{Spec}(T_f) = \{e^{\mu_p \cdot 2\pi i/h_1}\},\ \mathrm{Spec}(T_f) = \{e^{\nu_q \cdot 2\pi i/h_2}\}$$

then

$$\mathrm{Spec}(T_{f \oplus g}) = \{e^{(\mu_p h_2 + \nu_q h_1) \cdot 2\pi i/h_1 h_2}\}$$

2.4 Simple singularities

Cf. [1] (a), 15.1. They are:

$$x^{n+1},\ n \geq 1, \tag{A_n}$$

$$x^2 y + y^{n-1},\ n \geq 4 \tag{D_n}$$

$$x^4 + y^3 \tag{E_6}$$

$$xy^3 + x^3 \tag{E_7}$$

$$x^5 + y^3 \tag{E_8}$$

Their names come from the following facts:
— their lattices of vanishing cycles may be identified with the corresponding root lattices;
— the monodromy group is identified with the corresponding Weyl group;
— the classical monodromy T_f is a Coxeter element, therefore its order h is equal to the Coxeter number, and

$$\mathrm{Spec}(T_f) = \{e^{2\pi i k_1/h}, \ldots, e^{2\pi i k_r/h}\}$$

where the integers

$$1 = k_1 < k_2 < \ldots < k_r = h - 1,$$

are the exponents of our root system.

We will discuss the case of E_8 in some details below.

3 Cartan - Coxeter correspondence

3.1 Lattices, polarization, Coxeter elements

Let us call *a lattice* a pair (Q, A) where Q is a free abelian group, and

$$A : Q \times Q \to \mathbb{Z}$$

a symmetric bilinear map ("Cartan matrix"). We shall identify A with a map

$$A : Q \to Q^\vee := Hom(Q, \mathbb{Z}).$$

A polarized lattice is a triple (Q, A, L) where (Q, A) is a lattice, and

$$L : \ Q \xrightarrow{\sim} Q^\vee$$

("variation", or "Seifert matrix") is an isomorphism such that

$$A = A(L) := L + L^\vee \tag{10}$$

where

$$L^\vee : Q = Q^{\vee\vee} \xrightarrow{\sim} Q^\vee$$

is the conjugate to L.

The *Coxeter automorphism* of a polarized lattice is defined by

$$C = C(L) = -L^{-1}L^\vee \in GL(Q). \tag{11}$$

We shall say that the operators A and C are in a *Cartan - Coxeter correspondence.*

Example Let (Q, A) be a lattice, and $\{e_1, \dots, e_n\}$ an ordered $\mathbb{Z}$-base of Q. With respect to this base A is expressed as a symmetric matrix $A = (a_{ij}) = A(e_i, e_j) \in \mathfrak{gl}_n(\mathbb{Z})$. Let us suppose that all a_{ii} are even. We define the matrix of L to be the unique upper triangular matrix (ℓ_{ij}) such that $A = L + L^t$ (in particular $\ell_{ii} = a_{ii}/2$; in our examples we will have $a_{ii} = 2$.) We will call L the *standard polarization* associated to an ordered base. □

Polarized lattices form a groupoid:

an isomorphosm of polarized lattices $f : \ (Q_1, A_1, L_1) \xrightarrow{\sim} (Q_2, A_2, L_2)$ is by definition an isomorphism of abelian groups $f : Q_1 \xrightarrow{\sim} Q_2$ such that

$$L_1(x, y) = L_2(f(x), f(y))$$

(and whence $A_1(x, y) = A_2(f(x), f(y))$).

3.2 Orthogonality

Lemma 1 *(i) (orthogonality)*

$$A(x, y) = A(Cx, Cy).$$

(ii) (gauge transformations) For any $P \in GL(Q)$

$$A(P^\vee LP) = P^\vee A(L)P, \ C(P^\vee LP) = P^{-1}C(L)P.$$

□

3.3 Black/white decomposition and a Steinberg's theorem

Cf. [18], [4]. Let $\alpha_1, \ldots, \alpha_r$ be a base of simple roots of a finite reduced irreducible root system R (not necessarily simply laced).

Let

$$A = (a_{ij}) = (\langle \alpha_i, \alpha_j^\vee \rangle)$$

be the Cartan matrix.

Choose a black/white coloring of the set of vertices of the corresponding Dynkin graph $\Gamma(R)$ in such a way that any two neighbouring vertices have different colours; this is possible since $\Gamma(R)$ is a tree (cf. 5.2).

Let us choose an ordering of simple roots in such a way that the first p roots are black, and the last $r-p$ roots are white. In this base A has a block form

$$A = \begin{pmatrix} 2I_p & X \\ Y & 2I_{r-p} \end{pmatrix}$$

Consider a Coxeter element

$$C = s_1 s_2 \ldots s_r = C_B C_W, \tag{12}$$

where

$$C_B = \prod_{i=1}^{p} s_i, \; C_W = \prod_{i=p+1}^{r} s_i.$$

Here s_i denotes the simple reflection corresponding to the root α_i.

The matrices of C_B, C_W with respect to the base $\{\alpha_i\}$ are

$$C_B = \begin{pmatrix} -I & -X \\ 0 & I \end{pmatrix}, C_W = \begin{pmatrix} I & 0 \\ -Y & -I \end{pmatrix},$$

so that

$$C_B + C_W = 2I - A. \tag{13}$$

This is an observation due to R.Steinberg, cf. [18], p. 591.

We can also rewrite this as follows. Set

$$L = \begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}, \; U = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then $A = L + U$, and one checks easily that

$$C = -U^{-1} L, \tag{14}$$

so we are in the situation 3.1. This explains the name "Cartan - Coxeter coresspondence".

3.4 Eigenvectors' correspondence

Theorem 1 *Let*

$$L = \begin{pmatrix} I_p & 0 \\ Y & I_{r-p} \end{pmatrix}, \ U = \begin{pmatrix} I_p & X \\ 0 & I_{r-p} \end{pmatrix}$$

be block matrices. Set

$$A = L + U, \ C = -U^{-1}L.$$

Let $\mu \neq 0$ be a complex number, $\sqrt{\mu}$ be any of its square roots, and

$$\lambda = 2 - \sqrt{\mu} - 1/\sqrt{\mu}. \quad (15)$$

Then a vector $v_C = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is an eigenvector of C with eigenvalue μ if and only if

$$v_A = \begin{pmatrix} v_1 \\ \sqrt{\mu} v_2 \end{pmatrix}$$

is an eigenvector of A with the eigenvalue λ[1]*.*

Proof: a direct check. □

3.4.1 Remark

Note that the formula (15) gives two possible values of λ corresponding to $\pm\sqrt{\mu}$. On the other hand, λ does not change if we replace μ by μ^{-1}.

In the simplest case of 2×2 matrices the eigenvalues of A are $2 \pm (\sqrt{\mu} + \sqrt{\mu^{-1}})$, whereas the eigenvalues of C are $\mu^{\pm 1}$.

Corollary 1 *In the notations of 3.1, a vector*

$$x = \sum x_j \alpha_j$$

is an eigenvector of A with the eigenvalue $2(1 - \cos\theta)$ iff the vector

$$x_c := \sum e^{\pm i\theta/2} x_j \alpha_j$$

where the sign in $e^{\pm i\theta/2}$ is plus if i is a white vertex, and minus otherwise, is an eigenvector of C with eigenvalue $e^{2i\theta}$.

Cf. [7].

Proof Without loss of generality, we can suppose that A is expressed in a basis of simple roots such that the first $r - p$ ones are white, and the last p roots are black.

Then A has a block form

$$A = \begin{pmatrix} 2I_{r-p} & X \\ Y & 2I_p \end{pmatrix} = \begin{pmatrix} I_{r-p} & 0 \\ Y & I_p \end{pmatrix} + \begin{pmatrix} I_{r-p} & X \\ 0 & I_p \end{pmatrix} = L + U$$

Applying Theorem 1 with

[1] this formulation has been suggested by A.Givental.

$$v_1 = \begin{pmatrix} e^{i\theta/2}x_1 \\ .. \\ e^{i\theta/2}x_{r-p} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} e^{-i\theta/2}x_{r-p+1} \\ .. \\ e^{-i\theta/2}x_r \end{pmatrix}$$

and the well-known eigenvalues of the Cartan matrix A,

$$\lambda = 2 - 2\cos\theta_k, \text{ with } \theta_k = 2\pi k/h, k \in \mathrm{Exp}(R)$$

we obtain : $x_c := \sum e^{\pm i\theta/2} x_j \alpha_j$ is an eigenvector of C with the eigenvalue $e^{2i\theta_k}$ iff $e^{i\theta_k}x = e^{i\theta_k} \sum x_j \alpha_j$ is an eigenvector of A with the eigenvalue $2 - 2\cos\theta_k$. □

3.5 Example: the root systems A_n.

We consider the Dynkin graph of A_n with the obvious numbering of the vertices.

The Coxeter number $h = n + 1$, the set of exponents:

$$\mathrm{Exp}(A_n) = \{1, 2, \dots, n\}$$

The eigenvalues of any Coxeter element are $e^{i\theta_k}$, and the eigenvalues of the Cartan matrix $A(A_n)$ are $2 - 2\cos\theta_k$, $\theta_k = 2\pi k/h$, $k \in \mathrm{Exp}(A_n)$.

An eigenvector of $A(A_n)$ with the eigenvalue $2 - 2\cos\theta$ has the form

$$x(\theta) = (\sum_{k=0}^{n-1} e^{i(n-1-2k)\theta}, \sum_{k=0}^{n-2} e^{i(n-2-2k)\theta}, \dots, 1) \tag{16}$$

Denote by $C(A_n)$ the Coxeter element

$$C(A_n) = s_1 s_2 \dots s_n$$

Its eigenvector with the eigenvalue $e^{2i\theta}$ is:

$$X_{C(A_n)} = (\sum_{k=0}^{n-j} e^{2ik\theta})_{1\le j\le n}$$

For example, for $n = 4$:

$$C_{A_4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \text{ and } X_{C(A_4)} = \begin{pmatrix} 1 + e^{2i\theta} + e^{4i\theta} + e^{6i\theta} \\ 1 + e^{2i\theta} + e^{4i\theta} \\ 1 + e^{2i\theta} \\ 1 \end{pmatrix}$$

is an eigenvector with eigenvalue $e^{2i\theta}$.

Similarly, for $n = 2$:

$$C_{A_2} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \; X_{C(A_2)} = \begin{pmatrix} 1 + e^{2i\gamma} \\ 1 \end{pmatrix}$$

□

4 Sebastiani - Thom product; factorization of E_8 and E_6

4.1 Join product

Suppose we are given two polarized lattices (Q_i, A_i, L_i), $i = 1, 2$.

Set $Q = Q_1 \otimes Q_2$, whence

$$L := L_1 \otimes L_2 : Q \xrightarrow{\sim} Q^\vee,$$

and define

$$A := A_1 * A_2 := L + L^\vee : Q \xrightarrow{\sim} Q^\vee$$

The triple (Q, A, L) will be called the **join**, or **Sebastiani - Thom**, product of the polarized lattices Q_1 and Q_2, and denoted by $Q_1 * Q_2$.

Obviously

$$C(L) = -C(L_1) \otimes C(L_2) \in GL(Q_1 \otimes Q_2).$$

It follows that if $\mathrm{Spec}(C(L_i)) = \{e^{2\pi i k_i/h_i},\ k_i \in K_i\}$ then

$$\mathrm{Spec}(C(L)) = \{-e^{2\pi i(k_1/h_1 + k_2/h_2)},\ (k_1, k_2) \in K_1 \times K_2\} \tag{17}$$

4.2 E_8 versus $A_4 * A_2 * A_1$: elementary analysis

The ranks:

$$r(E_8) = 8 = r(A_4)r(A_2)r(A_1);$$

the Coxeter numbers:

$$h(E_8) = h(A_4)h(A_2)h(A_1) = 5 \cdot 3 \cdot 2 = 30.$$

It follows that

$$|R(E_8)| = 240 = |R(A_4)||R(A_2)||R(A_1)|.$$

The exponents of E_8 are:

$$1, 7, 13, 19, 11, 17, 23, 29.$$

All these numbers, except 1, are primes, and these are all primes ≤ 30, not dividing 30.

They may be determined from the formula

$$\frac{i}{5} + \frac{j}{3} + \frac{1}{2} = \frac{30 + k(i,j)}{30},\ 1 \le i \le 4,\ 1 \le j \le 2,$$

so

$$k(i, 1) = 1 + 6(i - 1) = 1, 7, 13, 19;$$

$$k(i, 2) = 1 + 10 + 6(i - 1) = 11, 17, 23, 29.$$

This shows that the exponents of E_8 are the same as the exponents of $A_4 * A_2 * A_1$.

The following theorem is more delicate.

4.3 Decomposition of $Q(E_8)$

Theorem 2 *(Gabrielov, cf. [8], Section 6, Example 3). There exists a polarization of the root lattice $Q(E_8)$ and an isomorphism of polarized lattices*

$$\Gamma : Q(A_4) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_8). \tag{18}$$

In the left hand side $Q(A_n)$ means the root lattice of A_n with the standard Cartan matrix and the standard polarization

$$A(A_n) = L(A_n) + L(A_n)^t$$

where the Seifert matrix $L(A_n)$ is upper triangular.

In the process of the proof, given in §4.4 - 4.6 below, the isomorphism Γ will be written down explicitly.

4.4 Beginning of the proof

For $n = 4, 2, 1$, we consider the bases of simple roots $e_1, \ldots, e_n$ in $Q(A_n)$, with scalar products given by the Cartan matrices $A(A_n)$.

The tensor product of three lattices

$$Q_* = Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$$

will be equipped it with the "factorizable" basis in the lexicographic order:

$$(f_1, \ldots, f_8) := (e_1 \otimes e_1 \otimes e_1, e_1 \otimes e_2 \otimes e_1, e_2 \otimes e_1 \otimes e_1, e_2 \otimes e_2 \otimes e_1,$$

$$e_3 \otimes e_1 \otimes e_1, e_3 \otimes e_2 \otimes e_1, e_4 \otimes e_1 \otimes e_1, e_4 \otimes e_2 \otimes e_1).$$

Introduce a scalar product (x, y) on Q_* given, in the basis $\{f_i\}$, by the matrix

$$A_* = A_4 * A_2 * A_1.$$

4.5 Gabrielov - Picard - Lefschetz transformations α_m, β_m

Let $(Q, (,))$ be a lattice of rank r. We introduce the following two sets of transformations $\{\alpha_m\}, \{\beta_m\}$ on the set $Bases - cycl(Q)$ of cyclically ordered bases of Q.

If $x = (x_i)_{i \in \mathbb{Z}/r\mathbb{Z}}$ is a base, and $m \in \mathbb{Z}/r\mathbb{Z}$, we set

$$(\alpha_m(x))_i = \begin{cases} x_{m+1} + (x_{m+1}, x_m)x_m & \text{if } i = m \\ x_m & \text{if } i = m+1 \\ x_i & \text{otherwise} \end{cases}$$

and

$$(\beta_m(x))_i = \begin{cases} x_m & \text{if } i = m-1 \\ x_{m-1} + (x_{m-1}, x_m)x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

We define also a transformation γ_m by

$$(\gamma_m(x))_i = \begin{cases} -x_m & \text{if } i = m \\ x_i & \text{otherwise} \end{cases}$$

```
1 − 2 − 3 − 5 − 6 − 7 − 8
        |
        4
```

Fig. 1 Gabrielov's ordering of E_8.

4.6 Passage from $A_4 * A_2 * A_1$ to E_8

Consider the base $f = \{f_1, \dots f_8\}$ of the lattice $Q_* := Q(A_4) \otimes Q(A_2) \otimes Q(A_1)$ described in §4.4, and apply to it the following transformation

$$G' = \gamma_2\gamma_1\beta_4\beta_3\alpha_3\alpha_4\beta_4\alpha_5\alpha_6\alpha_7\alpha_1\alpha_2\alpha_3\alpha_4\beta_6\beta_3\alpha_1, \tag{19}$$

cf. [8], Example 3.

Then the base $G'(f)$ has the intersection matrix given by the Dynkin graph of E_8, with the ordering indicated in Figure 1 below.

This concludes the proof of Theorem 2 □

4.7 The induced map of root sets

By definition, the isomorphism of lattices Γ, (21), induces a bijection between the bases

$$g: \ \{f_1, \dots, f_8\} \xrightarrow{\sim} \{\alpha_1, \dots, \alpha_8\} \subset R(E_8).$$

where in the right hand side we have the base of simple roots, and a map

$$G: \ R(A_4) \times R(A_2) \times R(A_1) \to R(E_8), \ G(x, y, z) = \Gamma(x \otimes y \otimes z)$$

of sets of the same cardinality 240 which is not a bijection however: its image consists of 60 elements.

Note that the set of vectors $\alpha \in Q(E_8)$ with $(\alpha, \alpha) = 2$ coincides with the root system $R(E_8)$, cf. [17], Première Partie, Ch. 5, 1.4.3.

4.8 Passage to Bourbaki ordering

The isomorphism G' (19) is given by a matrix $G' \in GL_8(\mathbb{Z})$ such that

$$A_G(E_8) = G'^t A_* G'$$

where we denoted

$$A_* = A(A_4) * A(A_2) * A(A_1),$$

the factorized Cartan matrix, and A_G denotes the Cartan matrix of E_8 with respect to the numbering of roots indicated on Figure 1.

Now let us pass to the numbering of vertices of the Dynkin graph of type E_8 indicated in [2] (the difference with Gabrielov's numeration is in three vertices $2, 3$, and 4).

The Gabrielov's Coxeter element (the full monodromy) in the Bourbaki numbering looks as follows:

$$C_G(E_8) = s_1 \circ s_3 \circ s_4 \circ s_2 \circ s_5 \circ s_6 \circ s_7 \circ s_8$$

```
1 − 3 − 4 − 5 − 6 − 7 − 8
        |
        2
```

Fig. 2 Bourbaki ordering of E_8.

Lemma 2 *Let $A(E_8)$ be the standard Cartan matrix of E_8 from [B]:*

$$A(E_8) = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Then

$$A(E_8) = G^t A_* G$$

and

$$C_G(E_8) = G^{-1} C_* G$$

where

$$C_* = C(Q(A_4) * Q(A_2) * Q(A_1)) = C(A_4) \otimes C(A_2) \otimes C(A_1),$$

is the factorized Coxeter element, and

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (3.8.1)$$

Here

$$G = G'P$$

where P is the permutation matrix of passage from the Gabrielov's ordering in Figure 1 to the Bourbaki ordering in Figure 2

4.9 Cartan eigenvectors of E_8

To obtain the Cartan eigenvectors of E_8, one should pass from $C_G(E_8)$ to the "black/white" Coxeter element (as in §3.3)

$$C_{BW}(E_8) = s_1 \circ s_4 \circ s_6 \circ s_8 \circ s_2 \circ s_3 \circ s_5 \circ s_7$$

Any two Coxeter elements are conjugate in the Weyl group $W(E_8)$.

The elements $C_G(E_8)$ and $C_{BW}(E_8)$ are conjugate by the following element of $W(E_8)$:

$$C_G(E_8) = w^{-1} C_{BW}(E_8) w$$

where

$$w = s_7 \circ s_5 \circ s_3 \circ s_2 \circ s_6 \circ s_4 \circ s_5 \circ s_1 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3 \circ s_2 \circ s_1 \circ s_2$$

This expression for w can be obtained using an algorithm described in [4], cf. also [3].

Thus, if x_* is an eigenvector of $C_*(E_8)$ then

$$x_{BW} = wG^{-1}x_*$$

is an eigenvector of $C_{BW}(E_8)$. But we know the eigenvectors of $C_*(E_8)$, they are all factorizable.

This provides the eigenvectors of $C_{BW}(E_8)$, which in turn have very simple relation to the eigenvectors of $A(E_8)$, due to Theorem 1.

Conclusion: an expression for the eigenvectors of $A(E_8)$.

Let $\theta = \frac{a\pi}{5},\ 1 \le a \le 4,\ \gamma = \frac{b\pi}{3},\ 1 \le b \le 2,\ \delta = \frac{\pi}{2}$,

$$\alpha = \theta + \gamma + \delta = \pi + \frac{k\pi}{30},$$

$$k \in \{1, 7, 11, 13, 17, 19, 23, 29\}.$$

The 8 eigenvalues of $A(E_8)$ have the form

$$\lambda(\alpha) = \lambda(\theta, \gamma) = 2 - 2\cos\alpha$$

An eigenvector of $A(E_8)$ with the eigenvalue $\lambda(\theta, \gamma)$ is

$$X_{E_8}(\theta,\gamma) = \begin{pmatrix} \cos(\gamma+\theta-\delta) + \cos(\gamma-3\theta-\delta) + \cos(\gamma-\theta-\delta) \\ \cos(2\gamma+2\theta) \\ \cos(2\gamma) + \cos(2\gamma+2\theta) + \cos(2\gamma-2\theta) + \cos(4\theta) + \cos(2\theta) \\ \cos(\gamma+3\theta-\delta) + \cos(\gamma+\theta-\delta) + \cos(-\gamma+3\theta-\delta) \\ 2\cos(2\gamma) + 2\cos(2\gamma+2\theta) + \cos(2\gamma-2\theta) + \cos(2\gamma+4\theta) + \cos(4\theta) + 2\cos(2\theta) + 1 \\ \cos(\gamma+3\theta-\delta) + \cos(\gamma+\theta-\delta) \\ \cos(2\gamma) + \cos(2\theta-2\delta) \\ \cos(\gamma-\theta-\delta) \end{pmatrix}$$

One can simplify it as follows:

$$X_{E_8}(\theta,\gamma) = -\begin{pmatrix} 2\cos(4\theta)\cos(\gamma-\theta-\delta) \\ -\cos(2\gamma+2\theta) \\ 2\cos^2(\theta) \\ -2\cos(\gamma)\cos(3\theta-\delta) - \cos(\gamma+\theta-\delta) \\ -2\cos(2\gamma+3\theta)\cos(\theta) + \cos(2\gamma) \\ -2\cos\theta\cos(\gamma+2\theta-\delta) \\ -2\cos(\gamma+\theta-\delta)\cos(\gamma-\theta+\delta) \\ -\cos(\gamma-\theta-\delta) \end{pmatrix} \tag{20}$$

4.10 Perron - Frobenius and all that

The Perron - Frobenius eigenvector corresponds to the eigenvalue

$$2 - 2\cos\frac{\pi}{30},$$

and may be chosen as

$$v_{PF} = \begin{pmatrix} 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{11\pi}{30} \end{pmatrix}$$

Ordering its coordinates in the increasing order, we obtain

$$v_{PF<} = \begin{pmatrix} \cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{5}\cos\frac{11\pi}{30} \\ 2\cos\frac{\pi}{30}\cos\frac{11\pi}{30} \\ \cos\frac{\pi}{15} \\ 2\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 2\cos^2\frac{\pi}{5} \\ 2\cos\frac{4\pi}{15}\cos\frac{\pi}{5}+\frac{1}{2} \\ 2\cos\frac{2\pi}{30}\cos\frac{\pi}{30} \end{pmatrix}$$

In the Ref. [19], A. B. Zamolodchikov obtains the following expression for the PF vector:

$$v_{Zam}(m) = \begin{pmatrix} m \\ 2m\cos\frac{\pi}{5} \\ 2m\cos\frac{\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 4m\cos\frac{\pi}{5}\cos\frac{2\pi}{15} \\ 4m\cos\frac{\pi}{5}\cos\frac{\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{7\pi}{30} \\ 8m\cos^2\frac{\pi}{5}\cos\frac{2\pi}{15} \end{pmatrix}$$

Setting $m = \cos\frac{11\pi}{30}$, we find indeed :

$$v_{PF<} = v_{Zam}(\cos\frac{11\pi}{30})$$

4.11 Factorization of E_6

Theorem 3 *(Gabrielov, cf. [8], Section 6, Example 2). There exists a polarization of the root lattice $Q(E_6)$ and an isomorphism of polarized lattices*

$$\Gamma_{E_6} : Q(A_3) * Q(A_2) * Q(A_1) \xrightarrow{\sim} Q(E_6). \tag{21}$$

The proof is exactly the same as for $Q(E_8)$. The passage from $A_3 * A_2 * A_1$ to E_6 is obtained by the following transformation

$$G'_{E_6} = \gamma_4\gamma_1\alpha_1\alpha_2\alpha_3\alpha_4\beta_6\beta_3\alpha_1$$

cf. [8], Example 2.

After a passage from Gabrielov's ordering to Bourbaki's, we obtain a transformation

$$G_{E_6} = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in GL_6(\mathbb{Z})$$

such that

$$A(E_6) = G^t_{E_6} A_* G_{E_6} \text{ and } C_G(E_6) = G^{-1}_{E_6} C_* G_{E_6}$$

where $A_* = A(A_3) * A(A_2) * A(A_1)$ and $C_* = C(A_3) \otimes C(A_2) \otimes C(A_1)$ and

$$C_G(E_6) = s_1 \circ s_3 \circ s_4 \circ s_2 \circ s_5 \circ s_6$$

$C_G(E_6)$ is the Gabrielov's Coxeter element in the Bourbaki numbering, cf. [2].

Let $C_{BW}(E_6) = s_1 \circ s_4 \circ s_6 \circ s_2 \circ s_3 \circ s_5$ be the "black/white" Coxeter element. $C_G(E_6)$ and $C_{BW}(E_6)$ are conjugated by the following element of the Weyl group $W(E_6)$:

$$v = s_5 \circ s_3 \circ s_2 \circ s_4 \circ s_1 \circ s_3 \circ s_3 \circ s_1 \circ s_2$$

Thus, if x_* is an eigenvector of $C_*(E_6)$ then $x_{BW} = vG^{-1}_{E_6}x_*$ is an eigenvector of $C_{BW}(E_6)$.

Finally, let $\theta = \frac{a\pi}{4}, 1 \le a \le 3$, $\gamma = \frac{b\pi}{3}, 1 \le b \le 2$, $\delta = \frac{\pi}{2}$ and

$$\alpha = \theta + \gamma + \delta$$

The 6 eigenvalues of $A(E_6)$ have the form $\lambda(\alpha) = \lambda(\theta,\gamma) = 2 - 2\cos\alpha$. An eigenvector of $A(E_6)$ with the eigenvalue $\lambda(\alpha)$ is

$$X_{E_6}(\theta,\lambda) = \begin{pmatrix} \cos(3\gamma + 3\theta - \delta) \\ 2\cos^2\theta \\ -2\cos(3\gamma + 3\theta - \delta)\cos(\gamma + \theta - \delta) \\ -4\cos^2\theta\cos(\gamma + \theta - \delta) \\ 1 - 2\cos(2\gamma + 3\theta)\cos\theta \\ -2\cos(\gamma)\cos(\theta - \delta) \end{pmatrix}$$

5 Givental's *q*-deformations

5.1 *q*-deformations of Cartan matrices

Let $A = (a_{ij})$ be a $n \times n$ complex matrix. We will say that A is a *generalized Cartan matrix* if

(i) for all $i \neq j$, $a_{ij} \neq 0$ implies $a_{ji} \neq 0$;
(ii) all $a_{ii} = 2$.

If only (i) is fulfilled, we will say that A is a *pseudo-Cartan matrix.*

We associate to a pseudo-Cartan matrix A an unoriented graph $\Gamma(A)$ with vertices $1, \ldots, n$, two vertices i and j being connected by an edge $e = (ij)$ iff $a_{ij} \neq 0$.

Let A be a generalized Cartan matrix. There is a unique decomposition

$$A = L + U$$

where $L = (\ell_{ij})$ (resp. $U = (u_{ij})$) is lower (resp. upper) triangular, with 1's on the diagonal.

We define a q-deformed Cartan matrix by

$$A(q) = qL + U$$

This definition is inspired by the q-deformed Picard - Lefschetz theory developed by Givental, [9].

Theorem 4 *Let A be a generalized Cartan matrix such that $\Gamma(A)$ is a tree.*

(i) The eigenvalues of $A(q)$ have the form

$$\lambda(q) = 1 + (\lambda - 2)\sqrt{q} + q \tag{22}$$

where λ is an eigenvalue of A.

(ii) There exist integers $k_1, \ldots, k_n$ such that if $x = (x_1, \ldots, x_n)$ is an eigenvector of A for the eigenvalue λ then

$$x(q) = (q^{k_1/2}x_1, \ldots, q^{k_n/2}x_n) \tag{23}$$

is an eigenvector of $A(q)$ for the eigenvalue $\lambda(q)$.

The theorem will be proved after some preparations.

5.2

Let Γ be an unoriented tree with a finite set of vertices $I = V(\Gamma)$.

Let us pick a root of Γ, and partially order its vertices by taking the minimal vertex i_0 to be the bottom of the root, and then going "upstairs". This defines an orientation on Γ.

Lemma 3 *Suppose we are given a nonzero complex number b_{ij} for each edge $e = (ij), i < j$ of Γ. There exists a collection of nonzero complex numbers $\{c_i\}_{i \in I}$ such that*

$$b_{ij} = c_j/c_i, \ i < j.$$

for all edges (ij).

We can choose the numbers c_i in such a way that they are products of some numbers b_{pq}.

Proof. Set $c_{i_0} = 1$ for the unique minimal vertex i_0, and then define the other c_i one by one, by going upstairs, and using as a definition

$$c_j := b_{ij}c_i, \ i < j.$$

Obviously, the numbers c_i defined in such a way, are products of b_{pq}. □

Lemma 4 *Let $A = (a_{ij})$ and $A' = (a'_{ij})$ be two pseudo-Cartan matrices with $\Gamma(A) = \Gamma(A')$. Set $b_{ij} := a'_{ij}/a_{ij}$. Suppose that*

$$b_{ij} = b_{ji}^{-1}. \tag{24}$$

for all $i \neq j$, and $a_{ii} = a'_{ii}$ for all i. Then there exists a diagonal matrix

$$D = \mathrm{Diag}(c_1, \ldots, c_r)$$

such that $A' = D^{-1}AD$.

Moreover, the numbers c_i may be chosen to be products of some b_{pq}.

Proof. Let us choose a partial order $<_p$ on the set of vertices $V(\Gamma)$ as in 5.2.

Warning. This partial order *differs* in general from the standard total order on $\{1, \ldots, n\}$.

Let us apply Lemma 3 to the collection of numbers $\{b_{ij},\ i <_p j\}$. We get a sequence of numbers c_{ij} such that

$$b_{ij} = c_j/c_i$$

for all $i <_p j$. The condition (24) implies that this holds true for all $i \neq j$.

By definition, this is equivalent to

$$a'_{ij} = c_i^{-1} a_{ij} c_j,$$

i.e. to $A' = D^{-1}AD$. □

5.3 **Proof of Theorem 4.**

Let us consider two matrices: $A(q) = (a(q)_{ij})$ with $a(q)_{ii} = 1 + q$

$$a(q)_{ij} = \begin{cases} a_{ij} & \text{if } i < j \\ qa_{ij} & \text{if } i > j \end{cases}$$

and

$$A'(q) = \sqrt{q}A + (1 - \sqrt{q})^2 I = (a(q)'_{ij})$$

with $a(q)'_{ii} = 1 + q$ and $a(q)'_{ij} = \sqrt{q}a(q)_{ij}$, $i \neq j$.

Thus, we can apply Lemma 4 to $A(q)$ and $A'(q)$. So, there exists a diagonal matrix D as above such that

$$A(q) = D^{-1}A'(q)D.$$

But the eigenvalues of $A'(q)$ are obviously

$$\lambda(q) = \sqrt{q}\lambda + (1 - \sqrt{q})^2 = 1 + (\lambda - 2)\sqrt{q} + q.$$

If v is an eigenvector of A for λ then v is an eigenvector of $A'(q)$ for $\lambda(q)$, and Dv will be an eigenvector of $A(q)$ for $\lambda(q)$. □

5.4 **Remark** (M.Finkelberg)

The expression (22) resembles the number of points of an elliptic curve X over a finite field $\mathbb{F}_q$. To appreciate better this resemblance, note that in all our examples λ has the form

$$\lambda = 2 - 2\cos\theta,$$

so if we set

$$\alpha = \sqrt{q}e^{i\theta}$$

(”a Frobenius root”) then $|\alpha| = \sqrt{q}$, and

$$\lambda(q) = 1 - \alpha - \bar{\alpha} + q,$$

cf. [10], Chapter 11, §1, [11], Chapter 10, Theorem 10.5.

So, the Coxeter eigenvalues $e^{2i\theta}$ may be seen as analogs of ”Frobenius roots of an elliptic curve over $\mathbb{F}_1$”.

5.5 **Examples.**

5.5.1 Standard deformation for A_n

Let us consider the following q-deformation of $A = A(A_n)$:

$$A(q) = \begin{pmatrix} 1+q & -1 & 0 & \dots & 0 \\ -q & 1+q & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & -q & 1+q \end{pmatrix}$$

Then

$$\mathrm{Spec}(A(q)) = \{\lambda(q) := 1 + (\lambda - 2)\sqrt{q} + q |\ \lambda \in \mathrm{Spec}(A(1))\}.$$

If $x = (x_1, \dots, x_n)$ is an eigenvector of $A = A(1)$ with eigenvalue λ then

$$x(q) = (x_1, q^{1/2}x_2, \dots, q^{(n-1)/2}x_n)$$

is an eigenvector of $A(q)$ with eigenvalue $\lambda(q)$.

5.5.2 Standard deformation for E_8

A q-deformation:

$$A_{E_8}(q) = \begin{pmatrix} 1+q & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1+q & 0 & -1 & 0 & 0 & 0 & 0 \\ -q & 0 & 1+q & -1 & 0 & 0 & 0 & 0 \\ 0 & -q & -q & 1+q & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q & 1+q & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -q & 1+q & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -q & 1+q & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q & 1+q \end{pmatrix}$$

Its eigenvalues are

$$\lambda(q) = 1 + q + (\lambda - 2)\sqrt{q} = 1 + q - 2\sqrt{q}\cos\theta$$

where $\lambda = 2 - 2\cos\theta$ is an eigenvalue of $A(E_8)$.

If $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ is an eigenvector of $A(E_8)$ for the eigenvalue λ, then

$$X = (x_1, \sqrt{q}x_2, \sqrt{q}x_3, qx_4, q\sqrt{q}x_5, q^2x_6, q^2\sqrt{q}x_7, q^3x_8) \tag{25}$$

is an eigenvector of $A_{E_8}(q)$ for the eigenvalue $\lambda(q)$.

6 A physicist's appendix

The following is a brief outline of how the above may be related to the physics of certain magnetic systems as observed in a beautiful neutron scattering experiment [20] and anticipated in a pioneering theoretical work [19].

6.1 The material and the model

The paper [20] reports the results of a magnetic neutron scattering experiment on cobalt niobate $CoNb_2O_6$, a material that to a good first approximation can be pictured as an array of uncoupled parallel chains with each site occupied by a spin that may point along or opposite to the "easy magnetization" axis z. In addition to this, each spin is subject to a small external magnetic field h_x in the transverse x direction, on top of which there is also an effective field h_z in the z direction, emerging due to weak action of spins at the neighboring chains. As a result, the effective Hamiltonian $\mathcal{H}$ of such a spin chain may be written as

$$\mathcal{H} = -J\sum_i \sigma_i^z\sigma_{i+1}^z - h_x\sum_i \sigma_i^x - h_z\sum_i \sigma_i^z, \tag{26}$$

where $J > 0$ is the Ising coupling constant, and $\sigma_i^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\sigma_i^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are the Pauli matrices, acting in the Hilbert space of the i-th spin of the chain, spanned by the two states $|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, commonly called "spin-up" and "spin-down". The Hamiltonian (26) acts on the tensor product of the single-spin Hilbert spaces of each spin of the chain.

6.2 The phase diagram

Qualitatively, magnetism of the cobalt niobate has been rather well understood. At low temperatures, the material has two phases. The ferromagnetic phase is realized at temperatures T below $T_C = 2.95$K in zero field, and fields h_x up to $h_c = 5.5$T at zero temperature. In the ground state of this phase, all the spins point the same way along the z axis. Qualitatively, the excited states can be most simply understood at zero field, where the energy cost $2J$ is associated with creating a boundary between two regions of differently oriented spins: a so-called "domain wall". Such a domain wall is indeed an elementary excitation of the ferromagnetic phase of the Ising chain. One may also observe, that flipping a single spin in the ground state is equivalent to creating two domain walls, which then can be separated by an arbitrary distance without incurring any additional energy cost.

The other phase, called the paramagnetic state, can be simply understood in the limit of strong field: here, the Ising coupling can be neglected, the spins decouple from each other and line

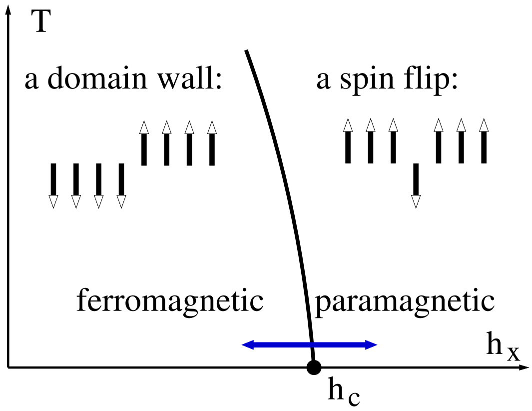

Fig. 3 A sketch of the low-temperature part of the phase diagram of $CoNb_2O_6$ near the critical transverse field $h_x = h_c$. The solid line stemming from the point $h_x = h_c$ at zero temperature is the transition line, separating the ferromagnetic and the paramagnetic phases. The horizontal blue line with two arrows indicates the range of the key neutron scattering measurements [20]. The figure also shows the simplest form of the elementary excitations in each of the two phases, as described in the text: the domain wall and the spin-flip.

up along the field. Even though all the spins point the same way, exactly as in the ferromagnetic state, their decoupling means that, in the paramagnetic phase it is a single spin flip that becomes an elementary excitation with the energy cost of $2h$.

In the Ising chain, subject to a purely transverse magnetic field ($h_z = 0$), the zero-temperature ("quantum") transition from the ferromagnetic to the paramagnetic phases takes place upon increasing the transverse field across the critical value $h_x = h_c$. At this point, the elementary excitations change their nature from domain walls to single spin flips. This has been schematically depicted in the Fig. (3).

This quantum phase transition at $h_x = h_c$ corresponds to the zero-field thermal phase transition at $T = T_C$ in a two-dimensional Ising model on the square lattice (see the Chapter 3 of [21]). As mentioned above, in the low-temperature experiment [20], in addition to the transverse field h_x, there is also a weak effective longitudinal field h_z. The corresponding classical transition is that in a two-dimensional Ising model on a square lattice in a small longitudinal field h_z, which is the subject matter of the Ref. [19].

In a pioneering work [19], A. B. Zamolodchikov showed that the critical ($T = T_C$) two-dimensional Ising model in a weak longitudinal field h_z is described by an integrable quantum field theory, defined via its purely elastic S-matrix. The words "purely elastic" mean that, in an arbitrary n-particle collision, the initial and the final sets of particle energies and momenta coincide. Moreover, the S-matrix of an arbitrary n-particle collision factorizes into the product of $\frac{n(n-1)}{2}$ two-particle S-matrices. The theory involves eight different stable excitations ("particles"), and thus its two-body S-matrix is an 8×8 matrix with elements, corresponding to the scattering amplitudes of the eight particles off each other. The theory has an infinite set of spin-s [2] integrals of motion with s having no common divisor with 30:

$$\{s\} = 1, 7, 11, 13, 17, 19, 23, 29 \ (mod\ 30).$$

These are precisely the exponents of the Lie algebra E_8, and 30 is the Coxeter number of the root system E_8. The sum of the first eight spin degeneracies

$$\sum_{s=1,7,11,\ldots 29} [2s+1] = 248$$

[2] Here, spin-s is meant with respect to the Lorentz group.

is equal to the dimension of the Lie algebra E_8, cf. [12] (a).

At low energies, all the eight particles have relativistic dispersion

$$\epsilon_a(\mathbf{p}) = \sqrt{m_a^2 + \mathbf{p}^2} \tag{27}$$

with different m_a ($a = 1, ..., 8$) called the "particle masses". Among other things, the theory [19] predicts the relative ratios of the m_a as described in the Section 4.10. Since $\epsilon_a(\mathbf{p}) \geq m_a$, the said "mass" m_a is also the energy gap in the a-th branch of the spectrum. Some of these gaps were measured in the neutron scattering experiment [20].

6.3 The neutron scattering experiment [20]

Neutron scattering is an extremely efficient experimental method of studying matter. It amounts to scattering neutrons off the sample to be studied, and is based on conservation laws: if the incident neutron has momentum $\mathbf{p}$ and energy ϵ, and scatters off with momentum $\mathbf{p'}$ and energy ϵ', the energy and momentum conservation laws imply that the difference, called the momentum transfer $\mathbf{q} = \mathbf{p} - \mathbf{p'}$ and energy transfer $\omega = \epsilon - \epsilon'$, have been absorbed by the sample. Similarly, if the scattering neutron changes its spin by a given amount, the said amount must have been taken by the sample.

Now, the key point is that the spin, energy and momentum transferred to the sample cannot be arbitrary, but rather are determined by the allowed states of the system, especially the low-energy states [3].

One of them is the so-called "ground state", that is the one with the lowest energy. For the system in question, there is one and only one such state, and the system assumes it at zero temperature.

The low-energy states (the ground state included) form a linear space, where one may define an orthogonal basis. The basis vectors of this space, other than the ground state, are called "elementary excitations", these states are coupled to each other so weakly that, for all practical purposes they can be considered non-interacting. Which implies that, together with the ground state, they can be viewed as eigenvectors of a Hermitian matrix (the effective Hamiltonian), with the corresponding eigenvalues being the energy of the states.

The system in question involves an underlying crystal lattice, and thus is invariant under the group of discrete spatial translations. Thus each eigenstate can be labeled by its momentum $\mathbf{p}$, which allows one to characterize the energy spectrum by a "dispersion law" $\epsilon = \epsilon(\mathbf{p})$ that defines the energy ϵ of an eigenstate (elementary excitation) with momentum $\mathbf{p}$, such as the dispersion law (27).

In the experiment of R. Coldea et al., the material sample was studied by means of magnetic neutron scattering. Neutron has spin-1/2, and zero charge. In a magnetic scattering act with momentum transfer $\mathbf{q}$ and energy transfer ω, a neutron flips its spin and, by virtue of the conservation laws, produces in the sample a spin flip with a given momentum $\mathbf{q}$ and energy ω.

If such a spin flip were an elementary excitation, then the scattering would be present only for its energies ω and momenta $\mathbf{q}$ such that satisfy the dispersion law $\omega = \epsilon(\mathbf{q})$. In other words, the neutron scattering intensity $\mathcal{S}(\mathbf{q}, \omega)$ would differ from zero only for $\omega = \epsilon(\mathbf{q})$. However, a spin flip does not necessarily correspond to a single elementary excitation, but rather to a superposition thereof. As a consequence, the experimentally measured neutron scattering intensity $\mathcal{S}(\mathbf{q}, \omega)$ may be non-zero for momentum transfers $\mathbf{q}$ and energy transfers ω outside the dispersion law $\omega = \epsilon(\mathbf{q})$. Put otherwise, this means that the dispersion $\omega = \epsilon(\mathbf{q})$ acquires a "width". Nevertheless, the

[3] The adjective "low" implies "by comparison with other relevant energy scales".

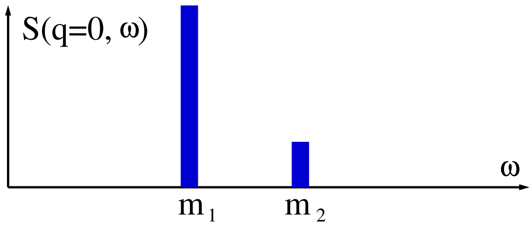

Fig. 4 A sketch of the zone center (zero-momentum) scattering intensity $\mathcal{S}(\mathbf{q}=0,\omega)$ peaks, centered at $\omega = m_1$ and $\omega = m_2$ and resolved in the experiment [20]. The found mass ratio m_2/m_1 is consistent with $\frac{m_2}{m_1} = \frac{1+\sqrt{5}}{2}$, as per the expression for the $v_{Zam}(m)$ in the Subsection 4.10.

scattering signal has a peak near $\omega = \epsilon(\mathbf{q})$, which allows experimentalists to extract the dispersion of the excitations.

In the Eq. (27) the $\epsilon_a(\mathbf{p}=\mathbf{0}) = m_a$, the zone center (zero-momentum) neutron scattering intensity $\mathcal{S}(\mathbf{q}=0,\omega)$ would be expected to have peaks at $\omega = m_a$. At the lowest temperatures, and in the immediate vicinity of $h_x = h_c$, the experiment [20] succeeded to resolve the first two excitations of the Eq. (27) and to extract their masses m_1 and m_2, as shown schematically in the Fig. 4. The mass ratio m_2/m_1 was found to be $\frac{m_2}{m_1} = 1.6 \pm 0.025$, consistent with $\frac{m_2}{m_1} = \frac{1+\sqrt{5}}{2} \approx 1.618$, predicted by the expression for the $v_{Zam}(m)$ in the Subsection 4.10.

A reader who would like to find out more about various facets of the story may turn to the references [22,23,24,25].

Acknowledgements We are grateful to Misha Finkelberg, Andrei Gabrielov, and Sabir Gusein-Zade for the inspiring correspondence, and to Patrick Dorey for sending us his thesis. Our special gratitude goes to Sasha Givental whose remarks enabled us to generalize some statements and to simplify the exposition. The third author thanks MPI in Bonn for hospitality; he was supported in part by NSF grant DMS-1362924 and the Simons Foundation.

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