Normal forms of analytic perturbations of quasihomogeneous vector fields : Rigidity, analytic invariants sets and exponentially small approximation

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This article is dedicated to Bernard Malgrange on the occasion of his 80th birthday

Abstract

In this article, we study germs of holomorphic vector fields which are "higher order" perturbations of a quasihomogeneous vector field in a neighborhood of the origin of $\mathbb{C}^n$, fixed point of the vector fields. We define a "diophantine condition" on the quasihomogeneous initial part S which ensures that if such a perturbation of S is formally conjugate to S then it is also holomorphically conjugate to it. We study the normal form problem relatively to S. We give a condition on S that ensure that there always exists an holomorphic transformation to a normal form. If this condition is not satisfied, we also show, that under some reasonable assumptions, each perturbation of S admits a Gevrey formal normalizing transformation. Finally, we give an exponentially good approximation by a partial normal form.

1 Introduction

The aim of this article is to study germs of holomorphic vector fields in a neighborhood of a fixed point, say 0, in $\mathbb{C}^n$. Lot of work is devoted to this problem mainly when the vector field is not too degenerate, that is when not all the eigenvalues of the linear part $DX(0)$ of X at the origin are zero. In this situation, the aim is to compare the vector field to its linear part. One way to achieve this, is to transform the vector field "as close as possible", in some sense, to its linear part by mean of regular change of variables.

In this article we shall focus on vector fields which are degenerate and which may not have of nonzero linear part at the origin.

We shall be given a "reference" polynomial vector field S to which we would like to compare a suitable perturbation of it. This means that we would like to know if some of the geometric or dynamical properties of the model can survive for the perturbation. For instance, the model $S_1 = y\frac{\partial}{\partial x}$ and $S_2 = y\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$ are quite different although they have

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the same linear part at the origin of $\mathbb{C}^2$. In fact, for S_1, each point of $\{y = 0\}$ is fixed whereas the "cusp" $\{2x^3 - 3y^2 = 0\}$ is globally invariant by S_2.

In this article, we shall assume that the **unperturbed vector field S is quasihomogeneous** with respect to some weight $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$. This means that each variable x_i has the weight p_i while $\frac{\partial}{\partial x_i}$ has the weight $-p_i$. Hence, the monomial x^Q is quasihomogeneous of quasidegree $(Q, p) := \sum_{i=1}^n q_i p_i$. In particular, the vector field $S = \sum_{i=1}^n S_i(x) \frac{\partial}{\partial x_i}$ is quasihomogeneous of quasidegree s if and only if S_i is a quasihomogeneous polynomial of degree $s + p_i$.

We shall then consider a germ of **holomorphic vector field X which is a good perturbation of a quasihomogeneous vector field** S, this means that the smallest quasidegree of nonzero terms in the Taylor expansion of $X - S$ is greater than s. In the homogeneous case ($p = (1, \ldots, 1)$), a linear vector field S is quasihomogeneous of degree 0 and a good perturbation is a nonlinear perturbation of S (i.e. the order at 0 of the components of $X - S$ is greater or equal than 2).

We shall develop an approach of these problems through **normal forms**. By this, we mean that the group of germs of holomorphic diffeomorphisms (biholomorphisms) of $(\mathbb{C}^n, 0)$ acts on the space of vector fields by conjugacy : if X (resp. Φ) is a germ of vector field (resp. biholomorphism) at 0 of $\mathbb{C}^n$, then the conjugacy of X by Φ is $\Phi_* X(y) := D\Phi(\Phi^{-1}(y))X(\Phi^{-1}(y))$. A normal form is special representant of this orbit which satisfy some properties. Although, the formal normal form theory of vector fields which are nonlinear perturbations of a semi-simple (resp. nilpotent, general) linear vector field is well known [Arn80] (resp. [CS86, Bel79, Mur03]), it is much more difficult to handle the problem when the vector field doesn't have a nonzero linear part. It might also be useful in problem with parameters to consider some of the parameters as a variable with a prescribed weight.

First of all, we shall define a special hermitian product $\langle ., . \rangle_{p,\delta}$ on each space $\mathcal{H}_\delta$ of quasihomogeneous vector fields of quasidegree δ (see (3)). It's main property is that the associated norm of a product is less or equal than the product of the norm. Let us consider the **cohomological operator** :

$$\begin{array}{rcl} d_0 : \mathcal{H}_\delta & \to & \mathcal{H}_{s+\delta} \\ U & \mapsto & [S, U] \end{array}$$

where $[., .]$ denotes the usual Lie bracket of vector fields. We emphasize that, contrary to the case where S is linear ($s = 0$), d_0 doesn't leave $\mathcal{H}_\delta$ invariant. Let $d_0^* : \mathcal{H}_{\delta+s} \to \mathcal{H}_\delta$ be the **adjoint** of d_0 with respect to the hermitian product. An element of the kernel of this operator will be called **resonant or harmonic**. The first result we have is the following :

Formal normal form transformation (see proposition 4.4) :
There exists a formal change of coordinates tangent to Id at the origin, such that, in the new coordinates, $X - S$ is resonant.

This means that there exists $\hat{\Phi} \in (\mathbb{C}[[x_1, \ldots, x_n]])^n$ such that $\hat{\Phi}(0)$ and $D\hat{\Phi}(0) = \mathrm{Id}$ and $d_0^*(\hat{\Phi}_* X - S) = 0$. When S is linear, this corresponds to classical normal forms [Arn80, Mur03]. In the homogeneous case, the first result in this direction is due to G. Belitskii [Bel79, Bel82] using a different scalar product. In the quasihomogeneous case, a general

scheme has been devised by H. Kokubu and al.[KOW96] to obtain a unique normal form. This scheme can be combined with our definition.

One of the main novelty of this article is to consider the **Box operator**

$$\begin{aligned} \Box_\delta : \mathcal{H}_\delta &\to \mathcal{H}_\delta \\ U &\mapsto \Box_\delta(U) := d_0 d_0^*(U) \end{aligned}$$

which is self-adjoint and which spectrum is non-negative. **Its nonzero spectrum is composed of the (squared) small divisors** of the problem. These are the numbers that we need to control. For instance in the homogeneous case, if $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, then the eigenvalues of $\Box_{k-1}$ is the set of $|(Q,\lambda) - \lambda_i|^2$, where $Q \in \mathbb{N}^n$, $|Q| = k$ and $1 \leq i \leq n$.

For each quasidegree $\lambda > s$, let us set

$$a_\delta := \min_{\lambda \in \mathrm{Spec}(\Box_\delta)\setminus\{0\}} \sqrt{\lambda}.$$

Then, we shall construct inductively a sequence of positive numbers η_δ from the a_δ's (see (12)). We shall say that S **is diophantine** if there exists positive constants M, c such that $\eta_\delta \leq M c^\delta$. Being diophantine a quantitative way of saying the the sequence $\{a_\delta\}$ doesn't accumulate the origin too quickly. Hence, we have defined a **small divisors condition for quasihomogeneous vector fields**. For instance in the homogeneous case, $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$ is diophantine if it satisfies Brjuno's small divisors condition [Bru72] :

$$(\omega) \qquad -\sum_{k\geq 1} \frac{\ln \omega_k}{2^k} < +\infty,$$

where

$$\omega_k := \inf\{|(Q,\lambda) - \lambda_i| \neq 0, Q \in \mathbb{N}^n, 2 \leq |Q| \leq 2^k, 1, \leq i \leq n\}.$$

Rigidity theorem (see theorem 5.8**) :**
In the general quasihomogeneous case, assume that the quasihomogeneous vector field S is diophantine. Let X be a good holomorphic deformation of S. If X is formally conjugate to S then it is holomorphically conjugate to it.

For instance in the homogeneous case and if $S = \sum_{i=1}^n \lambda_i x_i \frac{\partial}{\partial x_i}$, this is the classical Siegel-Brjuno linearization theorem : if S satisfies the diophantine condition (ω) and if an holomorphic nonlinear perturbation X is formally linearizable, then X is holomorphically linearizable.

Assume that the ring of polynomial first integrals of S is generated by some quasihomogeneous polynomials $h_1, \ldots, h_r$. Let us denote $\mathcal{I}$ (resp. $\widehat{\mathcal{I}}$) the ideal they generate in the ring of germs of holomorphic functions at the origin (resp. formal power series). The germ of the variety $\Sigma = \{h_1 = \cdots = h_r = 0\}$ at the origin is invariant by the flow of S. Does a good perturbation of S still have an invariant variety of this kind ?

Invariant variety theorem (see theorem 5.6**) :**
In the general quasihomogeneous case, assume that the quasihomogeneous vector field S is diophantine. Let X be a good holomorphic deformation of S. Essentially, if X is formally

conjugate to S modulo $\widehat{\mathcal{I}}$ then it is holomorphically conjugate to S modulo $\mathcal{I}$.

This means that there exists a germ of holomorphic diffeomorphism Φ such that

$$\Phi_* X = S + \sum_{i=1}^{n} g_i(x) \frac{\partial}{\partial x_i}, \quad \text{with } g_i \in \mathcal{I}.$$

Hence, in the new holomorphic coordinate system, Σ is an invariant variety of X since $g_{i|\Sigma} = 0$. The diophantine condition can eventually be relaxed a little bit taking into account the ideal $\mathcal{I}$. For instance in the homogeneous case and if $S = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$, this was proved by L. Stolovitch [Sto94].

What happens if instead of accumulating the origin, the sequence of the spectrum a_δ tends to infinity with δ ? Let us set $\nu := \max\left(1, \frac{\max p_i}{2}\right)$.

Poincaré's domain like theorem (see theorem 6.2) :
Assume that there exists a constant M such that for all $\delta > s$,

$$\min_{\lambda \in Spec(\Box_\delta)\setminus\{0\}} \sqrt{\lambda} \geq M(\delta - s)^\nu.$$

Then, any holomorphic good perturbation of S is holomorphically conjugate to a normal form.

For instance in the homogeneous case, if $S = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$ belongs to the Poincaré's domain [Arn80] then the convex hull of the λ_i in the complex plane doesn't contain the origin. This implies that $|(Q, \lambda)| \geq \varepsilon |Q|$ from which we infer that $|(Q, \lambda) - \lambda_i| \geq \varepsilon' |Q|$ if $|Q|$ is large enough. We refer to [Sto00, Sto08] for recent results and overview about the problem of holomorphic conjugacy to a normal form when S is a linear diagonal vector field.

Let $\hat{f} = \sum_{Q \in \mathbb{N}^n} f_Q x^q$ be a formal power series of $\mathbb{C}^n$ and $\alpha > 0$. We say that $\hat{f}$ is α-Gevrey if for all $Q \in \mathbb{N}^n$, $|f_Q| \leq M c^{|Q|} (|Q|!)^\alpha$. As we know from the linear diagonal case, normalizing transformations (that is formal transformation to a normal form) usually diverge. How bad can be this divergence ? We show that is the spectrum of $\Box$ is of Siegel type, then, at worst, there exists a **formal Gevrey normalizing transformation** :

Gevrey formal normal form theorem (see theorem 6.4):
Assume that there exists a positive constants M and nonnegative τ such that for all $\delta > s$,

$$\min_{\lambda \in Spec(\Box_\delta)\setminus\{0\}} \sqrt{\lambda} \geq \frac{M}{(\delta - s)^\tau}.$$

Then any good holomorphic perturbation of S admits a formal $\bar{p}(\tilde{b} + \tau)$-Gevrey normalizing transformation to a $\bar{p}(\tilde{b} + \tau)$-Gevrey formal normal form. Here, $\bar{p} = \max_i p_i$ and $\tilde{b}$ is a positive number depending only on p.

In the homogeneous case with S a linear vector field, this result was proved (but not stated!) by G. Iooss and E. Lombardi [IL05]. This kind of result was obtained in very peculiar case. Namely, in the case of two-dimensional saddle-node (resp. resonant saddle), X is a suitable perturbation of $x\frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y}$ (resp. $px\frac{\partial}{\partial x} - qx\frac{\partial}{\partial x} + x^q y^{p+1} \frac{\partial}{\partial y}$) (these are not

quasihomogeneous), the Gevrey character with respect to y (resp. to the monomial $x^q y^p$) was obtained by J. Ecalle [Eca], J. Martinet and J.-P. Ramis [MR82, MR83, Mal82] and S. Voronin [Vor81] (see also [IY08] for a general overview). In this case, there is no small divisor (i.e. $\tau = 0$). For general n-dimensional 1-resonant saddle, there are usually small divisors; the results was devised by J. Ecalle [Eca92], by L. Stolovitch [Sto96] and B. Braaksma and L. Stolovitch [BS07]. In the case of the "cusp", $S = 2y\frac{\partial}{\partial x} + 3x^2\frac{\partial}{\partial y}$ ($p = (2, 3)$), a normal form of vector fields tangent to the cusp was given by F. Loray. A very precise study of this case with sharp estimates of the Gevrey order was done by M. Canalis-Durand and R. Schäfke [CDS04]. T. Gramchev and M. Yoshino studied the cohomological equation (i.e. the linearized equation of the conjugacy equation) of a pair of commuting 4-dimensional vector fields having linear part with a Jordan block [YG08].

By applying a polynomial change of coordinates $\Psi_{\delta-s}$ of some quasidegree $\delta - s$, one can transform the perturbation X into a normal form $S + N_\delta$ up to some quasiorder δ, that is $(\Psi_{\delta-s})_* X - (S+N_\delta)$ is of quasiorder greater than δ. Hence, the norm of $(\Psi_{\delta-s})_* X - (S+N_\delta)$ on a ball of radius ε centered at the origin is bounded by a power of ε. Nevertheless, the formal normalizing diffeomorphism Φ we obtained from the previous theorem allow us to obtain much better estimate, that is an exponentially small estimate. Namely, let us consider the "twisted ball" $\tilde{B}_\varepsilon = \{(\sum_{i=1}^n p_i |x_i|^{2/p_i})^{1/2} < \varepsilon\}$.

Exponentially small approximation by a partial normal form theorem (see theorem 6.11**):**
For each $\varepsilon > 0$ sufficiently small, there exists a quasidegree δ_{opt} such that

$$\left\| (\Phi_{\delta_{opt}})_* X - \left(S + \mathcal{N}_{\delta_{opt}} \right) \right\|_{qh,\varepsilon} < M \exp\left(-\frac{A}{\varepsilon^b} \right)$$

for some exponent b that depends on τ, the order of small divisors. Here, $\|X\|_{gh,\varepsilon}$ is a "twisted norm" of the vector field X that measure its size on the twisted ball $\tilde{B}_\varepsilon$.

Some of these results were announced in [LS09].

2 Notations

Let us set some notation which will be used throughout this article.

- $\widehat{\mathcal{X}}_n$ denotes the $\mathbb{C}$-space of formal vector fields on $\mathbb{C}^n$,
- $\mathcal{X}_n$ denotes the $\mathbb{C}$-space of germs of holomorphic vector fields on $(\mathbb{C}^n, 0)$,
- $\widehat{\mathcal{O}}_n$ denotes the ring of formal power series in $\mathbb{C}^n$,
- $\mathcal{O}_n$ denotes the ring of germs at 0 of holomorphic functions in $\mathbb{C}^n$.

Let $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$. Let $|Q| := q_1 + \cdots + q_n$ be the length of Q. As usual, if $x = (x_1, \cdots, x_n)$, x^Q denotes the monomial $x_1^{q_1} \cdots x_n^{q_n}$. Let $n, k \in \mathbb{N}$ with $k \leq n$, we denotes by $C_n^k := \frac{n!}{k!(n-k)!}$ the binomial coefficients.

3 Quasihomogeneous vector fields and polynomials

3.1 Definitions and notations

Let $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$ be such that the largest common divisor of its components $p_1 \wedge \cdots \wedge p_n$ is equal to 1. Let us denote by

$$R_p := \sum_{i=1}^{n} p_i x_i \frac{\partial}{\partial x_i}$$

the p-**radial** vector field $\mathbb{C}^n$. Let $Q = (q_1, \ldots, q_n) \in \mathbb{N}^n$. Let (Q, p) stand for $\sum_{i=1}^{n} q_i p_i$. A polynomial will be called **quasihomogeneous** of degree δ if it can written as a finite sum

$$\sum_{(Q,p)=\delta} p_Q x^Q.$$

with complex coefficients. It is equivalent to say that the Lie derivative $R_p(f) := \sum_{i=1}^{n} p_i x_i \frac{\partial f}{\partial x_i} = \delta f$ since $R_p(x^Q) = (Q, p) x^Q$. The integer $\delta = (Q, p)$ is the p-degree of quasihomogeneity (or p-**quasidegree**) of x^Q. **When there is no confusion possible, we shall omit the reference to p, which is fixed once for all.** Let us define

$$\bar{p} := \max_{1 \leq i \leq n} p_i \quad \underline{p} := \min_{1 \leq i \leq n} p_i.$$

Let us define Δ to be the totally ordered set of p-quasihomogeneity degree of polynomials; that is to say $\Delta = \{\delta_1, \delta_2, \delta_3, \ldots\}$ where $\delta_1 < \delta_2 < \delta_3 < \cdots$. It is the set

$$\Delta = \{d \in \mathbb{N} / \ d = (\alpha, p)\,, \text{ with } \alpha \in \mathbb{N}^n\}.$$

An element of Δ will be called a **quasidegree**.

For $\delta \in \Delta$, we shall denote by $\mathcal{P}_\delta$ the vector space of p-quasihomogeneous polynomials of degree δ. If $\delta \notin \Delta$, we set $\mathcal{P}_\delta := \{0\}$. Hence, for any $\delta \in \mathbb{N}$,

$$\mathcal{P}_\delta := \left\{ f \in \mathbb{C}[x],\ f(x) = \sum_{(Q,p)=\delta} f_Q x^Q \right\} \quad \text{if } \delta \in \Delta,\ \mathcal{P}_\delta := \{0\} \quad \text{otherwiwse.}$$

A vector field $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ is **quasihomogeneous** of quasidegree $\delta \geq 0$ if, for each $1 \leq i \leq n$, X_i is belongs to $\mathcal{P}_{\delta + p_i}$. It is equivalent to say that $[R_p, X] = \delta X$ where $[.,.]$ denotes the Lie bracket. In other words, x_i has weight p_i and $\frac{\partial}{\partial x_i}$ has weight $-p_i$.

We shall denote by $\tilde{\Delta}$ the totally ordered set of p-quasihomogeneity degree of non zero polynomial vector fields. As a set, we have the quality

$$\tilde{\Delta} = \{\widetilde{\delta} \in \mathbb{Z} / \widetilde{\delta} = \delta - p_i, \text{with } \delta \in \Delta, 1 \leq i \leq n\}.$$

For $\delta \in \tilde{\Delta}$, we shall denote by $\mathcal{H}_\delta$ the complex vector space of p-quasihomogeneous polynomials of quasidegree δ. If $\delta \notin \tilde{\Delta}$, we shall set $\mathcal{H}_\delta := \{0\}$.

Remark 3.1. *Observe that if $\delta \in \widetilde{\Delta}$, then there exists $1 \leq j_0 \leq n$ such that $p_{j_0} + \delta \in \Delta$, but it may happen that for some $1 \leq j \leq n$, $p_j + \delta \notin \Delta$ which simply means that any polynomial vector fields $X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$ belonging in $\mathcal{H}_\delta$ has a j-th component X_j which is equal to 0.*

Remark 3.2. *There is only a finite number of elements of $\tilde{\Delta}$ which are negative. In fact, if $\delta \in \tilde{\Delta}$, then $\delta \geq -p_i$ for some i.*

The sets Δ and $\widetilde{\Delta}$ do not contain in general all the integers. However we have the following lemma (inspired by a remark of J.-C Yoccoz) :

Lemma 3.3. *Let $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$ as above.*

(a) *There exists δ_0 such that for every $\delta \geq \delta_0$, δ belongs to Δ.*

(b) *We have $\widetilde{\Delta} \supset \Delta$.*

(c) *Δ is stable by multiplication by any nonnegative integer and by addition which is usually not true for $\widetilde{\Delta}$.*

Proof. The following proof of (a) is due to Marc Revesat : let $N > 0$ be an integer. Then, we can write it as $N = p_1 u_1 + ... + p_n u_n$, where the u_i's are integers. For all i, there exists an integer k_i such that $0 \leq p_i u_i + k_i p_1 ... p_n < p_1 ... p_n$. Let us set $v_i = p_1 ... p_{i-1} p_{i+1} p_n$, $k = k_1 + ... + k_n$. Hence we have : $N + k p_1 ... p_n = p_1 v_1 + ... + p_n v_n$ with $0 \leq p_i v_i < p_1 ... p_n$.

Let us assume that $N \geq n p_1 ... p_n$. Therefore, according to previous computations, we have $N + k p_1 ... p_n = p_1 v_1 + ... + p_n v_n < n p_1 ... p_n$. Hence, k is negative. We obtain the result by changing, for instance, v_1 in $v_1 - k p_2 ... p_n$. Then observing that for any $\delta \in \Delta$, $\delta = (\alpha, p) = (\alpha + e_j, p) - pj$ where $\alpha \in \mathbb{N}^n$ and e_j is the j-th vector of the canonical basis of $\mathbb{R}^n$, we get that $\widetilde{\Delta} \supset \Delta$ holds. Finally statement(c) readily follows from the definition of Δ. □

Proposition 3.4. *Let $k, \ell \in \mathbb{Z}$ be two integers.*

(a) *Let f, g be two quasihomogeneous polynomials belonging respectively to $\mathcal{P}_k$ and $\mathcal{P}_\ell$. Then, fg belongs to $\mathcal{P}_{k+\ell}$.*

(b) *Let f be a quasihomogeneous polynomial belonging to $\mathcal{P}_k$ and let X be a quasihomogeneous polynomial vector field belonging to $\mathcal{H}_\ell$. Then,*

   (i) *the Lie derivative $X(f)$ belongs to $\mathcal{P}_{k+\ell}$;*

   (ii) *fX belongs to $\mathcal{H}_{k+\ell}$.*

(c) *let S, U be two quasihomogeneous vector fields belonging to respectively in $\mathcal{H}_k$ and $\mathcal{H}_\ell$. Then,*

   (i) *$DS.U$ belongs to $\mathcal{H}_{k+\ell}$;*

   (ii) *the Lie bracket $[S, U]$ belongs to $\mathcal{H}_{k+\ell}$.*

Proof. The proof readily follows from the definition of $\mathcal{P}_k$ and $\mathcal{H}_\ell$ observing that if f lies in $\mathcal{P}_k$ then $\frac{\partial f}{\partial x_j}$ lies in $\mathcal{P}_{k-pj}$. □

3.2 Decomposition of functions and vector fields as sum of homogeneous and quasi homogeneous components.

Let $f \in \mathbb{C}[[x_1, \cdots, x_n]]$ be a formal power series function. Hence f reads

$$f(x) = \sum_{Q \in \mathbb{N}^n} f_Q \; x^Q \qquad \text{where} \quad f_Q \in \mathbb{C}.$$

Classically, f admits a unique decomposition as a sum of homogeneous polynomials, $f_{\bullet,r}$, of different degree r, i.e.

$$f = \sum_{r \geq 0} f_{\bullet,r} \qquad \text{where} \quad f_{\bullet,r}(x) = \sum_{|Q|=r} f_Q \; x^Q.$$

where $|Q| = q_1 + \cdots + q_n$. In a similar way, f admits a unique decomposition as a sum of quasihomogeneous polynomials f_δ of different quasidegree δ, i.e.

$$f = \sum_{\delta \in \Delta} f_\delta \qquad \text{with} \quad f_\delta(x) = \sum_{(Q,p)=\delta} f_Q \; x^Q.$$

We shall say that f is of p-**order** δ_0 if $f_{\delta_0} \neq 0$ and $f_\delta = 0$ for all quasidegree $\delta < \delta_0$. Let μ be a quasidegree. We shall define the μ-**quasijet** of f to be

$$J^\mu(f) := \sum_{\delta \in \Delta,\, \delta \leq \mu} f_\delta.$$

Furthermore, if f is a germ of holomorphic function at the origin of $\mathbb{C}^n$, we will denote by $\{f\}_\mu := f_\mu$ the quasihomogeneous component of degree μ in the Taylor expansion of f at the origin.

Finally, f admits a unique decomposition as a sum of polynomials $f_{\delta,r}$ which are simultaneously quasihomogeneous of quasidegree δ and homogeneous of degree r, i.e.

$$f = \sum_{\delta \in \Delta} \sum_{\frac{\delta}{\overline{p}} \leq r \leq \frac{\delta}{\underline{p}}} f_{\delta,r} \qquad \text{with} \quad f_{\delta,r}(x) = \sum_{\substack{|Q|=r \\ (Q,p)=\delta}} f_Q \; x^Q.$$

In the last decomposition of f, r belongs to $\left[\frac{\delta}{\overline{p}}, \frac{\delta}{\underline{p}}\right]$, since for every $Q \in \mathbb{N}^n$, $\underline{p}|Q| \leq (Q,p) \leq \overline{p}|Q|$.

Any formal vector field V can be written as an element of $(\mathbb{C}[[x_1, \cdots, x_n]])^n$. Hence it can be decomposed along the quasihomogeneous filtration :

$$V = \sum_{\delta \in \widetilde{\Delta}} V_\delta$$

where V_δ is a quasihomogeneous vector field of quasidegree δ. By definition, we have $V_\delta = \sum_{i=1}^n V_{i,\delta} \frac{\partial}{\partial x_i}$ with $V_{i,\delta} \in \mathcal{P}_{\delta+p_i}$. We recall that $\mathcal{P}_{\delta+p_j}$ is equal to $\{0\}$ when $\delta + p_j \notin \Delta$.

Moreover, each quasihomogeneous component V_δ can be decomposed into homogeneous components $V_{\delta,r}$ of degree r:

$$V_\delta = \sum_{\delta_* \le r \le \delta^*} V_{\delta,r} \text{ with } V_{j,\delta,r}(x) = \sum_{\substack{|Q|=r \\ (Q,p)=\delta+p_j}} V_{j,Q}\ x^Q$$

where

$$\delta_* := \frac{\min\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\bar{p}} \text{ and } \delta^* := \frac{\max\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\underline{p}}. \tag{1}$$

Moreover, for any $q \ge 1$ and for any homogeneous polynomial $\phi \in (\mathbb{C}[x_1, \cdots, x_n])^q$ of degree r, there exists a unique r-linear, symmetric, operator $\widetilde{\phi} : (\mathbb{C}^n)^r \to \mathbb{C}^q$ such that $\widetilde{\phi}(\underbrace{x, \cdots, x}_{r \text{ times}}) = \phi(x)$ where $x = (x_1, \cdots, x_n)$ (see for instance the book of Cartan [Car67]). Moreover, for every $x^{(\ell)} \in \mathbb{C}^n$ with $1 \le \ell \le r$, $\widetilde{\phi}$ is given by

$$\widetilde{\phi}(x^{(1)}, \cdots, x^{(r)}) = \frac{1}{r!} D^r_x \phi(0).[x^{(1)}, \cdots, x^{(r)}] = \frac{1}{r!} \Delta_{x^{(1)}} \cdots \Delta_{x^{(r)}} \phi$$

where $\Delta_h \phi(x) = \phi(x+h) - \phi(x)$ and where one check that $\Delta_{x^{(1)}} \cdots \Delta_{x^{(r)}} \phi(x)$ does not depend on x [Car67].

The homogeneous and quasihomogeneous components of sum, products and derivatives of formal functions and vector fields can be computed with the standard rules (see Lemma A.1, A.2 in appendix A). Computation of quasihomogeneous components of the composition of a function or a vector field by a vector field is given by the following lemma :

Lemma 3.5 (Components of the composition)**.** *Let $f \in \mathbb{C}[[x_1, \cdots x_n]]$ and $U, V \in (\mathbb{C}[[x_1, \cdots x_n]])^n$. Then,*

(a) $$\{f \circ U\}_{\delta'} = \sum_{\substack{\delta \le \delta',\ \frac{\delta}{\bar{p}} \le r \le \frac{\delta}{\underline{p}} \\ \delta + \delta_1 + \cdots + \delta_r = \delta'}} \widetilde{f}_{\delta,r}(U_{\delta_1}, \cdots U_{\delta_r}),$$

(b) $$\{V \circ U\}_{\delta'} = \sum_{\substack{\delta \le \delta',\ \delta_* \le r \le \delta^* \\ \delta + \delta_1 + \cdots + \delta_r = \delta'}} \widetilde{V}_{\delta,r}(U_{\delta_1}, \cdots U_{\delta_r}).$$

where δ_* and δ^* are defined in (1).

The proof of this Lemma is given in appendix A.

3.3 Hermitian product for quasi homogeneous polynomials and vector fields

We shall provide on $\mathbb{C}[x_1, \ldots, x_n]$ an hermitian product suitable for the grading into quasi-homogeneous space. Moreover, on each $\mathcal{P}_\delta$, this hermitian product will induce a submultiplicative norm, i.e. a norm such that the norm of the product of two function is less or equal to the product of the norms. There are several ways for defining such an inner products with this property (see appendix A subsection A.2). In this paper, we choose one possible way given by the two following definitions :

- for quasi homogeneous functions $f, g \in \mathcal{P}_\delta$ we define the inner product given by

$$\langle f, g \rangle_{p,\delta} = \sum_{\substack{Q \in \mathbb{N}^n \\ (Q,p)=\delta}} \overline{f_Q} . g_Q \, \frac{(Q!)^p}{\delta!} \quad \text{where} \quad (Q!)^p = (q_1!)^{p_1} \cdots (q_n!)^{p_n}; \tag{2}$$

Let $|.|_{p,\delta}$ be the associated norm. If $p = (1, \ldots, 1)$ (i.e. in the homogeneous case), this is the Fischer scalar product [Sha89, Fis17, IL05]. Hence, we have

$$\left\langle x^R, x^Q \right\rangle_{p,\delta} := \begin{cases} \frac{(r_1!)^{p_1} \cdots (r_n!)^{p_n}}{\delta!} & \text{if } R = Q \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

- for quasi homogeneous vector field of degree delta $\delta \in \tilde{\Delta}$ we define the associated inner product and norm given by

$$\langle U, V \rangle_{p,\delta} := \sum_{i=1}^{n} \langle U_i, V_i \rangle_{p,\delta+p_i} \qquad \text{and} \qquad \|U\|^2_{p,\delta} := \sum_{i=1}^{n} |U_i|^2_{p,\delta+p_i} \tag{4}$$

where $U = \sum_{i=1}^{n} U_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$ and $V = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i} \in \mathcal{H}_\delta$.

One of the main features of these hermitian products is its good behavior with respect to the product. More precisely, we have

Proposition 3.6 (submultiplicativity of the norms)**.**

(a) *Let f, g be p-quasihomogeneous polynomials of δ, δ' respectively. Then,*

$$|fg|_{p,\delta+\delta'} \leq |f|_{p,\delta} \; |g|_{p,\delta'} .$$

(b) *Let $f_{\delta,r}$ be a function from $\mathbb{C}^n$ to $\mathbb{C}$. We assume that $f_{\delta,r}$ is simultaneously quasihomogeneous of degree δ and homogeneous of degree r. Denote by $\widetilde{f}_{\delta,r}$ the unique r-linear, symmetric form such that $\widetilde{f}_{\delta,r}(\underbrace{X, \cdots, X}_{r \text{ times}}) = f_{\delta,r}(X)$ where $X = (x_1, \ldots, x_n)$. Let U_{δ_i} be a p-quasihomogeneous vector field of degree δ_i, $i = 1, \ldots, r$.*

Then, $\widetilde{f}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r})$ is p-quasihomogeneous de degree $\delta + \delta_1 + \cdots + \delta_r$ and we have

$$\left| \widetilde{f}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r}) \right|_{p,\delta+\delta_1+\cdots+\delta_r} \leq N_1(\widetilde{f}_{\delta,r}) \; \|U_{\delta_1}\|_{p,\delta_1} \cdots \|U_{\delta_r}\|_{p,\delta_r} \tag{5}$$

with $N_1(\widetilde{R}_{\delta,r}) := \sum_{\substack{1 \leq i_\ell \leq n \\ 1 \leq \ell \leq r}} |\widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r})|$ where $(e_1, \cdots, e_n)$ is the canonical basis of $\mathbb{C}^n$.

(c) *Let $R_{\delta,r}$ be a vector field of $\mathbb{C}^n$ which is simultaneously quasihomogeneous of degree δ and homogeneous of degree r. Denote by $\widetilde{R}_{\delta,r}$ the unique r-linear, symmetric operator such that $\widetilde{R}_{\delta,r}(\underbrace{X, \cdots, X}_{r \text{ times}}) = R_{\delta,r}(X)$. Let U_{δ_i} be a p-quasihomogeneous vector field of degree δ_i, $i = 1, \ldots, r$.*

Then, $\widetilde{R}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})$ is p-quasihomogeneous de degree $\delta+\delta_1+\cdots+\delta_r$ and we have

$$\left\|\widetilde{R}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})\right\|_{p,\delta+\delta_1+\cdots+\delta_r} \leq N_{2,1}(\widetilde{R}_{\delta,r})\ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}. \tag{6}$$

with $N_{2,1}(\widetilde{R}_{\delta,r}) := \sqrt{\sum_{j=1}^{n}\left(N_1(\widetilde{R}_{\delta,r,j})\right)^2}$ where $\widetilde{R}_{\delta,r,j}$ is the j-th components of $\widetilde{R}_{\delta,r}$ in the canonical basis of $\mathbb{C}^n$.

(d) *Let U and N be two p-quasihomogeneous vector fields of quasi degree δ and α respectively. Then $DU.N$ is a p-quasihomogeneous vector field of degree $\delta+\alpha$ satisfying*

$$\begin{aligned}\|DU.N\|_{p,\delta+\alpha} &\leq n(\delta+\overline{p})^{\nu}\|U\|_{p,\delta}\|N\|_{p,\alpha} \quad \text{where} \quad \nu := \max(1,\tfrac{\overline{p}}{2})\\ &\leq \mathfrak{M}_p\ \delta^{\nu}\|U\|_{p,\delta}\|N\|_{p,\alpha} \quad \text{when } \delta>0 \text{ where} \quad \mathfrak{M}_p = n\sup_{\delta\in\widetilde{\Delta}}\left(\tfrac{\delta+\overline{p}}{\delta}\right)^{\nu}\end{aligned}$$

In the homogeneous case, this result is due to G. Iooss and E. Lombardi [IL05][lemma A.8]. The proof of this proposition is given in appendix A, subsection A.2.

Finally, the convergence of a formal power series is linked with the growth of the norms of its quasihomogeneous components. More precisely we have :

Proposition 3.7.

(a) *For a formal power series f, the following properties are equivalent:*

(i) *f is uniformly convergent in a neighborhood of the origin,*

(ii) *There exists $M,R>0$ such that for every $\delta\in\Delta$, $|f_\delta|_{p,\delta}\leq\frac{M}{R^\delta}$.*

(iii) *There exists $M,R>0$ such that for every $\delta\in\Delta$ and $r\geq 0$, $N_1(\widetilde{f}_{\delta r})\leq\frac{M}{R^r}$.*

(b) *For a formal vector field, V, the following properties are equivalent:*

(i) *V is uniformly convergent in a neighborhood of the origin,*

(ii) *There exist $M,R>0$ such that for every $\delta\in\widetilde{\Delta}$, $\|V_\delta\|_{p,\delta}\leq\frac{M}{R^\delta}$.*

(iii) *There exist $M,R>0$ such that for every $\delta\in\widetilde{\Delta}$ and $r\geq 0$, $N_{2,1}(\widetilde{V}_{\delta r})\leq\frac{M}{R^r}$.*

In the homogeneous case, this result is due to H. Shapiro [Sha89][lemma 1]. The proof of this lemma is given in appendix A, subsection A.2.

Lemma 3.8. *Let $f=\sum_{\delta\in\Delta}f_\delta=\sum_{Q\in\mathbb{N}}f_Qx^Q$ be a formal power series. If there exists a constant C such that, for all $\delta\in\Delta$, $|f_\delta|_{p,\delta}\leq C^\delta(\delta!)^b$, then f is a $(\overline{p}b)$-Gevrey formal power series. This means that there exists a positive constant D such that $|f_Q|\leq C^{|Q|}(|Q|!)^{\overline{p}b}$ for all multiindices $Q\in\mathbb{N}^n$.*

The proof of this lemma is given in the appendix A.

4 Normal forms for perturbation of quasihomogeneous vector fields

4.1 Good perturbations

Let $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$ be fixed such that the largest common divisor of its components $p_1 \wedge \cdots \wedge p_n$ is equal to 1. Let $n \geq 2$ be an integer. Let S be a quasihomogeneous vector field of $\mathbb{C}^n$ of quasidegree s. We are interested in suitable holomorphic perturbations of S.

Definition 4.1. *Let X be a germ of holomorphic vector field at the origin of $\mathbb{C}^n$. We shall say that X is a* **good perturbation** *of S if the Taylor expansion of $X - S$ at the origin is of quasiorder greater than s.*

Example 4.2. *Let us consider the germ of vector field at the origin of $\mathbb{C}^2$*

$$X = (2y + x^p U(x))\frac{\partial}{\partial x} - nx^{n-1}\frac{\partial}{\partial y}$$

where $U(0) = 1$. This example were considered by Cerveau and Moussu [MC88]. Let us define $S = 2y\frac{\partial}{\partial x} - nx^{n-1}\frac{\partial}{\partial y}$. If $n = 2m$ is even, then it is $(1, m)$-quasihomogeneous of degree $m - 1$. If $n = 2m + 1$ is odd, then it is $(2, n)$-quasihomogeneous of degree $n - 2$. In both case X is good perturbation of S whenever $2p > n$.

4.2 Formal normal form of a good deformation

In this section, we shall define of formal normal form of a good perturbation of a quasihomogeneous vector field S.

Let $\delta \in \tilde{\Delta}$. Let us define the **coboundary operator** $d_0 : \mathcal{H}_\delta \to \mathcal{H}_{s+\delta}$ to be the linear map

$$d_0(U) = [S, U]$$

where $[.,.]$ denotes de Lie bracket of vector fields.

For any quasidegree $\alpha \in \tilde{\Delta}$ such that $\alpha > s$, we consider the selfadjoint operator

$$\begin{array}{rcl} \Box_\alpha : \mathcal{H}_\alpha & \to & \mathcal{H}_\alpha \\ U & \mapsto & \Box_\alpha U := d_0 d_0^* U \end{array}$$

where d_0^* denotes the adjoint operator of d_0 relatively to the scalar product $|.|_{p,\delta}$ (defined by (2)). Let $\mathcal{S}_\alpha := \text{spec}\,(\Box_\alpha)$ be its spectrum. It is included in the nonnegative real axis.

Definition 4.3.

(a) *We shall say that a vector field of $\mathcal{H}_\alpha$ is* **resonant** *(or harmonic) if it belongs to the kernel $Ker\,\Box_\alpha$ of $\Box_\alpha$.*

(b) *A formal vector field will be said resonant if all of its quasihomogeneous component are resonant.*

(c) *A good perturbation $X = S + R$ of S is a* **normal form** *relatively to S if R is resonant.*

Proposition 4.4. *Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighborhood of the origin of $\mathbb{C}^n$ (i.e. the order of R at the origin is greater than s).*

Then, X can be formally normalized or equivalently partially normalized at any finite quasiorder. More precisely we have, the two following equivalent statements:

(a) **(Formal normal form)** *there exists a formal diffeomorphism $\hat{\Phi}$ tangent to the identity which conjugate X to a formal normal form; that is $\hat{\Phi}_* X - S$ is resonant. Moreover, there exists a unique normalizing diffeomorphism $\Phi = \mathrm{Id} + \mathcal{U}$ such that $\mathcal{U}$ has a zero projection on the kernel of $d_0 = [S, .]$.*

(b) **(Partial Normal Form)** *for every $\alpha \in \widetilde{\Delta}$, there exists a polynomial diffeomorphism tangent to identity $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \sum_{0<\delta\leq\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta \cap (Ker\, d_0)^\perp$ such that*

$$(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha}, \quad \text{where} \quad \mathcal{N}_\alpha = \sum_{s<\delta\leq\alpha} N_\delta, \quad N_\delta \in Ker\, \Box_\delta = Ker\, d^*_{0|\mathcal{H}_\delta}, \tag{7}$$

and where $\mathcal{R}_{>\alpha}$ is of quasiorder $> \alpha$.

Remark 4.5. *Observe that in the expansions of $\mathcal{N}_\alpha$ and $\mathcal{U}_\alpha$ given in statement (b), $U_\delta = 0$ and $N_\delta = 0$ for $\delta \notin \widetilde{\Delta}$ since $\mathcal{H}_\delta = \{0\}$.*

Proof. Firstly, observe that (a) follows directly from (b). Then, for proving (b), a basic identification of the quasihomogeneous components for $\delta \in \widetilde{\Delta}$ with $s < \delta \leq \alpha$ in (7) with $X = S + R$ leads to

$$\Big\{\mathcal{N}_\alpha + [S, \mathcal{U}_\alpha]\Big\}_\delta = \Big\{R(\mathrm{Id} + \mathcal{U}_\alpha) - D\mathcal{U}_\alpha.\mathcal{N}_\alpha + S(\mathrm{Id} + \mathcal{U}_\alpha) - S - DS.\mathcal{U}_\alpha\Big\}_\delta \tag{8}$$

Hence, using proposition 3.4, lemma 3.5 and (18), we get the following hierarchy of cohomological equations in $\mathcal{H}_\delta$ for $\delta \in \widetilde{\Delta}$ with $s < \delta \leq \alpha$:

$$N_\delta + d_0(U_{\delta-s}) = K_\delta \tag{9}$$

where K_δ depends only on R, S which are given and on N_β and $U_{\beta-s}$ for $s < \beta < \delta$ (the explicit formula of K_δ which is useless here is given in section 6 : see (23)). So the hierarchy of equations (9) for $s < \delta \leq \alpha$ can be solved by induction starting with the smallest $\delta \in \tilde{\Delta}$ greater than s.

If $\delta - s \notin \tilde{\Delta}$, then $\mathcal{H}_{\delta-s} = \{0\}$. Hence, $d_{0|\mathcal{H}_{\delta-s}} \equiv 0$ so that $K_\delta \in \mathrm{Ker}\, d^*_{0|\mathcal{H}_\delta} = \mathcal{H}_\delta$. Hence, if $\delta - s \notin \widetilde{\Delta}$, we set $U_{\delta-s} := 0$ and $N_\delta := K_\delta \in \mathrm{Ker}\, d^*_0$.

If $\delta - s \in \tilde{\Delta}$ (and $\delta \in \tilde{\Delta}$), then let us decompose H_δ along the direct sum

$$\mathcal{H}_\delta = \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}} \overset{\perp}{\bigoplus} \mathrm{Ker}\, d^*_{0|\mathcal{H}_\delta} = \mathrm{Im}\, \Box_\delta \overset{\perp}{\bigoplus} \mathrm{Ker}\, \Box_\delta$$

where $\mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}} = \mathrm{Im}\, \Box_\delta$ and $\mathrm{Ker}\, d^*_{0|\mathcal{H}_\delta} = \mathrm{Ker}\, \Box_\delta$. Then, denoting π_δ the orthogonal projection onto $(\mathrm{Ker}\, d^*_0)^\perp = (\mathrm{Ker}\, \Box_\delta)^\perp$, the cohomological equation (9) is equivalent to

$$N_\delta = (Id - \pi_\delta)(K_\delta) \in \mathrm{Ker}\, d^*_{0|\mathcal{H}_\delta}, \qquad d_0(U_{\delta-s}) = \pi_\delta(K_\delta) \in \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}}. \tag{10}$$

Then since d_0 induces an isomorphism from Ker $(d_{0|\mathcal{H}_{\delta-s}})^\perp$ on to Im $d_{0|\mathcal{H}_{\delta-s}}$, there exists a unique $U_{\delta-s} \in (\text{Ker } (d_{0|\mathcal{H}_{\delta-s}}))^\perp$ such that $d_0(U_{\delta-s}) = \pi_\delta(K_\delta) \in \text{Im } d_{0|\mathcal{H}_{\delta-s}}$. □

Example 4.6. *Let* $S = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i}$ *be a linear diagonal vector field. It is* $(1,\dots,1)$*-quasihomogeneous of degree* 0*. An easy computation shows that* $(ad_S)^* = ad_{\bar{S}}$ *where* $\bar{S} = \sum_{i=1}^{n} \bar{\lambda}_i x_i \frac{\partial}{\partial x_i}$*. Hence, Ker* $(ad_S)^* = Ker\, ad_{\bar{S}}$*. Moreover, the spectrum of* $\square_\delta$ *is the set* $\{|(Q,\lambda) - \lambda_i|^2, Q \in \mathbb{N}^n, |Q| = \delta + 1, 1 \le i \le n\}$*.*

Example 4.7. *Let* $S = y\frac{\partial}{\partial x}$ *in* $\mathbb{C}^2$*. It is* $(1,1)$*-quasihomogeneous of degree* 0*. The adjoint of the Lie derivative is* $\mathcal{L}^* = x\frac{\partial}{\partial y}$*; the adjoint of the Lie bracket with* S *is*

$$(ad_S)^* v = -x\frac{\partial v_1}{\partial y}\frac{\partial}{\partial x} + \left(v_1 - x\frac{\partial v_2}{\partial y}\right)\frac{\partial}{\partial y}.$$

Its formal kernel is the $\mathbb{C}[[x]]$*-module generated by the radial vector field* $R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ *and* $x\frac{\partial}{\partial y}$*. According to [IL05][p.36], the spectrum of* $\square_{k-1}$ *is composed of the following numbers*[1]

$$0,\ k+1,\ (\alpha-1)(\beta+1),\ \alpha(\beta+2), \quad \alpha = 1,\dots,k, \alpha+\beta = k.$$

An easy computation shows that the non-zero eigenvalues of $\square_{k-1}$ *are* $\ge k-1$*.*

A similar definition of normal form of perturbation of homogeneous vector fields was given by G. Belitskii [Bel79, Bel82] using a different scalar product. Another definition of normal form of perturbation of quasi-homogeneous vector fields was given by Kokubu and al. [KOW96]. It is a general scheme to provide a **unique** normal form. This scheme can be combined with our technics to provide a unique normal form as well.

The perturbation of a nilpotent linear vector field has been treated by R. Cushman and J.A Sanders [CS86] using $\mathfrak{sl}_2$-triple representation. Computational aspects with another definition of normal forms in any dimension was done by L. Stolovitch [Sto92]. Two dimensional aspects were initiated by R. Bogdanov and [Bog79] and F. Takens [Tak74]. Analytic conjugacy of perturbations of a nilpotent 2-dimensional to such a normal form was obtained in [SŻ02, Żoł06]

For very particular examples of S in dimension 2, normal forms has been obtained by V. Basov (see [Bas06] and references therein). When the perturbation of $S = y\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}$ is tangent to the germ of $x^2 = y^3$ at the origin, then a formal normal form of vector fields tangent to the cusp has been devised by F. Loray [Lor99]. It is described in terms of a basis of the local algebra of the function $x^2 - y^3$. This work has been improved by E. Paul [Pau04].

4.3 Vector fields with symmetries

In this section, we show how to adapt our normal form scheme in order to study vector fields that preserve a differential form or vector fields that are reversible. We shall show that we need to consider restrictions of the cohomological operator d_0 to some subspace of the

[1] in fact, it is the spectrum of $d_0^* d_0$ that is computed there.

space of quasihomogeneous vector fields with range in another subspace of a space of quasihomogeneous vector fields. On these subspaces, we shall consider the induced hermitian product.

- Vector fields leaving a differential form invariant

  One may be interested in studying vector fields leaving invariant a polynomial differential form ω (i.e. the Lie derivative $\mathcal{L}_X\omega = 0$) such as a **symplectic or a volume form**, for instance. First of all, we have to check that ω is also p-quasi-homogeneous (with the same p as for the vector fields). This means that $\mathcal{L}_{R_p}\omega = d\omega$ for some integer d. For instance, let $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ be the standard symplectic form of $\mathbb{C}^{2n}$. Let q_i (resp. r_i) be the weight of x_i (resp. y_i). If h_0 denotes a p-quasi-homogeneous polynomial of $\mathbb{C}^{2n}$. In order that the associated Hamiltonian vector field

  $$\sum_{i=1}^{n} -\frac{\partial h_0}{\partial y_i}\frac{\partial}{\partial x_i} + \frac{\partial h_0}{\partial x_i}\frac{\partial}{\partial y_i}$$

  be also p-quasihomogeneous, it is necessary and sufficient that $q_i + r_i = q_j + r_j$, for all i, j.

  In this situation, it is sufficient to work on the space $\mathcal{H}_{\delta,\omega} := \{X \in \mathcal{H}_\delta \mid \mathcal{L}_X(\omega) = 0\}$ of quasi-homogeneous vector fields preserving the form ω instead of $\mathcal{H}_\delta$. Indeed, the Lie bracket of the two vector fields preserving ω still leaves it invariant since $\mathcal{L}_{[S,Y]}(\omega) = \mathcal{L}_S\mathcal{L}_Y(\omega) - \mathcal{L}_Y\mathcal{L}_S(\omega) = 0$. Moreover, the flow $\exp(tX)$ of a vector field X that preserve ω leaves ω invariant : $\frac{d\exp(tX)^*\omega}{dt} = \exp(tX)^*(\mathcal{L}_X\omega) = 0$. Hence, we can consider the restriction maps $d_0 : \mathcal{H}_{\delta,\omega} \to \mathcal{H}_{\delta+s,\omega}$, $d_0^* : \mathcal{H}_{\delta+,\omega} \to \mathcal{H}_{\delta,\omega}$ and the box operator $\Box : \mathcal{H}_{\delta,\omega} \to \mathcal{H}_{\delta,\omega}$. The scheme goes as follow : assume that X is normalized up to order $\delta - 1$ and that $\mathcal{L}_X\omega = 0$. Let us conjugate X by $\exp U_{\delta-s}$ where $\mathcal{L}_{U_{\delta-s}}\omega = 0$ and $U_{\delta-s}$ is quasi-homogeneous of order $\delta - s$. As above, one has to solve the cohomological equation of the form $N_\delta + d_0(U_{\delta-s}) = K_\delta$. Since ω is p-quasi-homogeneous, it is easy to see that K_δ leaves ω invariant (see [Fra80] for a similar problem). Hence, we can appply our scheme on the spaces $\mathcal{H}_{\delta,\omega}$. As a consequence, if S and its good perturbation X preserve ω, then there is formal transformation (fixing ω) into a normal form (an element of Ker d_0*) which leaves invariant ω.

- Reversible vector fields

  Let $R : \mathbb{C}^n \to \mathbb{C}^n$ be a linear map such that $R^2 = Id$. A vector field Z is said to be **reversible** if it satisfies to $Z(Rx) = -RZ(x)$. Let U be a germ of holomorphic (or formal) vector field such that $R.U(x) = U(Rx)$ at the origin (a point at which it vanishes). Then, one can show that the transformation $y = x + U(x)$ conjugate a reversible vector field to a reversible vector field. As for the case of differential form, we require a compatibilty condition on R with respect to the weight p. Namely, we assume that the linear vector field Rx is p-quasihomogeneous of quasidegree 0. This implies that a formal vector field is reversible if and only if each of its quasihomogeneous component is reversible. Let us consider the space of quasihomogeneous transformations

  $$\mathcal{T}_\delta := \{U \in \mathcal{H}_\delta \mid R.U(x) = U(Rx)\}$$

and the spaces of quasihomogeneous reversible vector fields

$$\mathcal{R}_\delta := \{U \in \mathcal{H}_\delta \mid R.U(x) = -U(Rx)\}.$$

If S is reversible, then $d_0 : \mathcal{T}_\delta \to \mathcal{R}_{s+\delta}$. In fact, we have

$$\begin{aligned} R[S,U](x) &= RDS.U - RDU.S = -DS(Rx)RU - DU(Rx)R.S \\ &= -DS(Rx)U(Rx) + DU(Rx)S(Rx) = -[S,U](Rx) \end{aligned}$$

Hence, we will consider the operator $d_0^* : \mathcal{R}_{s+\delta} \to \mathcal{T}_\delta$ as well as the box operator $\Box : \mathcal{T}_\delta \to \mathcal{T}_\delta$. The normal form scheme goes as in the general case except that in equation (9), $N_\delta + d_0(U_{\delta-s}) = K_\delta$, we have $K_\delta, N_\delta \in \mathcal{R}_\delta$ and $U_{\delta-s} \in \mathcal{T}_{\delta-s}$.

4.4 Spectral properties of $\Box$

Lemma 4.8.

(a) *Let $f_\lambda \in \mathcal{H}_{s+\alpha}$ which belong to the λ-eigenspace of the operator $\Box_{s+\alpha}$, λ being a nonzero eigenvalue of $\Box_{s+\alpha}$. Let v_λ be such that $\Box_{s+\alpha} v_\lambda = f_\lambda$ (i.e. $f_\lambda = \lambda v_\lambda$) and let us set $U_\lambda := d_0^* v_\lambda \in \mathcal{H}_\alpha$. Then, we have*

$$\|U_\lambda\|_{p,\alpha} = \frac{1}{\sqrt{\lambda}} \|f_\lambda\|_{p,s+\alpha}. \tag{11}$$

Moreover, if λ and λ' are two different nonzero eigenvalues of $\Box_{s+\alpha}$, then U_λ and $U_{\lambda'}$ are orthogonal.

(b) *Let $f \in \mathcal{H}_{\alpha+s}$ belongs to* Im $d_{0|\mathcal{H}_\alpha}$ = Im $\Box_\alpha$ *and let $U \in \mathcal{H}_\alpha$ be such that $U \in$* Im $d^*_{0|H_{\alpha+s}} = (Ker\, d_{0|\mathcal{H}_\alpha})^\perp$ *and $d_0(U) = f$. Then*

$$\|U\|_{p,\alpha} \leq \frac{1}{\min\limits_{\lambda \in \mathcal{S}_{\alpha+s} \setminus \{0\}} \sqrt{\lambda}} \|f\|_{p,\alpha+s}$$

where $\mathcal{S}_{\alpha+s} =$ spec $\Box_{\alpha+s}$.

Proof. (a) : In fact, we have

$$\langle U_\lambda, U_\lambda \rangle_{p,\alpha} = \langle d_0^* v_\lambda, d_0^* v_\lambda \rangle_{p,\alpha} = \langle d_0 d_0^* v_\lambda, v_\lambda \rangle_{p,\alpha+s} = \langle f_\lambda, v_\lambda \rangle_{p,\alpha+s} = \frac{1}{\lambda} \langle f_\lambda, f_\lambda \rangle_{p,\alpha+s}$$

since $f_\lambda = \lambda v_\lambda$. About the second point, we have

$$\langle U_\lambda, U_{\lambda'} \rangle_{p,\alpha} = \langle d_0^* v_\lambda, d_0^* v_{\lambda'} \rangle_{p,\alpha} = \langle v_\lambda, d_0 d_0^* v_{\lambda'} \rangle_{p,\alpha+s} = \lambda' \langle v_\lambda, v_{\lambda'} \rangle_{p,\alpha+s} = 0.$$

(b) : Let $f \in \mathcal{H}_{\alpha+s} \cap$ Im $d_{0|\mathcal{H}_\alpha}$ and let $U \in \mathcal{H}_\alpha$ be such that $U \in$ Im $d^*_{0|H_{\alpha+s}} = ($Ker $d_{0|\mathcal{H}_\alpha})^\perp$ and $d_0(U) = f$. Then there exits $v \in ($Ker $d^*_{0|\mathcal{H}_{\alpha+s}})^\perp$ such that $d_0^*(v) = U$. Hence, $\Box_{\alpha+s} v = f$. Since $\Box_{\alpha+s}$ is a self adjoint operator, we have the spectral decomposition

$$\mathcal{H}_{s+\alpha} = \bigoplus_{\lambda \in \mathcal{S}_{\alpha+s}} \text{Ker } (\lambda \text{Id} - \Box_{\alpha+s}).$$

Moreover, since $f \in \text{Im } d_{0|\mathcal{H}_\alpha} = \text{Im } \Box_{\alpha+s}$, $v \in (\text{Ker } d^*_{0|\mathcal{H}_{\alpha+s}})^\perp$ and $\Box_{\alpha+s} v = f$, we also have the spectral decompositions

$$f = \bigoplus_{\lambda \in \mathcal{S}_{\alpha+s}\setminus\{0\}} f_\lambda, \qquad v = \bigoplus_{\lambda \in \mathcal{S}_{\alpha+s}\setminus\{0\}} v_\lambda, \qquad \Box_{\alpha+s} v_\lambda = f_\lambda.$$

Then, using (a) and setting $U_\lambda = d_0^*(v_\lambda)$, we finally obtain

$$\|U\|^2_{p,\alpha} = \sum_{\lambda \in \mathcal{S}_{\alpha+s}\setminus\{0\}} \|U_\lambda\|^2_{p,\alpha} \leq \sum_{\lambda \in \mathcal{S}_{\alpha+s}\setminus\{0\}} \tfrac{1}{\lambda} \|U_\lambda\|^2_{p,\alpha} \leq \left(\frac{1}{\min\limits_{\lambda \in \mathcal{S}_{\alpha+s}\setminus\{0\}} \sqrt{\lambda}} \right)^2 \|f\|^2_{p,\alpha+s}.$$

□

5 Rigidity of quasihomogenous vector fields

Let $\mathcal{I}$ be a quasihomogeneous ideal of $\mathcal{O}_n$ generated by quasihomogeneous polynomials $h_1, \ldots, h_r$ of p-quasidegree $e_1, \ldots, e_r$ respectively. We shall denotes $\widehat{\mathcal{I}} = \mathcal{I} \otimes \hat{\mathcal{O}}_n$ its formal completion, that is the ideal in the ring of formal power series $\hat{\mathcal{O}}_n$ generated by the h_i's. Let us denote by $\mathcal{M}_i$ the operator of multiplication by h_i in $\hat{\mathcal{O}}_n$ ($\mathcal{M}_i$ will also denote the multiplication operator, componentwise, on the space of formal vector field $\widehat{\mathcal{X}}_n$). Let us denote by $\mathcal{M} = \mathcal{M}_1\mathcal{X}_n + \cdots + \mathcal{M}_r\mathcal{X}_n$ (resp $\widehat{\mathcal{M}} = \mathcal{M}_1\widehat{\mathcal{X}}_n + \cdots + \mathcal{M}_r\widehat{\mathcal{X}}_n$) the submodule of germs of holomorphic (resp. formal) vector fields at the origin which components belong to the ideal generated by the h_i's.
Let $\delta \in \tilde{\Delta}$, let us set $\mathcal{M}_\delta := \mathcal{M} \cap \mathcal{H}_\delta$. Let $\mathcal{V}_\delta$ be the orthogonal complement of $\mathcal{M}_\delta$ in $\mathcal{H}_\delta$ and let $\pi_{\mathcal{I}^\perp}$ be the projection onto $\mathcal{V}_\delta$: $\mathcal{H}_\delta = \mathcal{V}_\delta \overset{\perp}{\bigoplus} \mathcal{M}_\delta$. We shall denote $\widehat{\mathcal{V}} := \bigoplus\limits_{\delta \in \tilde{\Delta}} \mathcal{V}_\delta$ as well as

$$\widehat{\mathcal{W}} := \left\{ U \in (\text{Ker } d_0)^\perp \mid [S, U] \in \widehat{\mathcal{V}} \right\}.$$

Lemma 5.1. *With the notation above, we have* $V_\delta = \bigcap\limits_{i=1}^{r} Ker\, \mathcal{M}^*_{i|\mathcal{H}_{\delta+e_i}}$ *where* $\mathcal{M}^*_{i|\mathcal{H}_{\delta+e_i}}$ *denotes the adjoint operator of* $\mathcal{M}_{i|\mathcal{H}_\delta} : \mathcal{H}_\delta \to \mathcal{H}_{\delta+e_i}$ *with respect to the family of hermitian products* $\langle ., . \rangle_{p,.}$.

Proof. Let $v \in \mathcal{V}_\delta$. By definition, we have, for all $w_i \in \mathcal{H}_{\delta-e_i}$, $(v, \mathcal{M}_1 w_i + \cdots + \mathcal{M}_r w_r) = 0$. In particular, we may chose $w_i = \mathcal{M}_i^* v$ for all i. We obtain $0 = \|\mathcal{M}_1^* v\|^2 + \cdots + \|\mathcal{M}_r^* v\|^2$. □

Let $\delta \in \tilde{\Delta}$ such that $\delta > s$. Let us denote by $\sigma_{\delta,\setminus\mathcal{I}}$ the set of nonzero eigenvalues of $\Box_\delta$ for which there exists an associated (quasihomogeneous of degree δ) eigenvector which is orthogonal to $\mathcal{M}_\delta$. Let us set

$$a_\delta := \min_{\lambda \in \sigma_{\delta,\setminus\mathcal{I}}} \sqrt{\lambda},$$

as well as

$$\delta_* := \frac{\min\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\bar{p}} \text{ and } \delta^* := \frac{\max\{\delta + p_i \mid \delta + p_i \in \Delta\}}{\underline{p}}.$$

Let us set

$$\tilde{\Delta}^- := \tilde{\Delta} \cap (\tilde{\Delta} - s), \quad \tilde{\Delta}^+ := \tilde{\Delta} \cap (\tilde{\Delta} + s), \quad \delta_0 := \max(\min_{\delta \in \tilde{\Delta}^-} \delta, 1).$$

The integer δ_0 is the smallest positive integer of $\tilde{\Delta}^-$. Let us define the sequence of positive real numbers $\{\eta_\delta\}_{\delta \in \tilde{\Delta}^- \cap \mathbb{N}^* \cup \{0\}}$ as follow : $\eta_0 = 1$; for any positive $\delta \in \tilde{\Delta}^-$ (i.e.$\delta \geq \delta_0$),

$$a_{s+\delta}\eta_\delta = \max_{s \leq \mu \leq s+\delta,\, \mu \in \tilde{\Delta}} \overset{*}{\max}_{\substack{\mu_* \leq r \leq \mu^* \\ \delta_1 + \cdots + \delta_r + \mu = s+\delta}} \eta_{\delta_1} \cdots \eta_{\delta_r}. \tag{12}$$

where if $\mu = s$ then the maximum is taken over the r-tuples $(\delta_1, \ldots, \delta_r)$ of nonnegative integers such that at least, two of the δ_i's are positive. Moreover, the maximum is taken over the indices δ_i (resp. μ) which belong $(\tilde{\Delta}^- \cap \mathbb{N}^*) \cup \{0\}$ (resp. $\tilde{\Delta}$). It can happen that $\delta_0 = 1$.

Remark 5.2. *The sequence η_δ is well defined by induction since the maximum only involves terms η_d's with $d < \delta$.*

Definition 5.3. *The quasihomogeneous vector field S will called* **diophantine with respect to the ideal** *$\mathcal{I}$ if the formal power series $\sum_{\delta > 0,\, \delta \in \tilde{\Delta}} \eta_\delta z^\delta$ converges in a neighborhood of the origin in $\mathbb{C}$; that is to say that there exists $c, M > 0$ such that $\eta_\delta \leq M c^\delta$. We shall say that S is* **diophantine** *is it is diophantine with respect to the zero ideal $\mathcal{I} = \{0\}$.*

Example 5.4. *Let return to example 4.6 where S is linear and diagonal. It is known [Sto94][lemma 2.3] that S is diophantine in the above sense if and only if it satisfies Brjuno condition :*

$$(\omega) \qquad -\sum_{k \geq 0} \frac{\ln(\omega_{k+1})}{2^k} < +\infty$$

where

$$\omega_k = \inf \left\{ |(Q, \lambda) - \lambda_i| \neq 0,\ i = 1, \ldots, n,\ Q \in \mathbb{N}^n,\ 2 \leq |Q| \leq 2^k \right\}.$$

Definition 5.5. *Let S be quasihomogeneous and let X be a good holomorphic perturbation of S at the origin. We shall say that X is formally (holomorphically)* **conjugate to** *S* **along** *$\widehat{\mathcal{I}}$ (resp. $\mathcal{I}$) if there exists a formal (resp. germ of holomorphic) diffeomorphism $\hat{\Phi}$ (resp. Φ) such that $\hat{\Phi}_* X - S \in \widehat{\mathcal{M}}$ (resp. $\Phi_* X - S \in \mathcal{M}$), i.e. in the new formal (resp. holomorphic) coordinates, X equals to the sum of S and a formal vector field which components belong to the ideal $\widehat{\mathcal{I}}$ (resp. $\mathcal{I}$).*

Theorem 5.6. *Let us assume that the quasihomogeneous vector field S is diophantine with respect to $\mathcal{I}$. Let X be a good holomorphic perturbation of S at the origin of $\mathbb{C}^n$. We assume that X is formally conjugated to S along $\widehat{\mathcal{I}}$ (by the mean of a formal diffeomorphism of the form $Id + U$, with $U \in \widehat{\mathcal{W}}$). Then, X is holomorphically conjugate to S along $\mathcal{I}$.*

Corollary 5.7. *Under the assumptions of the theorem, there exists a good holomorphic change of coordinates in which the germ at the origin of the zero locus $\Sigma := \{x \in \mathbb{C}^n,\ h_1(x) = \cdots = h_r(x) = 0\}$ at 0 is an* **invariant analytic set** *for X. Moreover, in these new coordinates, the restriction X to Σ equals to the restriction of S to Σ.*

Theorem 5.8. *If the quasihomogeneous vector field S is diophantine and if the holomorphic good perturbation X is formally conjugate to S, then X is holomorphically conjugate to S.*

Proof. We apply theorem 5.6 to the ideal $\mathcal{I} = \{0\}$. Moreover, we can assume that the normalizing diffeomorphism reads $\Phi := I + U$ with $U \in (\text{Ker } d_0)^{\perp}$. In fact, if $\Phi_* X = S$ then for any V commuting with S, we have

$$(\exp V)_* S = S + [V, S] + \frac{1}{2}[V, [V, S]] + \cdots = S.$$

□

The remaining of the section is devoted to the proof of the theorem 5.6.

First of all, let us write the conjugacy equations between the vector fields $X = S + R$ and $X' := \hat{\Phi}_* X = S + R'$ where the formal diffeomorphism is written as $\hat{\Phi}^{-1} = Id + U$ where $U \in \widehat{\mathcal{V}}$ stands for a formal vector field of positive quasiorder. Since, we have $D(\hat{\Phi})(\hat{\Phi}^{-1})X(\hat{\Phi}^{-1}) = X'$ we have $X(I + U) = D(I + U)X'$. Therefore, we obtain

$$\begin{aligned} R' + [S, U] &= R(Id + U) - DU.R' \\ &\quad + S(Id + U) - S - DS.U. \end{aligned} \tag{13}$$

For any positive integer δ such that $s + \delta \in \tilde{\Delta}$, let us project this equation onto the orthogonal space $\mathcal{V}_{s+\delta}$ to $\mathcal{M}_{s+\delta}$ in $\mathcal{H}_{s+\delta}$ and let us denote $\pi_{\mathcal{I}^{\perp}}$ this projection. Assume that $\hat{\Phi}$ conjugates X to S along $\widehat{\mathcal{M}}$. This means that R' belongs to $\widehat{\mathcal{M}}$. Therefore, we have

$$[S, U] = \pi_{\mathcal{I}^{\perp}}([S, U]) = \pi_{\mathcal{I}^{\perp}}(R(Id + U) + S(Id + U) - S - DS.U). \tag{14}$$

The first equality is due to the fact that $U \in \widehat{\mathcal{W}}$ whereas the second is due to the fact that $DU.R' \in \widehat{\mathcal{M}}$. We recall that U_δ denotes the quasihomogeneous component of (the Taylor expansion at the origin of) U of quasidegree δ of U. We emphasize that both side of the equation are reduced to zero if $\delta \notin \tilde{\Delta}$. **So, we will consider the case where $s + \delta \in \tilde{\Delta}$ and $\delta \in \tilde{\Delta}$.** We recall that

$$\tilde{\Delta}^- := \tilde{\Delta} \cap (\tilde{\Delta} - s), \quad \tilde{\Delta}^+ := \tilde{\Delta} \cap (\tilde{\Delta} + s).$$

By assumption, U_δ has also a zero projection on the kernel of the operator d_0. Since we have

$$\mathcal{H}_\delta = \text{Ker } d_0 \overset{\perp}{\bigoplus} \text{Im } d^*_{0|\mathcal{H}_{s+\delta}},$$

then we can write $U_\delta = d_0^* v_{s+\delta}$ for some $v \in \mathcal{H}_{s+\delta}$. Moreover, we can assume that v has a zero projection onto Ker d_0^*. The latter is nothing but the kernel of $\Box = d_0 d_0^*$. In fact, if $\Box v_{s+\delta} = 0$ then $0 = \langle \Box v, v \rangle_{p,s+\delta} = |d_0^* v_{s+\delta}|^2$, the converse being obvious. Let us decompose $v_{s+\delta}$ along the eigenspaces de $\Box_{s+\delta}$. Let λ be an eigenvalue of $\Box_{s+\delta}$ and let π_λ be the projection on the associated eigenspace. We shall say that λ **is quasihomogeneous of quasidegree** $s + \delta$ if $\Box$ has an λ-eigenvector in $\mathcal{H}_{s+\delta}$. We shall denote by $\pi_{\delta+s,\setminus\mathcal{I}}$ the projection onto the subspace of $\mathcal{H}_{s+\delta}$ generated by the eigenvectors of $\Box_{\delta+s}$ which are orthogonal to $\mathcal{M}_{s+\delta}$. Since $[S, U_\delta] = d_0 d_0^* v_{s+\delta}$, then, we have

$$\pi_{s+\delta,\setminus\mathcal{I}} \circ \pi_{\mathcal{I}^{\perp}}(d_0 d_0^* v_{s+\delta}) = \pi_{s+\delta,\setminus\mathcal{I}} \circ \pi_{\mathcal{I}^{\perp}} \left(\sum_{\lambda \in \sigma_{s+\delta,\setminus\mathcal{I}}} \lambda v_\lambda \right) = \sum_{\lambda \in \sigma_{s+\delta,\setminus\mathcal{I}}} \lambda v_\lambda,$$

where we have set $v_\lambda := \pi_\lambda(v)$. We recall that $\sigma_{\delta+s,\setminus\mathcal{I}}$ denotes the set of nonzero eigenvalues of $\Box_{\delta+s}$ for which there exists an associated (quasihomogeneous of degree $\delta+s$) eigenvector orthogonal $\mathcal{M}_{\delta+s}$. We can assume that $v_0 = 0$. Therefore

$$\sum_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} \lambda v_\lambda = \pi_{s+\delta,\setminus\mathcal{I}} \circ \pi_{\mathcal{I}^\perp}(R(Id+U) + S(Id+U) - S - DS.U).$$

Let us set $U_\lambda := d_0^* v_\lambda$ and let us denote by U_δ the sum of the U_λ's where λ ranges over $\sigma_{s+\delta,\setminus\mathcal{I}}$. According to the first point of lemma 4.8, we have $\|U_\lambda\|^2 = \lambda\|v_\lambda\|^2$. According to the second point,

$$\left\| \sum_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} \lambda v_\lambda \right\|^2_{p,s+\delta} = \sum_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} \lambda^2 \|v_\lambda\|^2_{p,s+\delta} = \sum_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} \lambda \|U_\lambda\|^2_{p,\delta} \geq \left(\min_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} \sqrt{\lambda} \right)^2 \|U_\delta\|^2_{p,\delta}.$$

Therefore, we obtain

$$\left(\min_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}} |\sqrt{\lambda}| \right) \|U_\delta\|_{p,\delta} \leq \|\pi_{s+\delta} \circ \pi_{\mathcal{I}^\perp}(R(Id+U) + S(Id+U) - S - DS.U)\|_{p,s+\delta}. \quad (15)$$

Let us estimate the right handside of last inequality. First of all, we have

$$\begin{aligned} &\|\pi_{s+\delta} \circ \pi_{\mathcal{I}^\perp}(R(Id+U) + S(Id+U) - S - DS.U)\|_{p,s+\delta} \\ &\qquad \leq \|\{R(Id+U) + S(Id+U) - S - DS.U\}_{p,s+\delta}\|_{p,s+\delta}. \end{aligned}$$

Then, let us decompose R into quasihomogeneous component $R = \sum_{\mu>s} R_\mu$. First of all, for any $d \in \mathbb{N}$, every quasihomogeneous polynomial of quasidegree d is either 0 or a polynomial of degree $\leq d/\underline{p}$ and of order $\geq d/\bar{p}$. In fact, if $d \in \Delta$, then we have $d = \alpha_1 p_1 + \cdots + \alpha_n p_n$ for some $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Hence, $\bar{p}|\alpha| \geq d \geq \underline{p}|\alpha|$. On the other hand, if $\mu \in \Delta$, the ith coordinate of the vector field R_μ is quasihomogeneous of quasidegree $\mu + p_i$. Hence, it is 0 if $\mu + p_i \notin \Delta$. Otherwise, it's a polynomial of degree $\leq (\mu+p_i)/\underline{p}$ and of order $\geq (\mu+p_i)/\bar{p}$. Therefore, R_μ can be written as a sum of homogeneous vector fields

$$R_\mu = \sum_{\mu_* \leq r \leq \mu^*} R_{\mu,r}$$

where $R_{\mu,r}$ is an homogeneous vector field of degree r (i.e. each component is an homogeneous polynomial of degree r or 0). We recall that we have set

$$\mu_* := \frac{\min\{\mu + p_i \mid \mu + p_i \in \Delta\}}{\bar{p}} \text{ and } \mu^* := \frac{\max\{\mu + p_i \mid \mu + p_i \in \Delta\}}{\underline{p}}.$$

Let $\tilde{R}_{\mu,r}$ be the associated r-linear map. Therefore, the $(s+\delta)$-quasihomogeneous component

of $R(Id+U)$ in its Taylor expansion at 0 is

$$\begin{aligned}\{R(Id+U)\}_{s+\delta} &= \left\{\sum_{\mu>s} R_\mu(Id+U)\right\}_{s+\delta} \\ &= \left\{\sum_{\mu>s}\sum_{r=\mu_*}^{\mu^*} \tilde{R}_{\mu,r}(\underbrace{Id+U,\dots,Id+U}_{r \text{ times}})\right\}_{s+\delta} \\ &= \sum_{\mu>s}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta} \tilde{R}_{\mu,r}(U_{\delta_1},\dots,U_{\delta_r})\end{aligned}$$

where the δ_i's are nonnegative elements of $\tilde{\Delta}^- = \tilde{\Delta}\cap(\tilde{\Delta}-s)$, $\mu\in\tilde{\Delta}$ is greater than s and where we have set $U_0 := Id$. Moreover, in the first sum of the last equality, μ is less or equal than $s+\delta$.

Moreover, according to proposition 3.6 and proposition 3.7, there exists positive constants M and ρ such that, for all $\mu > s$ belonging to $\tilde{\Delta}$, for all $\mu_* \le r \le \mu^*$, we have

$$\left\|\tilde{R}_{\mu,r}(U_{\delta_1},\dots,U_{\delta_r})\right\|_{p,\delta_1+\cdots+\delta_r+\mu} \le \frac{M}{\rho^r}\left\|U_{\delta_1}\right\|_{p,\delta_1}\cdots\left\|U_{\delta_r}\right\|_{p,\delta_r}.$$

As a consequence, we obtain the following estimate :

$$\left\|\{R(Id+U)\}_{s+\delta}\right\|_{p,s+\delta} \le \sum_{\mu>s}^{s+\delta}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta}\frac{M}{\rho^r}\left\|U_{\delta_1}\right\|_{p,\delta_1}\cdots\left\|U_{\delta_r}\right\|_{p,\delta_r} \tag{16}$$

On the other hand, we have

$$S(x) = \sum_{s_*\le r\le s^*}\tilde{S}_{\bullet,r}(\underbrace{x,\dots,x}_{\text{r times}})$$

where $\tilde{S}_{.,r}$ is a r-linear map. Therefore, we have

$$DS(x)U = \sum_{s_*\le r\le s^*} r\tilde{S}_{\bullet,r}(\underbrace{x,\dots,x}_{\text{r-1 times}},U).$$

Hence, the $s+\delta$-quasihomogeneous term in the Taylor expansion of $S(I+U)-S-DS(x)U$ is

$$\{S(I+U)-S-DS(x)U\}_{s+\delta} = \sum_{s_*\le r\le s^*}\sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}}\tilde{S}_{\bullet,r}(U_{\delta_1},\dots,U_{\delta_r}) \tag{17}$$

where

$$\Omega_r = \left\{(\delta_1,\dots,\delta_r)\in\left(\tilde{\Delta}^-\right)^r / \text{ at least, two of the index are positive}\right\}. \tag{18}$$

Therefore, we obtain the following estimate

$$\left\|\{S(I+U)-S-DS(x)U\}_{s+\delta}\right\|_{p,s+\delta} \le M'\sum_{s_*\le r\le s^*}\sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}}\left\|U_{\delta_1}\right\|_{p,\delta_1}\cdots\left\|U_{\delta_r}\right\|_{p,\delta_r}$$

where M' denote a constant depending only on S.

Let us define the sequence $\{\sigma_\delta\}_{\delta\in\tilde{\Delta}^-\cap\mathbb{N}^*\cup\{0\}}$ of positive numbers defined by $\sigma_0 := \|Id\|_{p,0}$ and if $\delta \in \tilde{\Delta}^-$ is positive,

$$\sigma_\delta := \sum_{\mu>s}^{s+\delta}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta}\frac{M}{\rho^r}\sigma_{\delta_1}\cdots\sigma_{\delta_r} + M'\sum_{s_*\le r\le s^*}\sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}}\sigma_{\delta_1}\cdots\sigma_{\delta_r}$$

where, in the first sum, the δ_i's are nonnegative elements of $\tilde{\Delta}^-$ and the μ's are elements of $\tilde{\Delta}$. This sequence is well defined. In fact, since $\mu > s$, then the δ_i's are all less than δ in the sum.

Lemma 5.9. *For all nonnegative $\delta \in \tilde{\Delta}^- \cap \mathbb{N}^* \cup \{0\}$, we have $\|U_\delta\|_{p,\delta} \le \eta_\delta\sigma_\delta$.*

Proof. We prove it by induction on nonnegative elements of $\tilde{\Delta}^- \cup \{0\}$. For $\delta = 0$, this is obviously true since $\eta_0 = 1$ and $\sigma_0 = \|Id\|_{p,0}$. Let us assume that the lemma is true for all $0 \le \delta' < \delta$ in $\tilde{\Delta}^-$. According to estimates (15) and (16), we have

$$\begin{aligned}
\left(\min_{\lambda\in\sigma_{s+\delta,\setminus\mathcal{I}}}\sqrt{\lambda}\right)\|U_\delta\|_{p,\delta} &\le \sum_{\mu>s}^{s+\delta}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta}\frac{M}{\rho^r}\|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\\
&\quad + M'\sum_{s_*\le r\le s^*}\sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}}\|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\\
&\le \sum_{\mu>s}^{s+\delta}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta}\frac{M}{\rho^r}\eta_{\delta_1}\sigma_{\delta_1}\cdots\eta_{\delta_r}\sigma_{\delta_r}\\
&\quad + M'\sum_{s_*\le r\le s^*}\sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}}\eta_{\delta_1}\sigma_{\delta_1}\cdots\eta_{\delta_r}\sigma_{\delta_r}\\
&\le \left(\max_{s\le\mu\le s+\delta,\,\mu\in\tilde{\Delta}}\ \max^{*}_{\substack{\delta_1+\cdots+\delta_r+\mu=s+\delta\\ \mu_*\le r\le\mu^*}}\eta_{\delta_1}\cdots\eta_{\delta_r}\right)\sigma_\delta.
\end{aligned}$$

The second inequality is a consequence of the induction assumption. The last one gives the desired result. □

Lemma 5.10. *The formal power series $\sigma(t) := \sum_{i\in\tilde{\Delta}^-\cup\{0\},i\ge0}\sigma_i t^i$ converges in a neighborhood of the origin of $\mathbb{C}$.*

Proof. First of all, we notice that we have

$$\sum_{\mu>s}^{\delta+s}\sum_{r=\mu_*}^{\mu^*}\sum_{\delta_1+\cdots+\delta_r+\mu=s+\delta}\frac{M}{\rho^r}\sigma_{\delta_1}\cdots\sigma_{\delta_r} = M\sum_{\mu>s}^{\delta+s}\left\{\left(\frac{\sigma(t)}{\rho}\right)^{\mu_*}+\cdots+\left(\frac{\sigma(t)}{\rho}\right)^{\mu^*}\right\}_{\delta+s-\mu}.$$

Let us set

$$P_\mu(z) := \sum_{r=\mu_*}^{\mu^*}\left(\frac{z}{\rho}\right)^r \quad\text{and}\quad F(z,t) = M\sum_{\mu\in\tilde{\Delta},\mu>s}P_\mu(z)t^{\mu-s}. \tag{19}$$

The power series F defines a germ of holomorphic function at the origin of $\mathbb{C}^2$ which satisfies to $F(z,0)=0$. Then, the coefficient of t^δ is the Taylor expansion of $F(\sigma(t),t)$ at the origin of $\mathbb{C}$ is given by

$$\begin{aligned}\{F(\sigma(t),t)\}_\delta &= M\left\{\sum_{\mu>s} P_\mu(\sigma(t))t^{\mu-s}\right\}_\delta = M\left\{\sum_{\mu>s}\{P_\mu(\sigma(t))\}_{\delta-\mu+s}t^{\delta-\mu+s}t^{\mu-s}\right\}_\delta \\ &= M\sum_{\mu>s}^{\delta+s}\{P_\mu(\sigma(t))\}_{\delta-\mu+s}.\end{aligned}$$

On the other hand, let us set

$$P(z) := \sum_{r=s_*}^{s^*}\left(z^r - \sigma_0^r - r\sigma_0^{r-1}(z-\sigma_0)\right). \tag{20}$$

We have $P(\sigma_0)=0$ and $DP(\sigma_0)=0$. Then, we notice that

$$\{P(\sigma(t))\}_\delta = M\sum_{s_*\le r\le s^*}\ \sum_{\substack{\delta_1+\cdots+\delta_r=\delta\\(\delta_1,\cdots,\delta_r)\in\Omega_r}} \sigma_{\delta_1}\cdots\sigma_{\delta_r}$$

where Ω_r is given by (18). Let us set $G(z,t) := F(z,t)+P(z)$. Therefore, we have $\sigma_\delta = \{F(\sigma(t),t)+P(\sigma(t))\}_\delta$.

As a consequence, the power series $\sigma(t)$ is solution of the problem $G(\sigma(t),t)=(\sigma(t)-\sigma_0)$ together with $\sigma(0)=\sigma_0$. Since $D_zG(\sigma_0,0)=0$, then, according to the implicit function theorem, this problem has a unique holomorphic solution satisfying the same initial condition. □

Remark 5.11. *The order of $F(z,t)$ at $t=0$ is $\delta_0 := \max(\min_{\delta\in\tilde\Delta^-}\delta, 1)$.*

Therefore, according to the diophantiness property of S, there exists $M,c>0$ such that $\eta_\delta \le Mc^\delta$ for all positive $\delta\in\tilde\Delta$. Moreover, according to the previous lemma and to proposition 3.7, there exists $M',d>0$ such that $\sigma_\delta \le M'd^\delta$ for all positive $\delta\in\tilde\Delta^-$. Hence, according to lemma 5.9, we have, for all positive $\delta\in\tilde\Delta^-$, $\|U_\delta\|_{p,\delta}\le Mc^\delta$ for some positive constants M and c. Therefore, according to proposition 3.7, U is holomorphic in a neighborhood of the origin in $\mathbb{C}^n$. This concludes the proof of the main theorem.

6 Conjugacy to normal forms and approximation up to an exponentially small remainder

In this section we shall study the conjugacy problem to normal form. We shall show that if the "small divisors" are actually big, then there is a convergent normalizing transformation. On the other hand, we shall show that, if the "small divisors" are not too small then there exists a formal normalizing transformation which is not worst than Gevrey. From this, we will be able to obtain an optimal choice of the quasidegree α of normalization such that discrepancy between the partial conjugate and the partial normal form of quasidegree α is exponentially small in some twisted ball.

6.1 Normalization and cohomological equations

Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighborhood of the origin of $\mathbb{C}^n$ (i.e. the quasiorder of R at the origin is greater than s). Proposition 4.4 ensures that for every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$, there exists a polynomial diffeomorphism tangent to identity $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \sum_{0<\delta\leq\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta$ such that $(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha}$, where $\mathcal{N}_\alpha = \sum_{s<\delta\leq\alpha} N_\delta$, $N_\delta \in \mathrm{Ker}\,\Box_\delta$, and where $\mathcal{R}_{>\alpha}$ is of quasiorder $> \alpha$. We recall that in the expansions of $\mathcal{N}_\alpha$ and $\mathcal{U}_\alpha$, $U_\delta = 0$ and $N_\delta = 0$ for $\delta \notin \widetilde{\Delta}$ since $\mathcal{H}_\delta = \{0\}$. A basic identification of the quasihomogeneous components for $\delta \in \widetilde{\Delta}$ with $s < \delta \leq \alpha$ leads to

$$\Big\{\mathcal{N}_\alpha + [S, \mathcal{U}_\alpha]\Big\}_\delta = \Big\{R(\mathrm{Id} + \mathcal{U}_\alpha) - D\mathcal{U}_\alpha.\mathcal{N}_\alpha + S(\mathrm{Id} + \mathcal{U}_\alpha) - S - DS.\mathcal{U}_\alpha\Big\}_\delta \tag{21}$$

Hence, using proposition 3.4, lemma 3.5 and (18), we get the following hierarchy of **cohomological equations** in $\mathcal{H}_\delta$ for $\delta \in \widetilde{\Delta}$ with $s < \delta \leq \alpha$:

$$N_\delta + d_0(U_{\delta-s}) = K_\delta \tag{22}$$

with

$$\begin{aligned} K_\delta = & \sum_{\mu>s,\ \mu\in\widetilde{\Delta}} \sum_{r=\mu_*}^{\mu^*} \sum_{\substack{\delta_1+\cdots+\delta_r+\mu=\delta \\ \delta_i\geq 0}} \tilde{R}_{\mu,r}(U_{\delta_1}, \ldots, U_{\delta_r}) - \sum_{\substack{\delta_1+\delta_2=\delta \\ \delta_1>0,\ \delta_2>s, \delta_2\in\tilde{\Delta}}} DU_{\delta_1}.N_{\delta_2} \\ & + \sum_{r=s_*}^{s^*} \sum_{\substack{\delta_1+\cdots+\delta_r+s=\delta \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} \tilde{S}_{\bullet,r}(U_{\delta_1}, \ldots, U_{\delta_r}) \end{aligned} \tag{23}$$

where by convention $U_0 = \mathrm{Id}$ and where Ω_r is given by (18). Moreover, if not specified, **the δ_i's belong to $\widetilde{\Delta}^- = \widetilde{\Delta} \cap (\tilde{\Delta} - s)$** in the previous sums.

Then, observe that (23) ensures that K_δ depends only on R and S which are given and on N_β and $U_{\beta-s}$ for $s < \beta < \delta$. So the hierarchy of equations (22) for $s < \delta \leq \alpha$ can be solved by induction starting with the smallest $\delta \in \tilde{\Delta}$ greater than s.

Let us denote π_δ the orthogonal projection on $(\mathrm{Ker}\,\Box_\delta)^\perp = (\mathrm{Ker}\, d^*_{0|\mathcal{H}_\delta})^\perp = \mathrm{Im}\, d_{0|\mathcal{H}_{\delta-s}}$. Since $N_\delta \in \mathrm{Ker}\,\Box_\delta$, (22) is equivalent to

$$N_\delta = (Id - \pi_\delta)(K_\delta), \qquad d_0(U_{\delta-s}) = \pi_\delta(K_\delta). \tag{24}$$

Remark 6.1. *Observe that if $\delta - s \notin \widetilde{\Delta}$, $U_{\delta-s} = 0$ and $N_\delta = K_\delta$ since $\mathcal{H}_{\delta-s} = \{0\}$.*

To compute by induction upper bounds of N_δ and $U_{\delta-s}$, we use the normalized norms

$$\begin{aligned} &\nu_s = 0, && \nu_\delta = \|N_\delta\|_{p,\delta} \quad \text{for } \delta \in \widetilde{\Delta},\ \delta > s, \\ &u_0 = \|U_0\|_{p,0} = \|\mathrm{Id}\|_{p,0} = \sqrt{\tfrac{1}{(p_1)!} + \cdots + \tfrac{1}{(p_n)!}}, && u_\delta = \|U_\delta\|_{p,\delta} \quad \text{for } \delta \in \widetilde{\Delta}, \delta > 0. \end{aligned}$$

We set $u_\delta = 0$ if $\delta + s \notin \tilde{\Delta}$ and $\nu_s = 0$. Then, since π_δ is orthogonal and using lemma 4.8, we deduce from (24) that, for all $\delta \in \tilde{\Delta}$,

$$\nu_\delta = \|N_\delta\|_{p,\delta} \leq \|K_\delta\|_{p,\delta}, \qquad u_{\delta-s} \leq \frac{1}{\min\limits_{\lambda\in\mathrm{spec}\,\Box_\delta\setminus\{0\}} \sqrt{\lambda}} \|K_\delta\|_{p,\delta}. \tag{25}$$

Finally, the submultiplicativity of the norms given by proposition 3.6, we get that there exist $M > 0$ such that for every $\delta \in \widetilde{\Delta}$ with $\delta > s$,

$$\|K_\delta\|_{p,\delta} \leq k_\delta \tag{26}$$

with

$$k_\delta = M\Bigg(\sum_{\mu>s,\ \mu\in\widetilde{\Delta}} \sum_{\substack{\mu_*\leq r\leq\mu^* \\ \delta_1+\cdots+\delta_r+\mu=\delta}} \frac{u_{\delta_1}\cdots u_{\delta_r}}{\rho^r} + \sum_{\substack{1\leq\delta_1\leq\delta-s-1 \\ \delta_1\in\tilde{\Delta}^-,\ \delta-\delta_1\in\tilde{\Delta}}} \delta_1^{\max(1,\frac{\bar{p}}{2})} u_{\delta_1}\nu_{\delta-\delta_1} + \sum_{\substack{s_*\leq r\leq s^* \\ \delta_1+\cdots+\delta_r+s=\delta \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} u_{\delta_1}\dots u_{\delta_r} \Bigg) \tag{27}$$

where in the first and the last sums, the δ_i's belong to $\tilde{\Delta}^-$ and where Ω_r is defined by (18).

6.2 Convergent conjugacy to a normal form

Let us set

$$\nu := \max\left(1, \tfrac{\bar{p}}{2}\right).$$

Theorem 6.2. *Assume that there exists constant $c > 0$ such that for all $\delta \in \tilde{\Delta}^+$,*

$$\min_{\lambda\in\mathrm{spec}\,\Box_\delta\setminus\{0\}} \sqrt{\lambda} > c^{-1}(\delta - s)^\nu.$$

Then, any good holomorphic perturbation X of S is holomorphically conjugate to a normal form.

Proof. Let us set $\gamma_0 = 1$ and if $\delta \in \tilde{\Delta}^+$ with $\delta > s$

$$\gamma_{\delta-s} := \tilde{M}\Bigg(\sum_{\mu>s,\ \mu\in\widetilde{\Delta}} \sum_{\substack{\mu_*\leq r\leq\mu^* \\ \delta_1+\cdots+\delta_r+\mu=\delta}} \left(\frac{u_0}{\rho}\right)^r \gamma_{\delta_1}\cdots\gamma_{\delta_r} + \sum_{\substack{1\leq\delta_1\leq\delta-s-1 \\ \delta_1\in\tilde{\Delta}^-,\ \delta-\delta_1\in\tilde{\Delta}}} \gamma_{\delta_1}\gamma_{\delta-\delta_1-s} + \sum_{\substack{s_*\leq r\leq s^* \\ \delta_1+\cdots+\delta_r+s=\delta \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} \gamma_{\delta_1}\cdots\gamma_{\delta_r} \Bigg).$$

Here, we have set $\tilde{M} := \max\left(Mcu_0, \frac{Mc}{u_0}, Mcu_0^{s^*-1}, Mcu_0^{s_*-1}\right)$. We claim that

$$\nu_\delta \leq u_0\ \gamma_{\delta-s}, \qquad (\delta-s)^\nu u_{\delta-s} \leq u_0\ \gamma_{\delta-s}. \tag{28}$$

Let us prove these inequalities by induction on $\delta \geq s$. This is obviously true for $\delta = s$. According to equations (27) and (26), we have for $\delta > s$

$$(\delta-s)^\nu u_{\delta-s} \leq M\Bigg(\sum_{\mu>s,\ \mu\in\widetilde{\Delta}} \sum_{\substack{\mu_*\leq r\leq\mu^* \\ \delta_1+\cdots+\delta_r+\mu=\delta}} \left(\frac{u_0}{\rho}\right)^r \gamma_{\delta_1}\cdots\gamma_{\delta_r} + \sum_{\substack{1\leq\delta_1\leq\delta-s-1 \\ \delta_1\in\tilde{\Delta}^-,\ \delta-\delta_1\in\tilde{\Delta}}} \gamma_{\delta_1}\gamma_{\delta-\delta_1-s} + \sum_{\substack{s_*\leq r\leq s^* \\ \delta_1+\cdots+\delta_r+s=\delta \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} \gamma_{\delta_1}\cdots\gamma_{\delta_r} \Bigg),$$

where, in the first and the last sum, we have used the fact that, if $\delta_i > s$, $u_{\delta_i-s} \leq (\delta_i - s)^\nu u_{\delta_i-s} \leq u_0\ \gamma_{\delta_i-s}$ as well as $u_0 \leq u_0\gamma_0$. Therefore, we obtain $(\delta-s)^\nu u_{\delta-s} \leq \gamma_{\delta-s}$. In the same way, we have $\nu_\delta \leq \gamma_{\delta-s}$. Let us define the formal power series

$$\gamma(t) := \sum_{i\in\tilde{\Delta}\cap(\tilde{\Delta}-s),i\geq 0} \gamma_i t^i,$$

Let $G(z,t) := F(z,t) + P(z)$ be the function defined by equation (19) and (20) where, in these formulas, ρ is replaced by ρ/u_0, M by M' and σ_0 by 1.

Let $\delta \in \tilde{\Delta}$ such that $\delta > s$. As we have seen above, we have

$$\sum_{\mu>s}^{\delta} \sum_{r=\mu_*}^{\mu^*} \sum_{\delta_1+\cdots+\delta_r+\mu=\delta} M' \left(\frac{u_0}{\rho}\right)^r \gamma_{\delta_1} \cdots \gamma_{\delta_r} + M' \sum_{\substack{s_* \le r \le s^* \\ \delta_1+\cdots+\delta_r+s=\delta \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} \gamma_{\delta_1} \cdots \gamma_{\delta_r} = \{G(\gamma(t),t)\}_{\delta-s}$$

where, in the first sum, the δ_i's are nonnegative elements of $\tilde{\Delta}^-$ and the μ's are elements of $\tilde{\Delta}$. We recall that $\{G(\gamma(t),t)\}_{\delta-s}$ denotes the coefficient of $t^{\delta-s}$ in the Taylor expansion at the origin of the formal power series $G(\gamma(t),t)$. Furthermore, we have,

$$\sum_{\substack{1\le\delta_1\le\delta-s-1 \\ \delta_1\in\tilde{\Delta}^-,\, \delta-\delta_1\in\tilde{\Delta}}} \gamma_{\delta_1}\gamma_{\delta-\delta_1-s} = \{(\gamma(t)-1)^2\}_{\delta-s}$$

Hence, $\gamma(t)$ is solution of the holomorphic implicit function problem:

$$G(\gamma(t),t) + (\gamma(t)-1)^2 = \gamma(t) - 1$$

with initial condition $\gamma(0) = 1$. Since $G(1,0) = 0$ and $D_zG(1,0) = 0$, γ is the unique holomorphic solution of this problem. Therefore, for all positive $\delta \in \tilde{\Delta}^+$, we have $u_\delta \le \gamma_\delta \le C^\delta$. Hence, the formal power series $\sum U_\delta$ converges in a neighborhood of the origin, that is to say the normalizing transformation Φ^{-1} is holomorphic in an neighborhood of the origin of $\mathbb{C}^n$.

□

Remark 6.3. *If S is a diagonal linear vector field, then the situation described by the previous theorem corresponds to the Poincaré domain [Arn80]. In fact, by definition, the closed convex hull of the eigenvalues λ_i in the complex plane does not contain the origin. Hence, if $Q \in \mathbb{N}^n$ is such that $|Q| = q_1+\cdots+q_n$ is large enough, then $|q_1\lambda_1+\cdots+q_n\lambda_n-\lambda_i| \ge m|Q|$.*

6.3 Formal Gevrey conjugacy to a normal form

Assume that S satisfies the following Siegel type condition : there exists $c \ge 1$ and $\tau \ge 0$ such that for every $\delta \in \tilde{\Delta}$ with $\delta \ge s$, we have

$$\frac{1}{(\delta-s)^\tau} \le c \min_{\lambda\in\mathrm{spec}\,\Box_\delta\setminus\{0\}} \sqrt{\lambda}. \tag{29}$$

Our aim is to is to show that both ν_δ and $u_{\delta-s}$ admit Gevrey estimates. Namely we prove in this section the following result :

Theorem 6.4. *Assume that S satisfies* (29)*. Any good holomorphic perturbation of S admits a formal transformation to a formal normal form both of which are $\bar{p}(\frac{a}{\delta_0}+\tau)$-Gevrey power series where $\delta_0 := \max(\min_{\delta\in\tilde{\Delta}^-}\delta, 1)$ and $a := \max\left(1, \left[\frac{(\bar{p}+1)}{2}\right]\right)$.*

The following lemma gives such an estimate using a common majorizing power series.

Lemma 6.5. *Let $\{\beta_{\delta-s}\}_{\delta\in\widetilde{\Delta}\cap(\tilde{\Delta}+s),\ \delta\geq s}$ be the sequence defined by induction with $\beta_0=1$ and for $\delta\in\widetilde{\Delta}^+,\ \delta>s$,*

$$\beta_{\delta-s}=M'\Bigg(\sum_{\mu>s,\ \mu\in\widetilde{\Delta}}\ \sum_{\substack{\mu_*\leq r\leq\mu^*\\ \delta_1+\cdots+\delta_r+\mu=\delta}}\Big(\frac{u_0}{\rho}\Big)^r\beta_{\delta_1}\cdots\beta_{\delta_r}$$
$$+\sum_{\substack{1\leq\delta_1\leq\delta-s-1\\ \delta_1\in\tilde{\Delta}\cap(\tilde{\Delta}-s),\ \delta-\delta_1\in\tilde{\Delta}}}\delta_1(\delta_1-1)\cdots(\delta_1-a+1)\beta_{\delta_1}\beta_{\delta-s-\delta_1}+\sum_{\substack{s_*\leq r\leq s^*\\ \delta_1+\cdots+\delta_r+s=\delta\\ (\delta_1,\cdots,\delta_r)\in\Omega_r}}\beta_{\delta_1}\dots\beta_{\delta_r}\Bigg)$$

where a is the smallest integer larger or equal to $\nu=\max(1,\frac{\overline{p}}{2})$ and where in the first and last sums the δ_i's belong to $\widetilde{\Delta}^-$, Ω_r is given by (18) and s^* is defined by (1). Here, we have set $M':=\max\Big(Mcu_0\mathfrak{m},\frac{Mc}{u_0},Mcu_0^{s^*-1},Mcu_0^{s_*-1}\Big)$ with $\mathfrak{m}=\sup\limits_{\delta\in\widetilde{\Delta}}\frac{\delta^a}{(\delta-1)\cdots(\delta-a+1)}$. Then for every $\delta\in\widetilde{\Delta}^+$ with $\delta\geq s$,

$$\nu_\delta\leq u_0\ ((\delta-s)!)^\tau\beta_{\delta-s},\qquad u_{\delta-s}\leq u_0\ ((\delta-s)!)^\tau\beta_{\delta-s}. \tag{30}$$

Remark 6.6. *$\delta\in\tilde{\Delta}^+$ if and only if $\delta-s\in\tilde{\Delta}^-$.*

Proof. The proof is made by induction. We first observe that (30) holds for $\delta=s$ since $\beta_0=1$ and $\nu_s=0$.

Then, let $\delta>s$ and assume that (30) holds for every $\alpha\in\widetilde{\Delta}^+$ satisfying $s\leq\alpha<\delta$. Our aim is now to prove that (30) holds for δ. We proceed in several steps.

Step 1. We start with $u_{\delta-s}$. Using (26) and (27), we get

$$\begin{aligned}\frac{u_{\delta-s}}{u_0((\delta-s)!)^\tau}\leq\ &\frac{Mc(\delta-s)^\tau}{u_0((\delta-s)!)^\tau}\sum_{1\leq\delta_1\leq\delta-s-1}u_0^2\ \delta_1^a\beta_{\delta_1}\beta_{\delta-s-\delta_1}\Big((\delta_1)!\ (\delta-\delta_1-s)!\Big)^\tau\\
&+\frac{Mc(\delta-s)^\tau}{u_0((\delta-s)!)^\tau}\sum_{\mu>s,\ \mu\in\widetilde{\Delta}}\ \sum_{\substack{\mu_*\leq r\leq\mu^*\\ \delta_1+\cdots+\delta_r+\mu=\delta}}\Big(\frac{u_0}{\rho}\Big)^r\beta_{\delta_1}\cdots\beta_{\delta_r}\ (\delta_1!)^\tau\cdots(\delta_r!)^\tau\\
&+\frac{Mc(\delta-s)^\tau}{u_0((\delta-s)!)^\tau}\sum_{\substack{s_*\leq r\leq s^*\\ \delta_1+\cdots+\delta_r+s=\delta\\ (\delta_1,\cdots,\delta_r)\in\Omega_r}}(u_0)^r\ \beta_{\delta_1}\dots\beta_{\delta_r}\ \ (\delta_1!)^\tau\cdots(\delta_r!)^\tau\\
\leq\ &Mcu_0\ \mathfrak{m}\sum_{1\leq\delta_1\leq\delta-s-1}\delta_1(\delta_1-1)\cdots(\delta-a+1)\beta_{\delta_1}\beta_{\delta-s-\delta_1}(D_{\delta-s,\delta_1,\delta-s-\delta_1})^\tau\\
&+\frac{Mc}{u_0}\sum_{\mu>s,\ \mu\in\widetilde{\Delta}}\ \sum_{\substack{\mu_*\leq r\leq\mu^*\\ \delta_1+\cdots+\delta_r+\mu=\delta}}\Big(\frac{u_0}{\rho}\Big)^r\beta_{\delta_1}\cdots\beta_{\delta_r}\ (D_{\delta-s,\delta_1,\cdots,\delta_r})^\tau\\
&+Mc\max((u_0)^{s^*-1},u_0^{s_*-1})\sum_{\substack{s_*\leq r\leq s^*\\ \delta_1+\cdots+\delta_r+s=\delta\\ (\delta_1,\cdots,\delta_r)\in\Omega_r}}\beta_{\delta_1}\dots\beta_{\delta_r}\ (D_{\delta-s,\delta_1,\cdots,\delta_r})^\tau\end{aligned} \tag{31}$$

where $D_{\delta-s,\delta_1,\cdots,\delta_r}=\frac{\delta_1!\cdots\delta_r!}{(\delta-s-1)!}$.

Step 1.1 Observe that if we set $M' = \max\left(Mcu_0\mathfrak{m}, \frac{Mc}{u_0}, Mcu_0^{s^*-1}, Mcu_0^{s_*-1}\right)$ then $Mcu_0\mathfrak{m} \leq M'$, $\frac{Mc}{u_0} \leq M'$, $Mc\max((u_0)^{s^*-1}, u_0^{s_*-1}) \leq M'$.

Step 1.2 Then, in the first sum of (31), $D_{\delta-s,\delta_1,\delta-s-\delta_1} = \frac{\delta_1!(\delta-s-\delta_1)!}{(\delta-s-1)!} = \frac{\delta-s}{C^{\delta_1}_{\delta-s}} \leq 1$ holds since $1 \leq \delta_1 \leq \delta - s - 1$.

Step 1.3 Our aim is now to prove that for every index in the second sum of (31), $D_{\delta-s,\delta_1,\cdots,\delta_r} \leq 1$. For that purpose when need to distinguish three cases.

case 1: $r \geq 2$ and $\delta_j \geq 1, 1 \leq j \leq r$. It is proved in [IL05], p.20, that for $r \geq 2$, $\delta_j \geq 1$, and $\delta_1 + \cdots + \delta_r = d$ $D_{d,\delta_1,\cdots,\delta_r} \leq 1$. So, in the second sum of (31) for $r \geq 2$ and $\delta_j \geq 1$, we have

$$D_{\delta-s,\delta_1,\cdots,\delta_r} = D_{\delta-\mu,\delta_1,\cdots,\delta_r}\frac{(\delta-\mu-1)!}{(\delta-s-1)!} \leq 1$$

since $\delta_1 + \cdots + \delta_r = \delta - \mu$ and $s < \mu \leq \delta$.

case 2. In the second sum of (31), if $r = 1$ (which implies $\delta_1 = \delta - \mu$) or if all the indexes vanishes except one, then

$$D_{\delta-s,\delta_1,\cdots,\delta_r} = \frac{(\delta-\mu)!}{(\delta-s-1)!} \leq 1$$

since $\mu > s$.

case 3. Finally, if some indexes δ_j vanish in the second sum of (31), then the computation of the corresponding $D_{\delta-s,\delta_1,\cdots,\delta_r}$ can be made by removing these indexes, i.e. by decreasing r.

So, for every index in the second sum of (31), $D_{\delta-s,\delta_1,\cdots,\delta_r} \leq 1$.

Step 1.4 Finally, in the third sum of (31), $D_{\delta-s,\delta_1,\cdots,\delta_r} \leq 1$ still holds for the same reasons as above, observing that in this case there are at least two positive indexes δ_j, i.e case 2 is not possible in the third sum..

Gathering the results of substeps 1.1,$\cdots$,1.4, we can conclude that

$$\frac{u_{\delta-s}}{u_0((\delta-s)!)^\tau} \leq M'\beta_{\delta-s}$$

where $M' = \max\left(Mcu_0, \frac{Mc}{u_0}, Mcu_0^{s^*-1}, Mcu_0^{s_*-1}\right)$ does not depend on δ.

Step 2 The the computation of the upper bound for ν_δ is performed exactly in the same way. □

Remark 6.7. *If the good perturbation is a formal α-Gevrey power series, then the estimate $\frac{1}{\rho^r}$ of $|\tilde{R}_{\mu,r}|$ has to be changed to $\frac{1}{\rho^r}(r!)^\alpha$. Then, inequality (26) is changed to $\|K_\delta\|_{p,\delta} \leq (\delta^*!)^\alpha k_\delta$. Since $\delta^* \leq \frac{\delta+\bar{p}}{p}$, then according to proof in section A.4, we have $(\delta^*!)^\alpha \leq (\delta+\bar{p})!^{\frac{\alpha}{p}}$. Hence, using lemma 6.8, we obtain estimates of the form*

$$\nu_\delta, u_{\delta-s} \leq MC^\delta((\delta-s)!)^{\frac{\alpha}{p}+\tau+\frac{a}{\delta_0}}$$

for some positive constants M, C. According to lemme 3.8, the formal normalizing transformation and the normal form are both $\bar{p}(\frac{\alpha}{\underline{p}} + \tau + \frac{a}{\delta_0})$-Gevrey.

In the homogeneous case, $p = (1, \dots, 1)$, the formal normalizing transformation and the normal form are both $(\alpha + \tau + 1)$-Gevrey.

6.3.1 Gevrey estimates for the β_i's

Let us define the formal power series

$$\beta(t) := \sum_{i \in \tilde{\Delta} \cap (\tilde{\Delta} - s), i \geq 0} \beta_i t^i,$$

We recall that $\beta_0 = 1$. Let δ_0 be the order of $\beta - \beta_0$ at the origin. We recall that $\delta_0 := \max(\min_{\delta \in \tilde{\Delta}^-} \delta, 1)$ from remark 5.11 and $a := \max\left(1, \left[\frac{(\bar{p}+1)}{2}\right]\right)$.

Lemma 6.8. *The formal power series β is a $\left(\frac{a}{\delta_0}\right)$-Gevrey power series. More precisely, there exists positive constants M_β and C such that $\beta_i \leq M_\beta C^i [(i - \delta_0)!]^{a/\delta_0}$, for all integers $i \geq \delta_0$ that belong to $\tilde{\Delta}^-$.*

Remark 6.9. *With no loss of generality we can assume that M_β is large enough so that $M_\beta \geq 1$ and $\frac{2M_\beta u_0}{\rho} \geq 1$ hold.*

Let $G(z,t) := F(z,t) + P(z)$ as defined by equation (19) and (20) where, in these formulas, ρ is replaced by ρ/u_0, M by M' and σ_0 by 1.

Let $\delta \in \tilde{\Delta}$ such that $\delta > s$. As we have seen above, we can write

$$\sum_{\mu > s}^{\delta} \sum_{r=\mu_*}^{\mu^*} \sum_{\delta_1 + \cdots + \delta_r + \mu = \delta} M' \left(\frac{u_0}{\rho}\right)^r \beta_{\delta_1} \cdots \beta_{\delta_r} + M' \sum_{\substack{s_* \leq r \leq s^* \\ \delta_1 + \cdots + \delta_r + s = \delta \\ (\delta_1, \cdots, \delta_r) \in \Omega_r}} \beta_{\delta_1} \cdots \beta_{\delta_r} = \{G(\beta(t), t)\}_{\delta - s}$$

where, in the first sum, the δ_i's (resp. μ) are nonnegative elements of $\tilde{\Delta}^-$ (resp. $\tilde{\Delta}$). We recall that $\{G(\beta(t), t)\}_{\delta - s}$ denotes the coefficient of $t^{\delta - s}$ in the Taylor expansion at the origin of the formal power series $G(\beta(t), t)$. On the other hand, we have

$$\sum_{\substack{1 \leq \delta_1 \leq \delta - s \\ \delta_1 \in \tilde{\Delta} \cap (\tilde{\Delta} - s),\ \delta - \delta_1 \in \tilde{\Delta}}} \delta_1(\delta_1 - 1) \cdots (\delta_1 - a + 1) \beta_{\delta_1} \beta_{\delta - s - \delta_1} = \left\{ \beta t^a \frac{d^a \beta}{dt^a} \right\}_{\delta - s}.$$

Hence, according to the definition of the sequence $\{\beta_{\delta - s}\}_{\delta \in \tilde{\Delta}^+, \delta \geq s}$ in lemma 6.5, the formal power series $\beta(t)$ satisfies to the following differential equation

$$\beta(t) - \beta_0 = M(\beta(t) - \beta_0) t^a \frac{d^a \beta}{dt^a} + G(\beta(t), t) \tag{32}$$

Let us set

$$\beta(t) = \beta_0 + t^{\delta_0} B(t).$$

We have $B(0) = \beta_{\delta_0} \neq 0$ and $\beta_0 = 1$. We have

$$\frac{d^a(t^{\delta_0}B(t))}{dt^a} = \sum_{l=0}^{\min(a,\delta_0)} C_a^l \delta_0(\delta_0-1)\cdots(\delta_0-l+1)t^{\delta_0-l}\frac{d^{a-l}(B(t))}{dt^{a-l}}$$

Then, B satisfies the following differential equation

$$t^{\delta_0}B = MBt^{\delta_0+a}\left[\sum_{l=0}^{\min(a,\delta_0)} C_a^l \delta_0(\delta_0-1)\cdots(\delta_0-l+1)t^{\delta_0-l}\frac{d^{a-l}(B(t))}{dt^{a-l}}\right] + G(\beta(t),t).$$

Dividing by MB leads to the the equation

$$t^{\delta_0+a}\left[\sum_{l=0}^{\min(a,\delta_0)} C_a^l \delta_0(\delta_0-1)\cdots(\delta_0-l+1)t^{\delta_0-l}\frac{d^{a-l}(B(t))}{dt^{a-l}}\right] = \tilde{G}(B(t),t) := \frac{t^{\delta_0}B - G(1+t^{\delta_0}B(t),t)}{MB}$$

and $\tilde{G}(z,t)$ is holomorphic in a neighborhood of $(\beta_{\delta_0},0)$. We have

$$G(1+t^{\delta_0}B(t),t) = F(1,t) + t^{\delta_0}BD_zF(1,t) + O(t^{2\delta_0})$$

since $G(z,t) = F(z,t) + P(z)$ and $D_zP(1) = 0$. We recall that the order of $F(1,t)$ at $t=0$ is δ_0 according to remark 5.11. Let us set

$$\tilde{G}'(z,t) := \frac{t^{\delta_0}z - G(1+t^{\delta_0}z,t)}{t^{\delta_0}Mz}.$$

This function is holomorphic in a neighborhood of $(\beta_{\delta_0},0)$. Moreover, by construction, we have $\lim_{t\to 0}\frac{F(1,t)}{t^{\delta_0}} = \beta_{\delta_{\delta_0}}$. Hence, we have $\tilde{G}'(\beta_{\delta_0},0) = 0$. Furthermore, we have

$$\begin{aligned}
-t^{\delta_0}M\frac{\partial\tilde{G}'}{\partial z}(z,t) &= \frac{1}{z^2}\left(t^{\delta_0}\frac{\partial G}{\partial z}(1+t^{\delta_0}z,t)z - G(1+t^{\delta_0}z,t)\right)\\
&= \frac{1}{z^2}\left(t^{\delta_0}z(D_zF(1+t^{\delta_0}z,t) + D_zP(1+t^{\delta_0}z)) - G(1+t^{\delta_0}z,t)\right)
\end{aligned}$$

$$\frac{\partial\tilde{G}'}{\partial z}(\beta_{\delta_0},0) = (M\beta_{\delta_0}^2)^{-1}.\lim_{t\to 0}\frac{F(1,t)}{t^{\delta_0}} = (M\beta_{\delta_0})^{-1} \neq 0.$$

Hence, $B(t)$ is solution of the following differential equation

$$t^a\left[\sum_{l=0}^{\min(a,\delta_0)} C_a^l \delta_0(\delta_0-1)\cdots(\delta_0-l+1)t^{\delta_0-l}\frac{d^{a-l}(B(t))}{dt^{a-l}}\right] = \tilde{G}'(B(t),t). \tag{33}$$

Let us consider the Newton polygon of the linearized of the differential operator (33) at B:

$$L\psi := t^a\left[\sum_{l=0}^{\min(a,\delta_0)} C_a^l \delta_0(\delta_0-1)\cdots(\delta_0-l+1)t^{\delta_0-l}\frac{d^{a-l}(\psi)}{dt^{a-l}}\right] - \frac{\partial\tilde{G}'}{\partial z}(B(t),t)\psi.$$

It is the convex hull of $\{0\}\cup\{(u,v)\in\mathbb{R}^2 | u\leq a, v = a+\delta_0-l-(a-l) = \delta_0\}$. It contains only one positive (not infinite) slope : $\frac{\delta_0}{a}$.

According to the main theorem of [Mal89] (or theorem A.2.4.2 of [Sib90][p.209], which are both nonlinear versions of theorem 1.5.17 of [Ram84]), then either B is holomorphic in a neighborhood of the origin or B is a $\left(\frac{a}{\delta_0}\right)$-Gevrey power series. Therefore, $B_k \leq Mc^k(k!)^{a/\delta_0}$ for some constants. The shift in the factorial in the bound of β is only due to the formula $\beta(t) = 1 + t^{\delta_0}B(t)$.

Therefore, we obtain an estimate of the form $\|U_\delta\|_{p,\delta} \leq C^\delta(\delta!)^{\tau+\frac{a}{\delta_0}}$ and $\|N_\delta\|_{p,\delta} \leq C^\delta(\delta!)^{\tau+\frac{a}{\delta_0}}$ for some constant $C > 0$. We just conclude using lemma 3.8.

6.4 Optimal partial normal form with exponentially small remainder

This section is devoted to the proof of theorem 6.11 below which ensures that an optimal choice of the quasiorder α of the partial normal form given by proposition 4.4 enables to obtain an exponentially small remainder.

To state a precise theorem, we need to introduce the following "quasi norms": for $x \in \mathbb{C}^n$, let us define

$$d_p(x) := \left(\sum_{i=1}^n p_i|x_i|^{2/p_i}\right)^{1/2}.$$

Then, for f a complex-valued function defined in a neighborhood of the twisted ball $d_p(x) < \varepsilon$ we shall set

$$|f|_{qh,\varepsilon} := \sup_{d_p(x)<\varepsilon} |f(x)|.$$

If X is a vector field defined in a neighborhood of the "twisted ball" $d_p(x) < \varepsilon$, we shall set

$$\|X\|^2_{qh,\varepsilon} := \sum_{i=1}^n \frac{1}{\varepsilon^{2p_i}}|X_i|^2_{qh,\varepsilon}$$

The subscript *qh* stands for *quasihomogeneous* as these norms are adapted to quasihomogeneous objects.

Remark 6.10. *Before stating the theorem, we recall that lemma 3.3-(a),(b) ensures that $\widetilde{\Delta}$ contains all the sufficiently large integers. In other words, there exists δ_* such that for every $\alpha \in \mathbb{N}$, if $\alpha \geq \delta_*$ then α belongs to $\widetilde{\Delta}$.*

Theorem 6.11. *Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighborhood of the origin of $\mathbb{C}^n$ (i.e. the quasiorder of R at the origin is greater than s). Proposition 4.4 ensures that for every $\alpha \in \widetilde{\Delta}$, there exists a polynomial diffeomorphism tangent to identity $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \sum_{0<\delta\leq\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta$ such that*

$$(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha},$$

where $\mathcal{N}_\alpha = \sum_{s<\delta\leq\alpha} N_\delta$, $N_\delta \in Ker\,\Box_\delta$, and where $\mathcal{R}_{>\alpha}$ is of quasiorder $> \alpha$.

Assume that there exist $c \geq 1$ and $\tau \geq 0$ such that for every $\delta \in \widetilde{\Delta}$ with $\delta \geq s$, we have

$$\frac{1}{\min_{\lambda\in\mathrm{spec}\,\Box_\delta\setminus\{0\}}\sqrt{\lambda}} \leq c(\delta - s)^\tau. \tag{34}$$

Then, there exists $\theta \geq 4$, $M_{\rm opt} > 0$, $w_{\rm opt} > 0$ *and* $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in]0, \varepsilon_0[$, $\alpha_{\rm opt} = \left[\frac{1}{(\theta C\varepsilon)^b}\right] + s - 2$ *satisfies*

$$\alpha_{\rm opt} > s \qquad \textit{and} \qquad \alpha_{\rm opt} \geq \delta_*, \tag{35}$$

and

$$\|\mathcal{R}_{>\alpha_{\rm opt}}\|_{qh,\varepsilon} \leq M_{\rm opt} e^{-\frac{w_{\rm opt}}{\varepsilon^b}} \tag{36}$$

where $\frac{1}{b} = \tau + \frac{a}{\delta_0}$ *and* δ_* *is defined in remark 6.10.*

Proof. The proof of this theorem is based on the following proposition which is proved in appendix B .

Proposition 6.12. *Let* $K \geq 2$ *and* $\gamma \geq 2$ *be fixed such that*

$$\rho_1(K) < 1 \qquad \textit{with} \qquad \rho_1(K) = \frac{u_0 M_\beta \mathfrak{M}_p}{K^{\delta_0}} \sum_{k=0}^{+\infty} (k+\delta_0)^a \left(\tfrac{1}{2}\right)^k$$

and

$$\frac{\chi}{\gamma C} < 1 \qquad \textit{with} \qquad \chi = \left(\frac{2M_\beta u_0}{\rho}\right)^{\frac{1}{p}}$$

where a, δ_0, C *and* M_β *are defined in lemma 6.8.*

Then there exist $M_R > 0$, *such that for every* $\varepsilon \in]0,1[$ *and every* $\alpha \in \widetilde{\Delta}$ *with* $\alpha > s$ *satisfying*

$$C\varepsilon \leq \frac{1}{\gamma K(\alpha - s)^{\frac{1}{b}}}, \tag{37}$$

we have

$$\|\mathcal{R}_{>\alpha}\|_{qh,\varepsilon} \leq M_{\mathcal{R}} \left((C\varepsilon)^{\alpha+1} ((\alpha - s + 2)!)^{\frac{1}{b}} \ \Delta_\alpha + \left(\tfrac{1}{K}\right)^{\alpha+1} \right) \tag{38}$$

where $\Delta_\alpha = 1$ *if* $\frac{1}{b} \geq a$ *and* $\Delta_\alpha = (\alpha - s)^{1-a-\frac{2}{b}}$ *otherwise.*

We now prove theorem 6.11 in the case $\frac{1}{b} \geq a$. The other case can be deduced from this one by an appropriate change of the value of M_R and C. The key idea is to choose an appropriate value $\alpha_{\rm opt}$ for α in (38) and to show using Stirling formula that for this value of α the upper bound is exponentially small.

So, we choose $\alpha_{\rm opt}$ such that

$$\alpha_{\rm opt} - s + 2 = \left[\frac{1}{(\gamma K C\varepsilon)^b}\right] \tag{39}$$

which ensures that $\alpha_{\rm opt} - s \leq \frac{1}{(\gamma K C\varepsilon)^b}$ and so that (37) is satisfied. Moreover, observe that for ε sufficiently small, (35) is satisfied. Then we compute the upper bound given by the right hand side of (38) with this choice of α. For that purpose let us denote

$$D_1 := (C\varepsilon)^{\alpha_{\rm opt}+1} ((\alpha_{\rm opt} - s + 2)!)^{\frac{1}{b}}, \qquad D_2 := \left(\tfrac{1}{K}\right)^{\alpha_{\rm opt}+1}.$$

Let us denote $x = (C\varepsilon)^b$, $M_S = \sup_{k\in\mathbb{N}} \frac{ek!}{k^{k+\frac{1}{2}}e^{-k}}$ and observe that Stirling formula ensures that $M_S < \infty$ holds. Then, using (39) we check

$$\begin{aligned}
\tfrac{(D_1)^b}{M_S} &\leq e^{-1}x^{\left[\frac{1}{x(\gamma K)^b}\right]+s-1}\ \exp\left(\left\{\left[\tfrac{1}{x(\gamma K)^b}\right]+\tfrac{1}{2}\right\}\ln\left[\tfrac{1}{x(\gamma K)^b}\right]-\left[\tfrac{1}{x(\gamma K)^b}\right]\right)\\
&= \frac{x^{s-1}}{e}\ \exp\left(\left\{\left[\tfrac{1}{x(\gamma K)^b}\right]+\tfrac{1}{2}\right\}\ln\left[\tfrac{1}{x(\gamma K)^b}\right]+\left[\tfrac{1}{x(\gamma K)^b}\right]\ln\tfrac{x}{e}\right)\\
&\leq \frac{x^{s-1}}{e}\ \exp\left(\left\{\left[\tfrac{1}{x(\gamma K)^b}\right]+\tfrac{1}{2}\right\}\ln\left(\tfrac{1}{x(\gamma K)^b}\right)+\left[\tfrac{1}{x(\gamma K)^b}\right]\ln\tfrac{x}{e}\right)\\
&= \frac{x^{s-1}}{e}\ \exp\left(-\left[\tfrac{1}{x(\gamma K)^b}\right]\left(1+\ln(\gamma K)^b\right)\right)\exp\left(\tfrac{1}{2}\ln\left(\tfrac{1}{x(\gamma K)^b}\right)\right)\\
&= \frac{x^{s-\frac{3}{2}}}{e(\gamma K)^{\frac{b}{2}}}\ \exp\left(-\left[\tfrac{1}{x(\gamma K)^b}\right]\left(1+\ln(\gamma K)^b\right)\right)\\
&= x^{s-\frac{3}{2}}(\gamma K)^{\frac{b}{2}}\ \exp\left(-\left(\left[\tfrac{1}{x(\gamma K)^b}\right]+1\right)\left(1+\ln(\gamma K)^b\right)\right)\\
&\leq x^{s-\frac{3}{2}}(\gamma K)^{\frac{b}{2}}\ \exp\left(-\tfrac{1+\ln(\gamma K)^b}{x(\gamma K)^b}\right).
\end{aligned}$$

Hence

$$D_1 \leq M_S^{1/b}\ (C\varepsilon)^{s-\frac{3}{2}}\sqrt{\gamma K}\ e^{-\frac{w_1}{\varepsilon^b}} \qquad \text{with} \qquad w_1 = \frac{1+\ln(\gamma K)^b}{b(\gamma KC)^b}. \tag{40}$$

On the other hand we check

$$\Delta_2 = \left(\frac{1}{K}\right)^{\left[\frac{1}{(\gamma KC\varepsilon)^b}\right]+s-1} \leq \frac{1}{K^{s-2}}\exp\left(-\frac{\ln K}{(\gamma KC\varepsilon)^b}\right).$$

Hence,

$$D_2 \leq \frac{1}{K^{s-2}}e^{-\frac{w_2}{\varepsilon^b}} \qquad \text{with} \qquad w_2 = \frac{\ln K}{(\gamma KC)^b}. \tag{41}$$

Observing that $w_1 > w_2$, we can conclude that (36) holds with $w_{\text{opt}} = w_2$ and

$$M_{\text{opt}} = M_{\mathcal{R}}\ \max\left(\frac{1}{K^{s-2}}, M_S^{1/b}\ \sqrt{\gamma K}\ \sup_{\varepsilon\in]0,1]}\left\{(C\varepsilon)^{s-\frac{3}{2}}e^{\frac{w_2-w_1}{\varepsilon^b}}\right\}\right).$$

□

It might be useful to consider in some problems, parameters as variables to which one prescribe a weight. This has been done implicitly in [IL04] for instance.

7 Computations and example

Let $p = (p_1, \ldots, p_n) \in (\mathbb{N}^*)^n$. Let S be a quasihomogeneous vector field of quasidegree s and let $\mathcal{H}_\delta$ be the space of quasihomogeneous vector fields of quasidegree $\delta > s$. We recall that for each positive quasidegree k, the map $d_0 : H_\delta \to H_{\delta+s}$ is defined to be $d_0(U) = [S, U]$ where $[.,.]$ denotes the Lie bracket of vector fields.

7.1 Computation of d_0^* and $\Box$

Let $U \in \mathcal{H}_\delta$ and $V \in \mathcal{H}_{\delta+s}$. We write $U = \sum_{i=1}^n U_i \frac{\partial}{\partial x_i}$. We have

$$d_0(U) = \sum_{i=1}^n (S(U_i) - U(S_i)) \frac{\partial}{\partial x_i} \tag{42}$$

where $S(U_i) := \sum_{j=1}^n S_j \frac{\partial U_i}{\partial x_j}$ denotes the Lie derivative of U_i along S. We have

$$\begin{aligned}
\langle d_0(U), V \rangle_{p,\delta+s} &= \sum_{i=1}^n \langle S(U_i) - U(S_i), V_i \rangle_{p,\delta+s+p_i} \\
&= \sum_{i=1}^n \langle U_i, S^*(V_i) \rangle_{p,\delta+p_i} - \langle U(S_i), V_i \rangle_{p,\delta+s+p_i} \\
&= \sum_{i=1}^n \langle U_i, S^*(V_i) \rangle_{p,\delta+p_i} - \sum_{j=1}^n \left\langle U_j, \left(\frac{\partial S_i}{\partial x_j}\right)^* V_i \right\rangle_{p,\delta+p_j} \\
&= \sum_{i=1}^n \left\langle U_i, S^*(V_i) - \sum_{j=1}^n \left(\frac{\partial S_j}{\partial x_i}\right)^* V_j \right\rangle_{p,\delta+p_i}.
\end{aligned}$$

Hence, we can write d_0^* in a matrix form as

$$d_0^*(V) = \begin{pmatrix}
S^* - \left(\frac{\partial S_1}{\partial x_1}\right)^* & -\left(\frac{\partial S_2}{\partial x_1}\right)^* & \cdots & -\left(\frac{\partial S_n}{\partial x_1}\right)^* \\
-\left(\frac{\partial S_1}{\partial x_2}\right)^* & S^* - \left(\frac{\partial S_2}{\partial x_2}\right)^* & \cdots & -\left(\frac{\partial S_n}{\partial x_2}\right)^* \\
\vdots & & \ddots & \vdots \\
-\left(\frac{\partial S_1}{\partial x_n}\right)^* & \cdots & -\left(\frac{\partial S_{n-1}}{\partial x_n}\right)^* & S^* - \left(\frac{\partial S_n}{\partial x_n}\right)^*
\end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ \vdots \\ V_n \end{pmatrix}$$

Let us set $A_i := S - \frac{\partial S_i}{\partial x_i}$. The operator $d_0 d_0^*$ can be viewed as a matrix $(P_{i,j})_{1 \le i,j \le n}$ of differential operators defined as follow :

$$P_{i,j} = \delta_{i,j} S S^* - S \left(\frac{\partial S_j}{\partial x_i}\right)^* - \frac{\partial S_i}{\partial x_j} S^* + \sum_{k=1}^n \frac{\partial S_i}{\partial x_k} \left(\frac{\partial S_j}{\partial x_k}\right)^*$$

where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise.

In the homogeneous case, that is $p = (1, \ldots, 1)$, the adjoint operator, with respect the the hermitian product $\langle ., . \rangle_H$ (see section A.2), of the multiplication by x_i is $\frac{\partial}{\partial x_i}$. Hence, the adjoint operator with respect to $\langle ., . \rangle_{p,\delta}$ is equal to $x_{i|H_\delta}^* = \frac{1}{\delta} \frac{\partial}{\partial x_i}$. Hence, the adjoint operator S^* of the Lie derivative along S is defined $S^* : P_{k+s} \to P_k$ with

$$S^*(f) := \frac{k!}{(k+s)!} \sum_{i=1}^n x_i \bar{S}_i(\frac{\partial}{\partial x})(f).$$

Here, if $S_i(x) = \sum_{|Q|=s+1} s_{i,Q} x^Q$ then $\bar{S}_i(\frac{\partial}{\partial x}) := \sum_{|Q|=s+1} \bar{s}_{i,Q} \frac{\partial^{|Q|}}{\partial x^Q}$.

7.2 Example

In this section we shall completely treat the case where $S = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$, $p = (1,1)$ and $s = 1$. We have $S^*_{|H_n} = \frac{1}{n}\left(x\frac{\partial^2}{\partial x^2} + y\frac{\partial^2}{\partial x \partial y}\right) =: \frac{1}{n}A$. If $v \in \mathcal{H}_{n-1}$ then,

$$nd_0^*(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}) = \begin{pmatrix} A - 2\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} \\ 0 & A - \frac{\partial}{\partial x} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

(a) The resonances

Let us compute the kernel of d_0^*. Let (v_1, v_2) be a couple of formal power series of order ≥ 3 such that $d_0^*(v_1, v_2) = 0$. Then,

$$\begin{cases} (A - \frac{\partial}{\partial x})v_2 &= 0 \\ (A - 2\frac{\partial}{\partial x})v_1 &= \frac{\partial v_2}{\partial y}. \end{cases}$$

First of all, for any $(p,q) \in \mathbb{N}^2$ with $p+q \geq 3$, we have $A(x^py^q) = p(p+q-1)x^{p-1}y^q$. Hence, a formal power series f of order ≥ 3 such that $A(f) = 0$ is of the form $f(y)$. Since $(A - \frac{\partial}{\partial x})(x^py^q) = p(p+q-2)x^{p-1}y^q$, any formal power series f of order ≥ 3 such that $(A - \frac{\partial}{\partial x})(f) = 0$ is the form $f(y)$.

As a consequence, we have $v_2 = f(y)$ for some power series $f = \sum_{k\geq 3} f_k y^k$ and $(A - 2\frac{\partial}{\partial x})v_1 = \frac{\partial f}{\partial y}$. Let us write $v_1 = \sum_{p+q\geq 3} v_{1,p,q}x^py^q$. Then, we have

$$\sum_{p+q\geq 3} v_{1,p,q}p(p+q-3)x^{p-1}y^q = \sum_{q\geq 2} f_{q+1}(q+1)y^q.$$

This means that $v_{1,1,q} = \frac{q+1}{q-2}f_{q+1}$ if $q > 2$, $v_{1,p,q}$ with $p+q = 3$ or $p = 0$ is unspecified and $f_3 = 0$.

Finally, any holomorphic perturbation $X = S + R$ of S of quasiorder > 1 (i.e the components of R are of order > 2) admits a formal normal form of the type :

$$\begin{aligned} \frac{dx}{dt} &= x^2 + P_3(x,y) + x\sum_{k\geq 3}\frac{k+1}{k-2}f_{k+1}y^k + \hat{h}_4(y) \\ \frac{dy}{dt} &= xy + \sum_{k\geq 3} f_{k+1}y^{k+1} \end{aligned} \tag{43}$$

for some power series $\hat{h}_4$ of order ≥ 4, some numbers f_k and some homogeneous polynomial P_3 of degree 3.

(b) "The small divisors"

Let us consider the differential operators $A_1(f) := S(f) - 2xf$ and $A_2(f) := S(f) - xf$. We have, if $f_n \in H_n$, $nA_1^*(f_n) := A(f_n) - 2\frac{\partial f_n}{\partial x}$ and $nA_2^*(f_n) := A(f_n) - \frac{\partial f_n}{\partial x}$. Then, if $V \in \mathcal{H}_{n-1}$ then

$$nd_0d_0^*(V) = \begin{pmatrix} A_1A_1^* & -A_1 \circ \frac{\partial}{\partial y} \\ -yA_1^* & A_2A_2^* + y\frac{\partial}{\partial y} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

For each $n \geq 3$, the 1-dimensional vector space generated by $x^n \frac{\partial}{\partial y}$ is left invariant by $d_0 d_0^*$ and we have

$$n d_0 d_0^*(x^n \tfrac{\partial}{\partial y}) = n(n-2)(n-3) x^n \tfrac{\partial}{\partial y}.$$

For each $Q = (p,q) \in \mathbb{N}^2$ with $p \geq 1$, the vector subspace E_Q generated by $e_{1,Q} = x^p y^q \frac{\partial}{\partial x}$ and $e_{2,Q} = x^{p-1} y^{q+1} \frac{\partial}{\partial y}$ is invariant by $d_0 d_0^*$. Its restriction to it given in the basis $\{e_{1,Q}, e_{2,Q}\}$ by

$$n d_0 d^*_{0|E_Q}(v,w) = \begin{pmatrix} p(p-3+q)^2 & -(q+1)(p-3+q) \\ -p(p-3+q) & (p-1)(p+q-2)^2 + (q+1) \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}.$$

Its smallest eigenvalue for $Q = (p, n-p)$ is

$$\begin{aligned} n\lambda_-(n,p) &:= (p - \tfrac{1}{2})n^2 + \tfrac{5}{2} n(1-2p) + (6p - \tfrac{3}{2}) \\ &\quad -(1/2)\sqrt{9 + 72p + (31+12p)n^2 - 30n(1-2p) - 10n^3 + n^4} \end{aligned}$$

Since $1 \leq p \leq n$, then for large enough n, we have

$$\begin{aligned} 9 + 72p + (31+12p)n^2 - 30n(1-2p) - 10n^3 + n^4 &\leq 9 + 72n + (31+12n)n^2 \\ &\quad -30n + 60n^2 - 10n^3 + n^4 \\ &\leq \tfrac{9}{4} n^4. \end{aligned}$$

Hence, we have

$$\begin{aligned} n\lambda_-(n,p) &\geq n^2 \left(p - \tfrac{1}{2}\left(1 + \tfrac{3}{2}\right)\right) \\ &\quad + \tfrac{5}{2} n(1-2p) + (6p - \tfrac{3}{2}) =: g(p) \end{aligned}$$

Let us find the smallest value of this lower bound $g(p)$ when p ranges from to 2 to n, n being a large enough fixed integer. We have $g'(p) = n^2 - 5n + 6 = (n-2)(n-3)$ which is positive if $n > 3$. Hence, g is an increasing function of p. Finally, we have for n large enough and $n \geq p \geq 2$,

$$n\lambda_-(n,p) \geq n^2 \left(2 - \tfrac{5}{4}\right) - \tfrac{15}{2} n + (12 - \tfrac{3}{2}) \tag{44}$$

and

$$n\lambda_-(n,1) = -5n + 9 + n^2. \tag{45}$$

Moreover, we have $H_n = \oplus_{p=1}^n E_{p,n-p} \oplus \mathbb{C} x^n \frac{\partial}{\partial y} \oplus \mathbb{C} y^n \frac{\partial}{\partial x}$ and $d_0 d_0^*(y^n \frac{\partial}{\partial x}) = 0$. As consequence, there exists a positive constant M such that, if n is large enough, then $\min_{\lambda \in Spec(\Box_n) \setminus \{0\}} \sqrt{\lambda} \geq M\sqrt{n}$.

A Inner products and analyticity

A.1 Decomposition as sum of quasihomogeneous components

This subsection is devoted of the computations of homogeneous and quasi homogeneous components of products, derivatives and composition of functions and vector fields.

Lemma A.1 (Components of the product)**.** *Let $f, g \in \mathbb{C}[[x_1, \cdots x_n]]$ and $U, S \in (\mathbb{C}[[x_1, \cdots x_n]])^n$. Then,*

(a) $\{fg\}_{\bullet,r} = \sum_{r_1+r_2=r} f_{\bullet,r_1} g_{\bullet,r_2}, \quad \{fV\}_{\bullet,r} = \sum_{r_1+r_2=r} f_{\bullet,r_1} V_{\bullet,r_2};$

(b) $\{fg\}_{\delta} = \sum_{\delta_1+\delta_2=\delta} f_{\delta_1} g_{\delta_2}, \quad \{fV\}_{\delta} = \sum_{\delta_1+\delta_2=\delta} f_{\delta_1} V_{\delta_2};$

Lemma A.2 (Components of the derivatives)**.** *Let $f \in \mathbb{C}[[x_1, \cdots x_n]]$ and $U, S \in (\mathbb{C}[[x_1, \cdots x_n]])^n$. Denote by $S(f)$ the Lie derivative of f along S and by $[S, U]$ the Lie brackets of S and U. Then,*

(a) $\{S(f)\}_{\bullet,r} = \sum_{r_1+r_2=r+1} S_{\bullet,r_1}(f_{\bullet,r_2}), \quad \{DS.U\}_{\bullet,r} = \sum_{r_1+r_2=r+1} DS_{\bullet,r_1}.U_{\bullet,r_2}, \quad \{[U,S]\}_{\bullet,r} = \sum_{r_1+r_2=r+1} [S_{\bullet,r_1}, U_{\bullet,r_2}];$

(b) $\{S(f)\}_{\delta} = \sum_{\delta_1+\delta_2=\delta} S_{\delta_1}(f_{\delta_1}), \quad \{DS.U\}_{\delta} = \sum_{\delta_1+\delta_2=\delta} DS_{\delta_1}.U_{\delta_2}, \quad \{[U,S]\}_{\delta} = \sum_{\delta_1+\delta_2=\delta} [S_{\delta_1}, U_{\delta_2}];$

Proof. The proofs of the above three lemmas follow directly from definition and from proposition 3.4 □

The following lemma gives a characterization of quasihomogeneous polynomial and vector fields of given quasidegree. This characterization happens to be very convenient to compute the quasihomogeneous components of compositions.

Lemma A.3. *Let us define $t^p.x := (t^{p_1}x_1, \ldots, t^{p_n}x_n)$. Then, a polynomial P is p-quasihomogeneous of degree δ if and only if $P(t^p.x) = t^\delta P(x)$. Furthermore, a vector field is p-quasihomogeneous of degree δ if and only if $X(t^p.x) = t^\delta (t^p.X(x))$.*

Proof. The proof is immediate. □

Lemma A.4 (Components of the composition)**.** *Let $f \in \mathbb{C}[[x_1, \cdots x_n]]$ and $U, V \in (\mathbb{C}[[x_1, \cdots x_n]])^n$. Then,*

(a) $\{f \circ U\}_{\delta'} = \sum_{\substack{\delta \le \delta',\ \frac{\delta}{\underline{p}} \le r \le \frac{\delta}{\underline{p}} \\ \delta+\delta_1+\cdots+\delta_r=\delta'}} \widetilde{f}_{\delta,r}(U_{\delta_1}, \cdots U_{\delta_r}),$

(b) $\{V \circ U\}_{\delta'} = \sum_{\substack{\delta \le \delta',\ \delta_* \le r \le \delta^* \\ \delta+\delta_1+\cdots+\delta_r=\delta'}} \widetilde{V}_{\delta,r}(U_{\delta_1}, \cdots U_{\delta_r}),$

where δ_* and δ^* are defined in (1).

Proof. The proof is based on the characterization of the δ- quasihomogeneous components given by lemma A.3. Indeed, using that f_δ is quasihomogeneous of quasidegree δ and that $\widetilde{f}_{\delta,r}$ is r-linear, we have

$$\begin{aligned}
f(U)(t^p.x) &= \sum_{\delta\in\Delta} f_\delta\Big(\Big(\sum_{d\in\widetilde{\Delta}} U_d(t^p.x)\Big)\Big)\\
&= \sum_{\delta\in\Delta} f_\delta\Big(t^p.\Big(\sum_{d\in\widetilde{\Delta}} t^d U_d(x)\Big)\Big)\\
&= \sum_{\delta\in\Delta} t^\delta f_\delta\Big(\sum_{d\in\widetilde{\Delta}} t^d U_d(x)\Big)\\
&= \sum_{\delta\in\Delta} t^\delta \sum_{\frac{\delta}{p}\le r\le\frac{\delta}{\underline{p}}} \widetilde{f}_{\delta,r}\Big(\sum_{\delta_1\in\widetilde{\Delta}} t^{\delta_1}U_{\delta_1}(x),\cdots,\sum_{\delta_r\in\widetilde{\Delta}} t^{\delta_r}U_{\delta_r}(x)\Big)\\
&= \sum_{\delta\in\Delta}\sum_{\frac{\delta}{p}\le r\le\frac{\delta}{\underline{p}}}\sum_{\delta_1,\cdots,\delta_r\in\widetilde{\Delta}} t^{\delta+\delta_1+\cdots+\delta_r}\widetilde{f}_{\delta,r}(U_{\delta_1}(x),\cdots,U_{\delta_r}(x)).
\end{aligned}$$

Hence,

$$\{f\circ U\}_{\delta'} = \sum_{\substack{\delta\le\delta',\ \frac{\delta}{p}\le r\le\frac{\delta}{\underline{p}}\\ \delta+\delta_1+\cdots+\delta_r=\delta'}} \widetilde{f}_{\delta,r}(U_{\delta_1},\cdots U_{\delta_r}).$$

For vector fields the proof is the same. □

A.2 Inner products for quasihomogeneous polynomials and vector fields

In this section, to avoid any confusion we denote by $\mathcal{P}_\delta(\mathbb{C}^n)$ the space of p-quasihomogeneous polynomials from $\mathbb{C}^n$ to $\mathbb{C}$ of quasidegree δ and by $\mathcal{H}_\delta(\mathbb{C}^n)$ the space of p-quasihomogeneous vector fields of quasidegree δ in $\mathbb{C}^n$.

In a similar way, let us denote by $P_d(\mathbb{C}^N)$ the space of standard homogeneous polynomials from $\mathbb{C}^N$ to $\mathbb{C}$ of degree d and by $H_d(\mathbb{C}^N)$ the space of standard homogeneous vector fields of degree d in $\mathbb{C}^N$.

The aim of this subsection is to build on $\mathcal{P}_\delta(\mathbb{C}^n)$ and $\mathcal{H}_\delta(\mathbb{C}^n)$ inner products which lead to norms such that the norm of the product is less or equal to the product of the norms. In the homogeneous case for $P_\delta(\mathbb{C}^N)$ and $H_\delta(\mathbb{C}^N)$, the Fisher's inner products $\langle\cdot,\cdot\rangle_H$ given by

$$\left\langle x^R, x^Q\right\rangle_H := \begin{cases} R! \text{ if } R=Q \\ 0 \text{ otherwise}\end{cases} \qquad \text{where } R! = r_1!\cdots r_n! \text{ if } R=(r_1,\ldots,r_n) \tag{46}$$

for monomials and by

$$\langle U,V\rangle_H := \sum_{j=1}^n \langle U_j, V_j\rangle_H . \tag{47}$$

for polynomial vector fields $U=\sum\limits_{j=1}^n U_j\frac{\partial}{\partial x_j}$ and $V=\sum\limits_{j=1}^n V_j\frac{\partial}{\partial x_j}$, lead to multiplicative norms given by

$$|\phi|_{H,\delta} = \sqrt{\frac{\langle\phi,\phi\rangle_H}{\delta!}}.$$

One can check that

$$\left\langle \frac{\partial}{\partial x_j} f(x), g(x) \right\rangle_H = \langle f(x), x_j g(x) \rangle_H .$$

In the homogeneous case, $p = (1, \ldots, 1)$. Let $f \in H_{\delta-1}$, $g \in H_\delta$, then we have

$$\langle x_i f, g \rangle_{p,\delta} = \frac{1}{\delta!} \langle x_i f, g \rangle_H = \frac{1}{\delta!} \left\langle f, \frac{\partial g}{\partial x_i} \right\rangle_H = \frac{(\delta-1)!}{\delta!} \left\langle f, \frac{\partial g}{\partial x_i} \right\rangle_{p,\delta-1} .$$

In the quasihomogeneous case, a natural idea to build inner products which lead to multiplicative norms is based on the following proposition:

Proposition A.5. *Let N be and integer and s be a morphism of algebra from $\mathbb{C}[x_1, \cdots, x_n]$ to $\mathbb{C}[x_1, \cdots, x_N]$ which is injective (i.e. Ker $\mathsf{s} = 0$) and which maps $\mathcal{P}_\delta(\mathbb{C}^n)$ into $P_\delta(\mathbb{C}^N)$ for every $\delta \in \Delta$. Then,*

(a) *The bilinear form $\langle f, g \rangle_p = \langle \mathsf{s}(f), \mathsf{s}(g) \rangle_H$ is an inner product on $\mathcal{P}_\delta(\mathbb{C}^n)$;*

(b) *the renormalized norm $|f|_{p,\delta} = \sqrt{\frac{\langle f,f \rangle_p}{\delta!}}$ satisfies for every $f \in \mathcal{P}_\delta$ and $g \in \mathcal{P}_{\delta'}$*

$$|fg|_{p,\delta+\delta'} \leq |f|_{p,\delta} \ |g|_{p,\delta'} . \tag{48}$$

(c) *Let $f_{\delta,r} : \mathbb{C}^n \to \mathbb{C}$ be simultaneously quasihomogeneous of degree δ and homogeneous of degree r. Denote by $\widetilde{f}_{\delta,r}$ the unique r-linear, symmetric form such that $\widetilde{f}_{\delta r}(\underbrace{x, \cdots, x}_{r \text{ times}}) = f_{\delta,r}(x)$ where $x = (x_1, \ldots, x_n)$. For $1 \leq \ell \leq r$, let U_{δ_ℓ} be a p-quasihomogeneous vector field of quasidegree δ_ℓ. Then, $\widetilde{f}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r})$ is p-quasihomogeneous de degree $\delta + \delta_1 + \cdots + \delta_r$ and we have*

$$\left| \widetilde{f}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r}) \right|_{p,\delta+\delta_1+\cdots+\delta_r} \leq N_1(\widetilde{f}_{\delta,r}) \ \|U_{\delta_1}\|_{p,\delta_1} \cdots \|U_{\delta_r}\|_{p,\delta_r} . \tag{49}$$

with

$$\|U\|_{p,\delta}^2 = \left\| \sum_{i=1}^n U_i \frac{\partial}{\partial x_i} \right\|_{p,\delta}^2 := \sum_{i=1}^n |U_i|_{p,\delta+p_i}^2$$

and

$$N_1(\widetilde{f}_{\delta,r}) := \sum_{\substack{1 \leq i_\ell \leq n \\ 1 \leq \ell \leq r}} \left| \widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r}) \right|$$

where $(e_1, \cdots, e_n)$ is the canonical basis of $\mathbb{C}^n$.

(d) *Let $R_{\delta,r}$ be a vector field of $\mathbb{C}^n$. We assume that $R_{\delta,r}$ is simultaneously quasihomogeneous of degree δ and homogeneous of degree r. Denote by $\widetilde{R}_{\delta,r}$ the unique r-linear, symmetric operator of $\mathbb{C}^n$ such that $\widetilde{R}_{\delta,r}(\underbrace{x, \cdots, x}_{r \text{ times}}) = R_{\delta r}(x)$ where $x = (x_1, \ldots, x_n)$.*

For $1 \leq \ell \leq r$, let U_{δ_ℓ} be p-quasihomogeneous vector field of degree δ_ℓ. Then, $\widetilde{R}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r})$ is p-quasihomogeneous de degree $\delta + \delta_1 + \cdots + \delta_r$ and we have

$$\left\| \widetilde{R}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r}) \right\|_{p,\delta+\delta_1+\cdots+\delta_r} \leq N_{2,1}(\widetilde{R}_{\delta,r}) \ \|U_{\delta_1}\|_{p,\delta_1} \cdots \|U_{\delta_r}\|_{p,\delta_r} . \tag{50}$$

with

$$N_{2,1}(\widetilde{R}_{\delta,r}) := \sqrt{\sum_{j=1}^{n} N_1^2(\widetilde{R}_{\delta,r,j})}$$

where $\widetilde{R}_{\delta,r,j}$ is the j-th components of $\widetilde{R}_{\delta,r}$ in the canonical basis of $\mathbb{C}^n$.

Proof.
(a) : Property (a) directly follows from the fact that s is linear and injective.

(b) : using that s is a morphism of algebra and that the renormalized norm for homogeneous polynomials $|\phi|_{H,\delta} = \sqrt{\frac{\langle \phi, \phi \rangle_H}{\delta!}}$ is multiplicative we get

$$|fg|_{p,\delta+\delta'} = |\mathsf{s}(fg)|_{H,\delta+\delta'} = |\mathsf{s}(f)\mathsf{s}(g)|_{H,\delta+\delta'} \leq |\mathsf{s}(f)|_{H,\delta} \ |\mathsf{s}(g)|_{H,\delta'} = |f|_{p,\delta} \ |g|_{p,\delta'} .$$

Hence the renormalized norm for quasihomogeneous polynomial is multiplicative.

(c) : The proof is made in three steps.

Step (c)-1 : *Explicit formula for $\widetilde{f}_{\delta,r}$.* For $1 \leq \ell \leq n$, let $x^{(\ell)}$ be a vector of $\mathbb{C}^n$ with $x^{(\ell)} = (x_1^{(\ell)}, \cdots, x_n^{(\ell)})$. Then denoting by $(e_i)_{1 \leq i \leq n}$ the canonical basis of $\mathbb{C}^n$, we get

$$\widetilde{f}_{\delta,r}(x^{(1)}, \cdots, x^{(r)}) = \sum_{\substack{1 \leq i_\ell \leq n \\ 1 \leq \ell \leq r}} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)} \widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r}).$$

since $\widetilde{f}_{\delta,r}$ is r-linear. Hence, for $x = (x_1, \cdots, x_n)$,

$$f_{\delta,r}(x) = \widetilde{f}_{\delta,r}(x, \cdots, x) = \sum_{\substack{1 \leq i_\ell \leq n \\ 1 \leq \ell \leq r}} x_{i_1} \cdots x_{i_r} \widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r}).$$

Then since the quasi degree of $x_{i_1} \cdots x_{i_r}$ is $p_{i_1} + \cdots + p_{i_r}$ and since $f_{\delta,r}$ is of quasi degree δ we get that for every $x^{(\ell)} \in \mathbb{C}^n$, we have

$$\widetilde{f}_{\delta,r}(x^{(1)}, \cdots, x^{(r)}) = \sum_{\substack{1 \leq i_\ell \leq n, \ 1 \leq \ell \leq r \\ p_{i_1} + \cdots + p_{i_r} = \delta}} x_{i_1}^{(1)} \cdots x_{i_r}^{(r)} \widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r}).$$

Step c-2 : *Quasidegree of $f_{\delta,r}(U_{\delta_1}, \cdots, U_{\delta_r})$..* For $1 \leq \ell \leq r$, let U_{δ_ℓ} be in $\mathcal{H}_{\delta_\ell}$. Denote by $U_{\delta_\ell,i}$ the i-th coordinate of U_{δ_ℓ} in the canonical basis of $\mathbb{C}^n$. Then, $U_{\delta_\ell,i}$ belongs to $\mathcal{P}_{\delta_\ell+p_i}$ and $U_{\delta_1,i_1} \cdots U_{\delta_r,i_r}$ belongs to $\mathcal{P}_{\delta_1+\cdots+\delta_r+p_{i_1}+\cdots+p_{i_r}}$. Hence since

$$\widetilde{f}_{\delta,r}(U_{\delta_1}, \ldots, U_{\delta_r}) = \sum_{\substack{1 \leq i_\ell \leq n, \ 1 \leq \ell \leq r \\ p_{i_1} + \cdots + p_{i_r} = \delta}} U_{\delta_1,i_1} \cdots U_{\delta_r,i_r} \ \widetilde{f}_{\delta,r}(e_{i_1}, \cdots, e_{i_r}) \tag{51}$$

$\widetilde{f}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})$ belongs to $\mathcal{P}_{\delta'}$ with $\delta' := \delta_1+\cdots+\delta_r+\delta$.

Step c-3 : *Majorization of* $\left|\widetilde{f}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})\right|_{p,\delta'}$. Using (51), (48) and observing that for a polynomial vector field $U=\sum_{j=1}^{n} U_j\frac{\partial}{\partial x_j}\in\mathcal{H}_\delta$ we have $|U_j|_{p,\delta+p_j}\le\|U\|_{p,\delta}$ we get,

$$\begin{aligned}\left|\widetilde{f}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})\right|_{p,\delta'} &\le \sum_{\substack{1\le i_\ell\le n,\ 1\le\ell\le r\\ p_{i_1}+\cdots+p_{i_r}=\delta}} |\widetilde{f}_{\delta,r}(e_{i_1},\cdots,e_{i_r})|\ \ |U_{\delta_1,i_1}|_{p,\delta_1+p_1}\cdots|U_{\delta_r,i_r}|_{p,\delta_r+p_r}\\ &\le \sum_{\substack{1\le i_\ell\le n,\ 1\le\ell\le r\\ p_{i_1}+\cdots+p_{i_r}=\delta}} |\widetilde{f}_{\delta,r}(e_{i_1},\cdots,e_{i_r})|\ \ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\\ &= N_1(\widetilde{f}_{\delta,r})\ \ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\end{aligned}$$

(d) : For a polynomial vector field $R_{\delta,r}:=\sum_{j=1}^{n} R_{\delta,r,j}\frac{\partial}{\partial x_j}$, (d) ensures that for every $1\le j\le n$, $\widetilde{R}_{\delta,r,j}(U_{\delta_1},\dots,U_{\delta_r})$ belongs to $\mathcal{P}_{\delta_1+\cdots+\delta_r+\delta+p_j}$ and that

$$\left|\widetilde{R}_{\delta,r,j}(U_{\delta_1},\dots,U_{\delta_r})\right|_{p,\delta'+p_j}\le N_1(\widetilde{R}_{\delta,r,j})\ \ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}$$

where $\delta'=\delta_1+\cdots+\delta_r+\delta$. Hence, $\widetilde{R}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})$ belongs to $\mathcal{H}_{\delta'}$ and we have

$$\begin{aligned}\left\|\widetilde{R}_{\delta,r}(U_{\delta_1},\dots,U_{\delta_r})\right\|^2_{p,\delta'} &= \sum_{j=1}^{n}\left|\widetilde{R}_{\delta,r,j}(U_{\delta_1},\dots,U_{\delta_r})\right|^2_{p,\delta'+p_j}\\ &\le \sum_{j=1}^{n} N_1^2(\widetilde{R}_{\delta,r,j})\ \ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\\ &= N_{2,1}(\widetilde{R}_{\delta,r})\ \ \|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r}\end{aligned}$$

□

The following lemma and corollary give four examples of morphism of algebra from $P_\delta(\mathbb{C}^n)$ into $P_\delta(\mathbb{C}^N)$ which lead to four different inner products on $P_\delta(\mathbb{C}^n)$. The first example is the one used throughout this paper (see (2), Lemma A.5 and (52)).

Lemma A.6. *Let us define*

$$\begin{aligned}\mathsf{s}_1:\ \mathcal{P}_\delta(\mathbb{C}^n) &\to P_\delta(\mathbb{C}^{|p|})\\ f &\mapsto \mathsf{s}_1(f)(x_{1,1},\cdots,x_{1,p_1},\cdot,x_{n,1},\cdots x_{n,p_n}) := f\Big((x_{1,1}\cdots x_{1,p_1}),\cdots,(x_{n,1}\cdots x_{n,p_n})\Big);\\ \mathsf{s}_2:\ \mathcal{P}_\delta(\mathbb{C}^n) &\to P_\delta(\mathbb{C}^n)\\ f &\mapsto \mathsf{s}_2(f)(x_1,\cdots,x_n) := f(x_1^{p_1},\cdots,x_n^{p_n});\\ \mathsf{s}_3:\ \mathcal{P}_\delta(\mathbb{C}^n) &\to P_\delta(\mathbb{C}^{2n})\\ f &\mapsto \mathsf{s}_3(f)(x_1,\cdots,x_n,\eta_1,\cdots,\eta_n) := f(x_1\eta_1^{p_1-1},\cdots,x_n\eta_n^{p_n-1});\\ \mathsf{s}_4:\ \mathcal{P}_\delta(\mathbb{C}^n) &\to P_\delta(\mathbb{C}^{n+1})\\ f &\mapsto \mathsf{s}_4(f)(x_1,\cdots,x_n,\varepsilon) := f(x_1\varepsilon_1^{p_1-1},\cdots,x_n\varepsilon_n^{p_n-1});\end{aligned}$$

Then,

(a) *For $1 \leq k \leq 4$, s_k is an injective morphism of algebra. So it induces on P_δ an inner product given by $\langle f, g\rangle_{k,p} := \langle \mathsf{s}(f), \mathsf{s}(g)\rangle_H$*

(b) *For every $Q = (q_1, \cdots, q_n)$ and $R = (r_1, \cdots, r_n)$,*

$$\begin{aligned}
\left\langle x^Q, x^R \right\rangle_{1,p} &:= \delta_{Q,R}\ (q_1!)^{p_1} \cdots (q_n!)^{p_n} \\
\left\langle x^Q, x^R \right\rangle_{2,p} &:= \delta_{Q,R}\ (p_1 q_1)! \cdots (p_n q_n)! \\
\left\langle x^Q, x^R \right\rangle_{3,p} &:= \delta_{Q,R}\ (q_1)! \cdots (q_n)!\ \ ((p_1-1)q_1)! \cdots ((p_n-1)q_n)! \\
\left\langle x^Q, x^R \right\rangle_{4,p} &:= \delta_{Q,R}\ (q_1)! \cdots (q_n)!\ ((Q,p) - |Q|).
\end{aligned} \tag{52}$$

where $\delta_{Q,R} := 1$ if $Q = R$ and $\delta_{Q,R} := 0$ otherwise.

The proof of this lemma follows directly from proposition A.5. The details are left to the reader.

Lemma A.7. *Assume that $\mathcal{P}_\delta$ is endowed with the scalar product $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{1,p}$ defined in lemma A.6 and that $\mathcal{P}_\delta$ and $\mathcal{H}_\delta$ are normed with the two corresponding norms.*

(a) *Let f be in $\mathcal{P}_\delta$ and N in $\mathcal{H}_\alpha$. Then $Df.N$ belongs to $\mathcal{P}_{\delta+\alpha}$ and*

$$|Df.U|_{p,\delta+d} \leq \mathfrak{m}_p\ \delta^{\max(1,\frac{\overline{p}}{2})}\, \|f\|_{p,\delta}\, \|U\|_{p,d} \qquad \text{where} \quad \mathfrak{m}_p = n.$$

(b) *Let U be in $\mathcal{H}_\delta$ and N in $\mathcal{H}_\alpha$. Then $DU.N$ lie in $\mathcal{H}_{\delta+\alpha}$ and*

$$\|DU.N\|_{p,\delta+\alpha} \leq \mathfrak{m}_p\ (\delta + \overline{p})^{\max(1,\frac{\overline{p}}{2})}\, \|U\|_{p,\delta}\, \|N\|_{p,\alpha}\,.$$

Proof. (a) : Proposition 3.4 ensures that $Df.N$ lie in $\mathcal{H}_{\delta+d}$. Moreover denoting $N_j := \pi_j(N)$ the jth component of U in $\mathbb{C}^n$ we have

$$|Df.N|_{p,\delta+\alpha} = \left| \sum_{j=1}^n \frac{\partial f}{\partial x_j} N_j \right|_{p,\delta+\alpha} \leq \sum_{j=1}^n \left| \frac{\partial f}{\partial x_j} \right|_{p,\delta-p_j} |N_j|_{p,\alpha+p_j} \leq \sqrt{\sum_{j=1}^n \left| \frac{\partial f}{\partial x_j} \right|^2_{p,\delta-p_j}}\ \|N\|_{p,\alpha}\,.$$

Then denoting $f = \sum_{(\alpha,p)=\delta} f_Q x^Q$ we have

$$\sum_{j=1}^n \left| \frac{\partial f}{\partial x_j} \right|^2_{p,\delta-p_j} = \sum_{j=1}^n \sum_{(Q,p)=\delta} |f_Q|^2 |q_j|^2 \frac{(Q!)^p}{(q_j)^{p_j}(\delta - p_j)!} = \sum_{(\alpha,p)=\delta} \frac{|f_Q|^2 (Q!)^p}{\delta!} \sum_{j=1}^n \frac{|q_j|^2 \delta!}{(q_j)^{p_j}(\delta-p_j)!}.$$

Moreover, we check that

$$\sum_{j=1}^n \frac{|q_j|^2 \delta!}{(q_j)^{p_j}(\delta - p_j)!} \leq \sum_{j=1}^n \left(\frac{\delta}{q_j} \right)^{p_j} |q_j|^2$$

Then, using that for $(Q,p)=\delta$, we have $\underline{p}\ q_j \le \underline{p}|Q| \le \delta \le |Q|\overline{p}$,we that

$$\begin{array}{lll} \text{for} & p_j=1, & \left(\frac{\delta}{q_j}\right)^{p_j}|q_j|^2 = \delta\ |q_j| \le \frac{\delta^2}{\underline{p}} \le \delta^2 \\ \text{for} & p_j=2, & \left(\frac{\delta}{q_j}\right)^{p_j}|q_j|^2 = \delta^2 \\ \text{for} & p_j\ge 3, & \left(\frac{\delta}{q_j}\right)^{p_j}|q_j|^2 \le \delta^{\overline{p}}\frac{1}{q_j^{p_j-2}} \le \delta^{\overline{p}} \end{array}$$

Hence

$$|Df.N|_{p,\delta+\alpha} \le n\ \delta^{\max(1,\frac{\overline{p}}{2})}\ \|f\|_{p,\delta}\ \|U\|_{p,\alpha}$$

(b) : Proposition 3.4 ensures that $Df.N$ lie in $\mathcal{H}_{\delta+\alpha}$. Moreover denoting $S_j := \pi_j(S)$ the j-th component of S in $\mathbb{C}^n$ and using (b), we get

$$\begin{aligned} \|DU.N\|^2_{p,\delta+\alpha} &= \sum_{j=1}^n |DU_j.N|^2_{p,\delta+\alpha+p_j} \\ &\le \sum_{j=1}^n n^2\ (\delta+p_j)^{2\max(1,\frac{\overline{p}}{2})}\ |U_j|^2_{p,\delta+p_j}\ \|N\|^2_{p,\alpha} \\ &\le n^2\ (\delta+\overline{p})^{2\max(1,\frac{\overline{p}}{2})}\ \|U\|^2_{p,\delta}\ \|N\|^2_{p,\alpha}\,. \end{aligned}$$

Hence

$$\|DN.U\|_{p,\delta+\alpha} \le n\ (\delta+\overline{p})^{\max(1,\frac{\overline{p}}{2})}\ \|U\|_{p,\delta}\ \|N\|_{p,\alpha}\,.$$

□

A.3 Quasihomogeneous decomposition and analyticity

In subsection 3.2 we introduced several decompositions of a formal power series $f \in \mathbb{C}[[x_1,\cdots,x_n]]$ as sum of homogeneous and quasihomogeneous components. We now prove that f converges uniformly in a neighborhood of the origin if and only if its homogeneous or quasihomogeneous components grow at most geometrically. In this subsection, we use the normalized norm $|f|_{p,\delta} := \sqrt{\frac{\langle \mathsf{s}_1(f),\mathsf{s}_1(f)\rangle_H}{\delta!}}$ (see proposition A.5 and lemmas A.6). More precisely we have

Proposition A.8. *For a formal power series , $f = \sum_{Q\in\mathbb{N}^n} f_Q x^Q \in \mathbb{C}[[x_1,\cdots,x_n]]$, the following properties are equivalent:*

(a) *f is uniformly convergent in a neighborhood of the origin;*

(b) *There exists $M,R>0$ such that for every $Q\in\mathbb{N}^n$, $\quad |f_Q| \le \frac{M}{R^{|Q|}}$.*

(c) *There exists $M,R>0$ such that for every $Q\in\mathbb{N}^n$, $\quad |f_{\bullet,r}|_{0,r} := \sup_{x\in\mathbb{C}^n} \frac{|f_{\bullet,r}(x)|}{|x|^r} \le \frac{M}{R^r}$.*

(d) *There exists $M,R>0$ such that for every $Q\in\mathbb{N}^n$, $|||\widetilde{f}_{\bullet,r}||| := \sup_{x^{(\ell)}\in\mathbb{C}^n} \frac{|\widetilde{f}_{\bullet,r}(x^{(1)},\cdots,x^{(r)})|}{|x^{(1)}|\cdots|x^{(r)}|} \le \frac{M}{R^r}$.*

(e) *There exists* $M, R > 0$ *such that for every* $\delta \in \Delta$, $\quad |f_\delta|_{p,\delta} \leq \frac{M}{R^\delta}$;

(f) *There exists* $M, R > 0$ *such that for every* $\delta \in \Delta$ *and* $r \geq 0$, $\quad N_1(\widetilde{f}_{\delta,r}) \leq \frac{M}{R^r}$.

We have a similar proposition for vector fields. Statements (a), (b), (c), (d) are still equivalent for vector fields. Statements (e) and (f) should be be modified with appropriate norms for vector fields. More precisely we have

Proposition A.9. *For a formal vector field,* $V \in (\mathbb{C}[[x_1, \cdots, x_n]])^n$*, the following properties are equivalent:*

(a) V *is uniformly convergent in a neighborhood of the origin;*

(b) *There exists* $M, R > 0$ *such that for every* $\delta \in \widetilde{\Delta}$, $\quad \|V_\delta\|_{p,\delta} \leq \frac{M}{R^\delta}$;

(c) *There exists* $M, R > 0$ *such that for every* $\delta \in \widetilde{\Delta}$ *and* $r \geq 0$, $\quad N_{21}(\widetilde{V}_{\delta,r}) \leq \frac{M}{R^r}$.

A.3.1 Proof of Proposition A.8

The proof of the equivalence of statements (a),(b),(c),(d) of proposition A.8 which correspond to the homogeneous decompositions is due to H. Shapiro [Sha89][lemma 1]. The equivalence of (c) and (d) rely on the equivalence of the norms $|\cdot|_{0,r}$ and $|||\cdot|||$ which can be found in the book of Cartan [Car67]. More precisely we have

Lemma A.10. *For an homogeneous polynomial* ψ *of degree* r*, let us denote by* $\widetilde{\psi}$ *the unique* r*-linear form such that for every* $x \in \mathbb{C}^n$, $\widetilde{\psi}(x, \cdots, x) = \psi(x)$*. Then there exists* $M > 0$ *such that for every* $r \geq 0$ *and every homogeneous polynomial* ψ *of degree* r

$$|\psi|_{0,r} \leq |||\widetilde{\psi}||| \leq M(2e)^r |\psi|_{0,r}.$$

The prof of the equivalence of statements (a) and (e) of proposition A.8 is based on the following lemma:

Lemma A.11. *Let* f *be in* $\mathbb{C}[[x_1, \cdots, x_n]]$*. The the following properties are equivalent*

(a) f *is uniformly convergent in a neighborhood of the origin;*

(b) $F := \mathsf{s}_1(f) \in \mathbb{C}[[x_{1,1}, \cdots, x_{1,p_1}, \cdots, x_{n,1}, \cdots, x_{n,p_n]]}$ *is uniformly convergent in a neighborhood of the origin;*

(c) *There exists* $M, R > 0$ *such that for every* $\delta \in \Delta$, $\quad |f_\delta|_{p,\delta} \leq \frac{M}{R^\delta}$;

Proof. The proof is performed in three steps.

Step 1. We prove that (a)$\Leftrightarrow$(b).

Let us decompose f and $F = \mathsf{s}_1(f)$ as sum of monomials. We have

$$f = \sum_{Q \in \mathbb{N}^n} f_Q \, x^Q, \quad F = \sum_{Q \in \mathbb{N}^n} f_Q (x_{1,1})^{q_1} \cdots (x_{1,p_1})^{q_1} \cdots (x_{n,1})^{q_n} \cdots (x_{1,p_n})^{q_n} = \sum F_A X^A$$

with

$$X = (x_{1,1}, \cdots, x_{1,p_1}, \cdots, x_{n,1}, \cdots, x_{1,p_n}),$$
$$A = (\underbrace{q_1, \cdots\cdots, q_1}_{p_1 \text{times}}, \cdots\cdots, \underbrace{q_n, \cdots\cdots, q_n}_{p_n \text{times}}),$$
$$F_A = f_Q.$$

Hence, we have $|A| = p_1 q_1 + \cdots p_n q_n = (Q, p)$. Thus $\underline{p}\,|Q| \leq |A| \leq \overline{p}\,|Q|$.

Hence, on one hand if f is uniformly convergent in a neighborhood of the origin, then there exists $M, R > 0$ such that for every $Q \in \mathbb{N}^n$

$$|F_A| = |f_Q| \leq M \frac{1}{R^{|Q|}} \leq M \left(\frac{1}{R^{\frac{1}{\underline{p}}}} \right)^{|A|}.$$

Hence F, is uniformly convergent in a neighborhood of the origin.

On the other hand if F is uniformly convergent in a neighborhood of the origin, then there exists $M', R' > 0$ such that for every $A \in \mathbb{N}^n$

$$|F_Q| = |f_A| \leq M' \left(\frac{1}{R'} \right)^{|A|} \leq M' \left(\frac{1}{(R')^{\overline{p}}} \right)^{|Q|}.$$

Hence f, is uniformly convergent in a neighborhood of the origin.

Step 2. We prove that (b)⇒(c). PropositionA.8-(b) applied to $F = \mathsf{s}_1(f)$ ensures that if F is uniformly convergent in a neighborhood of the origin, then there exists $M_0, R_0 > 0$ such that for every δ,

$$|F_{\bullet\delta}|_{0,\delta} \leq \frac{M_0}{R_0^\delta},$$

where $F_{\bullet\delta}$ is the homogeneous component of F of degree δ. Moreover it is proved in [IL05][lemma A.5] that

$$|F_{\bullet\delta}|_{H,\delta} = \sqrt{\frac{\langle F_{\bullet\delta}, F_{\bullet\delta} \rangle_H}{\delta!}} \leq \sqrt{C^{n-1}_{\delta+n-1}}\, |F_{\bullet\delta}|_{0,\delta}$$

where

$$C^{n-1}_{\delta+n-1} = \frac{(\delta+n-1)!}{(n-1)!\ \delta!} = \frac{(\delta+n-1)\cdots(\delta+1)}{(n-1)!}$$

Hence, there exists $M' > 0$ such that for every δ,

$$|F_{\bullet\delta}|_{H,\delta} \leq M' \delta^{\frac{n}{2}}\ \frac{M_0}{R_0^\delta} \leq M \frac{1}{R^\delta},$$

where R is any number in $]0, R_0[$ and where $M = M' M_0 \sup_{\delta \geq 0} \left[\delta^{\frac{n}{2}}\ \left(\frac{R}{R_0} \right)^\delta \right]$. So we can conclude that if F is uniformly convergent in a neighborhood of the origin, then there exits $M, R > 0$ such that for every δ,

$$|f_\delta|_{p,\delta} = \sqrt{\frac{\langle \mathsf{s}_1(f_\delta), \mathsf{s}_1(f_\delta) \rangle_H}{\delta!}} = |F_{\bullet\delta}|_{H,\delta} =\leq M \frac{1}{R^\delta}.$$

Step 3. We prove that (c)⇒(b). Assume that there exist $M, R > 0$ such that for every δ,

$$|f_\delta|_{p,\delta} \leq M \frac{1}{R^\delta}.$$

Then, $F = \mathsf{s}_1(f)$ satisfies

$$|F_{\bullet\delta}|_{H,\delta} = |f_\delta|_{p,\delta} \leq M \frac{1}{R^\delta}$$

Moreover it is proved in [IL05][lemma A.3] that $|F_{\bullet\delta}|_{0,\delta} \leq |F_{\bullet\delta}|_{H,\delta}$. Hence, propositionA.8-(b) applied to F ensures that F is uniformly convergent in a neighborhood of the origin. □

To prove statement (f) of proposition A.8, we first need a technical lemma giving the equivalence of the norms $|||\cdot|||$ and $N_1(\cdot)$.

Lemma A.12.

(a) *For every r-linear form* $\widetilde{\varphi} : \mathbb{C}^n \to \mathbb{C}$*, we have* $\quad |||\widetilde{\varphi}||| \leq N_1(\widetilde{\varphi}) \leq n^r |||\widetilde{\varphi}|||.$

(b) *For every r-linear operator* $\widetilde{R} : \mathbb{C}^n \to \mathbb{C}^n$*, we have* $\quad |||\widetilde{R}||| \leq N_1(\widetilde{R}) \leq n^r |||\widetilde{R}|||.$

Proof. (a) : For $x^{(\ell)} = \sum_{i=1}^{n} x_i^{(\ell)}\ e_i$ where $(e_i)_{1\leq i\leq n}$ is the canonical basis of $\mathbb{R}^n$ we have

$$\widetilde{\varphi}(x^{(1)}, \cdots, x^{(r)}) = \sum_{\substack{1\leq \ell\leq r \\ 1\leq i_\ell\leq n}} \widetilde{\varphi}(e_{i_1}, \cdots, e_{i_r})\ x_{i_1}^{(1)} \cdots x_{i_r}^{(r)}$$

Using that $|x_i^{(\ell)}| \leq |x^{(\ell)}|$ we get that $|\widetilde{\varphi}(x^{(1)}, \cdots, x^{(r)})| \leq |x^{(1)}| \cdots |x^{(r)}|\ N_1(\widetilde{\varphi})$. Hence $|||\widetilde{\varphi}||| \leq N_1(\widetilde{\varphi})$.

Reciprocally,

$$N_1(\widetilde{\varphi}) := \sum_{\substack{1\leq \ell\leq r \\ 1\leq i_\ell\leq n}} |\widetilde{\varphi}(e_{i_1}, \cdots, e_{i_r})| \leq \sum_{\substack{1\leq \ell\leq r \\ 1\leq i_\ell\leq n}} |||\widetilde{\varphi}|||1 \cdots 1 = n^r |||\widetilde{\varphi}|||.$$

(b) :Let us denote $\widetilde{R}(x^{(1)}, \cdots, x^{(r)}) := \sum_{i=1}^{n} \widetilde{R}_i(x^{(1)}, \cdots, x^{(r)})\ e_i$. Then using (a) we have

$$\begin{aligned} |\widetilde{R}(x^{(1)}, \cdots, x^{(r)})|^2 = \sum_{i=1}^{n} |\widetilde{R}_i(x^{(1)}, \cdots, x^{(r)})|^2 &\leq \sum_{i=1}^{n} |||\widetilde{R}_i|||^2\ |x^{(1)}|^2 \cdots |x^{(r)}|^2 \\ &\leq |x^{(1)}|^2 \cdots |x^{(r)}|^2 \sum_{i=1}^{n} N_1^2(\widetilde{R}_i) \end{aligned}$$

Hence $|||\widetilde{R}||| \leq N_{21}(\widetilde{R})$. Reciprocally, using Cauchy schwartz inequality we get

$$\begin{aligned}
N_{21}^2(\widetilde{R}) = \sum_{j=1}^{n} N_1^2(\widetilde{R}_j) &= \sum_{j=1}^{n} \Big(\sum_{\substack{1\leq \ell \leq r \\ 1\leq i_\ell \leq n}} |\widetilde{R}_j(e_{i_1}, \cdots, e_{i_r})| \Big)^2 \\
&\leq \sum_{j=1}^{n} \Big(\sum_{\substack{1\leq \ell \leq r \\ 1\leq i_\ell \leq n}} 1 \Big) \Big(\sum_{\substack{1\leq \ell \leq r \\ 1\leq i_\ell \leq n}} |\widetilde{R}_j(e_{i_1}, \cdots, e_{i_r})|^2 \Big) \\
&= n^r \sum_{j=1}^{n} \sum_{\substack{1\leq \ell \leq r \\ 1\leq i_\ell \leq n}} |\widetilde{R}_j(e_{i_1}, \cdots, e_{i_r})|^2 \\
&= n^r \sum_{\substack{1\leq \ell \leq r \\ 1\leq i_\ell \leq n}} |\widetilde{R}(e_{i_1}, \cdots, e_{i_r})|^2 \\
&\leq n^{2r} \; |||\widetilde{R}|||^2
\end{aligned}$$

Hence, $N_{21}(\widetilde{R}) \leq n^r \; |||\widetilde{R}|||$.

□

Finally, equivalents of statements (a) and (f) of proposition A.8 directly follows from

Lemma A.13. *Let* $f = \sum_{Q\in\mathbb{N}^n} f_Q x^Q \in \mathbb{C}[[x_1, \cdots, x_n]]$. *Then,*

(a) *for every* $Q \in \mathbb{N}^n$, $\quad |f_Q| \leq n^{\frac{r}{2}} \; N_1(\widetilde{f}_{\delta,r})$ *where* $r = |Q|$ *and* $\delta = (Q, p)$.

(b) *There exists* $M, R > 0$ *such that for every* $r \geq 0$ *and* $\delta \in \Delta$, $N_1(\widetilde{f}_{\delta,r}) \leq M r^n (2en^{\frac{3}{2}})^r \, |f_{\bullet,r}|_{0,r}$

Proof. (a) : Using Cauchy's formula, for $Q = (q_1, \cdots, q_n)$, we get

$$f_Q = \frac{1}{2\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f_{\delta,r}(\mathrm{e}^{\mathrm{i}\theta_1}, \cdots, \mathrm{e}^{\mathrm{i}\theta_n}) \; \mathrm{e}^{-\mathrm{i}q_1\theta_1} \cdots \mathrm{e}^{-\mathrm{i}q_n\theta_n} \; d_{\theta_1} \cdots d_{\theta_n}.$$

Hence, using that $|f_{\delta,r}(x_1, \cdots, x_n)| \leq |f_{\delta,r}|_{0,r} \left(x_1^2 + \cdots + x_n^2\right)^{\frac{r}{2}}$ and using lemma A.10 and A.12, we get

$$\begin{aligned}
|f_Q| &\leq \frac{1}{2\pi^n} \int_0^{2\pi} \cdots \int_0^{2\pi} |f_{\delta,r}|_{0,r} \left(\sqrt{|\mathrm{e}^{\mathrm{i}\theta_1}|^2 + \cdots + |\mathrm{e}^{\mathrm{i}\theta_n}|^2} \right)^r \; d_{\theta_1} \cdots d_{\theta_n} \\
&\leq n^{\frac{r}{2}} \, |f_{\delta,r}|_{0,r} \leq n^{\frac{r}{2}} |||\widetilde{f}_{\delta,r}||| \leq n^{\frac{r}{2}} N_1(\widetilde{f}_{\delta,r}).
\end{aligned}$$

(b): Using lemma A.10 and A.12, we get that for every $\delta, r \geq 0$

$$N_1(\widetilde{f}_{\delta,r}) \leq n^r |||\widetilde{f}_{\delta,r}||| \leq M n^r (2\mathrm{e})^r \, |f_{\delta,r}|_{0,r} \, .$$

Moreover we have,

$$|f_{\delta,r}|_{0,r} = \sup_{x\in\mathbb{C}^n} \frac{|f_{\delta,r}(x)|}{|x|^r} \leq \sup_{x\in\mathbb{C}^n} \sum_{\substack{|Q|=r \\ (Q,p)=r}} |f_Q| \frac{|x_1|^{q_1} \cdots |x_r|^{q_r}}{|x|^r} \leq \sum_{\substack{|Q|=r \\ (Q,p)=r}} |f_Q| \leq \sum_{|Q|=r} |f_Q|.$$

Since using Cauchy's formula we get for any Q such that $|Q| = r$, $|f_Q| \leq n^r \left|f_{\bullet,r}\right|_{0,r}$, and since $\#\{Q/|Q| = r\} = C_{r+n-1}^{n-1}$ (see [IL05][lemma A.2], we obtain that there exists $M' > 0$ such that for every $r \geq 0$

$$\left|f_{\delta,r}\right|_{0,r} \leq C_{r+n-1}^{n-1} n^{\frac{r}{2}} \left|f_{\bullet,r}\right|_{0,r} \leq M' r^n n^{\frac{r}{2}} \left|f_{\bullet,r}\right|_{0,r}.$$

So, we finally obtain that for every $\delta, r \geq 0$, $N_1(\widetilde{f}_{\delta,r}) \leq M r^n (2en^{\frac{3}{2}})^r | \left|f_{\bullet,r}\right|_{0,r}$.

□

A.3.2 Proof of Proposition A.9

The proof of equivalence of statements (a), (c) is exactly the same as for functions. The equivalence of statements (a) and (b) directly follows from the case of functions. Indeed,

$V = \sum_{j=1}^{n} \pi_j(V) \frac{\partial}{\partial_{x_j}}$ is uniformly convergent in a neighborhood of the origin,

$\Leftrightarrow$ for all $1 \leq j \leq n$, $\pi_j(V)$ is uniformly convergent in a neighborhood of the origin,

$\Leftrightarrow$ for all $1 \leq j \leq n$, there exits $M_j, R_j > 0$, such that for every δ, $\left|\{\pi_j(V)\}_{\delta+p_j}\right|_{p,\delta+p_j} \leq \frac{M_j}{R_j^{\delta+p_j}}$,

$\Leftrightarrow$ there exits $M, R > 0$, such that for every δ and all $1 \leq j \leq n$, $\left|\{\pi_j(V)\}_{\delta+p_j}\right|_{p,\delta+p_j} \leq \frac{M}{R^\delta}$,

$\Leftrightarrow$ there exits $M, R > 0$, such that for every δ and all $1 \leq j \leq n$, $\|V_\delta\|_{p,\delta} \leq \frac{M}{R^\delta}$,

since $\{\pi_j(V)\}_{\delta+p_j} = \pi_j(V_\delta)$ and $\|V_\delta\|_{p,\delta} = \sum_{j=1}^{n} |\pi_j(V_\delta)|_{p,\delta+p_j}^2$.

A.4 Proof of lemma 3.8

First of all, using Sterling formula, it is easy to show that there exists a positive constant C (depending on p) such that, for all multiindices $Q \in \mathbb{N}^n$,

$$(p|Q|)! \leq C^{|Q|} (|Q|!)^p.$$

Hence, we have

$$\frac{(Q,p)!}{Q!^p} \leq C^{|Q|} \frac{(Q,p)!}{(p_1 q_1)! \cdots (p_n q_n)!}.$$

Furthermore, since $2^k = (1+1)^k = \sum_{m=0}^{k} C_k^m$, we have $\frac{a+b!}{a!b!} \leq 2^{a+b}$. We have

$$\frac{(Q,p)!}{(p_1 q_1)! \cdots (p_n q_n)!} = \frac{q_1 p_1 + (q_2 p_2 + \cdots + q_n p_n)!}{(p_1 q_1)! (q_2 p_2 + \cdots + q_n p_n)!} \frac{(q_2 p_2 + \cdots + q_n p_n)!}{(p_2 q_2)! \cdots (p_n q_n)!} \leq 2^{(Q,p)} \frac{(q_2 p_2 + \cdots + q_n p_n)!}{(p_2 q_2)! \cdots (p_n q_n)!}.$$

Hence, the same argument by induction, we obtain that there exist a constant C such that for all multiindices $Q \in \mathbb{N}^n$, $\frac{(Q,p)!}{Q!^p} \leq C^{|Q|}$. Let $f = \sum_{\delta \in \Delta} f_\delta$ be a formal power series.

Let $\delta \in \Delta$ and let $Q \in \mathbb{N}^n$ such that $(Q,p) = \delta$. By definition of the norm and using the previous argument, we have

$$|f_Q| \leq |f_\delta|_{p,\delta} \sqrt{\frac{\delta!}{Q!^p}} \leq C^{|Q|} |f_\delta|_{p,\delta}$$

Hence, if $|f_\delta|_{p,\delta} \leq D^\delta (\delta!)^b$ then

$$|f_Q| \leq \tilde{C}^{|Q|} (\delta!)^b \leq \tilde{D}^{|Q|} (Q!^p)^b \leq E^{|Q|} |Q|!^{\bar{p}b}$$

for some constants $\tilde{C}, \tilde{D}, E$.

B Proof of proposition 6.12

Let S be a p-quasihomogeneous vector field of $\mathbb{C}^n$. Let $X := S + R$ be a good holomorphic perturbation of S in a neighborhood of the origin of $\mathbb{C}^n$ (i.e. the quasiorder of R at the origin is greater than s). Proposition 4.4 ensures that for every $\alpha \in \widetilde{\Delta}$, there exists a polynomial diffeomorphism tangent to identity $\Phi_\alpha^{-1} = \mathrm{Id} + \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \sum_{0<\delta\leq\alpha-s} U_\delta$, with $U_\delta \in \mathcal{H}_\delta$ such that

$$(\Phi_\alpha)_*(X) = S + \mathcal{N}_\alpha + \mathcal{R}_{>\alpha},$$

where $\mathcal{N}_\alpha = \sum_{s<\delta\leq\alpha} N_\delta$, $N_\delta \in \mathrm{Ker}\ \Box_\delta$, and where $\mathcal{R}_{>\alpha}$ is of quasiorder $> \alpha$. The aim of this appendix is to prove proposition 6.12 which gives a kind of "gevrey estimates" of the remainder $\mathcal{R}_{>\alpha}$. We first check that the remainder is explicitly given by

Lemma B.1.

$$\mathcal{L}_\alpha \mathcal{R}_{>\alpha} = \mathcal{Q}^1_{>\alpha} + \mathcal{Q}^2_{>\alpha} + \mathcal{Q}^3_{>\alpha},$$

with

$$\mathcal{L}_\alpha = \mathrm{Id} + D\mathcal{U}_\alpha = \mathrm{Id} + \sum_{0<\delta\leq\alpha-s} DU_\delta,$$

and

$$\mathcal{Q}^1_{>\alpha} = \sum_{\substack{\delta_1+\delta_2>\alpha \\ 0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ s<\delta_2\leq\alpha,\ \delta_2\in\widetilde{\Delta}}} DU_{\delta_1}.N_{\delta_2}, \tag{53}$$

$$\mathcal{Q}^2_{>\alpha} = \sum_{\mu>s,\ \mu\in\widetilde{\Delta}} \sum_{r=\mu_*}^{\mu^*} \sum_{\substack{\delta_1+\cdots+\delta_r+\mu>\alpha \\ 0\leq\delta_i\leq\alpha-s,\ \delta_i\in\widetilde{\Delta}^-}} \tilde{R}_{\mu,r}(U_{\delta_1},\ldots,U_{\delta_r}), \tag{54}$$

$$\mathcal{Q}^3_{>\alpha} = \sum_{r=s_*}^{s^*} \sum_{\substack{\delta_1+\cdots+\delta_r+s>\alpha \\ 0\leq\delta_i\leq\alpha-s \\ (\delta_1,\cdots,\delta_r)\in\Omega_r}} \tilde{S}_{\bullet,r}(U_{\delta_1},\ldots,U_{\delta_r}). \tag{55}$$

Then to compute upper bounds of $\mathcal{L}_\alpha^{-1}\mathcal{Q}^j_{>\alpha}$ we introduce the following family of norms and Banach spaces:

Definition B.2. *For $\varepsilon > 0$, let us denote $\mathcal{B}_\varepsilon$ the Banach space of all formal vector fields $V = \sum_{\delta\in\tilde{\Delta}} V_\delta$ of $\mathbb{C}^n$ such that*

$$N_\varepsilon(V) := \sum_{\delta\in\widetilde{\Delta}} \varepsilon^\delta \left\|V_\delta\right\|_{p,\delta} < +\infty.$$

Remark B.3. *Statement (c) Lemma A.9 ensures that any analytic vector field of $\mathbb{C}^n$ belongs to $\mathcal{B}_\varepsilon$ for ε sufficiently small.*

We first prove the following lemma which which gives comparisons of the different norms

Lemma B.4. *Let ε be fixed in $]0,1[$.*

(a) *Let $f(x_1,\dots,x_n) \in \mathcal{P}_\delta$ be a quasihomogeneous polynomial of degree δ. Let $F \in P_\delta(\mathbb{C}^{|p|})$ be given by $F(X_{1,1},\cdots,X_{1,p_1},\cdots,X_{n,1},\cdots,X_{n,p_n}) := \mathsf{s}_1(f)$ as defined in lemma A.6. Then, we have*

$$|f|_{qh,\varepsilon} := \sup_{d_p(x)<\varepsilon} |f(x)| \leq \sup_{\|X\|<\varepsilon} |F(X)|\,. \tag{56}$$

(b) *For every $f \in \mathcal{P}_\delta$, $\quad |f|_{qh,\varepsilon} \leq \varepsilon^\delta \left|f\right|_{p,\delta}$ holds.*

(c) *For every $V \in \mathcal{H}_\delta$,*

$$\|V\|^2_{qh,\varepsilon} := \sum_{i=1}^n \frac{1}{\varepsilon^{2p_i}} |V_i|^2_{qh,\varepsilon} \leq \varepsilon^\delta \left\|V\right\|_{p,\delta} \tag{57}$$

holds.

(d) *For every $V \in \mathcal{B}_\varepsilon$,*

$$\|V\|^2_{qh,\varepsilon} \leq N_\varepsilon(V) \tag{58}$$

holds.

Remark B.5. *In fact it is possible to prove more accurate results for statements (a), (b), (c). Indeed, for $f \in \mathcal{P}_\delta$ and $V \in \mathcal{H}_\delta$, we have*

$$\begin{aligned}
&|f|_{qh,\varepsilon} = \sup_{\|X\|<\varepsilon} |F(X)|\,,\\
&|f|_{qh,\varepsilon} \leq \varepsilon^\delta \left|f\right|_{p,\delta} \leq \sqrt{C^{|p|}_{\delta+|p|+1}}\ \ |f|_{qh,\varepsilon},\\
&\|V\|_{qh,\varepsilon} \leq \varepsilon^\delta \left\|V\right\|_{p,\delta} \leq \sqrt{C^{|p|}_{\overline{p}+\delta+|p|+1}}\ \ \|V\|_{qh,\varepsilon}.
\end{aligned}$$

Proof of Lemma B.4. Let $f(x_1,\dots,x_n) \in \mathcal{P}_\delta$ be a quasihomogeneous polynomial of degree δ. Let $F \in P_\delta(\mathbb{C}^{|p|})$ be given by $F(X_{1,1},\cdots,X_{1,p_1},\cdots,X_{n,1},\cdots,X_{n,p_n}) := \mathsf{s}_1(f)$ as defined in lemma A.6

Proof of (a): Let $x = (x_1,\cdots,x_n)$ be in $\mathbb{C}^n$ and let us denote $x_k = r_k e^{i\theta_k}$ where $r_k, \theta_k \in \mathbb{R}$. Then, setting $X_{k,j} = (r_k)^{\frac{1}{p_k}} e^{i\frac{\theta_k}{p_k}}$, we get

$$f(x_1,\cdots,x_n) = F(X_{1,1},\cdots,X_{1,p_1},\cdots,X_{n,1},\cdots,X_{n,p_n}).$$

Moreover,

$$(d_p(x))^2 := \sum_{k=1}^{n} p_k |x_k|^{\frac{2}{p_k}} = \sum_{k=1}^{n} \sum_{j=1}^{p_n} |X_{k,j}|^2 = \|X\|^2.$$

Thus, if $d_p(x) < \varepsilon$, then $\|X\| < \varepsilon$ and

$$|f(x_1, \cdots, x_n)| = |F(X_{1,1}, \cdots, X_{1,p_1}, \cdots, X_{n,1}, \cdots, X_{n,p_n})| \leq \sup_{\|X\|<\varepsilon} |F(X)|.$$

Hence,

$$|f|_{qh,\varepsilon} = \sup_{d_p(x)<\varepsilon} |f(x)| \leq \sup_{\|X\|<\varepsilon} |F(X)|.$$

Proof of (b): Let f be in $\mathcal{P}_\delta$ and let us denote $F := \mathsf{s}_1(f)$. The homogeneous polynomial $F \in P_\delta(\mathbb{C}^{|p|})$ is an homogeneous polynomial of degree δ. It is proved in [IL05]-Lemma A.3 that

$$\|F\|_{0,\delta} := \sup_{X \neq 0} \frac{|F(X)|}{\|X\|^\delta} \leq \|F\|_{H,\delta}$$

Then since $\|F\|_{H,\delta} := \|\mathsf{s}_1(f)\|_{H,\delta} := \sqrt{\frac{\langle \mathsf{s}_1(f), \mathsf{s}_1(f) \rangle_H}{\delta!}} = |f|_{p,\delta}$, using (a) we finally get

$$|f|_{qh,\varepsilon} \leq \sup_{\|X\|<\varepsilon} |F(X)| \leq \|F\|_{0,\delta}\ \varepsilon^\delta \leq \|F\|_{H,\delta}\ \varepsilon^\delta = |f|_{p,\delta}\ \varepsilon^\delta.$$

Proof of (c): Let V be in $\mathcal{H}_\delta$. Using the previous result, we directly get

$$\|V\|^2_{qh,\varepsilon} := \sum_{i=1}^{n} \frac{1}{\varepsilon^{2p_i}} |V_i|^2_{qh,\varepsilon} \leq \sum_{i=1}^{n} \frac{\varepsilon^{2(\delta+pi)}}{\varepsilon^{2p_i}} |V_i|^2_{p,\delta+p_i} = \varepsilon^{2\delta} \|V\|^2_{p,\delta}.$$

Proof of (d): Let V be in $\mathcal{B}_\varepsilon$. Writing V as the sum of its quasi homogeneous components, $V = \sum_{\delta \in \widetilde{\Delta}} V_\delta$ we get

$$\|V\|_{qh,\varepsilon} \leq \sum_{\delta \in \widetilde{\Delta}} \|V_\delta\|_{qh,\varepsilon} \leq \sum_{\delta \in \widetilde{\Delta}} \varepsilon^\delta \|V\|_{p,\delta} = N_\varepsilon(V).$$

□

Then we prove that $\mathcal{L}_\alpha$ is invertible and we compute the operator norm of its inverse.

Proposition B.6. *Let $K \geq 2$ be fixed such that*

$$\rho_1(K) < 1 \qquad \text{with} \qquad \rho_1(K) = \frac{u_0 M_\beta \mathfrak{M}_p}{K^{\delta_0}} \sum_{k=0}^{+\infty} (k + \delta_0)^a \left(\frac{1}{2}\right)^k \tag{59}$$

where a, δ_0 and M_β are defined in lemma 6.8.

Then for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$ satisfying

$$C\varepsilon \leq \frac{1}{K(\alpha - s)^{\frac{1}{b}}}, \tag{60}$$

we have :

(a) *The operator* $\mathcal{T}_\alpha$ *given by* $\mathcal{T}_\alpha.V = D\mathcal{U}_\alpha.V$ *maps* $\mathcal{B}_\varepsilon$ *into* $\mathcal{B}_\varepsilon$ *and for every* $V \in \mathcal{B}_\varepsilon$ *we have*

$$N_\varepsilon(\mathcal{T}_\alpha.V) \leq \rho_1(K)\ N_\varepsilon(V).$$

(b) *The operator* $\mathcal{L}_\alpha = \mathrm{Id} + \mathcal{T}_\alpha$ *is invertible and for for every* $V \in \mathcal{B}_\varepsilon$,

$$\|\mathcal{L}_\alpha^{-1}V\|_{qh,\varepsilon} \leq N_\varepsilon(\mathcal{L}_\alpha^{-1}V) \leq \tfrac{1}{1-\rho_1(K)} N_\varepsilon(V).$$

Proof. Statement (b) directly follows from (a) since it ensures that $|||\mathcal{T}_\alpha|||_{\mathcal{L}(\mathcal{B}_\varepsilon)} < \rho_1(K) < 1$ and so $\mathcal{L}_\alpha^{-1} = (\mathrm{Id} + \mathcal{T}_\alpha)^{-1} = \sum\limits_{n=0}^{\infty} (-\mathcal{T}_\alpha)^n$ holds. We now prove statement (a). Observing that

$$\mathcal{T}_\alpha.V = \sum_{\substack{0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ \delta_2\in\widetilde{\Delta}}} DU_{\delta_1}.V_{\delta_2},$$

recalling that $a := \max\left(1, \left[\frac{(\bar{p}+1)}{2}\right]\right)$ and using Proposition 3.6-(d) we get

$$\begin{aligned} N_\varepsilon(\mathcal{T}_\alpha.V) &= \sum_{\substack{0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ \delta_2\in\widetilde{\Delta}}} \left\| \sum_{\delta_1+\delta_2=\delta} DU_{\delta_1}.V_{\delta_2} \right\|_{p,\delta} \varepsilon^\delta, \\ &\leq \mathfrak{M}_p \sum_{\substack{0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ \delta_2\in\widetilde{\Delta}}} \delta_1^a \left\|U_{\delta_1}\right\|_{p,\delta_1} \varepsilon^{\delta_1} \left\|V_{\delta_2}\right\|_{p,\delta_2} \varepsilon^{\delta_2} \\ &\leq \eta\ \ N_\varepsilon(V) \end{aligned} \tag{61}$$

where

$$\eta = \mathfrak{M}_p \sum_{0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^-} \delta_1^a \left\|U_{\delta_1}\right\|_{p,\delta_1} \varepsilon^{\delta_1}$$

Using (30) and Lemma 6.8 and recalling that $\frac{1}{b} = \tau + \frac{a}{\delta_0}$ we get

$$\eta \leq \mathfrak{M}_p M_\beta u_0 \sum_{\delta_0\leq\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^-} \delta_1^a (C\varepsilon)^{\delta_1} (\delta_1!)^\tau \big((\delta_1-\delta_0)!\big)^{\frac{a}{\delta_0}} \leq \mathfrak{M}_p M_\beta u_0 \sum_{\delta_0\leq\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^-} \delta_1^a (C\varepsilon)^{\delta_1} (\delta_1!)^{\frac{1}{b}}.$$

Then for every $K \geq 2$ and every ε, α satisfying (60), we obtain

$$\begin{aligned} \eta &\leq \mathfrak{M}_p M_\beta u_0 \sum_{\delta_0\leq\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^-} \delta_1^a \left(\frac{1}{K}\right)^{\delta_1} \left(\frac{\delta_1!}{(\alpha-s)^{\delta_1}}\right)^{\frac{1}{b}} \\ &\leq \mathfrak{M}_p M_\beta u_0 \sum_{\delta_0\leq\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^-} \delta_1^a \left(\frac{1}{K}\right)^{\delta_1} \\ &\leq \frac{\mathfrak{M}_p M_\beta u_0}{K^{\delta_0}} \sum_{\delta_1=\delta_0}^{+\infty} \delta_1^a \left(\frac{1}{K}\right)^{\delta_1-\delta_0} \\ &\leq \frac{\mathfrak{M}_p M_\beta u_0}{K^{\delta_0}} \sum_{k=0}^{+\infty} (k+\delta_0)^a \left(\frac{1}{2}\right)^k \\ &= \rho_1(K) \end{aligned} \tag{62}$$

In conclusion, gathering (61) and (62) we get that for every $V \in \mathcal{B}_\varepsilon$, every $K \geq 2$ and every ε, α satisfying (60),

$$N_\varepsilon(\mathcal{T}_\alpha.V) \leq \rho_1(K)\ N_\varepsilon(V).$$

□

Before computing upper bounds of $\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}$, we prove a last lemma giving an estimates of the norm of $\mathrm{Id} + \mathcal{U}_\alpha$:

Lemma B.7. *Let $K \geq 2$ be fixed such that (59) is satisfied. Then for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ such that $\alpha > s$ satisfying (60), we have*

$$N_\varepsilon(\mathrm{Id} + \mathcal{U}_\alpha) = \sum_{0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} \varepsilon^\delta u_\delta \leq 2M_\beta u_0.$$

Remark B.8. *The key point in the above estimate is that the upper bound does not depend on α nor on ε.*

Proof. Using (30) and Lemma 6.8 and recalling that $\frac{1}{b} = \tau + \frac{a}{\delta_0}$ and that $M_\beta \geq 1$, we get that for every $K \geq 2$, every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$ satisfying (60), we have

$$\begin{aligned}
N_\varepsilon(\mathrm{Id} + \mathcal{U}_\alpha) &= \sum_{0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} \varepsilon^\delta u_\delta \\
&\leq u_0 + M_\beta u_0 \sum_{\delta_0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} (C\varepsilon)^\delta(\delta!)^\tau\big((\delta-\delta_0)!\big)^{\frac{a}{\delta_0}} \\
&\leq M_\beta u_0 \sum_{0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} (C\varepsilon)^\delta(\delta!)^{\frac{1}{b}}. \\
&\leq M_\beta u_0 \sum_{0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} \left(\tfrac{1}{K}\right)^\delta \left(\frac{\delta!}{(\alpha-s)^\delta}\right)^{\frac{1}{b}} \\
&\leq M_\beta u_0 \sum_{0\leq\delta\leq\alpha-s,\ \delta\in\widetilde{\Delta}^-} \left(\tfrac{1}{K}\right)^\delta \\
&\leq M_\beta u_0 \sum_{\delta=0}^{\infty} \left(\tfrac{1}{2}\right)^\delta = 2M_\beta u_0
\end{aligned}$$

□

We have now enough material to be able to compute an upper bound for $\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}^j$. We estimate each of them separately in the three following lemmas.

Lemma B.9. *Let $K \geq 2$ be fixed such that (59) is satisfied. Then, there exists $M_1 > 0$ such that for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$, satisfying (60), we have*

(a) *when $\frac{1}{b} = \tau + \frac{a}{\delta_0} \geq a$,*

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}^1\|_{qh,\varepsilon} \leq M_1(C\varepsilon)^{\alpha+1}((\alpha+2-s)!)^{\frac{1}{b}};$$

(b) *when* $\frac{1}{b} = \tau + \frac{a}{\delta_0} \leq a$,

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}^1_{>\alpha}\|_{qh,\varepsilon} \leq M_1(C\varepsilon)^{\alpha+1}((\alpha+2-s)!)^{\frac{1}{b}}(\alpha-s)^{1+a-\frac{2}{b}}.$$

Proof. Lemma B.7-(b) ensures that

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}^1_{>\alpha}\|_{qh,\varepsilon} \leq \tfrac{1}{1-\rho_1(K)} N_\varepsilon(\mathcal{Q}^1_{>\alpha}). \tag{63}$$

So to get the desired result we only need to compute an upper bound of $N_\varepsilon(\mathcal{Q}^1_{>\alpha})$.

Recalling that $a := \max\left(1, \left[\frac{(\bar{p}+1)}{2}\right]\right)$ and using Proposition 3.6-(d), (30) and Lemma 6.8 we get

$$\begin{aligned}
N_\varepsilon(\mathcal{Q}^1_{>\alpha}) &\leq \mathfrak{M}_p M_\beta^2 u_0^2 \sum_{\substack{\delta_1+\delta_2>\alpha \\ 0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ s<\delta_2\leq\alpha,\ \delta_2\in\widetilde{\Delta}}} \delta_1^a (C\varepsilon)^{\delta_1+\delta_2}(\delta_1!)^\tau\big((\delta_1-\delta_0)!\big)^{\frac{a}{\delta_0}}\big((\delta_2-s)!\big)^\tau\big((\delta_2-s-\delta_0)!\big)^{\frac{a}{\delta_0}} \\
&\leq \mathfrak{M}_p M_\beta^2 u_0^2 \sum_{\substack{\delta_1+\delta_2>\alpha \\ 0<\delta_1\leq\alpha-s,\ \delta_1\in\widetilde{\Delta}^- \\ s<\delta_2\leq\alpha,\ \delta_2\in\widetilde{\Delta}}} \delta_1^a (C\varepsilon)^{\delta_1+\delta_2}(\delta_1!(\delta_2-s)!)^\tau\big((\delta_1)!(\delta_2-s)!\big)^{\frac{a}{\delta_0}} \\
&\leq \mathfrak{M}_p M_\beta^2 u_0^2 \sum_{\substack{\delta_1+\delta_2\geq\alpha+1 \\ 1\leq\delta_1\leq\alpha-s \\ s+1\leq\delta_2\leq\alpha}} \delta_1^a (C\varepsilon)^{\delta_1+\delta_2}(\delta_1!(\delta_2-s)!)^{\frac{1}{b}}
\end{aligned}$$

In the above estimate, one can obtain a sharper result using a smaller set of index in the last sum, i.e. $\{(\delta_1,\delta_2)\in\mathbb{N}^2/\delta_1+\delta_2\geq\alpha^+,\ 0^+\leq\delta_1\leq\alpha-s,\ s^+\leq\delta_2\leq\alpha\}$ (α^+ is the small integer of $\tilde{\Delta}$ greater than α). However, it leads to far more intricate computations, for a not so better estimate. This is why we have chosen this more rough estimate corresponding to a larger set of index.

So now, performing the change of indexes $(\delta_1,\delta_2)\mapsto(\delta_1,\delta=\delta_1+\delta_2)$ we get that for every $K\geq 2$, every $\varepsilon\in]0,1[$ and every $\alpha\in\widetilde{\Delta}$ with $\alpha>s$ satisfying (60), we have

$$\begin{aligned}
\frac{N_\varepsilon(\mathcal{Q}^1_{>\alpha})}{\mathfrak{M}_p M_\beta^2 u_0^2} \leq\ & \sum_{\delta_1=1}^{\alpha-s}\sum_{\delta=\alpha+1}^{\delta_1+\alpha} \delta_1^a (C\varepsilon)^{\delta}(\delta_1!(\delta-\delta_1-s)!)^{\frac{1}{b}} \\
&\leq (C\varepsilon)^{\alpha+1}\sum_{\delta_1=1}^{\alpha-s}\delta_1^a(\delta_1!)^{\frac{1}{b}}\sum_{\delta=\alpha+1}^{\delta_1+\alpha}(C\varepsilon)^{\delta-(\alpha+1)}((\delta-\delta_1-s)!)^{\frac{1}{b}} \\
&\leq (C\varepsilon)^{\alpha+1}\sum_{\delta_1=1}^{\alpha-s}\delta_1^a(\delta_1!)^{\frac{1}{b}}\sum_{\delta=\alpha+1}^{\delta_1+\alpha}(\tfrac{1}{K})^{\delta-(\alpha+1)}\left(\frac{(\delta-\delta_1-s)!}{(\alpha-s)^{\delta-(\alpha+1)}}\right)^{\frac{1}{b}}
\end{aligned}$$

Then observing that for $0\leq\delta_1\leq\alpha-s$, $\alpha+1\leq\delta\leq\delta_1+\alpha$, we have $\delta-\delta_1-s\leq\alpha-s$ and

$$\frac{(\delta-\delta_1-s)!}{(\alpha-s)^{\delta-(\alpha+1)}} = (\alpha+1-\delta_1-s)!\frac{(\alpha+2-\delta_1-s)\cdots(\delta-\delta_1-s)}{(\alpha-s)^{\delta-(\alpha+1)}} \leq (\alpha+1-\delta_1-s)!$$

we get

$$\begin{aligned}\frac{N_\varepsilon(\mathcal{Q}^1_{>\alpha})}{\mathfrak{M}_p M_\beta^2 u_0^2} \leq \quad &\leq (C\varepsilon)^{\alpha+1} \sum_{\delta_1=1}^{\alpha-s} \delta_1^a \Big(\delta_1!(\alpha+1-\delta_1-s)!\Big)^{\frac{1}{b}} \sum_{\delta=\alpha+1}^{\delta_1+\alpha} (\frac{1}{2})^{\delta-(\alpha+1)} \\ &\leq 2(C\varepsilon)^{\alpha+1} \sum_{\delta_1=1}^{\alpha-s} \delta_1^a \Big(\delta_1!(\alpha+1-\delta_1-s)!\Big)^{\frac{1}{b}}.\end{aligned}$$

When $\frac{1}{b} \geq a$, we obtain

$$\begin{aligned}\frac{N_\varepsilon(\mathcal{Q}^1_{>\alpha})}{\mathfrak{M}_p M_\beta^2 u_0^2} &\leq 2(C\varepsilon)^{\alpha+1} \sum_{\delta_1=1}^{\alpha-s} \Big((\delta_1+1)!(\alpha+1-\delta_1-s)!\Big)^{\frac{1}{b}} \\ &\leq 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}} \sum_{\delta_1=1}^{\alpha-s} \left(\frac{1}{C^{\delta_1+1}_{\alpha+2-s}}\right)^{\frac{1}{b}} \\ &\leq 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}} \sum_{\delta_1=1}^{\alpha-s} \left(\frac{1}{\alpha+2-s}\right)^{\frac{1}{b}} \\ &= 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}} \frac{\alpha-s}{(\alpha+2-s)^{\frac{1}{b}}} \\ &\leq 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}}\end{aligned}$$

since $\frac{1}{b} \geq a \geq 1$. Hence, when $\frac{1}{b} \geq a$,

$$N_\varepsilon(\mathcal{Q}^1_{>\alpha}) \leq 2\ \mathfrak{M}_p M_\beta^2 u_0^2\ (C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}}$$

On the other hand, when $\frac{1}{b} \leq a$ we get

$$\begin{aligned}\frac{N_\varepsilon(\mathcal{Q}^1_{>\alpha})}{\mathfrak{M}_p M_\beta^2 u_0^2} &= 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}} (\alpha-s)^{a-\frac{1}{b}} \frac{\alpha-s}{(\alpha+2-s)^{\frac{1}{b}}} \\ &\leq 2(C\varepsilon)^{\alpha+1} \Big((\alpha+2-s)!\Big)^{\frac{1}{b}}\ (\alpha-s)^{a+1-\frac{2}{b}}.\end{aligned}$$

This achieves the proof of lemma B.9 with

$$M_1 = \frac{2\mathfrak{M}_p M_\beta^2 u_0^2}{1-\rho_1(K)}.$$

□

Lemma B.10. *Let $K \geq 2$ be fixed such that (59) holds. Then let $\gamma \geq 2$ be fixed such that*

$$q = \frac{\chi}{\gamma C} < 1 \qquad \textit{with} \qquad \chi = \left(\frac{2M_\beta u_0}{\rho}\right)^{\frac{1}{p}} \tag{64}$$

where C and M_β are defined in lemma 6.8.

Then there exists $M_2 > 0$, such that for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$ satisfying

$$C\varepsilon \leq \frac{1}{\gamma K(\alpha - s)^{\frac{1}{b}}}, \tag{65}$$

we have

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}^2\|_{qh,\varepsilon} \leq M_2 \left(\tfrac{1}{K}\right)^{\alpha+1}.$$

Proof. Like for $\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}^1$, Lemma B.7-(b) ensures that

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}_{>\alpha}^2\|_{qh,\varepsilon} \leq \tfrac{1}{1-\rho_1(K)} N_\varepsilon(\mathcal{Q}_{>\alpha}^2). \tag{66}$$

So, to get the desired result we only need to compute an upper bound of $N_\varepsilon(\mathcal{Q}_{>\alpha}^2)$.

According to proposition 3.6 and proposition 3.7, there exists positive constants M_R and ρ such that, for all $\mu > s$ belonging to $\tilde{\Delta}$, for all $\mu_* \leq r \leq \mu^*$, we have

$$\left\|\tilde{R}_{\mu,r}(U_{\delta_1},\dots,U_{\delta_r})\right\|_{p,\delta_1+\cdots+\delta_r+\mu} \leq \frac{M_R}{\rho^r}\|U_{\delta_1}\|_{p,\delta_1}\cdots\|U_{\delta_r}\|_{p,\delta_r} = \frac{M_R}{\rho^r}\, u_{\delta_1}\cdots u_{\delta_r}. \tag{67}$$

Hence, using (30) we get

$$\begin{aligned} N_\varepsilon(\mathcal{Q}_{>\alpha}^2) &\leq \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \sum_{r=\mu_*}^{\mu^*} \sum_{\substack{\delta_1+\cdots+\delta_r+\mu>\alpha \\ 0\leq\delta_j\leq\alpha-s,\ \delta_j\in\widetilde{\Delta}^-}} \frac{M_R}{\rho^r}\, u_{\delta_1}\cdots u_{\delta_r}\varepsilon^{\delta_1+\cdots+\delta_r+\mu} \\ &\leq \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \sum_{r=\mu_*}^{\mu^*} \sum_{\substack{\delta_1+\cdots+\delta_r+\mu>\alpha \\ 0\leq\delta_j\leq\alpha-s,\ \delta_j\in\widetilde{\Delta}^-}} \frac{M_R\, u_0^r}{\rho^r}\, \varepsilon^{\delta_1+\cdots+\delta_r+\mu} \prod_{j=1}^r (\delta_j!)^\tau \beta_{\delta_j} \end{aligned}$$

Then lemma 6.8 and remark 6.9 ensure that for every $\delta \geq 0$ lying in $\delta \in \widetilde{\Delta}^-$, we have

$$\beta_\delta \leq M_\beta C^\delta (\delta!)^{\frac{a}{\delta_0}}.$$

Thus, for every for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$ satisfying (65), we have

$$\begin{aligned} N_\varepsilon(\mathcal{Q}_{>\alpha}^2) &\leq \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \sum_{r=\mu_*}^{\mu^*} \sum_{\substack{\delta_1+\cdots+\delta_r+\mu>\alpha \\ 0\leq\delta_j\leq\alpha-s,\ \delta_j\in\widetilde{\Delta}^-}} M_R \left(\frac{M_\beta u_0}{\rho}\right)^r \frac{(C\varepsilon)^{\delta_1+\cdots+\delta_r+\mu}}{C^\mu} \prod_{j=1}^r (\delta_j!)^{\frac{1}{b}} \\ &\leq M_R \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \tfrac{1}{C^\mu} \sum_{r=\mu_*}^{\mu^*} \left(\frac{M_\beta u_0}{\rho}\right)^r \sum_{\substack{\delta_1+\cdots+\delta_r+\mu>\alpha \\ 0\leq\delta_j\leq\alpha-s,\ \delta_j\in\widetilde{\Delta}^-}} \left(\frac{1}{\gamma K(\alpha-s)^{\frac{1}{b}}}\right)^{\delta_1+\cdots+\delta_r+\mu} \prod_{j=1}^r (\delta_j!)^{\frac{1}{b}} \\ &\leq \tfrac{M_R}{K^{\alpha+1}} \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \left(\frac{1}{\gamma C(\alpha-s)^{\frac{1}{b}}}\right)^\mu \sum_{r=\mu_*}^{\mu^*} \left(\tfrac{M_\beta u_0}{\rho}\right)^r \sum_{\substack{(\delta_1,\dots,\delta_r)\in\mathbb{N}^r \\ 0\leq\delta_j\leq\alpha-s}} \prod_{j=1}^r (\delta_j!)^{\frac{1}{b}} \left(\frac{1}{\gamma(\alpha-s)^{\frac{1}{b}}}\right)^{\delta_j} \\ &\leq \tfrac{M_R}{K^{\alpha+1}} \sum_{s<\mu,\ \mu\in\widetilde{\Delta}} \left(\frac{1}{\gamma C(\alpha-s)^{\frac{1}{b}}}\right)^\mu \sum_{r=\mu_*}^{\mu^*} \left(\tfrac{M_\beta u_0}{\rho}\right)^r \left(\sum_{\delta=0}^{\alpha-s} (\delta!)^{\frac{1}{b}} \left(\frac{1}{\gamma(\alpha-s)^{\frac{1}{b}}}\right)^\delta\right)^r \end{aligned}$$

Then, observe that

$$\sum_{\delta=0}^{\alpha-s}(\delta!)^{\frac{1}{b}}\left(\frac{1}{\gamma(\alpha-s)^{\frac{1}{b}}}\right)^{\delta}=\sum_{\delta=0}^{\alpha-s}\left(\frac{\delta!}{(\alpha-s)^{\delta}}\right)^{\frac{1}{b}}\ \frac{1}{\gamma^{\delta}}\leq 1+\sum_{\delta=1}^{\alpha-s}\frac{1}{\gamma^{\delta}}\leq\frac{1}{1-\frac{1}{\gamma}}\leq 2$$

since $\gamma\geq 2$. So, we can conclude

$$\begin{aligned}N_{\varepsilon}(\mathcal{Q}^2_{>\alpha}) &\leq \frac{M_R}{K^{\alpha+1}}\sum_{s<\mu,\ \mu\in\widetilde{\Delta}}\left(\frac{1}{\gamma C(\alpha-s)^{\frac{1}{b}}}\right)^{\mu}\sum_{r=\mu_*}^{\mu^*}\left(\frac{2M_\beta u_0}{\rho}\right)^r\\ &\leq \frac{M_R}{K^{\alpha+1}}\sum_{s<\mu,\ \mu\in\widetilde{\Delta}}\left(\frac{1}{\gamma C}\right)^{\mu}\sum_{r=\mu_*}^{\mu^*}\left(\frac{2M_\beta u_0}{\rho}\right)^r. \qquad (68)\end{aligned}$$

Now, observe that (1) ensures that

$$\frac{\mu}{\overline{p}}+\frac{p}{\overline{\overline{p}}}\leq\mu_* \quad\text{and}\quad \mu^*\leq\frac{\mu}{\underline{p}}+\frac{\overline{p}}{\underline{p}}.$$

Then, since according to remark 6.9, we can assume that $\frac{2M_\beta u_0}{\rho}\geq 1$, we get

$$\sum_{r=\mu_*}^{\mu^*}\left(\frac{2M_\beta u_0}{\rho}\right)^r\leq(\mu^*-\mu_*+1)\left(\frac{2M_\beta u_0}{\rho}\right)^{\mu^*}\leq(A\mu+B)\chi^{\mu} \qquad (69)$$

where χ is given by (64) and where

$$A=\left(\frac{1}{\underline{p}}-\frac{1}{\overline{p}}\right)\ \chi^{\overline{p}} \quad\text{and}\quad B=\left(\frac{\overline{p}}{\underline{p}}-\frac{p}{\overline{\overline{p}}}+1\right)\ \chi^{\overline{p}}.$$

Finally, (64), (68) and (69) ensures that that for every $\varepsilon\in]0,1[$ and every $\alpha\in\widetilde{\Delta}$ with $\alpha>s$ satisfying (65), we have

$$\begin{aligned}N_{\varepsilon}(\mathcal{Q}^2_{>\alpha}) &\leq \frac{M_R}{K^{\alpha+1}}\sum_{s<\mu,\ \mu\in\widetilde{\Delta}}\left(\frac{\chi}{\gamma C}\right)^{\mu}(A\mu+B)\\ &\leq \frac{M_R}{K^{\alpha+1}}\sum_{\mu=s+1}^{\infty}\left(\frac{\chi}{\gamma C}\right)^{\mu}(A\mu+B)\\ &= M_R\ q^{s+1}\left(\frac{B}{1-q}+\frac{qA}{(1-q)^2}\right)\ \frac{1}{K^{\alpha+1}}. \qquad (70)\end{aligned}$$

This achieves the proof of lemma B.10 with

$$M_2=\frac{1}{1-\rho_1(K)}M_R\ q^{s+1}\left(\frac{B}{1-q}+\frac{qA}{(1-q)^2}\right).$$

□

Lemma B.11. *Let $K \geq 2$ and $\gamma \geq 2$ be fixed such that (59) and (64) holds.*

Then there exist $M_3 > 0$, such that for every $\varepsilon \in]0,1[$ and every $\alpha \in \widetilde{\Delta}$ with $\alpha > s$ satisfying (65), we have

$$\|\mathcal{L}_\alpha^{-1}\mathcal{Q}^3_{>\alpha}\|_{qh,\varepsilon} \leq M_3 \left(\tfrac{1}{K}\right)^{\alpha+1}.$$

Proof. The proof is very similar to the one of lemma B.10 and we get an estimate analogous to (68) which read

$$N_\varepsilon(\mathcal{Q}^3_{>\alpha}) \leq \tfrac{M_S}{K^{\alpha+1}} \left(\tfrac{1}{\gamma C}\right)^s \sum_{r=s_*}^{s^*} \left(\tfrac{2M_\beta u_0}{\rho_S}\right)^r \leq M_S \left(\tfrac{1}{C}\right)^s \sum_{r=s_*}^{s^*} \left(\tfrac{2M_\beta u_0}{\rho_S}\right)^r \frac{1}{K^{\alpha+1}}.$$

The details are left to the reader. This which achieves the proof of lemma B.11 with

$$M_3 = M_S \left(\tfrac{1}{C}\right)^s \sum_{r=s_*}^{s^*} \left(\tfrac{2M_\beta u_0}{\rho_S}\right)^r$$

□

Considering lemma B.1, proposition 6.12 directly follows from lemmas B.9, B.10, B.11.

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