SATURATION SPACES FOR REGULARIZATION METHODS IN INVERSE PROBLEMS

JEAN-MICHEL LOUBES AND ANNE VANHEMS

Abstract. The aim of this article is to characterize the saturation spaces that appear in inverse problems. Such spaces are defined for a regularization method and a rate of convergence of the estimation part of the inverse problem depends on their definition. Here we prove that it is possible to define these spaces as regularity spaces, independent of the choice of the approximation method. Moreover, this intrinsec definition enables us to provide minimax rate of convergence under such assumptions.

1. Introduction

An inverse problem deals with the estimation of an unknown function φ which is not observed directly but through an implicit relation to solve. Generally speaking, let φ be our functional interest parameter which belongs to a Hilbert space Φ. We denote S a random variable and the associated cumulative distribution function $F \in \mathcal{F}$. Our objective is to study the solution of the relation:

$$A(\varphi, F) = 0 \tag{1.1}$$

where A is an operator defined on $\Phi \times \mathcal{F}$.

The main feature of this presentation is that φ is implicitely related to F. Therefore, the problem to deal with is to check whether or not there exists a unique solution that is stable under small perturbation of the initial condition of the problem. If there exists a unique stable solution φ to (**??**), then we can define the operator B such that:

$$\varphi = B(F)$$

Moreover, let $S_1, ..., S_n$ be realizations of the random variable X. Since F is unknown, we have to replace it by an estimator F^δ and the associated estimated solution φ^δ is defined through:

$$\varphi^\delta = B\left(F^\delta\right)$$

Since the solution is stable, we know that the perturbation involved in the solution $\left\|\varphi^\delta - \varphi\right\|$ will be controlled by the initial perturbation $\left\|F^\delta - F\right\|$. One example in econometrics of well-posed inverse problem is illustrated by Vanhems (2002). Another classical example is the GMM estimation. For example, let assume $S = (X, Y) \in I\!R^m$ a random vector and F

1991 *Mathematics Subject Classification.* Primary: 60G17, Secondary:62G07.

Key words and phrases. Inverse Problems, Regularization Methods, Structural Econometrics.

the associated cumulative distribution function; let h be an operator defined on $I\!R^m \times \Phi$ and valued in $I\!R^r$. We assume that h is integrable for any φ and consider the following problem:

$$I\!E_F\left[h(Y, X, \varphi)\right] = 0$$

When φ is finite dimensional, we obtain the usual moment conditions of the GMM method. It has been extensively studied (Hansen 1982, Hall 1993) and extended to infinite dimensional spaces through Carrasco and Florens 2000.

In this paper, we will focus on linear inverse problem and we want to characterize solutions of:

$$y = L\varphi^+ \quad \Leftrightarrow \quad \varphi^+ = L^+ y \tag{1.2}$$

for a specific situation where the exact data y are not known, but only an approximation y^δ such that $\|y - y^\delta\| \le \delta$. L is a linear operator that is supposed to be known. For example, we may think of an observation model $y_i = y_i^\delta + \varepsilon_i$ where ε_i are observation errors. This is also the case when L is unknown and is estimated by $\widehat{L}_n$. In that case,the obervable data are given by the relation:

$$y^\delta = \hat{L}_n \varphi = y + (\hat{L_n} - L)\varphi$$

Moreover, we will suppose that our inverse problem is ill-posed. Then, $L^+ y^\delta$ is not a good approximation of $L^+ y$ due to the unboundedness of the inverse operator L^+.

Therefore, we cannot directly inverse the operator L but we try to approximate it by a regularization operator which inverse is continuous and which converges to the true operator L. In what follows, we define a regularization operator R_α which converges to L^+ as α decreases to zero (but not too fast in order to ensure the stability of the solution). Then construct $\varphi_\alpha^\delta = R_\alpha y^\delta$, the regularized estimator of the solution of the ill-posed inverse problem. Write also $\varphi_\alpha = R_\alpha y$ the regularized of the real data y. The estimator should verify $\varphi_\alpha^\delta -> \varphi$ when α and δ go to zero.

This regularization operator depends on a smoothing parameter α which converges to 0. Moreover, in order to prove the convergence of φ_α^δ to φ, we usually have to impose another constraint: $\|\varphi_\alpha - \varphi\| = O(\alpha^\beta)$ where the parameter β controls the convergence of the regularised solution to the true one. We define the space Φ_β such that: $\varphi^+ \in \Phi_\beta = \left\{\varphi; \|\varphi_\alpha - \varphi\| = O(\alpha^\beta)\right\}$.

In what follows, the sub-space defined by this condition is called saturation space. As a matter of fact, such spaces determine the longest sets where a regularization scheme provide estimators converging at an optimal rate of convergence. The objective of our work is then to characterize this condition in terms of regularity assumptions of both the function φ and the operator L. Moreover, under classical smoothness assumptions for the operator L, the space will only depend on φ.

For general references, we refer to Chow, Ibragimov and Haasminski 1999, Cavalier and Tsybakov 2000, Ermakov 1989, O'Sullivan 1996. Such kind of ill-posed linear inverse problems occurs frequently in econometrics.

A very classical example is the Fredholm integral equation of the first kind with the kernel function $K(x,s)$:

$$\int_a^b K(x,s)\varphi(s)ds = u(x), x \in [a,b]$$

where u is a known function. This inverse problem is ill-posed. The associated regularized equation is the Fredholm integral equation of the second kind:

$$\varphi(x) - \int_a^b K(x,s)\varphi(s)ds = u(x), x \in [a,b]$$

The Fredholm type one integral equation is in particular used in Carrasco Florens with the extension of GMM to a continuous number of moment conditions. The formalisation of the inversion of the variance of the moment conditions then lead to a linear integral equation which is part of the implementation of optimal GMM.

Let us detail for example the case developed by Darolles, Florens Renault 2002.

Note $S = (Y, Z, W)$ a random vector; the probability distribution on S is characterized by its joint cumulative distribution function F. For a given F, we consider the Hilbert space L^2_F of square integrable functions of S and we denote $L^2_F(Y)$, $L^2_F(Z)$, $L^2_F(W)$ the subspaces of L^2_F of functions depending on Y, Z or W only. Then, the objective is to study the function $\varphi \in L^2_F(Z)$ solution of the functional equation:

$$\mathbb{E}\left[Y - \varphi(Z) \,|W\right] = 0 \tag{1.3}$$

This relation can be rewritten in the following way:

$$L\varphi = y$$

where $y = \mathbb{E}\left[Y\,|W\right]$ and $L\varphi = \mathbb{E}\left[\varphi(Z)\,|W\right]$. More precisely:

$$\begin{array}{rcll} L & : & L^2_F(Z)- > L^2_F(W) & \varphi \longmapsto L\varphi = \mathbb{E}\left[\varphi(Z)\,|W\right] \\ L^* & : & L^2_F(W)- > L^2_F(Z) & \psi \longmapsto L^*\psi = \mathbb{E}\left[\psi(W)\,|Z\right] \end{array}$$

where L^* is the adjoint of L.

Darolles Florens Renault show that this inverse problem (**??**) is ill-posed and they transform it the following way. First, instead of studying L, they consider L^*L and regularize it by using Tikhonov regularisation:

$$(\alpha I + L^*L)\varphi = L^*y \tag{1.4}$$

where α is a smoothing parameter.

The problem defined by (**??**) is now well-posed and we can define the approximated solution:

$$\begin{array}{rcl} \varphi_\alpha & = & (\alpha I + L^*L)^{-1}L^*y \\ & = & R_\alpha\left(L^*y\right) \end{array}$$

When replacing the second part y by an estimator y^δ, it becomes:

$$\varphi_\alpha^\delta = R_\alpha\left(L^* y^\delta\right)$$

In order to prove the convergence of φ_α^δ to the true function φ, they have to restrict the set of functions $\varphi \in L_F^2(Z) \cap \Phi_\beta$.Therefore, under assumptions on α and δ, they prove that:

Theorem 1.1. *If $\varphi \in \Phi_\beta$, then there exists a choice of α and δ such that*

$$n^{\frac{\beta}{2+\beta}} \left\|\varphi_\alpha^\delta - \varphi\right\|^2 = O(1)$$

The condition $\varphi \in \Phi_\beta$ is crucial for the demonstration and also appears in many ill-posed inverse problems (see for example Loubes Vanhems 2002) but up to now, the link between the space Φ_β, the regularity of the function φ and the operator L was not clearly established.

Therefore, the main goal of this paper is to try to characterize this space Φ_β and we show that its definition is independant of the type of regularization; moreover we can characterize this space only through regularity assumptions on φ, which enables us to check the minimax properties of Darolles Florens Renault estimator.

Even if in this work we only consider linear inverse problems, it is possible to study in a similar way the nonlinear case, when replacing the assumptions over L by assumptions over $DL(\varphi^+)$ (the differential of L with respect to φ^+). For a close study of nonlinear inverse problem, we refer to Ludena Loubes 2003.

2. Minimax rate of convergence for inverse problems

The scheme of our study is the classical inverse problem defined in (**??**).

$$y = L\varphi^+$$

where φ^+ is the true functional interest parameter which belongs to an Hilbert space $\Phi \subset L^2(X)$, where $L^2(X)$ is the Hilbert space of square integrable real valued functions depending on X, a random real-valued variable. Moreover L is a linear operator defined on $L^2(X)$ to $L^2(Y)$ (with Y a real-valued random variable). At last we define the function y which belongs to an Hilbert space $\Psi \subset L^2(Y)$. Then, $L^* : L^2(Y) -> L^2(X)$ will be the adjoint of L.
We assume that L^*L is a compact operator. This assumption is a natural assumption, common in all the work about this topic. Then, we can write (σ_n^2, v_n, u_n), $\forall n \in \mathbb{N}$ the associated spectral value decomposition and E_λ the spectrum of the compact operator L^*L. (σ_n^2, v_n), $n \in \mathbb{N}$ are respectively the eigenvalues and the eigenvectors of the compact operator L^*L and (u_n) are chosen by $u_n = \frac{Lv_n}{\|Lv_n\|}$, $\forall n \in \mathbb{N}$. As a result we have the following notations for all integer n:

$$Lv_n = \sigma_n u_n, \quad L^* u_n = \sigma_n v_n,$$

Then, for all functions ϕ and y, we can write:

$$L\varphi = \sum_n \sigma_n < \varphi, v_n > u_n, \quad L^* y = \sum_n \sigma_n < y, u_n > v_n,$$

$$L^*L\varphi = \sum_n \sigma_n^2 < \varphi, v_n > v_n = \int \lambda dE_\lambda \varphi.$$

Using the spectral measure of the operator, we define the following notations. For every continuous function g, set:

$$g(L^*L)\varphi = \int g(\lambda)dE_\lambda\varphi = \sum_n g(\sigma_n^2) < \varphi, v_n > v_n. \tag{2.1}$$

As a result, for every regularization scheme R_α, there exists a function g_α such that

$$\varphi_\alpha = R_\alpha L^* y = \int g_\alpha(\lambda)dE_\lambda L^* y.$$

For example, the Tikhonov's regularized estimator is defined by the function

$$g_\alpha = \frac{1}{\lambda + \alpha}.$$

See for instance Tikhonov and Arsenin (1977).
Now, we want to quantify the regularity of the inverse operator L^+. Due to the ill-posedness of the operator, we can not study directly this regularity directly. That is the reason why, $\forall \delta > 0$, and for all subspace $\mathcal{M}$ of $L^2(X)$, we define

$$\Omega(\delta, \mathcal{M}) = \sup\{\|\varphi\|, : \varphi \in \mathcal{M}, : \|L\varphi\| \leq \delta\}. \tag{2.2}$$

This quantity is a way of measuring the action of the pseudo-inverse L^+ over a ball $\|L\phi\| \leq \delta$. Set also, for a regularization operator $R = R_\alpha$ and a regularization sequence α,

$$\Delta(\delta, \mathcal{M}, R) = \sup\{\|R_\alpha\varphi^\delta - \varphi\|, : \varphi \in \mathcal{M}, : y^\delta \in \mathcal{Y}, : \|y - y^\delta\| \leq \delta\}. \tag{2.3}$$

This quantity measures the quality of approximation of the regularization method R for functions in the set $\mathcal{M}$. The following inequality links these two quantities.

Lemma 2.1.

$$\Delta(\delta, \mathcal{M}, R) \geq \Omega(\delta, \mathcal{M}).$$

Proof. Let $\varphi \in \mathcal{M}$, such that $\|L\varphi\| \leq \delta$. As a result, for a choice of $y^\delta = 0$, we get $y = L\varphi$ is such that $\|y\| \leq \delta$. Hence, taking the supremum over all $x \in \mathcal{M}$, we get

$$\Delta(\delta, \mathcal{M}, R) \geq \Omega(\delta, \mathcal{M}).$$

□

Define, for $\beta \geq 0$ the set X_β as the range of the operator $(L^*L)^\beta$, i.e:

$$X_\beta = \mathcal{R}((L^*L)^\beta).$$

Indeed

$$X_\beta = \{\varphi \in \Phi, : \exists\omega \in L^2(X), : \varphi = (L^*L)^\beta\omega\}.$$

The set X_β can be written using the following decomposition for $\rho > 0$:

$$X_\beta = \cup_{\rho>0} X_{\beta,\rho}, \text{with}$$

$$X_{\beta,\rho} = \{\varphi \in X, : \exists\omega \in L^2(X), : \|\omega\| \leq \rho, : \varphi = (L^*L)^\beta\omega\}.$$

Using Lemma (**??**), a lower bound for $\Omega(\delta, \mathcal{M})$ will give the lower rate of convergence for the approximation method R. This rate determines the difficulty of the issue. The following proposition gives this rate of convergence, which, of course, depends on δ, the approximation of the real data y by y^δ.

Proposition 2.2.

$$\Omega(\delta, X_{\beta,\rho}) = \delta^{\frac{2\beta}{2\beta+1}} \rho^{\frac{1}{2\rho+1}}.$$

Proof. The proof of the previous result falls into 2 parts.
First, using the definition of X_β we get

$$\begin{aligned}
\|\varphi\| &= \|(L^*L)^\beta \omega\| \\
&\leq \|(L^*L)^{\beta+\frac{1}{2}} \omega\|^{\frac{2\beta}{2\beta+1}} \|\omega\|^{\frac{1}{2\beta+1}} \\
&\leq \|(L^*L)^{\frac{1}{2}} \varphi\|^{\frac{2\beta}{2\beta+1}} \|\omega\|^{\frac{1}{2\beta+1}} \\
&\leq \|L\varphi\|^{\frac{2\beta}{2\beta+1}} \|\omega\|^{\frac{1}{2\beta+1}} \\
&\leq \delta^{\frac{2\beta}{2\beta+1}} \rho^{\frac{1}{2\rho+1}}.
\end{aligned}$$

Here we have used the interpolation inequality with $r = \beta$ and $\rho = \beta + 1/2$.

$$\forall q > r \geq 0, \ \ \|(L^*L)^r \phi\| \leq \|(L^*L)^q \phi\|^{\frac{r}{q}} \|\phi\|^{1-\frac{r}{q}}.$$

As a consequence we get the upper bound

$$\Omega(\delta, X_{\beta,\rho}) \leq \delta^{\frac{2\beta}{2\beta+1}} \rho^{\frac{1}{2\rho+1}}. \tag{2.4}$$

Then, recall that the eigenvalues σ_n are decreasing towards 0, as n increases. Hence, set $\delta_n = \rho\sigma_n^{2\beta+1}$. As a result

$$\left(\frac{\delta_n}{\rho}\right)^{\frac{2}{2\beta+1}} = \sigma_n^2$$

is an eigenvalue of the operator L^*L. Hence, the associated eigenvector v_n satisfies $\|v_n\| = 1$. Set now

$$\varphi_n = \rho(L^*L)^\beta v_n \in X_{\beta,\rho}.$$

We have

$$\begin{aligned}
\varphi_n &= \rho(L^*L)^\beta v_n \\
&= \rho\sigma_n^{2\beta} v_n \\
&= \delta_n^{\frac{2\beta}{2\beta+1}} \rho^{\frac{1}{2\beta+1}} v_n \\
&= \delta_n^{\frac{2\beta+2}{2\beta+1}} \rho^{-\frac{1}{2\beta+1}} v_n.
\end{aligned}$$

So, we get

$$\|L\varphi_n\|^2 =< L^*L\varphi_n, \varphi_n >= \delta_n^2.$$

As a consequence

$$\Omega(\delta_n, X_{\beta,\rho}) \geq \|\phi_n\| = \delta_k^{\frac{2\beta}{2\beta+1}} \rho^{\frac{1}{2\beta+1}}, \tag{2.5}$$

Then inequalities (**??**) and (**??**) conclude the proof. ☐

3. Characterization of saturation spaces for regularization method

Recall that the regularized function φ_α and the true function φ^+ ared defined by the following relations:

$$\varphi_\alpha = \int g_\alpha(\lambda) dE_\lambda L^* y$$
$$\varphi^+ = \int \frac{1}{\lambda} dE_\lambda L^* y$$

As a consequence, the difference between the two functions can be expressed using the spectral measure E_λ as:

$$\begin{aligned}\varphi^+ - \varphi_\alpha &= \varphi^+ - g_\alpha(L^*L)L^*y \\ &= (I - g_\alpha(L^*L)L^*L)\varphi^+ \\ &= \int (1 - \lambda g_\alpha(\lambda)) dE_\lambda \varphi^+ \\ &= r_\alpha(L^*L)\varphi^+.\end{aligned}$$

We have also the following usefull equality:

$$\|\varphi^+ - \varphi_\alpha\|^2 = \int_0^{\|L\|^2} r_\alpha^2(\lambda) d\|E_\lambda \varphi^+\|^2. \tag{3.1}$$

In the following theorem, we give the conditions that enable to identify the saturation spaces Φ_β and the spaces X_β. The condition depends on the regularization scheme and the decay of the eigenvalues of the operator.

Theorem 3.1.
- *If $\lambda^\beta |r_\alpha(\lambda)| \leq \alpha^\beta$, then*

$$\varphi^+ \in X_{\beta,\rho} \quad \Rightarrow \|\varphi_\alpha - \varphi^+\| \leq \rho\alpha^\beta. \tag{3.2}$$

- *If there exists a constant γ such that $\forall \lambda \in [c\alpha, \|L\|^2]$, $\lambda^\beta |r_\alpha(\lambda)| \geq \gamma\alpha^\beta$, then we get the following proposition:*

$$\|\varphi_\alpha - \varphi^+\| = O(\alpha^\beta) \quad \Rightarrow \varphi^+ \in X_\beta. \tag{3.3}$$

Proof. For the first part, note that we have:

$$\begin{aligned}\|\varphi^+ - \varphi_\alpha\| &= \|r_\alpha(L^*L)(L^*L)^\beta \omega\| \\ &\leq |r_\alpha(\lambda)|\lambda^\beta \|\omega\| \\ &\leq \alpha^\beta \rho.\end{aligned}$$

For the second part, using (**??**) we get:

$$\begin{aligned}\|\varphi^+ - \varphi_\alpha\|^2 &= \int_0^{\|L\|^2} r_\alpha^2(\lambda) d\|E_\lambda \varphi^+\|^2 \\ &\geq \int_{c\alpha}^{\|L\|^2} r_\alpha^2(\lambda) d\|E_\lambda \varphi^+\|^2 \\ &\geq \gamma^2 \alpha^{2\beta} \int_{c\alpha}^{\|L\|^2} \lambda^{-2\beta} d\|E_\lambda \varphi^+\|^2 \\ &= O(\alpha^{2\beta}).\end{aligned}$$

As a result, $\int_{c\alpha}^{\|L\|^2} \lambda^{-2\beta} d\|E_\lambda \varphi^+\|^2 = O(1)$, hence we can define

$$w = \int_0^{\|L\|^2} \lambda^{-\beta} dE_\lambda \varphi^+ \in L^2(X).$$

We now can see that

$$\varphi^+ = (L^*L)^\beta w.$$

□

As a consequence, under the assumption of Theorem (**??**), we have the equality of the two sets

$$\Phi_\beta = \{\varphi, : \|\varphi - \varphi_\alpha\| = O(\alpha^\beta)\} = X_\beta = \{\varphi, : \exists \omega \in L^2, \varphi = (L^*L)^\beta \omega\}. \tag{3.4}$$

The equality (**??**) provides a characterization of the saturation spaces Φ_β in terms of functionnal spaces, independent of the chosen regularization method. As a consequence, the sets Φ_β can be characterized as functional sets, where the regularity of the operator L is linked with the regularity of the function φ. These two regularities can be expressed by the decay of the Fourier coefficients of φ, $< \varphi, v_n >$ and of the eigenvalues σ_n^2 of the operator L^*L as it appears in the following corollary.

Corollary 3.2. *Using the spectral values decomposition, the sets X_β can be rewritten as:*

$$\Phi_\beta = X_\beta = \{\varphi = \sum_n < \varphi, v_n > v_n, : \sum_{k \geq n} | < \varphi, v_n > |^2 = \sigma_n^{-2\beta}\}.$$

Now, we aim at giving a definition of the sets Φ_β that only involves the regularity of the parameter of interest ϕ^+. It will enables us to check the optimality of the estimation procedures, used in econometrics, in terms of minimax rate of convergence. For this assume regularity conditions for the operator L. Let H^t be a Sobolev space. We recall that the spaces H^p are defined by the following relation:

$$\phi \in H^p(X) \Leftrightarrow \forall 0 \leq m \leq p,\ \phi^{(m)} \in L^2(X).$$

Hence the dual space of a space H^t is $(H^t)' = H^{-t}$. Here, we consider that L is a smoothing operator of order t. Indeed, there exists a real t such that, for all $\varphi \in L^2$ we get

$$< L\varphi, \varphi > \sim \|\varphi\|_{H^{-\frac{t}{2}}}.$$

This assumption is standard in linear inverse problems, see for instance Cohen, Hoffmann and Reiss (2002), Cavalier and Tsybakov (2000) or Johnstone and Silverman (1990). Hence, Corollary (**??**) shows that the condition $\varphi \in \Phi_\beta$ is equivalent to $\varphi \in H^{t\beta}$, the Sobolev space of order $t\beta$. As a consequence, for $\phi^+ \in \Phi_\beta$, we get that $\phi^+ \in H^{t\beta}$ and the operator L is such that:

$$L : H^{t\beta} \longrightarrow H^{t(1+\beta)}. \tag{3.5}$$

For every $\varphi \in \Phi_\beta$, we get the following rate of convergence:

$$\begin{aligned} \|\varphi_\alpha^\delta - \varphi^+\|^2 &\le \|\varphi_\alpha^\delta - \varphi_\alpha\|^2 + \|\varphi_\alpha - \varphi^+\|^2 \\ &\le O(\alpha^{2\beta}) + \|R_\alpha(y^\delta - y)\|^2 \\ &\le O(\alpha^{2\beta}) + \frac{\delta}{\alpha}. \end{aligned}$$

An optimal choice for the regularization parameter is $\alpha \sim \delta^{\frac{1}{2\beta+1}}$. So, for estimating a function $\varphi \in \Phi_\beta$, an upper bound for the rate of convergence is given by $\delta^{\frac{2\beta}{2\beta+1}}$. This result, together with Proposition (**??**), prove that the rate of convergence in $\delta^{\frac{2\beta}{2\beta+1}}$ is a minimax rate of convergence for the inverse problem (**??**) and for the quadratic loss function.
The approximation rate δ has now to be made more precise. When L is not observed, we consider an estimate $\hat{L}_n \rightarrow L$. The observable data are then given by the relation

$$y^\delta = \hat{L}_n\varphi = y + (\hat{L_n} - L)\varphi.$$

As a result we get the following correspondance

$$\delta_n = \|(\hat{L_n} - L)\varphi\|.$$

In the example studied by Darolles, Florens and Renault (2002), the operator is estimated by a kernel estimator and δ_n is the optimal choice for the smoothing parameter h_n of the kernel.

References

[Am74] T. Amemiya. Multivariate regression and simultaneous equation models when the dependent variables are truncated normal. *Econometrica*, 42:999–1012, 1974.

[CT00] L. Cavalier and A. Tsybakov. Sharp adaptation for inverse problems with random noise. *preprint*, 2000.

[CF00] M. Carasco and J-P. Florens. Generalization of GMM to A Continuum of Moments Conditions. *Econometrics Theory*, 16, 797-834, 2000.

[CHR03] A. Cohen, M. Hoffmann and M. Reiss. Adaptive wavelet Galerkin methods for linear inverse problems. *preprint*, 2003.

[CIK99] P. Chow, Ibragimov I. and Khasminskii R. Statistical approach to some ill-posed problems for partial differential equations. *Prob. Theory and Related Fields*, 113, 421–441, 1999.

[Erm89] M. Ermakov. Minimax estimation of the solution of an ill-posed convolution type problem. *Problems of Information Transmission*, 25, 191–200, 1989.

[DFR01] S. Darolles, J-P. Florens, and E. Renault. Nonparametric instrumental regression. *preprint*, 2002.

[Flo00] J-P. Florens. Inverse problems and structural econometrics: the example of instrumental variables. *Invited lecture at the 8th World Congress of the Econometric Society*, 2000.

[H82] L. Hansen. Large Sample Properties of Generalized Method of Moments. *Econometrica*, 50, 1029-1054, 1982.

[HN] J. Hausman and W. K. Newey. Nonparametric estimation of exact consumers surplus and deadweight loss. *Econometrica*, 63:1445–1476.

[JS90] I. Johnstone and B. Silverman. Speed of estimation in positron emission tomography and related inverse problems. *Ann. Stat.*, 18:251–280, 1990.

[LL03] C. Ludena and J-M. Loubes. Penalized estimators for solving non linear inverse problems. *preprint*, 2003.

[LvdG02] J-M. Loubes and S. van de Geer. Adaptive estimation with soft thresholding penalties. *Statistica Neerlandica*, 56, 1-26, 2002.

[LV02] J-M. Loubes and A. Vanhems. Estimating the solution of a differential equation with endogeneous effects. *Prépublications of Université Paris Sud*, 12-2002.

[N89] A. Neubauer. Tikhonov regularization for non linear ill-posed problems: optimal convergence rates and finite dimensional approximation. *Inverse Problems*, 5, 541-557, 1989.

[Osu96] F. O'Sullivan. A statistical perspective on ill-posed problems. *Statist. Science*, 1, 502–527, 1996.

[QN99] J. Qi-nian. A convergence analysis for Tikhonov regularization of non linear ill posed problems. *Inverse Problems*, 15, 1087-1098, 1999.

[S58] J.D. Sargan. The Estimation of Economic Relationship using Instrumental Variables. *Econometrica.* 1958.

[DS88] N. Dunford and J. Schwartz. *Linear operators. Part III.* John Wiley & Sons Inc., New York, 1988. Spectral operators, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1971 original, A Wiley-Interscience Publication.

[TA77] A. Tikhonov and V. Arsenin. *Solutions of ill-posed problems.* V. H. Winston & Sons, Washington, D.C.: John Wiley & Sons, New York, 1977. Translated from the Russian, Preface by translation editor Fritz John, Scripta Series in Mathematics.

[Van01] A. Vanhems. Nonparametric estimation of differential equation. *preprint*, 2001.

CNRS - Laboratoire de Mathématiques, UMR 8628, Université Paris-Sud, Bât 425, 91405 Orday cedex

ESC Toulouse, 20 boulevard Lascrosses, BP 7010, 31068 Toulouse cedex

E-mail address: `Jean-Michel.Loubes@math.u-psud.fr`
E-mail address: `a.vanhems@esc-toulouse.fr`