ON THE LOCAL CONVERGENCE OF THE DOUGLAS–RACHFORD ALGORITHM

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Abstract. We discuss the Douglas–Rachford algorithm to solve the feasibility problem for two closed sets A, B in $\mathbb{R}^d$. We prove its local convergence to a fixed point when A, B are finite unions of convex sets. We also show that for more general nonconvex sets the scheme may fail to converge and start to cycle, and may then even fail to solve the feasibility problem.

Keywords nonconvex feasibility problem · fixed-point · discrete dynamical system · convergence · stability

1. Introduction

The Douglas–Rachford iterative scheme, originally introduced in [6] to solve nonlinear heat flow problems, aims to find a point x^* in the intersection of two closed constraint sets A, B in $\mathbb{R}^d$ or in Hilbert space. As a consequence of more general results in monotone operator theory by Lions and Mercier [12] it is known that the scheme converges weakly for two closed convex sets A, B in Hilbert space with non-empty intersection. A rather comprehensive analysis of the convex case is given in [3].

Due to its success in applications, the Douglas–Rachford scheme is frequently used in the nonconvex setting despite the lack of a satisfactory convergence theory. Recently Hesse and Luke [10] made progress by proving local convergence of the scheme for B an affine subspace intersecting the set A transversally, where A is no longer convex, but satisfies a regularity hypothesis called superregularity. Numerical experiments in the nonconvex case indicate, however, that the Douglas–Rachford scheme should converge in much more general situations. Very frequently one observes that the iterates settle for convergence after a chaotic transitory phase; see [1, 7] and the references therein. Here we prove local convergence of the Douglas–Rachford scheme when A, B are finite unions of convex sets. Our result is complementary to [10], because no transversality hypothesis is required. This result is proved in section 3.

We will also show that for nonconvex sets A, B the Douglas–Rachford scheme may fail to converge and start to cycle without solving the feasibility problem. We show that this may even lead to continuous limiting cycles. These are more delicate to construct, because in that case the Douglas–Rachford sequence $x_{n+1} \in T(x_n)$ is bounded and satisfies $x_{n+1} - x_n \to 0$, but fails to converge. Our construction is given in section 4.

2. Preparation

Given a nonempty closed subset A of $\mathbb{R}^d$, the projection onto A is the set-valued mapping P_A associating with $x \in \mathbb{R}^d$ the nonempty set

$$P_A(x) = \big\{a \in A : \|x - a\| = \operatorname{dist}(x, A)\big\},$$

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where $\|\cdot\|$ is the Euclidean norm, and $\text{dist}(x, A) = \min\{\|x - a\| : a \in A\}$. The reflection of x in A is then the set-valued operator

$$R_A = 2P_A - I.$$

Given two nonempty closed sets A, B in $\mathbb{R}^d$, the Douglas–Rachford iterative scheme, starting at x_0, generates a sequence x_n by the recursion

$$x_{n+1} \in T(x_n), \qquad T := \tfrac{1}{2}\left(R_B R_A + I\right),$$

$n \in \mathbb{N}$. We call T the Douglas–Rachford operator, or shortly, DR operator. Suppose $x^+ \in T(x)$ is one step of the Douglas–Rachford scheme. Then x^+ is obtained as

$$x^+ = x + b - a,$$

where $a \in P_A(x)$, $y = 2a - x$, and $b \in P_B(y)$. We call a the shadow of iterate x on A, b the reflected shadow of x in B, both used to produce x^+. We write $x^+ = T(x)$ if the DR-operator is single-valued, and similarly for projectors P_A, P_B and reflectors R_A, R_B.

The fixed point set of T is defined as $\text{F}(T) = \{x \in \mathbb{R}^d : x \in T(x)\}$. Note that if $x^* \in \text{F}(T)$, and if $a^* \in A$, $b^* \in B$ are the shadow and reflected shadow of x^* used to produce $x^* \in T(x^*)$, then $a^* = b^* \in A \cap B$, so every fixed point gives rise to a solution $a^* \in A \cap B$ of the feasibility problem. However, in the set-valued case, it may happen that $x^* \in \text{F}(T)$ has other shadow-reflected shadow pairs $(\tilde{a}, \tilde{b})$ leading away from x^*, i.e., where $\tilde{a} \neq \tilde{b}$, so that $\tilde{x} = x^* + \tilde{b} - \tilde{a} \in T(x^*) \setminus \{x^*\}$. We therefore introduce the set of strong fixed-points of T as

$$\mathcal{F}(T) = \left\{x \in \mathbb{R}^d : T(x) = \{x\}\right\}.$$

Note that $A \cap B \subset \mathcal{F}(T) \subset \text{F}(T)$. If T is single-valued, then $\text{F}(T) = \mathcal{F}(T)$.

These concepts are linked to discrete dynamical system theory, where fixed points are steady states, and where any sequence $x_{n+1} \in T(x_n)$ is called an orbit or a trajectory. We recall that a steady state x^* is stable in the sense of Lyapunov if for every $\epsilon > 0$ there exists $\delta > 0$ such that every trajectory $x_{n+1} \in T(x_n)$ with starting point $x_0 \in B(x^*, \delta)$ satisfies $x_n \in B(x^*, \epsilon)$ for all n. Here and throughout $B(x, r)$ means the closed Euclidian ball with centre x and radius r. It is clear that $x^* \in \text{F}(T) \setminus \mathcal{F}(T)$ can never be stable, because $x_0 = x^*$ produces trajectories going away from x^*.

3. Unions of convex sets

In this section $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$ are finite unions of closed convex sets, a case which is of interest in a number of practical applications like rank or sparsity optimization [11], or even road design [5], where finite unions of linear or affine subspaces are used. For every $i \in I$ and $j \in J$ let T_{ij} be the Douglas–Rachford operator associated with the sets A_i, B_j. By convexity of A_i, B_j, the operators T_{ij} are single-valued, and $T(x) \subset \{T_{ij}(x) : i \in I, j \in J\}$.

Since A, B are finite unions of convex sets, every DR step is realized as the DR step of one of the operators T_{ij}, and in that case, we say that this operator is active. To make this precise, we define the set of active indices at x as

$$K(x) = \left\{(i,j) \in I \times J : P_{A_i}(x) \in P_A(x), P_{B_j}\left(R_{A_i}(x)\right) \in P_B\left(R_{A_i}(x)\right)\right\}. \tag{1}$$

Note that if $(i, j) \in K(x)$, then $T_{ij}(x) \in T(x)$. Conversely, for every $x^+ \in T(x)$, there exists $(i, j) \in K(x)$ such that $x^+ = T_{ij}(x)$. However, be aware that $T_{ij}(x) \in T(x)$ may be true without (i, j) being active at x.

Theorem 1. (Stable local attractor). *Let $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$ be finite unions of closed convex sets, and let $x^* \in \mathcal{F}(T)$ be a strong fixed point. Then x^* has a radius of attraction $r > 0$ with the following property: For arbitrary fixed $\epsilon \in (0, r)$,* *suppose a Douglas–Rachford trajectory* $x_{n+1} \in T(x_n)$ *enters the ball* $B(x^*, \epsilon)$. *Then it stays there and converges to some fixed point* $\bar{x} \in \mathrm{F}(T)$. *Moreover, every accumulation point of the shadow sequence* $a_n \in P_A(x_n)$ *is a solution of the feasibility problem. The radius of attraction can be computed as*

$$r = \sup\left\{\epsilon > 0 : K\left(B(x^*, \epsilon)\right) \subset K(x^*)\right\}. \tag{2}$$

Proof. 1) The fact that $x^* \in \mathcal{F}(T)$ is a strong fixed point has the following consequence. Whenever $(i, j) \in K(x^*)$ is active, then $P_{A_i}(x^*) = P_{B_j}\left(R_{A_i}(x^*)\right) \in A \cap B$. Therefore, for every $(i, j) \in K(x^*)$, the operator T_{ij} has x^* as a fixed point.

2) We now show the following. There exists $\epsilon > 0$ such that every $x \in B(x^*, \epsilon)$ has $K(x) \subset K(x^*)$.

Let us consider the set $I(x) = \{i \in I : \text{there exists } j \in J \text{ such that } (i, j) \in K(x)\}$ of active indices $i \in I$ at x. Then by definition

$$\delta_1 := \min\left\{\mathrm{dist}(x^*, A_i) : i \notin I(x^*)\right\} - \mathrm{dist}(x^*, A) > 0. \tag{3}$$

Similarly, we have

$$\delta_2 := \min\left\{\mathrm{dist}\left(R_{A_i}(x^*), B_j\right) - \mathrm{dist}\left(R_{A_i}(x^*), B\right) : i \in I(x^*), (i, j) \notin K(x^*)\right\} > 0. \tag{4}$$

Choose $\epsilon > 0$ such that $2\epsilon < \min\{\delta_1, \delta_2\}$. We show that ϵ is as claimed. Pick $(i, j) \notin K(x^*)$. We have to show that $(i, j) \notin K(x)$. First consider the case where $i \in I \setminus I(x^*)$. We show that $i \in I \setminus I(x)$. Indeed,

$$\begin{aligned} \mathrm{dist}(x, A) &\leq \|x - x^*\| + \mathrm{dist}(x^*, A) \\ &\leq \epsilon + \mathrm{dist}(x^*, A_i) - \delta_1 \qquad \text{(using (3))} \\ &\leq 2\epsilon + \mathrm{dist}(x, A_i) - \delta_1 < \mathrm{dist}(x, A_i), \end{aligned}$$

showing that $i \notin I(x)$. We now discuss the case where $i \in I(x^*)$, but $(i, j) \notin K(x^*)$. That means $\mathrm{dist}(R_{A_i}(x^*), B_j) - \mathrm{dist}(R_{A_i}(x^*), B) \geq \delta_2$. Therefore

$$\begin{aligned} \mathrm{dist}(R_{A_i}(x), B) &\leq \|R_{A_i}(x) - R_{A_i}(x^*)\| + \mathrm{dist}(R_{A_i}(x^*), B) \\ &\leq \epsilon + \mathrm{dist}(R_{A_i}(x^*), B_j) - \delta_2 \qquad \text{(using (4))} \\ &\leq 2\epsilon + \mathrm{dist}(R_{A_i}(x), B_j) - \delta_2 < \mathrm{dist}(R_{A_i}(x), B_j), \end{aligned}$$

proving $(i, j) \notin K(x)$.

3) As an immediate consequence of 2) we have the following: If $x \in B(x^*, \epsilon)$ and $x^+ \in T(x)$ is realized as $x^+ = T_{ij}(x)$ for some active operator T_{ij}, that is, for some $(i, j) \in K(x)$, then this operator T_{ij} has x^* as a fixed point. Namely, by 2) x satisfies $K(x) \subset K(x^*)$, hence $(i, j) \in K(x^*)$, and therefore $P_{A_i}(x^*) = P_{B_j}(R_{A_i}(x^*))$, which proves what we claimed.

5) We next show that as soon as a DR sequence $x_{n+1} \in T(x_n)$ enters the ball $B(x^*, \epsilon)$, then it stays there and converges.

This can be seen as follows. Suppose the trajectory enters $B(x^*, \epsilon)$ at stage n. Then the active operator $T_{i_n j_n}$ used to produce $x_{n+1} = T_{i_n j_n}(x_n) \in T(x_n)$ has x^* as a fixed point, because $(i_n, j_n) \in K(x_n) \subset K(x^*)$. Therefore, by [2, Prop. 4.21], this operator satisfies $\|x_{n+1} - x^*\| = \|T_{i_n j_n}(x_n) - x^*\| \leq \|x_n - x^*\| \leq \epsilon$. The conclusion is that from index n onward the sequence x_n stays in the ball $B(x^*, \epsilon)$, and all operators $T_{i_m j_m}$ used from here on have x^* as a common fixed point.

Now we invoke Elsner *et al.* [8, Thm. 1], who show that x_n converges to a common fixed point $\bar{x}$ of the operators $T_{i_m j_m}$, $m \geq n$. But $\bar{x}$ is then also a fixed point of T, as follows from the continuity of the distance functions. One has $\bar{x} \in B(x^*, \epsilon)$, and moreover, if $a_n = P_{i_n}(x_n) \in A_{i_n} \subset A$, then every accumulation point of the sequence a_n is a solution of the feasibility problem. Namely, if we consider $b_n = P_{B_j}\left(R_{A_i}(x_n)\right) \in B$, and if we take accumulation points a^* of a_n and b^* of b_n, then $a^* \in P_A(x^*)$, $b^* \in P_B\left(2a - x^*\right))$, hence $a^* = b^*$, because x^* is a strong fixed point.

6) To conclude let us now define the radius of attraction r as in formula (2). In 1) – 5) above we have shown that there exists $\epsilon > 0$ such that $K\left(B(x^*, \epsilon)\right) \subset K(x^*)$, and that this inclusion alone already implies convergence of every trajectory entering $B(x^*, \epsilon)$. This means that the supremum in (2) is over a nonempty set, and that is all that we need. ☐

Remark 1. (Stable steady state.) The dynamic system interpretation of Theorem 1 is that a strong fixed point $x^* \in \mathcal{F}(T)$ is a stable steady state of the Douglas–Rachford dynamical system $x^+ \in T(x)$ when A, B are finite unions of convex sets. Note also that we do not claim that $\bar{x} \in \mathrm{F}(T)$ is strong, nor do we claim that the iterates converge to x^* itself. ☐

The following observation is also of the essence.

Remark 2. (Strong fixed point needed). Theorem 1 is not true if $x^* \in \mathrm{F}(T) \setminus \mathcal{F}(T)$, that is, if x^* is not a strong fixed point. Indeed, let $A = \{-1, 1\}$ and $B = \{-2, 1\}$. Then $x^* = 0$ is a fixed-point of T, but not a strong one. Now there exist arbitrarily small $\epsilon \in (0, 1)$ such that trajectories starting in $B(0, \epsilon) = (-\epsilon, \epsilon)$ will not stay in that ball. Indeed, for $x \in (-\epsilon, 0)$, x^+ will move away from 0 and will not stay in $B(0, \epsilon)$, while for $x \in (0, \epsilon)$, x^+ stays in that interval. ☐

Remark 3. Note that if A, B are convex sets, then all $K(x)$ are identical singleton sets, so formula (2) gives $r = \infty$, which means the DR scheme converges globally.

Remark 4. Formula (2) allows to compute the radius of attraction of a strong fixed-point $x^* \in \mathcal{F}(T)$ in certain cases. For illustration, consider $A = \{(x, 0) : x \in \mathbb{R}\} \cup \{(0, y) : y \in \mathbb{R}\}$ a union of two lines and $B = \{(x, y) : y = -\frac{y^*}{x^*}x + y^*\}$ a line, where $x^* > 0$, $y^* > 0$. Then $(0, y^*)$ and $(x^*, 0)$ are the two only fixed points of T, both strong, and one easily finds $r(x^*, 0) = x^*/\sqrt{2}$ and $r(0, y^*) = y^*/\sqrt{2}$.

Remark 5. (Asymptotic stability). Let $x^* \in \mathcal{F}(T)$ be a strong fixed point, and suppose there exists $\delta > 0$ such that $B(x^*, \delta)$ contains no further fixed point of T. Then it follows from Theorem 1 that we can find $\epsilon \in (0, \delta]$ such that every trajectory $x_{n+1} \in T(x_n)$ entering $B(x^*, \epsilon)$ stays there and converges to x^*. In the dynamical system terminology, x^* is then asymptotically stable in the sense of Lyapunov. Note that this still fails for an isolated fixed point $x^* \in \mathrm{F}(T) \setminus \mathcal{F}(T)$.

Theorem 2. (Local convergence). *Let $A = \bigcup_{i \in I} A_i$ and $B = \bigcup_{j \in J} B_j$ be finite unions of convex sets. Let $x_{n+1} \in T(x_n)$ be a bounded Douglas–Rachford sequence satisfying $x_{n+1} - x_n \to 0$. Then x_n converges to a fixed-point $\bar{x} \in \mathrm{F}(T)$. Moreover, every accumulation point of the shadow sequence $a_n \in P_A(x_n)$ is a solution to the feasibility problem.*

Proof. 1) For every $n \in \mathbb{N}$ let us choose an active index pair $(i_n, j_n) \in K(x_n)$ such that $x_{n+1} = T_{i_n j_n}(x_n)$. Put $a_n = P_{A_{i_n}}(x_n)$ and $b_n = P_{B_{j_n}}\left(R_{A_{i_n}}(x_n)\right) = P_{B_{j_n}}(2a_n - x_n)$, so that $x_{n+1} = x_n + b_n - a_n$. Note that $a_n - b_n = x_n - x_{n+1} \to 0$ by hypothesis.

2) Let x^* be any accumulation point of the sequence x_n. We define a subset $K_0(x^*)$ of the active set $K(x^*)$ as

$$K_0(x^*) = \left\{(i, j) \in K(x^*) : P_{A_i}(x^*) = P_{B_j}\left(R_{A_i}(x^*)\right)\right\}.$$

Note that every T_{ij} with $(i, j) \in K_0(x^*)$ has x^* as a fixed point.

3) We now claim that for every accumulation point x^* of the sequence x_n there exists $\epsilon > 0$ and an index n_0 such that for every x_n with $n \geq n_0$ and $x_n \in B(x^*, \epsilon)$, we have $(i_n, j_n) \in K_0(x^*)$.

To prove this, assume on the contrary that for every $\epsilon = \frac{1}{k}$ there exists n_k such that $x_{n_k} \in B(x^*, \frac{1}{k})$, but $(i_{n_k}, j_{n_k}) \notin K_0(x^*)$. Moreover, let $n_k < n_{k+1} \to \infty$. Then $x_{n_k} \to x^*$. Passing to another subsequence which we also denote by x_{n_k}, we may assume that $i_{n_k} = i$, $j_{n_k} = j$. Then $a_{n_k} = P_{A_i}(x_{n_k}) \to a^* \in A$, $b_{n_k} = P_{B_j}(R_{A_i}(x_{n_k})) \to b^* \in B$. Since $a_n - b_n \to 0$ by part 1), we deduce that $a^* = b^* \in A \cap B$. Since we also have $a_{n_k} = P_{A_i}(x_{n_k}) \in P_A(x_{n_k})$ and $b_{n_k} \in P_B(R_{A_i}(x_{n_k})) \subset P_B(R_A(x_{n_k}))$, we get $a^* \in P_A(x^*)$ and $b^* \in P_B(R_A(x^*))$, hence $(i, j) \in K(x^*)$. Since $a^* = b^*$, we have $(i, j) \in K_0(x^*)$. This contradiction proves the claim.

4) Since x^* is an accumulation point of the sequence x_n, there exist infinitely many indices with $x_n \in B(x^*, \epsilon)$. Choose one with $n \geq n_0$. Then $x_{n+1} = T_{i_n j_n}(x_n)$, and by part 3) we have $(i_n, j_n) \in K_0(x^*)$. By part 2), $T_{i_n j_n}$ has therefore x^* as a fixed point. Using [2, Prop. 4.21], we deduce $\|x_{n+1} - x^*\| = \|T_{i_n j_n}(x_n) - x^*\| \leq \|x_n - x^*\| \leq \epsilon$, hence x_{n+1} stays in the ball $B(x^*, \epsilon)$. This means we can repeat the argument, showing that the entire sequence x_m, $m \geq n$, stays in $B(x^*, \epsilon)$. By part 3) the operators $T_{i_m j_m}$, $m \geq n$, have the common fixed point x^*, hence we conclude again using [8, Thm. 1] that x_m converges to some $\bar{x}$, which must be a fixed point of T. The second part of the statement follows now from $a_n - b_n \to 0$. □

Remark 6. (Discrete limit cycle). Let $A = \{(x, y) : y = 0\} \subset \mathbb{R}^2$ be the x-axis, fix $1 \geq \eta > 0$, and put $B = \{(0, 0), (7 + \eta, \eta), (7, -\eta)\}$. When started at $x_1 = (7, \eta)$, the method cycles between the four points x_1, $x_2 = (7 + \eta, 0)$, $x_3 = (7 + \eta, -\eta)$, $x_4 = (7, 0)$. Note that B is a finite union of bounded convex sets and A is convex, the iterates x_2 and x_4 reach A, but the method fails to converge, and it also fails to solve the feasibility problem.

Remark 7. (Several shadows). Let B be a circle in $\mathbb{R}^2$, and let A consist in the union of two circles which touch B from outside in a_1^*, a_2^*. Then the centre x^* of B is a fixed-point of the Douglas–Rachford operator T, and the two points $a_i^* \in A \cap B$ are both shadows of x^*. This shows that even in the case of convergence of x_n we do not expect the shadows a_n to converge.

Remark 8. (More than two sets.) It is a standard procedure in applications to extend the Douglas–Rachford scheme to solve the feasibility problem for a finite number of constraint sets $C_1, \dots, C_m$ in $\mathbb{R}^d$. One defines A to be the diagonal in $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (m times), and chooses as $B = C_1 \times \cdots \times C_m$ in the product space. Then if $\bigcap_{i=1}^m C_i \neq \emptyset$, the Douglas–Rachford algorithm in product space can be used to compute a point in this intersection. The interesting observation is that if each C_i is a finite union of convex sets, then this remains true for the set B, hence our convergence theory applies.

4. Existence of a continuous limit cycle

In this section we construct two closed bounded sets A, B such that the Douglas–Rachford iteration $x_{n+1} \in T(x_n)$ with $T = \frac{1}{2}(R_B R_A + I)$ fails to converge and produces a continuum of accumulation points $F \subset \mathrm{F}(T)$ forming a continuous limit cycle. We let A be the cylinder mantle

$$A = \left\{(\cos t, \sin t, h) : 0 \leq t \leq 2\pi, 0 \leq h \leq 1\right\},$$

and B a double spiral consisting of two logarithmic spirals in 3D winding down against the cylinder, one from inside, one from outside. That is,

$$B = \left\{((1 \pm e^{-t}) \cos t, (1 \pm e^{-t}) \sin t, e^{-t/2}) : 0 \leq t\right\} \cup F,$$

where $F = \{(\cos \alpha, \sin \alpha, 0) : \alpha \in [0, 2\pi]\}$. Note that $A \cap B = F$. We will construct a Douglas–Rachford sequence $x_{n+1} = T(x_n)$, whose set of accumulation points is the entire set F. It will be useful to divide the spiral in its outer and inner part

$$B_\pm = \left\{((1 \pm e^{-t})\cos t, (1 \pm e^{-t})\sin t, e^{-t/2}) : 0 \leq t\right\} \cup F$$

so that $B = B_- \cup B_+$ and $B_- \cap B_+ = F$ with these notations.

Theorem 3. (Continuous limit cycle). *Let $x_{n+1} = T(x_n)$ be any Douglas–Rachford sequence between A and B with starting point $x_1 \in B_- \setminus F$. Then the sequence x_n is bounded, satisfies, $x_{n+1} - x_n \to 0$, but fails to converge. Its set of accumulation points is $F = A \cap B \subset \mathcal{F}(T)$.*

Proof. 1) For $t \geq 0$ let us introduce the notations

$$a(t) = \big(\cos t, \sin t, e^{-t/2}\big) \in A,$$

and

$$b_\pm(t) = \big((1 \pm e^{-t})\cos t, (1 \pm e^{-t})\sin t, e^{-t/2}\big) \in B_\pm \setminus F.$$

The set $\{a(t) : t \geq 0\} \subset A$ is the shadow of the spiral on the cylinder mantle. Namely, it is clear that for $t > 0$,

$$P_A\left(b_+(t)\right) = P_A\left(b_-(t)\right) = a(t). \tag{5}$$

In particular,

$$\|b_\pm(t) - P_A\left(b_\pm(t)\right)\| = e^{-t}. \tag{6}$$

In consequence

$$R_A\left(b_+(t)\right) = b_-(t), \quad R_A\left(b_-(t)\right) = b_+(t).$$

In words, the two branches $B_\pm$ of the double spiral are the reflections of each others in the cylinder mantle.

2) Let us now analyze the projection of $a(t)$ on the double spiral B. We consider the projections of $a(t)$ on each of the branches $B_\pm$. We start with the analysis of $b \in P_{B_+}\left(a(t)\right)$. We first claim that $b \notin F$. Indeed, the point $v \in F$ closest to $a(t)$ is $v = (\cos t, \sin t, 0) = P_F\left(a(t)\right)$, so $\|v - a(t)\| = e^{-t/2}$. But $\|b_+(t) - a(t)\| = e^{-t} < e^{-t/2}$, hence there are points on $B_+ \setminus F$ closer to $a(t)$ than v. This shows that any projected point $b \in P_{B_+}(a(t))$ has to be of the form $b_+(\tau)$ for some $\tau \geq 0$. Now consider some such $b_+(\tau) \in P_{B_+}\left(a(t)\right)$, then

$$e^{-t} = \|a(t) - b_+(t)\| \geq \|a(t) - b_+(\tau)\| \geq \|a(\tau) - b_+(\tau)\| = e^{-\tau}, \tag{7}$$

which shows $\tau \geq t$. Here the second estimate follows from $a(\tau) = P_A\left(b_+(\tau)\right)$.

3) Let us further observe that $\tau > t$. Namely, if we had $\tau = t$, then we would have a fixed point pair for the method of alternating projections between A and B_+ in the sense that $a(t) = P_A\left(b_+(t)\right)$, $b_+(t) \in P_{B_+}\left(a(t)\right)$. That would mean the distance squared $\tau \mapsto \frac{1}{2}\|a(t) - b_+(\tau)\|^2$ had a local minimum at $\tau = t$. But the derivative of this function at $\tau = t$ is $-e^{-2t} < 0$, so $\tau = t$ is impossible, and we deduce $\tau > t$.

4) Using $b_+(\tau) \in P_{B_+}\left(a(t)\right)$ and (7), we find

$$e^{-t} \geq \|a(t) - b_+(\tau)\| \geq |e^{-t/2} - e^{-\tau/2}| = e^{-t/2} - e^{-\tau/2} = e^{-t}\left(e^{t/2} - e^{t-\tau/2}\right),$$

which shows

$$0 < e^{t/2} - e^{t-\tau/2} \leq 1.$$

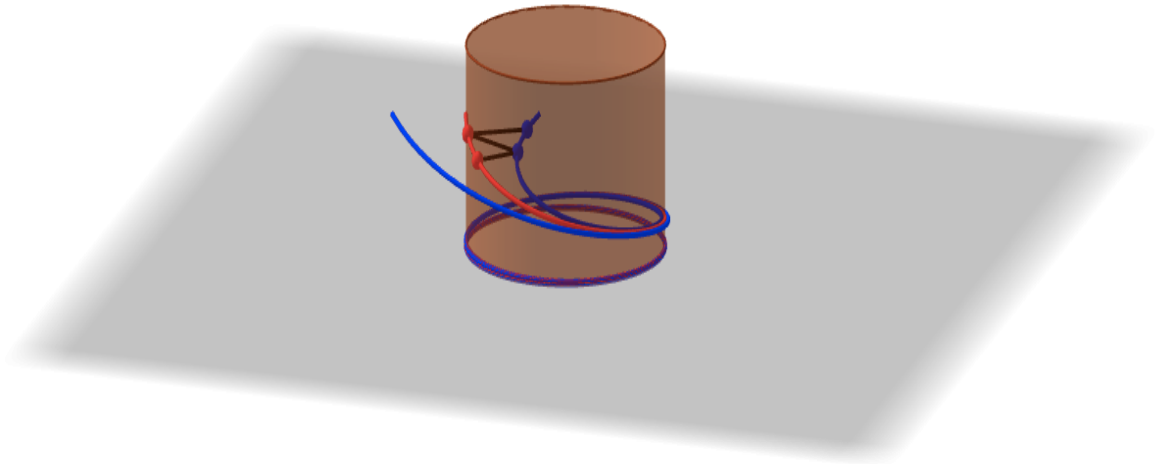
This can be re-arranged as

$$0 < 1 - e^{t/2-\tau/2} \leq e^{-t/2}. \tag{8}$$

In particular, for $t \to \infty$ we must have $\tau - t \to 0$.

5) Let us next show that the projection $b_+(\tau) = P_{B_+}\left(a(t)\right)$ is unique for t sufficiently large. Indeed, suppose we find $t < \tau_1 < \tau_2$ such that $b_+(\tau_1), b_+(\tau_2) \in P_{B_+}\left(a(t)\right)$. Then

by (8) we have $t < \tau_1 < \tau_2 < t - 2\log\left(1 - e^{-t/2}\right)$. Define the function $d_+(x) = \frac{1}{2}\|a(t) - b_+(x)\|^2$, then $d_+(t) = \frac{1}{2}e^{-2t}$, $d_+'(t) = -e^{-2t} < 0$. Since τ_1, τ_2 are local minima, we have $d_+'(\tau_1) = d_+'(\tau_2) = 0$. But $d_+''(x) = \cos(t-x) + 2e^{-2x} + 2e^{-x}\sin(t-x) + \frac{3}{2}e^{-x} - \frac{1}{4}e^{-x/2-t/2}$. In consequence, for t large and x moving in the interval $x \in \left(t, t - 2\log\left(1 - e^{-t/2}\right)\right)$, we have $d_+''(x) \approx \cos(t-x) \approx 1$, so certainly $d_+''(x) > 0$ for these x, and since the local minima τ_i are in that interval for t large, $d_+'(\tau_2) = 0$ is impossible. This proves $b_+(\tau) = P_{B_+}(a(t))$ for t sufficiently large.



6) Let us now consider the point $b_-(\tau) \in B_- \setminus F$ on the inner spiral. We claim that $b_-(\tau)$ is closer to $a(t)$ than $b_+(\tau)$. Indeed, the set of points w having equal distance to $b_-(\tau)$ and $b_+(\tau)$ is the tangent plane to the cylinder at the point $a(\tau) = \frac{1}{2}\left(b_-(\tau) + b_+(\tau)\right)$. But the cylinder lies in one of the half-spaces associated with this plane, namely the one containing $b_-(\tau)$, $\|b_-(\tau) - a(t)\| < \|b_+(\tau) - a(t)\|$. Since $b_+(\tau)$ is the nearest point to $a(t)$ in B_+, we deduce that $P_B(a(t)) \subset P_{B_-}(a(t))$ for all t. In other words, projections from the shadow of the spiral onto the double spiral always go to the inner spiral.

We could also use an analytic argument to prove this. Let $d_-(x) = \frac{1}{2}\|a(t) - b_-(x)\|^2$ and consider the function $f(x) = d_+(x) - d_-(x)$, then $f(x) = \frac{1}{2}e^{-x}\left(1 - \cos(x-t)\right)$, so $f \geq 0$, and $f = 0$ for $x = t + 2k\pi$. Since $t < \tau < t - 2\log(1 - e^{-t/2}) \ll t + 2\pi$, this proves $f(\tau) > 0$.

7) Let $b \in P_B(a(t)) = P_{B_-}(a(t))$ a projected point of $a(t)$ in the inner spiral. We know already that $b \notin F$, hence $b = b_-(\sigma)$ for some $\sigma \geq 0$. Repeating the argument in part 2), it follows that $\sigma > t$. Indeed, like in (7) we have

$$e^{-t} = \|a(t) - b_-(t)\| \geq \|a(t) - b_-(\sigma)\| \geq \|a(\sigma) - b_-(\sigma)\| = e^{-\sigma},$$

and the same argument as in part 2) shows $\sigma > t$. But then again

$$e^{-t} \geq \|a(t) - b_-(\sigma)\| \geq |e^{-t/2} - e^{-\sigma/2}| = e^{-t/2} - e^{-\sigma/2} = e^{-t}\left(e^{t/2} - e^{t-\sigma/2}\right),$$

which shows

$$0 < e^{t/2} - e^{t-\sigma/2} \leq 1.$$

This can be re-arranged to

$$0 < 1 - e^{t/2-\sigma/2} \leq e^{-t/2}. \tag{9}$$

In particular, for $t \to \infty$ we must have $\sigma - t \to 0$, and in particular $0 < \sigma - t \ll 2\pi$. Therefore projected points $b_-(\sigma) \in P_B(a(t))$ lie on the same tour of the spiral as $a(t), b_-(t)$, and one does not take shortcuts by jumping down a full turn of the spiral B_-

or more. Repeating the argument of 5), we also see that $b_-(\sigma) = P_{B_-}(a(t))$ is unique for $t > 0$ large enough.

8) Let us now generate our Douglas–Rachford sequence x_n, starting at $x_1 = b_-(t_1) \in B_-$ with $t_1 > 0$, excluding $t_1 = 0$ for simplicity to have a unique projection on the cylinder mantle at the start.

We get $a_1 = P_A(x_1) = a(t_1)$, hence $R_A(x_1) = b_+(t_1)$. Since $b_+(t_1) \in B$, it is its own reflection in B, and we get $b_1 = b_+(t_1)$. Averaging then gives $x_2 = (b_1 + x_1)/2 = a_1 = a(t_1)$, which concludes the first DR-step.

The second DR-step proceeds as follows. Since $x_2 = a(t_1) \in A$, it is its own reflection in A, so $a_2 = x_2 = a_1 = R_A(x_1)$. Now let $b_2 = P_B(R_A(x_2))$, then $b_2 = P_B(a(t_1)) = P_{B_-}(a(t_1))$, so $b_2 = b_-(t_2)$ for some $t_2 > t_1$, where t_2 is for t_1 what σ was for t in part 7). So the reflected point is $2b_-(t_2) - a(t_2)$ and averaging then gives $x_3 = b_-(t_2) \in B_-$.

Proceeding in this way, we generate a strictly increasing sequence t_n such that

$$x_{2k-1} = b_-(t_k) \in B_-, \quad x_{2k} = a(t_k) \in A. \tag{10}$$

Moreover, the shadow and reflected shadow are

$$P_A(x_{2k-1}) = a_{2k-1} = a(t_k), \quad P_B(R_A(x_{2k-1})) = b_{2k-1} = b_+(t_k)$$

respectively,

$$P_A(x_{2k}) = a_{2k} = a(t_k), \quad P_B(R_A(x_{2k})) = b_{2k} = b_-(t_k).$$

Furthermore, note that we generate a sequence of alternating projections between A and B_-. Namely

$$b_0 := x_1 \stackrel{P_A}{\to} a_1 = a_2 \stackrel{P_{B_-}}{\to} b_2 \stackrel{P_A}{\to} a_3 = a_4 \stackrel{P_{B_-}}{\to} b_4 \ldots \tag{11}$$

9) We now argue that t_n so constructed tends to ∞. Suppose on the contrary that $t_n < t_{n+1} \to t^* < \infty$. Then from the construction we see that we create a pair $a(t^*) \in A$, $b_-(t^*) \in B_-$ such that $a(t^*) \in P_A(b_-(t^*))$ and $b_-(t^*) \in P_{B_-}(a(t^*))$. Arguing as before, this would imply that $\tau \mapsto \frac{1}{2}\|a(t^*) - b_-(\tau)\|^2$ had a local minimum at $\tau = t^*$, which it does not because its derivative at t^* is $-e^{-2t^*} < 0$. Hence $t^* < \infty$ is impossible, and we have $t_n \to \infty$. As a consequence, the statements about uniqueness of the operators and the estimate (7) are now satisfied from some counter n_0 onward.

10) To conclude, observe that $x_n - x_{n+1} \to 0$ by (9), and that the $a(t_n)$ are $2e^{-t_n}$-dense in the interval $[t_n, t_n + 2\pi]$, because of

$$\|a(t_n) - a(t_{n+1})\| \le \|a(t_n) - b_-(t_{n+1})\| + \|b_-(t_{n+1}) - a(t_{n+1})\| \le e^{-t_n} + e^{-t_{n+1}} \le 2e^{-t_n}.$$

Using (6) and the fact that every $a(t)$ with $t \in [t_n, t_n + 2\pi]$ is at distance $\le e^{-t_n/2}$ to the set F, we deduce that every point in F is an accumulation point of both the DR-sequence (10) and the MAP sequence (11). □

Remark 9. (Strong fixed points need not be stable). The system-theoretic interpretation of this result is that a strong fixed-point $x^* \in F \subset \mathcal{F}(T)$ need not be a stable steady state. This is in contrast with Theorem 1, where this was shown to be true when A, B are finite unions of convex sets. A second interpretation is that F is a stable attractor for the dynamical system $x^+ = T(x)$.

Remark 10. (Shadows need not converge). We note that not only does the DR sequence x_n fail to converge in Theorem 3, also the sequences $a_n = P_A(x_n) \in A$, $b_n = P_B(R_A(x_n)) \in B$ fail to converge and have the same continuum set of accumulation points F. Presently no example of failure of convergence of a bounded DR-sequence x_n satisfying $x_n - x_{n+1} \to 0$ is known, where the shadow sequence a_n converges to a single limit $a^* \in A \cap B$. It is clear that this could only happen when $F \subset \{x : \|x - a^*\| = \epsilon\}$ for some $\epsilon > 0$.

Corollary 1. (Continuous limit cycle for MAP). *Let x_n be the Douglas–Rachford sequence constructed above, and let $a_n \in A$, $b_n \in B$ be the shadows associated with x_n. Then b_{2n}, a_{2n+1} is a sequence of alternating projections between the cylinder A and the inner spiral B_-. This sequence also fails to converge and has the same set of accumulation points $F = A \cap B$.* □

Corollary 2. (Limit cycle for MAP with one set convex). *Every sequence of alternating projections a_n, b_n between the outer spiral B_+ and the solid cylinder $\operatorname{conv}(A)$ started at $b_1 \in B_+ \setminus F$ is bounded, satisfies $a_n - b_n \to 0$, but fails to converge and has the set $F = B_+ \cap \operatorname{conv}(A)$ as its set of accumulation points.*

Proof. In part 2) of the proof of Theorem 3 we analyzed this sequence, which is generated by the building blocks $a(t) \to b_+(\tau) \to a(\tau)$. □

Remark 11. Here we have an example of a semi-algebraic convex set $\operatorname{conv}(A)$, and the spiral B_+, which is the projection of a semi-analytic set in $\mathbb{R}^4$, where the MAP sequence fails to converge and leads to a continuous limit cycle. The first example with a continuous limit cycle appears in [4], but with more pathological sets A, B. The fact that B_+ is not subanalytic can be deduced from [13, Cor. 7]. Currently we do not have an example where the DR-algorithm fails to converge and creates a continuous limit cycle with one of the sets convex.

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