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RESEARCH ARTICLE

Parabolic tools

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In this paper we define a notion of parabolic ray which is analogous to external/internal ray in super-attracting basins. It is used to construct parabolic puzzles similar to Yoccoz puzzles, but modified near the parabolic fixed point. As an application we show that any non renormalizable quadratic rational map in the connectedness locus $\mathbf{M_1}$ of $Per_1(1)$ has locally connected Julia set.

Keywords: parabolic rays, parabolic Yoccoz puzzle, parabolic Mandelbrot set $\mathbf{M_1}$, Yoccoz' Combinatorial Analytic invariant.

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1. Introduction

Douady and Hubbard introduced the notion of external rays of polynomials. These are used in several ways in the analysis of the dynamics of polynomials. They serve to connect the dynamics of Julia points with the dynamics of nearby points in the basin of attraction of infinity. Also if two or more rays land (or converge) to the same point, then they define a dynamical partition of the Julia set which possibly may be used like a Markov partition of the Julia set. This was exploited by Yoccoz, who constructed so called Yoccoz puzzles with exactly this feature. Using puzzles he proved combinatorial rigidity for almost all parameters with respect to harmonic measure on the boundary of the Mandelbrot set $\mathbf{M}$.

External and internal rays are defined for super attracting dynamics. The aim of this paper is to introduce parabolic rays in certain parabolic basins and corresponding parabolic puzzles. These are used to study maps with a fixed parabolic basin, e.g. the family $Per_1(1)$ of (Möbius conjugacy classes) of quadratic rational maps with a parabolic fixed point of multiplier 1. In particular, if $\mathbf{M_1}$ denotes the connectedness locus in $Per_1(1)$, we prove that

THEOREM 1.1. *There exists a natural dynamics preserving projection* $\Psi^1 : \mathbf{M_1} \longrightarrow \mathbf{M}$.

THEOREM 1.2. *Suppose $g \in \mathbf{M_1}$ possesses a repelling fixed point. If g is non-renormalizable then J_g is locally connected and the two maps g and Q_c with $c = \Psi^1(g)$ are topologically conjugate on their respective Julia sets.*

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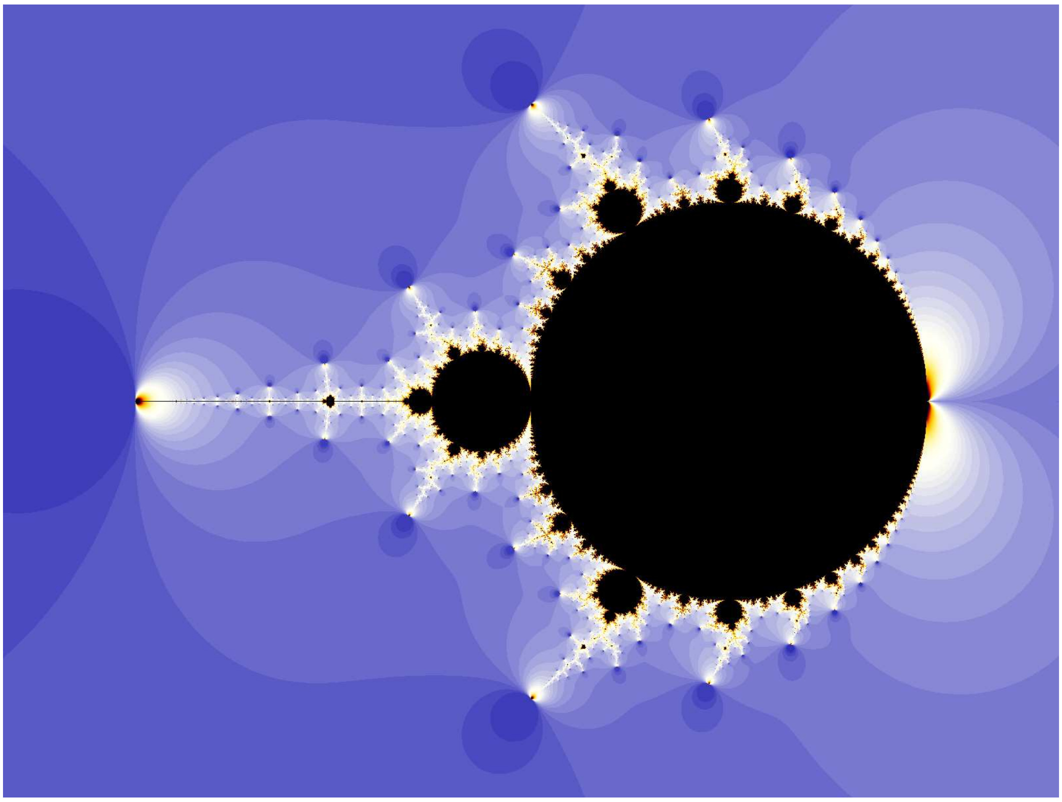

Figure 1. The parabolic Mandelbrot set $\mathbf{M_1}$.

2. Parabolic rays.

We construct parabolic rays first in the basin of attraction of a model map, the degree d parabolic Blaschke product :

$$P_d(z) = \frac{z^d + v_d}{1 + v_d z^d} \qquad \text{with } v_d = \frac{d-1}{d+1} \text{ and } d > 1.$$

It has critical points at 0 and ∞, critical values at v_d and $1/v_d$ and a double parabolic fixed point at 1.

Before we construct the parabolic rays, let us remark that the combinatorics of P_d on $\mathbb{S}^1$ is the "same as" the combinatorics of z^d on $\mathbb{S}^1$:

Remark 1. There exists a unique homeomorphism $h = h_d : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ fixing 1 and conjugating $z \mapsto z^d$ to P_d, i.e. $h_d(z^d) = P_d \circ h_d$.

Proof. This is a classical theorem for strongly expanding maps, for which the proof passes over to the weakly expanding case without any essential changes. Here the maps P_d are weakly expanding on $\mathbb{S}^1$, as $|P_d'(z)| \geq 1$ on $\mathbb{S}^1$ with equality if and only if $z^d = 1$. The idea is as follows. Let $h_0 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ denote the identity and define recursively $h_n : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ by $P_d \circ h_n = h_{n-1}(z^d)$ and $h_n(1) = 1$. The maps h_n converge to an order-preserving bijection between the two sets of iterated preimages of 1 and by the weakly expanding property both sets of iterated preimages are dense in $\mathbb{S}^1$ so that the limit of the h_n exists on all of $\mathbb{S}^1$ and is the required topological conjugacy.

For the map z^d, the internal ray in $\mathbb{D}$ of argument $\theta \in [0,1]$ is naturally the half line $\{\exp(t + 2\pi i\theta), t < 0\}$. Similarly the external ray of argument $\theta \in [0,1]$ for z^d

is the half line $\mathcal{R}_\theta = \{\exp(t+2\pi i\theta), t > 0\}$. Note that the external ray of argument θ is mapped onto the internal ray of argument $-\theta$ by the conformal isomorphism $1/z : \overline{\mathbb{C}}\backslash\overline{\mathbb{D}} \longrightarrow \mathbb{D}$ and onto the internal ray of argument θ by the conformal, but anti-holomorphic map $\tau(z) = 1/\overline{z} : \overline{\mathbb{C}}\backslash\overline{\mathbb{D}} \longrightarrow \mathbb{D}$ (the reflection in the unit circle).

The object of this section is to define in a dynamical way a similar notion of rays for the parabolic Blaschke products P_d, *internal parabolic rays* in $\mathbb{D}$ and *external parabolic rays* in $\overline{\mathbb{C}}\backslash\overline{\mathbb{D}}$. Then afterwards we define *parabolic rays* in more general parabolic basins by transportation by conformal isomorphisms. When labeling rays one should of course be careful to indicate wether the rays are considered internal or external rays. In general rays should be internal rays, but in the particular cases of basins whose attracting or parabolic point is the point at infinity, e.g., polynomials the use of external rays is usually the most appealing.

2.1 The shift encoding

Let $\Sigma_d := \{0, 1, \ldots, d-1\}^{\mathbb{N}}$ denote the one-sided shift space on $d > 1$ symbols with $\sigma_d : \Sigma_d \longrightarrow \Sigma_d$ the shift map: $\sigma_d(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) = (\epsilon_2, \epsilon_3, \ldots, \epsilon_{n-1}, \ldots)$.

Write $I_j = [\omega^j, \omega^{j+1}] \subset \mathbb{S}^1$ for $j = 0, \ldots, d-1$, where $\omega = \exp(i2\pi/d)$. An itinerary of a point $z \in \mathbb{S}^1$ under z^d is a sequence $\underline{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots)$ with the property that for all $n \in \mathbb{N}$: $z^{d^n} \in I_{\epsilon_{n+1}}$.

Denote by $\Pi_d : \Sigma_d \longrightarrow \mathbb{S}^1$ the d-ary projection map:

$$\Pi_d(\epsilon_1, \epsilon_2, \ldots, \epsilon_n, \ldots) = \exp(2\pi i\theta), \qquad \text{where } \theta = \sum_{n=1}^{\infty} \frac{\epsilon_n}{d^n}, \tag{1}$$

$\underline{\epsilon}$ is called a d-ary expansion of θ. Obviously Π_d semi-conjugates the shift σ_d to $z \mapsto z^d$ on $\mathbb{S}^1$. The reader shall easily verify that for each $\underline{\epsilon} \in \Sigma_d$ the point $\Pi_d(\underline{\epsilon})$ is the unique point of itinerary $\underline{\epsilon}$ under z^d.

We equip Σ_d with the lexicographic order : $\underline{\epsilon}^1 = (\epsilon_1^1, \ldots, \epsilon_n^1, \ldots) < (\epsilon_1^2, \ldots, \epsilon_n^2, \ldots) = \underline{\epsilon}^2$ if and only if for some $m \in \mathbb{N}$: $\epsilon_k^1 = \epsilon_k^2$ for $k < m$ and $\epsilon_m^1 < \epsilon_m^2$. Two sequences $\underline{\epsilon}^1 < \underline{\epsilon}^2$ are the common itineraries of a point z if and only if $z^{d^n} = 1$ for some minimal $n \geq 0$; equivalently for this n $\epsilon_k^1 = \epsilon_k^2$ for $k < n$, $0 < \epsilon_n^2 = \epsilon_n^1 + 1 < d$ and $\epsilon_k^1 = d-1$, $\epsilon_k^2 = 0$ for $k > n$.

Defining itineraries for P_d by the same algorithm as for z^d above, i.e. $P_d^n(z) \in I_{\epsilon_{n+1}}$, we obtain exactly the same statements for P_d. (Note that by symmetry h fixes each of the d-th roots of unity ω^j, for $j = 0, \ldots, d-1$.) For example : $h \circ \Pi$ conjugates the shift σ_d to P_d, any itinerary for P_d determines a unique point of $\mathbb{S}^1$ and a point has two itineraries if and only if $P_d^n(z) = 1$ for some n.

2.2 The d-adic tree and parabolic rays for P_d

The parabolic rays are paths in a d-adic tree (to be defined below) that have the prescribed combinatorics.

Let $z_\emptyset = 0$, where $\emptyset$ denotes the empty set and let $z_j = v_d^{\frac{1}{d}} e^{\frac{i\pi}{d}} \omega^j$ be all the preimages of $z_\emptyset$ by P_d. Define $T_\emptyset := P_d^{-1}([0, v_d])$, it contains $z_\emptyset$ and is equal to $\cup_{j=0}^{d-1}[0, z_j]$; then denote by T_j the connected component of $P_d^{-1}(T_\emptyset)$ containing z_j (see also Fig. 2). For $\underline{\epsilon} \in \Sigma_d$, define recursively $z_{\epsilon_1\epsilon_2\cdots\epsilon_n}$ as the unique point of the preimage $P_d^{-1}(z_{\epsilon_2\cdots\epsilon_n})$ belonging to $T_{\epsilon_1\epsilon_2\cdots\epsilon_{n-1}}$, so that $P_d(z_{\epsilon_1\epsilon_2\cdots\epsilon_n}) = z_{\sigma(\epsilon_1\epsilon_2\cdots\epsilon_n)}$. Then the twig $T_{\epsilon_1\epsilon_2\cdots\epsilon_n}$ is defined as the connected component of the preimage $P_d^{-1}(T_{\epsilon_2\ldots\epsilon_n})$ containing $z_{\epsilon_1,\epsilon_2,\ldots,\epsilon_n}$, so that $P_d(T_{\epsilon_1\epsilon_2\cdots\epsilon_n}) = T_{\sigma(\epsilon_1\epsilon_2\cdots\epsilon_n)}$. (Interpreting $n = 0$ as index $\emptyset$ in $z_\emptyset$ or $T_\emptyset$.)

Define, for each $n \in \mathbb{N}$, the d-adic tree $\mathcal{T}_n := \bigcup_{k=0}^{n} P_d^{-k}(T_\emptyset)$ and the infinite d-adic tree $\mathcal{T} := \bigcup_{k=0}^{\infty} P_d^{-k}(T_\emptyset)$ with boundary (in) $\mathbb{S}^1$, so that $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$ and

$$\mathcal{T}_n = \mathcal{T}_{n-1} \cup \bigcup_{(\epsilon_1\epsilon_2\cdots\epsilon_n)\in\{0,1,\ldots,(d-1)\}^n} T_{\epsilon_1\epsilon_2\cdots\epsilon_n}.$$

Denote by S_j the open sector spanned by the arc $I_j = [\omega^j, \omega^{j+1}]$, for $0 \leq j \leq d-1$, *i.e.*, the interior of the convex hull of the union of I_j and 0. It contains z_j and it is mapped univalently onto $\mathbb{D}\backslash[v_d, 1] \supset T_\emptyset$ (with the boundary arcs $[0, \omega_j]$ and $[0, \omega_{j+1}]$ each mapped (homeomorphically) onto $[v_d, 1]$). It easily follows by induction that both the point $z_{j\epsilon_2\cdots\epsilon_n}$ and the twig $T_{j\epsilon_2\cdots\epsilon_n}$ are contained in S_j for any n and any $(\epsilon_2\cdots\epsilon_n)$, so that $\mathcal{T}_n \cap S_j$ is connected.

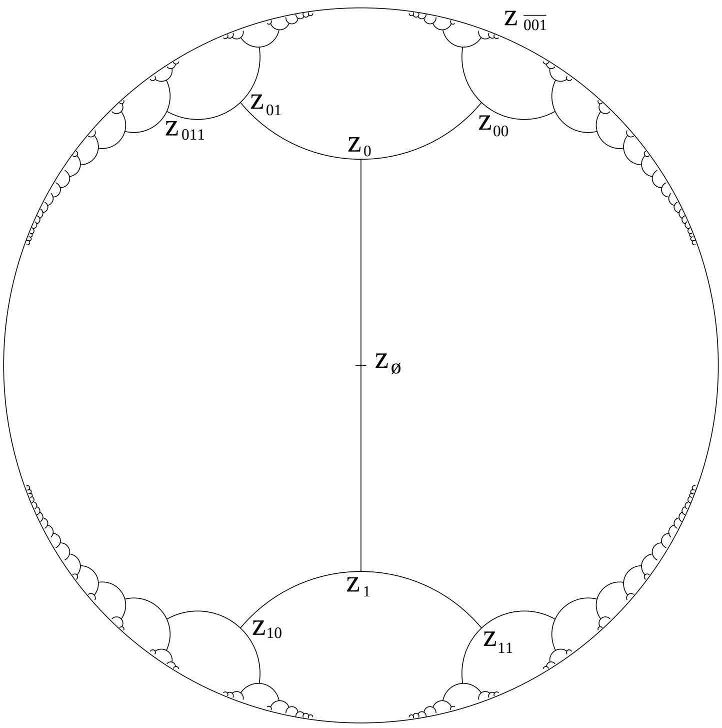

Figure 2. The infinite dyadic tree $\mathcal{T}$ containing the internal parabolic rays of P_2.

Definition 2.1. For $\underline{\epsilon} \in \Sigma_d$ define the parabolic ray $R_{\underline{\epsilon}}$ as the minimal connected subset of $\mathcal{T}$ containing the sequence of points $(z_{\epsilon_1\epsilon_2\cdots\epsilon_n})_{n\in\mathbb{N}}$ (interpreting it as $z_\emptyset$ for $n = 0$).

Remark 2. For any $\underline{\epsilon} \in \Sigma_d$, the combinatorics of $R_{\underline{\epsilon}}$ is determined by $\underline{\epsilon}$ since :

- $P_d(R_{\underline{\epsilon}}) = R_{\sigma(\underline{\epsilon})} \cup [0, v_d]$;
- any ray $R_{\underline{\epsilon}}$ with $\epsilon_1 = j$ belongs to $S_j \cup \{0\}$.

Let $\Phi_+ = \Phi_{d,+} : \mathbb{D} \longrightarrow \mathbb{C}$ be the attracting Fatou coordinate for P_d on the basin $\mathbb{D}$, normalized by $\Phi_+(0) = 0$. It maps $[0, 1[$ homeomorphically onto $[0, +\infty[$. Therefore, the ray $R_{\underline{\epsilon}}$ is mapped homeomorphically onto $\mathbb{R}_- =]-\infty, 0]$. Note that the map Φ_+ has a degree d critical point at each $z_{\epsilon_1\epsilon_2\ldots\epsilon_n} \in R_{\underline{\epsilon}}$ with critical value $-n$ for $n \geq 0$.

Definition 2.2. For every $\underline{\epsilon} \in \Sigma_d$, let $\widehat{R}_{\underline{\epsilon}}$ denote the extension of $R_{\underline{\epsilon}}$ by the segment $[0,1[$. For $t \in \mathbb{R}$, there is a unique point on it of "potential" t (since $\widehat{R}_{\underline{\epsilon}}$ is mapped homeomorphically to $\mathbb{R}$ by Φ_+). Denote by $\widehat{R}_{\underline{\epsilon}}(t)$ this point : $\widehat{R}_{\underline{\epsilon}}(t) = \Phi_+^{-1}(t) \cap \widehat{R}_{\underline{\epsilon}}$ and by $R_{\underline{\epsilon}}(t)$ the restriction to $t \leq 0$.

By construction,

$$\forall\, \underline{\epsilon} \in \Sigma_d, \quad \forall\, t \in \mathbb{R} : P_d(\widehat{R}_{\underline{\epsilon}}(t)) = \widehat{R}_{\sigma(\underline{\epsilon})}(t+1). \tag{2}$$

Note that every ray $R_{\underline{\epsilon}}(t)$ converges to $h_d \circ \Pi_d(\underline{\epsilon}) =: z_{\underline{\epsilon}}$, as t tends to $-\infty$, because any limit point of $R_{\underline{\epsilon}}(t)$ belongs to $\mathbb{S}^1$ and has itinerary $\underline{\epsilon}$. The external ray, resp. extended external ray, of itinerary $\underline{\epsilon}$ is the arc $\tau(R_{\underline{\epsilon}})$, resp. $\tau(\widehat{R}_{\underline{\epsilon}})$ (where $\tau(z) = 1/\overline{z}$).

The restriction of the usual Euclidean Hausdorff distance $\mathrm{d}_{\mathbb{C}}(\cdot,\cdot)$ between compact sets in the plane, to the set of closed parabolic rays $\overline{R}_{\underline{\epsilon}} = R_{\underline{\epsilon}} \cup \{z_{\underline{\epsilon}}\}$, $\underline{\epsilon} \in \Sigma_d$ is a metric $\mathrm{d}(\cdot,\cdot)$ on this set. Note that for this metric $\mathrm{d}(\overline{R}_{\overline{0}}, \overline{R}_{\overline{d-1}}) > 0$, where $\overline{j} := (j, j, \ldots, j, \ldots)$ for $0 \leq j \leq d-1$, even though the two rays land on the same point 1 and even though in the super attracting classical case the two rays are identical. In fact $R_{\overline{0}} \cup \{0\} \cup R_{\overline{d-1}}$ bounds a topological disk D_0, whose image under Φ_+ is the slit plane $\mathbb{C} \setminus]-\infty, 0]$.

2.3 Parabolic rays

Let $U \subset \overline{\mathbb{C}}$ be an open subset and $f : U \longrightarrow \overline{\mathbb{C}}$ a holomorphic map, which has a fixed proper simply connected parabolic basin $\Lambda \subset U$ with a single critical point c of order $d-1$. Then the Riemann map $\phi : \Lambda \longrightarrow \mathbb{D}$ with $\phi(c) = 0$ can be normalized so that the conjugate map $\phi \circ f \circ \phi^{-1}$ equals P_d.

Definition 2.3. The parabolic ray of argument $\underline{\epsilon} \in \Sigma_d$ in Λ is $R^{\Lambda}_{\underline{\epsilon}} := \phi^{-1}(R_{\underline{\epsilon}})$ and the extended parabolic ray is $\widehat{R}^{\Lambda}_{\underline{\epsilon}} := \phi^{-1}(\widehat{R}_{\underline{\epsilon}})$.

For correspondence with polynomials we shall in the case Λ is unbounded consider instead external rays defined via $z \mapsto 1/\phi(z)$, where ϕ is as above. As for polynomials this amounts to a change of orientation, because $z \mapsto 1/z$ is an automorphism of both z^d and P_d and $1/\mathrm{e}^{i2\pi\theta} = \mathrm{e}^{i2\pi(1-\theta)}$. In terms of the labeling by elements of Σ_d it corresponds to the labeling change by the automorphism of σ_d induced by $\epsilon \mapsto d-1-\epsilon$ on $\{0, 1, \ldots, d-1\}$.

We call the rays in Λ external rays, and if not internal rays. As for the classical rays we say that a q cycle for f^k of rays $R_0, \ldots R_{q-1}$ landing on a common k periodic point $z \in \mathbb{C}$ and numbered in the counter clockwise order around z defines *combinatorial rotation number* p/q, $(p,q) = 1$ if and only if $f^k(R_j) = R_{(j+p) \mod q}$. The following is a classical result initially proved for external rays of polynomials, see also Theorem 3.1:

THEOREM 2.4. *Assume that $\overline{\Lambda} \subset U$. Then, for any (pre-)periodic argument $\underline{\epsilon} \in \Sigma_d$, i.e. $\sigma^k(\sigma^l(\underline{\epsilon})) = \sigma^l(\underline{\epsilon})$, the parabolic ray $R = R^{\Lambda}_{\underline{\epsilon}}$ converges to an f (pre-)periodic point $z \in \partial\Lambda$, satisfying $f^k(f^l(z)) = f^l(z)$.*

If $\underline{\epsilon}$ is periodic (i.e. $l = 0$), the point z is repelling or parabolic of period k' dividing k. The ray R defines a combinatorial rotation number p/q at the point z, where $q = k/k'$ (and $(p,q) = 1$). Moreover, if z is parabolic it has multiplier $\mathrm{e}^{i2\pi p/q}$. Finally, any other ray in Λ landing at z is also k-periodic and defines the same rotation number.

This is a standard result which in its initial form is due to Sullivan, Douady and Hubbard, see also [P, Th. A and Prop. 2.1].

And conversely

THEOREM 2.5 . *Suppose $z \in \partial\Lambda$ is any periodic point such that z does not belong to the closure of any other connected component of $f^{-1}(\Lambda)$. Then there is at least one periodic parabolic ray landing at z. In particular z has a combinatorial rotation number.*

For a proof see e.g. [P, Th. B, for the case of rational maps, the general case is identical].

Remark 3. From now on and during the rest of this paper we consider only quadratic maps.

3. The Quadratic Universal Yoccoz Puzzles.

3.1 Rays for quadratic polynomials

We recall here some classical results of Douady and Hubbard (see [DH1] and [DH2]) concerning the family of quadratic polynomials $Q_c(z) = z^2 + c$. The basin of attraction of ∞ is denoted by $B_c(\infty) := \{z \in \mathbb{C} \mid Q_c^n(z) \to \infty\}$ and its Julia set by $J_c := \partial B_c(\infty)$. The Mandelbrot set $\mathbf{M}$ corresponds to the parameters $c \in \mathbb{C}$ such that J_c is connected. The Riemann map $\phi_c : B_c(\infty) \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ tangent to identity at ∞ conjugates Q_c to Q_0. Denote by ψ_c its inverse. The external ray of angle θ is defined by $\mathcal{R}_\theta^c = \psi_c(\mathcal{R}_\theta)$ (where $\mathcal{R}_\theta$ is the straight line of angle θ, see section 2). The map $Q_c(z)$ has (counting multiplicity) two fixed points. The beta-fixed point $\beta(c)$ is by definition the landing point of the unique fixed ray, $\mathcal{R}_0^c$. The other fixed point $\alpha(c)$ can be attracting, neutral or repelling.

THEOREM 3.1 Douady-Hubbard-Sullivan-Yoccoz *For $c \in \mathbf{M}$ we have :*

(1) Every Q_c-(pre-)periodic external ray $\mathcal{R}_\theta^c$, say with $2^{k+l}\theta \equiv 2^l\theta \mod 1$, lands at (or converges to) a (pre-)periodic repelling or parabolic point $z = z(c)$ satisfying $Q_c^{k+l}(z) = Q_c^l(z)$. Moreover the landing of the ray $\mathcal{R}_\theta^c$ at z is locally stable at c when $Q_c^l(z)$ is repelling and 0 is not in the orbit of z. In particular the landing is globally stable in any connected open set for which $Q_c^l(z)$ remains repelling and 0 does not enter the forward orbit of $\overline{\mathcal{R}_\theta^c}$.

(2) Any Q_c-periodic repelling or parabolic point is the landing point of at least one cycle of external rays.

If $\alpha(c)$ is repelling or parabolic there exists a q-cycle of rays $\mathcal{R}_{\theta_0}^c, \mathcal{R}_{\theta_1}^c, \ldots, \mathcal{R}_{\theta_{q-1}}^c$, with $0 < \theta_0 < \theta_1 < \ldots < \theta_{q-1} < 1$ landing on $\alpha(c)$. It has some combinatorial rotation number p/q with $(p, q) = 1$, i.e. $2\theta_j \equiv \theta_{(j+p) \mod q} \mod 1$. Denote by H_0 the set of parameters $c \in \mathbf{M}$ such that $\alpha(c)$ is attracting; it corresponds to the main cardioid of $\mathbf{M}$. The following decomposition of $\mathbf{M}$ is originally due to Douady and Hubbard (see [M]).

THEOREM 3.2 . $\mathbf{M} = \overline{H_0} \cup \bigcup_{\frac{p}{q} \neq \frac{0}{1}} L_{p/q}^{\star}$, *where the uprooted limb $L_{p/q}^{\star}$ consists of those parameters $c \in \mathbf{M}$ for which the separating fixed point $\alpha(c)$ is repelling and has combinatorial rotation number p/q.*

For the rest of this section we fix the reduced rational p/q, so all the introduced quantities will depend on p/q, but we shall only occasionally make reference to p/q.

3.2 The Classical Yoccoz Puzzle.

For $c \in \mathbf{M} \setminus \overline{H_0}$, we define the Yoccoz puzzle as follows. We shall fix an arbitrary choice of equipotential (say of level 1), $\mathcal{E}_1 = \phi_c(\{z \mid |z| = e\}$.

Let $\mathcal{GY}_c^0$ denote the union of the equipotential $\mathcal{E}_1$ together with $\alpha(c)$, $-\alpha(c)$ (its preimage) and the segments, inside $\mathcal{E}_1$, of the external rays of the p/q cycle $(\mathcal{R}^c_{\theta_j})$ (landing at it) together with the segments inside $\mathcal{E}_1$ of the preimages by Q_c of the segments of those rays. (Note that the original construction involved only the cycle of rays and not the preimages that we add here for convenience).

The *level-*0 *puzzle pieces* are the closures of each of the bounded connected components of $\mathbb{C}\setminus\mathcal{GY}_c^0$. Denote by $\mathcal{Y}_c^0$ the level-0 puzzle : the collection of these $2q-1$ puzzle pieces. Define the level-$n \in \mathbb{N}$ puzzle $\mathcal{Y}_c^n$ as the collection of closures of connected components of $Q_c^{-n}(\overset{\circ}{Y})$, where Y ranges over all of the level-0 puzzle pieces. The (p/q-Yoccoz) Puzzle for Q_c is the union $\mathcal{Y}_c = \cup_{n\geq 0}\mathcal{Y}_c^n$ of the puzzles at all levels.

Denote by $\mathcal{GY}_c^n$ the union $\bigcup_{n\geq 0} Q_c^{-n}(\mathcal{GY}_c^0)$; it coincides with the union of the boundaries of puzzle pieces of all levels up to and including n. Let $\mathcal{GY}_c$ be the union of these graphs of all levels.

Any two puzzle pieces $Y \in \mathcal{Y}_c^n$ and $Y' \in \mathcal{Y}_c^m$, $m \leq n$ are either interiorly disjoint or nested with $Y \subseteq Y'$ (because the potential is multiplied by two under the dynamics and the set of rays in the construction of $\mathcal{Y}_c^0$ is forward invariant).

A *nest*, i.e. a sequence $\mathcal{N} = \{Y^n\}_n$, $Y^n \in \mathcal{Y}_c^n$ with $Y^{n+1} \subseteq Y^n$, is called *convergent* iff $\mathrm{End}(\mathcal{N}) := \bigcap_{n\in\mathbb{N}} Y^n$ is a singleton set and is called *divergent* otherwise. A nest $\mathcal{N}$ is called *critical* iff $0 \in \mathrm{End}(\mathcal{N})$ and called a *critical value nest* iff $c \in \mathrm{End}(\mathcal{N})$.

In his proof of local connectivity of Julia sets, Yoccoz considers the sequence of disjoint annuli of level-n : $A_n = Y_n\setminus\overset{\circ}{Y}_{n+1}$ where $\mathcal{N} = \{Y^n\}_n$is some nest. By classical analysis the $\mathrm{End}(\mathcal{N})$ reduces to a point if and only if $\mod (Y^0 \setminus \mathrm{End}(\mathcal{N})) = \infty$. Moreover by the Grötzsch inequality for moduli of annuli,

$$\sum_{n=0}^{\infty} \mod (A_n) \leq \mod (Y^0 \setminus \mathrm{End}(\mathcal{N}))$$

(The definition of the modulus of an annulus is recalled in section 5.6). To have positive modulus, we need that A_n is non degenerate *i.e.* that $Y_{n+1} \subset \overset{\circ}{Y}_n$ in which case we say that the annulus A_n is *good*. The annulus is called degenerate otherwise. (That is an annulus is good if and only if the inner and outer boundaries are disjoint.)

A main idea of the proof of Yoccoz' theorem is that there are infinitely many higher level annuli A_n in the critical nest, which are non ramified covering spaces of some good lower level critical annulus A_{n_0} via the dynamics. And moreover that the sum of the moduli of those annuli, called descendants of A_{n_0}, is infinite whenever Q_c is non renormalizable. The fundamental properties here are 1) that $Q_c(Y_n) = Y'_{n-1}$, where Y'_{n-1} is the puzzle piece containing $f(x)$ for some $x \in Y_n$; and 2) that if two annuli are related by a degree $d' > 0$ non ramified holomorphic covering then the modulus of the image is d' times the modulus of the domain. It easily follows that the divergence or non divergence of the sum of the moduli of the descendants of A_{n_0} in the critical nest does not depend on the actual value of $\mathrm{mod}(A_{n_0}) > 0$, rather it is a property of the combinatorics of the dynamics.

So apart from the combinatorics, the question is: When is an annulus $A_n = Y_n\setminus\overset{\circ}{Y}_{n+1}$ degenerate? Answer: A_n is good if and only if the ray segments on the boundary of Y_{n+1} are disjoint from the ray segments on the boundary of Y_n. This is something which is visible already in the Böttcher coordinate.

The graph $\mathcal{GY}_c^0$ is stable (by Theorem 3.1) in any domain for which $\alpha(c)$ remains repelling and 0 does not enter the closure of the cycle of rays. That is any domain on which $\alpha(c)$ has a stable non-zero rotation number. In fact, when viewed in the Böttcher coordinates at ∞ any two Yoccoz puzzles $\mathcal{Y}_c$ and $\mathcal{Y}_{c'}$ for which $\alpha(c)$ and $\alpha(c')$ have the same rotation number, look "identical". Thus, we can define a "Yoccoz puzzle in the Böttcher coordinate" called the Universal Yoccoz Puzzle.

3.3 The Universal p/q-Yoccoz Puzzle.

In Böttcher coordinates, the graph $\mathcal{GU}^0 := \phi_c(\mathcal{GY}_c^0 \backslash \{\alpha, -\alpha\}) \cup \mathbb{S}^1$ is the union of the circles of radii 1 and e^1 together with the segments of the straight rays $\mathcal{R}_{\theta_j}, \mathcal{R}_{\theta_j+\frac{1}{2}}$, $0 \leq j \leq q-1$ between these two circles.

The *level*-0 *universal puzzle* $\mathcal{U}^0$ consists of the closures of the $2q$ connected components of $\mathbb{C} \setminus \mathcal{GU}^0$ between the two circles. The critical puzzle piece $Y_0^c(0)$, i.e. the puzzle piece containing the critical point 0, has two connected components in $B_c(\infty)$, whereas the other puzzle pieces have connected intersection with $B_c(\infty)$. This is why the level-0 universal puzzle has $2q$ pieces, whereas the level-0 puzzle of Q_c only has $2q-1$ pieces, since ϕ_c is defined only on $B_c(\infty)$.

Let $\mathcal{U}^n$ be the union of the $2^n 2q$ pieces $Q_0^{-n}(U)$, where U ranges over the $2q$ pieces of the level-0 puzzle $\mathcal{U}^0$. Since ϕ_c is a conjugacy, the universal puzzle $\mathcal{U}^n$ is the "image" of the level-n puzzle $\mathcal{Y}_c^n$.

Define $\mathcal{U} = \cup_{n \in \mathbb{N}} \mathcal{U}^n$. We call $\mathcal{U}$ *the universal* p/q*-Yoccoz puzzle.* For a universal puzzle piece U, we call the intersection $U \cap \mathbb{S}^1$ the *base* of U, the opposite edge the *top* and the two remaining edges the *sides* of U.

Note that a Yoccoz puzzle piece $Y \in \mathcal{Y}_n^c$ can correspond to several universal puzzle pieces of $\mathcal{U}_n$ through the map ϕ_c : each preimage of $Y_0^c(0)$ corresponds to two universal puzzle pieces of $\mathcal{U}_0$ and this number is doubled each time the iterated preimage contains the critical point 0.

Let $\mathcal{Z}^0 = \displaystyle\bigcup_{j=0}^{q-1} \mathrm{e}^{i2\pi\theta_j} \cup \mathrm{e}^{i2\pi(\theta_j+\frac{1}{2})}$ be the unique q-cycle for Q_0 of combinatorial rotation number p/q in $\mathbb{S}^1$ and its preimage. Note that $\mathcal{Z}^0$ coincides with the endpoints on $\mathbb{S}^1$ of the rays in $\mathcal{U}^0$. Define $\mathcal{Z}^n := Q_0^{-n}(\mathcal{Z}^0)$ so that $\mathcal{Z}^n$ consists of the endpoints on $\mathbb{S}^1$ of the rays in $\mathcal{U}^n$. Notice that $\mathcal{Z}^n \subset \mathcal{Z}^{n+1}$ for all $n \geq 0$ and let $\mathcal{Z}$ denote the increasing union $\mathcal{Z} := \cup_{n \geq 0} \mathcal{Z}^n$.

Yoccoz' theorem on combinatorial rigidity can be interpreted as follows: Though the common model $\mathcal{U} = \phi_c(\mathcal{Y}_c) = \phi_{c'}(\mathcal{Y}_{c'})$ does not detect directly any differences between c and c' both from $L^{\star}_{p/q}$, the following natural equivalence $\sim_\infty^c$ on $\mathcal{Z}$ does.

Definition 3.3. The equivalence $\sim_\infty^c$ on $\mathcal{Z}$ is given by declaring τ and τ' in $\mathcal{Z}$ equivalent if and only if the corresponding rational rays for Q_c land at a common point.

Remark 1. We can detect a degenerate annulus already through the universal p/q-Yoccoz puzzle $\mathcal{U}$ and $\sim^c$: Let Y, Y' be two consecutive puzzle pieces with $A = Y \backslash \overline{Y'}$. Then $\phi_c(Y \cap B_c(\infty))$ consists of a certain number of level n puzzle pieces $U_1, \ldots, U_k$ in the universal p/q-Yoccoz puzzle (as noted before) and $\phi_c(Y' \cap B_c(\infty))$ consists of some number of level-$(n+1)$ puzzle pieces $U'_{j,1}, \ldots, U'_{j,i_j}$, with $U'_{j,i} \subset U_j$ for each $0 < j \leq k$ and $1 \leq i \leq i_j$. Now A is degenerate if and only if at least one of the semi-annuli $A_{j,i} = \overline{U_j \backslash U'_{j,i}}$ is degenerate.

Definition 3.4. A semi annulus A is non-degenerate if the union of A with its reflection $\tau(A)$ in $\mathbb{S}^1$ is non-degenerate.

Via the equivalence relation $\sim_\infty^c$, the critical nest can readily be identified in the universal puzzle and hence its convergence and the convergence of any other nest can be detected there with $\sim_\infty^c$.

4. Quadratic Universal Parabolic Puzzles.

For our purpose we define in a similar spirit a universal Parabolic puzzle $\mathcal{P}$. The parabolic puzzle in a parabolic basin is then obtained via the conjugacy ϕ. Given an irreducible non-zero rational p/q we construct the universal external p/q Parabolic Yoccoz Puzzle. The corresponding internal p/q Parabolic Yoccoz Puzzle is obtained by reflection in $\mathbb{S}^1$. So in this section $R_{\underline{\epsilon}}$ refers to the *external parabolic ray* of P_2 with argument $\underline{\epsilon} \in \Sigma_2$, as defined in section 2.

4.1 Shortcuts

Given $\underline{\epsilon}^0, \underline{\epsilon}^1 \in \Sigma_2$ the two rays $R_{\underline{\epsilon}^0}$ and $R_{\underline{\epsilon}^1}$ coincide down to and including $-n$ where $\epsilon_1^0, \ldots, \epsilon_n^0 = \epsilon_1^1, \ldots, \epsilon_n^1$. Thus it is natural to consider the minimal sub-arc $\gamma(\underline{\epsilon}^0, \underline{\epsilon}^1)$ of $R_{\underline{\epsilon}^0} \cup R_{\underline{\epsilon}^1}$ connecting the two endpoints $z_{\underline{\epsilon}^0}, z_{\underline{\epsilon}^1} \in \mathbb{S}^1$. However even the arc $\gamma(\underline{\epsilon}^0, \underline{\epsilon}^1)$ will be disproportionably large compared to the length of the arc on $\mathbb{S}^1$ connecting the end points, when $\underline{\epsilon}^0$ and $\underline{\epsilon}^1$ are both close to some dyadic, but on opposite sides of it. As a prototype example consider the case, where $\underline{\epsilon}^0$ has $n_0 > 1$ leading 0's, but the (n_0+1)-th digit is 1 and that $\underline{\epsilon}^1$ has $n_1 > 1$ leading 1's, but the (n_1+1)-th digit is 0, so that the corresponding angles satisfy $0 < \theta_0 < \frac{1}{4}$ and $\frac{3}{4} < \theta_1 < 1$. Then $R_{\underline{\epsilon}^0} \cap R_{\underline{\epsilon}^1} = \{0\}$, but $R_{\underline{\epsilon}^0}(t) = R_{\overline{0}}(t)$ for $-n_0 \leq t$ and $R_{\underline{\epsilon}^1}(t) = R_{\overline{1}}(t)$ for $-n_1 \leq t$. As $R_{\overline{0}} \cup R_{\overline{1}}$ bounds the disk D_0, the end points of $R_{\underline{\epsilon}^0}$ and $R_{\underline{\epsilon}^1}$ converge to 1 as n_0, n_1 diverge to ∞. For this reason we modify $\gamma(\underline{\epsilon}^0, \underline{\epsilon}^1)$ by replacing the subarc $\delta = \delta_{0,n_0,n_1} = R_{\underline{\epsilon}^0}([-n_0, 0]) \cup R_{\underline{\epsilon}^1}([-n_1, 0])$ by a shortcut across D_0. Recall that $\Phi_+ : D_0 \longrightarrow \mathbb{C}\backslash]-\infty, 0]$ is a conformal isomorphism extending continuously to the boundary. As a shortcut to replace δ we choose the arc which is mapped by Φ_+ to the Archimedean spiral of center 0 connecting the two points $-n_0$ and $-n_1$ through $\mathbb{C}\backslash\mathbb{R}_-$. We denote the new arc by $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$. The new shortcutting sub-arc of $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$ corresponding to the Archimedean spiral is denoted the *top* of $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$ and the remaining two sub-arcs, which are sub-arcs of the original two rays, are denoted the *sides*.

We shall port the construction to the iterated preimages of $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$. Consider first the case $\underline{\epsilon}^0 = 1\underline{\epsilon}'^0$ and $\underline{\epsilon}^1 = 0\underline{\epsilon}'^1$, where $\underline{\epsilon}'^0$, resp. $\underline{\epsilon}'^1$ is starting with 0, resp. with 1, but not equal to $\overline{0}$, resp. $\overline{1}$, so that $\frac{1}{4} < \theta_1 < \frac{1}{2} < \theta_0 < \frac{3}{4}$ for the corresponding angles. The map $z \mapsto -z$ sends $\gamma(\underline{\epsilon}_0, \underline{\epsilon}_1)$ to the arc $\gamma(0\underline{\epsilon}'_0, 1\underline{\epsilon}'_1)$ of the type discussed above. Therefore, define $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$ as $-\widehat{\gamma}(0\underline{\epsilon}'^0, 1\underline{\epsilon}'^1)$. Hence, $P_2(\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)) = P_2(\widehat{\gamma}(0\underline{\epsilon}'^0, 1\underline{\epsilon}'^1))$. Note that the shortcutting subarc of $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$ is contained in $D_{1/2} = -D_0$.

Finally suppose that $\underline{\epsilon}^0, \underline{\epsilon}^1$ have a common initial segment $(\epsilon_1, \ldots, \epsilon_n)$ of length $n > 0$ and that $\sigma^n(\underline{\epsilon}^0) = \underline{\epsilon}'^0$, $\sigma^n(\underline{\epsilon}^1) = \underline{\epsilon}'^1$ satisfy the second condition, i.e. $\epsilon_{n+1}^0, \epsilon_{n+2}^0 = 1, 0$ and $\epsilon_{n+1}^1, \epsilon_{n+2}^1 = 0, 1$. Then define $\widehat{\gamma}(\underline{\epsilon}^0, \underline{\epsilon}^1)$ as the unique connected component of $P_2^{-n}(\widehat{\gamma}(\sigma^n(\underline{\epsilon}^0), \sigma^n(\underline{\epsilon}^1)))$ that coincides with $\gamma(\underline{\epsilon}^0, \underline{\epsilon}^1)$ near $\mathbb{S}^1$. Note that any iterated preimage of 1 other than 1 itself is also an (iterated) preimage of -1.

4.2 The Universal Parabolic Yoccoz Puzzle.

The set $h_2(\mathcal{Z}^0)$ corresponds to the unique p/q orbit of P_2 together with its preimage under P_2. Let $\overline{0} < \underline{\epsilon}_0 < \underline{\epsilon}_1 < \ldots < \underline{\epsilon}_{2q-1} < \overline{1}$ denote the unique itineraries of these points and let $\mathcal{GP}^0$ denote the graph

$$\mathcal{GP}^0 = \mathbb{S}^1 \cup \bigcup_{i=0}^{2q-1} \widehat{\gamma}(\underline{\epsilon}_i, \underline{\epsilon}_{(i+1) \mod 2q})$$

and define the parabolic (p/q-Yoccoz) puzzle $\mathcal{P}^0$ as the set consisting of the $2q$ closures of bounded connected components of the complement of $\mathcal{GP}^0$ in $\mathbb{C} \setminus \mathbb{D}$.

Define $\mathcal{P}^n$ recursively as follows:

$$\mathcal{P}^n = \{P_2^{-1}(P) \mid P \in \mathcal{P}^{n-1}, 1 \notin P\} \cup P_{1,n} \cup P_{-1,n},$$

where $P_{1,n}$, resp. $P_{-1,n}$, is the closure of the component bounded by

$$\widehat{\gamma}((\underbrace{0,\ldots,0}_{n \text{ times}}, \underline{\epsilon}_0), (\underbrace{1,\ldots,1}_{n \text{ times}}, \underline{\epsilon}_{2q-1})) \quad \text{resp. by} \quad \widehat{\gamma}((0, \underbrace{1,\ldots,1}_{(n-1) \text{ times}}, \underline{\epsilon}_{2q-1}), (1, \underbrace{0,\ldots,0}_{(n-1) \text{ times}}, \underline{\epsilon}_0))$$

together with the corresponding arc on the unit circle.

Finally define $\mathcal{P} = \cup_{n\in\mathbb{N}} \mathcal{P}^n$. We call $\mathcal{P}$ *the (quadratic) universal parabolic p/q-Yoccoz puzzle*. It defines *level-n puzzle pieces*, *level-n semi-annuli* and *non degenerate* semi-annuli as in previous section (see definition 3.4).

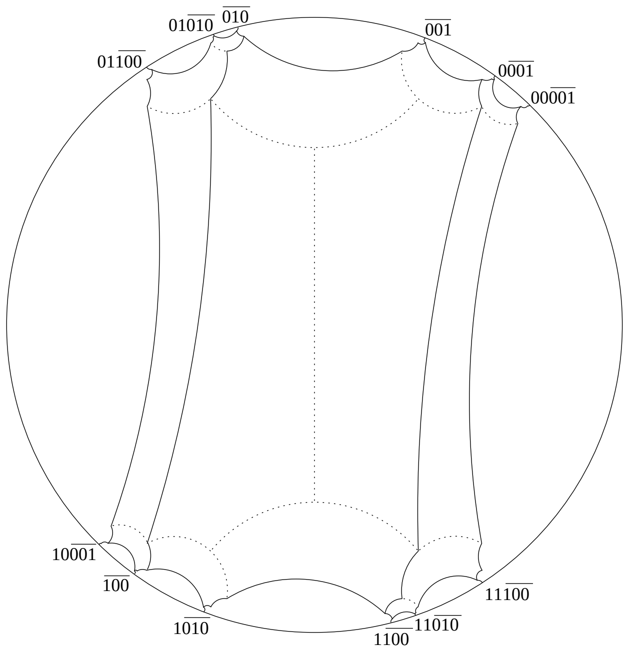

Figure 3. The first two levels of the Universal Parabolic 2/3-Yoccoz puzzle drawn in $\mathbb{D}$ (instead of $\mathbb{C} \setminus \overline{\mathbb{D}}$).

Definition 4.1. For any level n universal Yoccoz puzzle piece U set $\chi_{\mathcal{P}}(U) = P$, where P is the unique level n universal parabolic puzzle piece, whose vertices on $\mathbb{S}^1$ have the same itineraries as the vertices of U on $\mathbb{S}^1$ have. Let $\chi_{\mathcal{P}} : \mathcal{U} \longrightarrow \mathcal{P}$ be the map defined this way on the puzzles.

Remark 1. The map $\chi_{\mathcal{P}}$ is a bijection from $\mathcal{U}$ to $\mathcal{P}$ which allows us to compare the two puzzle pieces. Nevertheless to transfer Yoccoz' result, we need that :

(1) the critical puzzle pieces correspond in the different puzzles,
(2) the non-degeneracy property on the annuli is preserved through the bijection.

It would be nice and easy if the universal parabolic p/q-Yoccoz puzzle was homeomorphic to the universal p/q-Yoccoz puzzle. But it is not. However we can modify the universal p/q-Yoccoz puzzle slightly with slanted equipotentials to obtain a modified universal p/q-Yoccoz puzzle, for which the Yoccoz theorem can still be deduced and which is homeomorphic to the universal parabolic p/q-Yoccoz puzzle. This is the aim of the next subsection. Then to find the critical puzzle pieces one needs to compare the landing points in the puzzles. This is done in the last section.

4.3 The Modified Universal p/q-Yoccoz puzzle.

As a bridge for passing from the Universal Yoccoz puzzle to the Universal Parabolic Yoccoz puzzle, we introduce the following Modified Universal Yoccoz puzzles. They appear only in the following two subsections and are used to prove that a semi-annulus of the Universal Parabolic Yoccoz puzzle is degenerate if and only if the corresponding semi-annulus is already degenerate for the Universal Yoccoz puzzle (Corollary 4.3).

Let $p/q \neq 0/1$, $(p,q) = 1$ be a given rotation number. Denote by $\tau_0, \tau_1, \ldots, \tau_{2q-1} \in \mathbb{S}^1$ the points of $\mathcal{Z}^0$ with τ_k of itinerary $\underline{\epsilon}_k$ (ordered as in the previous subsection) and corresponding angle θ_k. Let n_0, resp. n_1, denote the number of leading zeros, resp. leading one's, in $\underline{\epsilon}^0$, resp. in $\underline{\epsilon}^{2q-1}$.

Definition 4.2. Define the level-0 modified p/q-puzzle $\mathcal{M}^0 = \mathcal{M}^0_{p/q}$ as the set of "trapezoids"

$$\mathcal{M}^0 = \{M_0, M_1, \ldots, M_{q-1}, \widehat{M_0}, \ldots \widehat{M}_{q-1}\} \quad \text{where}$$

(1) the puzzle piece $\widehat{M_0}$ is the image under e^z of the straight trapezoid of vertices $\theta_+ = \mathrm{Log}\,\tau_0 = i2\pi\theta_0$, $\theta_- = \mathrm{Log}\,\tau_{2q-1} = i2\pi(\theta_{2q-1} - 1)$, and $\widehat{\theta}_+ = \theta_+ + 2^{2-n_0}$, $\widehat{\theta}_- = \theta_- + 2^{2-n_1}$,
(2) the puzzle piece M_0 as $-\widehat{M_0}$,
(3) recursively the puzzle pieces M_j and $\widehat{M_j}$ as the two preimages of M_{j-1} under Q_0, so labelled that for $0 < j < q-1$, M_j is the preimage for which both corner points on $\mathbb{S}^1$ belongs to the p/q-cycle. For $j = q-1$ the labelling is no longer needed, but to fix the ideas M_{q-1} is in the upper half plane.

Then, define recursively the level-n modified p/q puzzle $\mathcal{M}^n$ as the n-th preimage of $\mathcal{M}^0$ under Q_0.

As before denote by *bottom* of M the arc $M \cap \mathbb{S}^1$ and by *top* of M the opposite edge of the "trapezoid" boundary of M and finally by *sides* of M the remaining two edges of M. We remark that :

a. any level-n puzzle piece M and any other level-m puzzle piece M' with $n \leq m$ are either interiorly disjoint or $M' \subseteq M$ (only the equipotentials differ from the classical case),
b. any level-n puzzle piece M is mapped diffeomorphically by Q_0^n to one of the $2q$ puzzle pieces M_j or $\widehat{M_j}$ of the level-0 and consequently mapped diffeomorphi-

cally to either M_0 or $\widehat{M}_0$ by Q_0^{n+j},

c. A set M can belong to up to q consecutive levels of puzzle pieces, this happens e.g. for M_{q-1} and $\widehat{M}_{q-1}$.

4.4 Comparing the Puzzles

Actually, we need to compare the semi-annuli of the different puzzles (Remark 1). Note that the bijection $\chi_{\mathcal{P}}$ (and $\chi_{\mathcal{M}}$ below) clearly extends to the set of all semi-annuli.

PROPOSITION 4.3. *A semi-annulus A of $\mathcal{U}$ is non-degenerate iff $\chi_{\mathcal{P}}(A)$ is a non-degenerate.*

This Proposition is a corollary of Proposition 4.5 and Theorem 4.6, both below.

Definition 4.4. Let $\chi_{\mathcal{M}} : \mathcal{U} \longrightarrow \mathcal{M}$ be the bijection defined by $\chi_{\mathcal{M}}(U) = M$ where U and M are the level-n universal puzzle piece of respectively Yoccoz puzzle and the Modified Yoccoz puzzle whose vertices on $\mathbb{S}^1$ have the same itineraries, i.e. are the same.

PROPOSITION 4.5. *A semi-annulus A of $\mathcal{M}$ is non-degenerate iff $\chi_{\mathcal{M}}^{-1}(A)$ is also a non-degenerate semi-annulus.*

Proof. By point b. in the previous subsection and the $z \mapsto -z$ symmetry of Q_0 it suffices to consider the case where A is obtained from $\widehat{M}_0$ and a level 1 puzzle piece $M' \subset \widehat{M}_0$. In this particular case we see by inspection that A is good iff M' is not adjacent, i.e. does not contain one of the bottom corners (τ_0 or τ_{2q-1}) of $\widehat{M}_0$ or equivalently is not a preimage of one of the two puzzle pieces $M_{q-1}, \widehat{M}_{q-1}$. However exactly this also holds for the Universal Yoccoz puzzle. □

As a preparation for Theorem 4.6 let us define the accumulated graphs $\mathcal{GM}^n$, $\mathcal{GM}$, $\mathcal{GP}^n$ and $\mathcal{GP}$ of the universal puzzles $\mathcal{M}$ and $\mathcal{P}$ similarly to the accumulated graphs for the (universal) Yoccoz puzzle as the union of the boundaries of all puzzle pieces up to and including level n, respectively all puzzle pieces in that puzzle. As above a puzzle piece is called *fixed*, resp. *prefixed*, if it contains 1, resp. -1.

THEOREM 4.6. *For each non-zero p/q, $(p,q) = 1$ the conjugacy $h : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ between Q_0 and P_2 extends to a homeomorphism $\mathcal{H} : \mathcal{GM} \longrightarrow \mathcal{GP}$ between the graph of the modified universal Yoccoz puzzle and the graph of the universal parabolic Yoccoz puzzle. The homeomorphism $\mathcal{H}$ conjugates dynamics except on the tops of the fixed and the prefixed puzzle pieces.*

Proof. For any $\underline{\epsilon} \in \Sigma_2$ let

$$\mathcal{R}_{\underline{\epsilon}}(t) = \exp(i2\pi\theta + 2^t) : \mathbb{R} \to \mathbb{C}$$

be a parameterization of the ray (or half-line) $\mathcal{R}_{\underline{\epsilon}} := \mathcal{R}_\theta$ where $\theta = \theta(\underline{\epsilon}) \in [0,1[$ is given by (1). Then (analogously to the parabolic case, see (2)) :

$$\forall\, t \in \mathbb{R}\,, Q_0(\mathcal{R}_{\underline{\epsilon}}(t)) = \mathcal{R}_{\sigma(\underline{\epsilon})}(t+1). \tag{3}$$

Hence we shall define the extension $\mathcal{H}$ on the sides of the puzzle pieces through these parameterizations. We define the homeomorphism $\mathcal{H}$ recursively defining $\mathcal{H}_n : \mathcal{GM}^n \longrightarrow \mathcal{GP}^n$:

As $\mathcal{H}_0 = h$ on $\mathbb{S}^1$, we define $\mathcal{H}_0$ on the sides of $\widehat{M}_0$ by $\mathcal{H}_0(\mathcal{R}_{\underline{\epsilon}_0}(t)) = R_{\underline{\epsilon}_0}(t - 2)$, for $t \le 2 - n_0$ and $\mathcal{H}_0(\mathcal{R}_{\underline{\epsilon}_{2q-1}}(t)) = R_{\underline{\epsilon}_{2q-1}}(t-2)$, for $t \le 2 - n_1$. Then, we extend $\mathcal{H}_0$ as some homeomorphism between the tops of $\widehat{M}_0$ and $\widehat{P}_0$. To fix the ideas let us define $\mathcal{H}_0$ so that it is affine in the natural angular coordinates given by arg on $\widehat{M}_0$ and $\arg\circ\Phi_+$ on $\widehat{P}_0$ (where $z \mapsto \arg z$ is a continuous choice of the argument function). Extend $\mathcal{H}_0$ as a homeomorphism between ∂M_0 and ∂P_0 by $\mathcal{H}_0(-z) = -\mathcal{H}_0(z)$. Next extend $\mathcal{H}_0$ as a homeomorphism from $\partial M_j \cup \partial\widehat{M}_j$ onto $\partial P_j \cup \partial\widehat{P}_j$ for $0 < j < q$ using the dynamics, i.e. by $\mathcal{H}_0 \circ Q_0^j(\partial M_j) = P_2^j \circ \mathcal{H}_0(\partial M_j)$, $\mathcal{H}_0 \circ Q_0^j(\partial\widehat{M}_j) = P_2^j \circ \mathcal{H}_0(\partial\widehat{M}_j)$ and by requiring $\mathcal{H}_0$ to be an extension of h. This extends h to a homeomorphism $\mathcal{H}_0 : \mathcal{GM}^0 \longrightarrow \mathcal{GP}^0$. We shall recursively define homeomorphic extensions $\mathcal{H}_n : \mathcal{GM}^n \longrightarrow \mathcal{GP}^n$ of $\mathcal{H}_{n-1} : \mathcal{GM}^{n-1} \longrightarrow \mathcal{GP}^{n-1}$. Suppose the homeomorphic extension $\mathcal{H}_{n-1}$ has been constructed and define $\breve{\mathcal{H}}_n$ to be the unique homeomorphism which satisfies $P_2 \circ \breve{\mathcal{H}}_n = \mathcal{H}_{n-1} \circ Q_0$ and agrees with h on $\mathbb{S}^1$. Then $\breve{\mathcal{H}}_n(\mathcal{GM}^n)$ equals $\mathcal{GP}^n$ except that the tops $\pm top$ of $\pm M$, where M is the fixed level n puzzle piece, do not map to the tops of the corresponding level n puzzle pieces in $\mathcal{P}$. To encompass this problem define $\mathcal{H}_n = \breve{\mathcal{H}}_n$ on $\mathcal{GY}^n \backslash (top \cup -top)$ and define $\mathcal{H}_n$ to be the homeomorphism of top onto the arc in D_0 whose image is the Archimedean spiral in $\mathbb{C}\backslash\mathbb{R}_-$ connecting $-(n_0 + n)$ and $-(n_1 + n)$ and which is affine in the respective angular coordinates and extend $\mathcal{H}_n$ to $-top$ using the symmetry. Then $\mathcal{H}_n$ is the required homeomorphic extension.

Finally define $\mathcal{H} : \mathcal{GM} \longrightarrow \mathcal{GP}$ by $\mathcal{H} = \mathcal{H}_n$ on $\mathcal{GM}^n$. Then $\mathcal{H}$ is a bijection by construction. Moreover both $\mathcal{H}$ and $\mathcal{H}^{-1}$ are continuous at any point of $\mathbb{D}\cap\mathcal{GM}$ and $\mathbb{D} \cap \mathcal{GP}$ respectively. Thus to prove that $\mathcal{H}$ is a homeomorphism we need to check that both $\mathcal{H}$ and $\mathcal{H}^{-1}$ are continuous on $\mathbb{S}^1$. As both $\mathcal{GM}$ and $\mathcal{GP}$ are compact and $\mathcal{H}$ is a bijection, we need only check that $\mathcal{H}$ or $\mathcal{H}^{-1}$ continuous, since any continuous bijection between compact sets in metric spaces is a homeomorphism. It is easy to see that $\mathcal{H}^{-1}$ is continuous on $\mathbb{S}^1$, because the maximal diameter of a level n modified puzzle piece decreases geometrically with n. (In the log-coordinate the diameter of the largest level n puzzle piece is $2^{-n}d$, where d denotes the diameter of M_0 in the log-coordinate.) $\square$

5. Quadratic rational maps : applications of the theory in the parabolic case.

A quadratic rational map has three fixed points counted with multiplicity. The space of Möbius conjugacy classes of quadratic rational maps with a fixed point of multiplier $\lambda \in \mathbb{C}$ has been named $\mathrm{Per}_1(\lambda)$ by Milnor. Each of these spaces are naturally isomorphic to $\mathbb{C}$ with the isomorphism given by the product σ of the two remaining fixed point eigenvalues.

The special case of $Per_1(0)$ has been discussed at length above. It consists of Möbius conjugacy classes of quadratic polynomials. For this family we have the Douady-Hubbard normal form $Q_c(z) = z^2 + c$, in which $\sigma = 4c$.

We focus now on the other special case of $Per_1(1)$. It consists of Möbius conjugacy classes of quadratic rational maps with a parabolic fixed point of multiplier 1. Note that at least one of the two other fixed points has multiplier 1 and coincides with the first. Hence, in this special case σ equals the third fixed point multiplier. Therefore, if $\sigma = 1$, the third fixed point equals the two others. The family $Per_1(1)$ at hand has no normal form, which univalently parameterizes it. We shall thus be content with the normal form $g_A(z) = z + 1/z + A$, for which ∞ is the parabolic fixed point with multiplier 1. The third fixed point for g_A is $\alpha = \alpha_A = -1/A$ with eigenvalue $\sigma = \sigma_A = 1 - 1/A^2$. It follows that $A \mapsto g_A$ double covers $Per_1(1)$ with branch point 0 and branch value 1, as also follows from the fact that $z \mapsto -z$ conjugates g_A to g_{-A}. For a map g we shall in the following write $g \in Per_1(1)$ though strictly speaking it is the conjugacy class $[g]$ of g, which belongs to $Per_1(1)$. Also we shall write $\sigma \in Per_1(1)$, where $\sigma = \sigma_g$.

The parabolic Mandelbrot set $\mathbf{M_1}$ is the connectedness locus of $Per_1(1)$:

$$\mathbf{M_1} = \{g \in Per_1(1) \mid J_g \text{ is connected } \}.$$

We define a projection Ψ^1 from $\mathbf{M_1}$ into $\mathbf{M}$ with the property that, it preserves the puzzles' dynamics. We shall not prove here that this projection is a bijection, this will be done in a parallel publication. From this construction, we obtain on one hand for parabolic Yoccoz parameters (i.e. those whose projection to $\mathbf{M}$ have two repelling fixed points and are non-renormalizable) that the Julia set is locally connected (Theorem 1.2) and on the other hand that the dynamics are conjugated except on the main cardioid of $\mathbf{M}$ (Theorem 1.1).

The proof of these two theorems are interwoven. Note that the proof of the convergence of the fixed nest of the parabolic puzzle associated to g is more technical, so is postponed until the end of the section. We shall recall the notion towers of equivalence relations and Yoccoz Combinatorial Analytic Invariants defined in [PR]; they are used to compare the parabolic and classical puzzles.

5.1 Parabolic puzzle.

Let $g \in \mathbf{M_1}$, denote by $\Lambda = \Lambda_g$ the basin of the parabolic fixed point $\beta = \beta_g (= \infty)$ with eigenvalue 1. Since J_g is connected and g has degree 2, each component of Λ is completely invariant, is isomorphic to $\mathbb{D}$ and contains a unique critical point. In the particular case $\sigma = \sigma_g = 1$ the parabolic fixed point has two attracting petals, the Julia set J_g is a circle in $\overline{\mathbb{C}}$ passing through β and the basin Λ_g consists of the two complementary round disks. In the general case $\sigma \neq 1$ the parabolic fixed point β has only one attracting petal, so there is only one component of Λ and one critical point denoted c_g in Λ.

We will focus now on the case $|\sigma| > 1$. It means that the third fixed point α is repelling. Since Λ is totally invariant, $J_g = \partial\Lambda$. It follows from Theorem 2.5 that α is periodically accessible from Λ with a combinatorial rotation number p/q, $(p, q) = 1$. Note that $q > 1$, since the two fixed parabolic rays define the same access and thus both converge to β.

The *parabolic p/q-Yoccoz puzzle* $\mathcal{P}_g$ for the map g is defined as follows: For $n \geq 0$, the *level-n parabolic p/q-Yoccoz puzzle graph* is the graph $\mathcal{GP}_g^n := \eta^{-1}(\mathcal{GP}^n) \cup g^{-(n+1)}(\alpha)$ where $\eta : \Lambda \longrightarrow \overline{\mathbb{C}} \backslash \overline{\mathbb{D}}$ denotes the Riemann map which conjugates g to P_2 such that $\eta(c_g) = \infty$. Then, let *the level-n parabolic p/q-Yoccoz puzzle* be the set of closures of those connected components of the complement which do not contain c_g. Define the accumulated parabolic p/q-Yoccoz graph and puzzle $\mathcal{GP}_g$, $\mathcal{P}_g$ as for the previous puzzles (as the union of the boundaries of all puzzle pieces up to and including level n, respectively all puzzle pieces in that puzzle). As in Remark 1 note that we can detect a degenerate annulus already through the universal parabolic p/q-Yoccoz puzzle $\mathcal{P}$:

Let $E_0 \subset \mathbb{S}^1$ denote the unique q-cycle for Q_0 with rotation number p/q so that $\mathcal{Z}^0 = E_0 \cup (-E_0)$ (see also page 8). Similarly to the discussion for quadratic polynomials, we consider $\sim_\infty^g$ the equivalence relation on $\mathcal{Z} = \bigcup_n Q_0^{-n}(\mathcal{Z}^0)$, induced by the puzzle $\mathcal{P}_g$ through η and h_2, where $h_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is the conjugacy between Q_0 and P_2. That is two points u, v of $\mathcal{Z}$ are equivalent if and only if the parabolic rays of $\mathcal{GP}_g^n$ corresponding to $h_2(u)$ and $h_2(v)$ co-land at the same iterated preimage of α.

LEMMA 5.1. *If there is a parameter $c \in \mathbf{M}$ such that $\sim_\infty^c = \sim_\infty^g$, then the map $\chi_{\mathcal{P}}$ induces a $1:1$ correspondence χ_g between the puzzle pieces of $\mathcal{Y}_c$ and $\mathcal{P}_g$. Moreover*

(1) any annulus of the parabolic puzzle $\mathcal{P}_g$ is non degenerate if and only if the corresponding annulus in the Yoccoz puzzle $\mathcal{Y}_c$ is non degenerate,

(2) the correspondence preserves the dynamics of puzzle pieces and annuli (except for the fixed and prefixed nests of $\mathcal{P}_g$),

(3) critical puzzle pieces correspond to critical puzzle pieces.

Proof. The definition of the map χ_g is clear from $\chi_{\mathcal{P}}$. The first item is a direct consequence of Proposition 4.3 and Remark 1. The second and third ones follow from the definition of the equivalence relations and the fact that $\sim_\infty^c = \sim_\infty^g$. □

To find a parameter $c \in \mathbf{M}$ such that $\sim_\infty^c = \sim_\infty^g$ one needs to know which equivalence relations are realized. This has been sorted out in [PR], it deals with the Yoccoz Combinatorial Analytic Invariants. We recall in the next subsection the results needed.

5.2 Towers

The following presentation is an excerpt from [PR]. For more details the reader is referred to this paper.

As above let E_0 denote the unique p/q cycle for Q_0, $\mathcal{Z}^n = Q_0^{-n}(\mathcal{Z}^0)$ for $n \geq 0$ and $\mathcal{Z} = \cup_{n\geq 0}\mathcal{Z}^n$. For $E \subset \mathbb{S}^1$ we let $H(E)$ denote E union its hyperbolic convex hull in $\mathbb{D}$.

Definition 5.2. A tuple of equivalence relations $(\sim_n)_{0\leq n\leq N}$, with $N \in \mathbb{N}\cup\{\infty\}$, is called a tower if it satisfies the following admissibility conditions (see also [K]):

i) For each n: $\sim_n$ is an equivalence relation on $\mathcal{Z}^n$.

ii) $\sim_0$ has the two classes E_0 and $-E_0$.

iii) For any class E of $\sim_n$ with $0 \leq n \leq N$ the set $Q_0(E)$ is a class of $\sim_{(n-1)}$;

iv) $\sim_N = \bigcup_{n=0}^{N} \sim_n$ so that $\sim_n |_{\mathcal{Z}^m} = \sim_m$ for any m, n with $0 \leq m < n \leq N$;

v) For any two classes E and E' of $\sim_n$, with $0 \leq n \leq N$, $H(E) \cap H(E') = \emptyset$.

By property iv) $\sim_N$ imposes $\sim_n$ for $n \leq N$. We shall thus abbreviate and write simply $\sim_N$ for the tower $(\sim_n)_{0\leq n\leq N}$.

The *level* of a class E is the minimal $n \geq 0$ for which $E \subset \mathcal{Z}^n$.

The finite towers $\sim_N$ are the nodes of a tree with root $\sim_0$ and with a branch connecting each child $\sim_N$ back to its parent $\sim_{N-1}$. We denote this tree by $\mathcal{T}$. The infinite towers $\sim_\infty$ on the other hand are the infinite branches of this tree starting at $\sim_0$. We denote the set or space of all infinite branches $\mathcal{T}^\infty$.

For a tower $\sim_N$, we denote by the *graph of* $\sim_N$ the set

$$\mathcal{G}_{\sim_N} = \bigcup_{E \text{ a class of } \sim_N} H(E) \subset \overline{\mathbb{D}}$$

A *gap* G of a finite tower $\sim_n$ is any connected component of $\overline{\mathbb{D}}\backslash\mathcal{G}_{\sim_n}$. We denote by *essential boundary* of a gap G the set $\delta G = G \cap \mathbb{S}^1$. The image of the gap G_n of $\sim_n$ is defined as the gap G_{n-1} of $\sim_{n-1}$ with $\delta G_{n-1} = Q_0(\delta G_n)$.

A class E or a gap G is said to be *critical* iff $0 \in H(E)$, resp. $0 \in G$. Clearly any finite tower has either a (unique) critical class or gap. We shall denote the critical class/gap of $\sim_n$ by E_n^*/G_n^* (or just E^*/G^*). The image of the critical class or gap will be called the *critical value class or gap of* $\sim_n$ and denoted E_n'/G_n'. Note that the critical value class or gap is a class or gap of $\sim_{n-1}$ and (provided the level of the critical class is n) is a subset of the critical value gap of $\sim_{n-1}$.

For a finite tower $\sim_N$ with critical gap G_N^* define the *critical period* $k \geq 1$ of $\sim_N$ as the minimal $k \geq 1$ for which $Q_0^k(G_N^*)$ is again a critical gap (of $\sim_{N-k}$). Note that in fact $k \geq q$ always. Also in order to ensure that a critical gap always has a critical period, we may formally define $\mathcal{Z}^n = E_0$ and $\sim_n$ as the equivalence relation with only one class E_0 for any n with $-q < n < 0$.

Let $\sim_n$ be a finite tower. If $\sim_n$ has a critical class E it has a unique child and we say that $\sim_n$ is a *terminal* tower.

If $\sim_n$ has a critical gap with critical value gap G'_n and if $E \subset G'_n$ or $G \subseteq G'_n$ is any class or gap of $\sim_n$ within G'_n. Then $\sim_n$ has a unique extension $\sim_{n+1}$ with critical value class E respective critical value gap G. For this reason we say $\sim_n$ is a *fertile* tower, when it has a critical gap.

An infinite tower $\sim_\infty$ is said to be *renormalizable with combinatorics* $\sim_N$ and renormalization period k if for every $n \geq N$, $\sim_n$ has critical period k and N is the minimal height with this period.

It is a classical result of Thurston that any finite height tower $\sim_N$ is realized by a (unique) post critically finite quadratic polynomial Q_c (see also [PR]). In other words there exists $c \in \mathbf{M}$ such that $\sim_\infty^c |\mathcal{Z}^N = \sim_N$. However, not all infinite towers are realized. To describe exactly which are realized we defined the space of p/q-equivalences.

5.3 The p/q-equivalence relations.

Suppose $\sim_\infty^T$ is an infinite terminal tower with *critical value class* E'_n, that is $\sim_n^T = \sim_\infty^T |_{\mathcal{Z}^n}$ has a critical class E_n^* with image E'_n and $\sim_{n-1} = \sim_\infty^T |_{\mathcal{Z}^{n-1}}$ has a critical gap G_{n-1}^* with image the *critical value gap* G'_{n-1} containing $H(E'_n)$. Then G'_{n-1} contains exactly q gaps $G_n^1, \ldots, G_n^q$ of $\sim_{n-1}$, which are adjacent to E'_n, i.e. with $H(E'_n) \cap \partial G_n^j \neq \emptyset$, because $H(E'_n)$ is a q-gon. In the light of the above discussion let $\sim_n^j$ denote the unique extensions of $\sim_{n-1}$ with critical value gaps G_n^j for $j = 1, \ldots, q$. Define recursively for $m > n$ unique extensions $\sim_m^j$ of $\sim_{m-1}^j$ with critical value gap $G_m^j \subset G_{m-1}^j$ adjacent to E'_n. Finally denote by $\sim_\infty^j = \cup_{m \geq n} \sim_m^j$ the corresponding infinite towers for $j = 1, \ldots, q$. It turns out (see Lemma 5.7) that these q towers are in general not realized. Rather they are like infinitesimal variations of $\sim_\infty^T$. We shall henceforth say that $\sim_\infty^T$ is *adjacent* to any of the q towers $\sim_\infty^1, \ldots, \sim_\infty^q$ and vice versa.

Definition 5.3. We consider the smallest equivalence relation on the set $\mathcal{T}^\infty = \mathcal{T}_{p/q}^\infty$ of infinite towers such that any two adjacent towers are equivalent. The equivalence classes will be called p/q-equivalences and will generically be denoted by F. The space of all p/q-equivalences will be called $\mathcal{F}_{p/q}$.

In [PR, Proposition 3.18] we proved the following characterization of the p/q-equivalences:

PROPOSITION 5.4. *There are three types of p/q-equivalences $F \in \mathcal{F}_{p/q}$:*

(1) There is one p/q-equivalence $F_\bigstar = F_\bigstar(p/q)$ with countably many elements. It consists of $\sim_\infty^\bigstar = \sim_\infty^\bigstar (p/q)$, the unique q-renormalizable tower with combinatorics $\sim_0 = \sim_0 (p/q)$, and countably infinitely many terminal towers $\sim_\infty^T$ adjacent to $\sim_\infty^\bigstar$ and for each such terminal tower $\sim_\infty^T$ the $q-1$ other infinite towers adjacent to it.

(2) There are countably many p/q-equivalences F consisting of just one terminal tower $\sim_\infty^T$ and its q adjacent infinite towers $\sim_\infty^1, \ldots, \sim_\infty^q$.

(3) There are uncountably many p/q-equivalences F consisting of just one in-

finite tower.

Definition 5.5. For $p/q \neq 0/1$ with $(p,q)=1$ define the space $\mathcal{C}_{p/q}$ of all p/q (quadratic) combinatorial analytic invariants as

$$\begin{aligned}\mathcal{C}_{p/q} &= \{F|F \text{ is a non-renormalizable } p/q \text{ equivalence}\}\\ &\cup \{(F,c)|F \text{ is a } k>q \text{ renormalizable } p/q \text{ equivalence and } c\in\mathbf{M}\}\\ &\cup \{(F_\bigstar,c)|F_\bigstar = F_\bigstar(p/q) \text{ and } c\in \mathbf{M}\backslash\{\frac{1}{4}\}\}\end{aligned}$$

Moreover define the space $\mathcal{C}$ of all quadratic combinatorial analytic invariants as

$$\mathcal{C} = \overline{\mathbb{D}} \cup \bigcup_{p/q\neq 0/1} \mathcal{C}_{p/q}.$$

THEOREM 5.6. *There exists a dynamically defined bijective mapping $\Xi : \mathbf{M} \longrightarrow \mathcal{C}$ given by:*
$\Xi(c)=\lambda$ if Q_c has a non-repelling fixed point of multiplier λ,
$\Xi(c)=F(\sim_\infty^c)\in\mathcal{C}_{p/q}$ if $c\in L_{p/q}^\bigstar$ is non-renormalizable and
$\Xi(c)=(F(\sim_\infty^c),\chi_{F(\sim_\infty^c)}(c))$ if Q_c is renormalizable, where $\chi_{F(\sim_\infty^c)}$ is the straightening map associated to the renormalization of Q_c.

5.4 Proof of Theorem 1.1

As a preparation for the proof we need the following Lemma.

LEMMA 5.7. *Let $\sim_\infty$ be an infinite tower obtained either from a polynomial Q_c or a map $g\in\mathbf{M_1}$. If the p/q-equivalence $F=F(\sim_\infty)$ contains a terminal tower $\sim_\infty^T$ and $\sim_\infty \neq \sim_\infty^\bigstar$, then $\sim_\infty = \sim_\infty^T$. That is for parabolics, as for polynomials, the infinite towers $\sim_\infty$ other than $\sim_\infty^\bigstar$ that are adjacent to a terminal tower $\sim_\infty^T$ are not realized.*

Proof. Suppose to the contrary that $\sim_\infty\neq\sim_\infty^\bigstar$ is adjacent to a terminal tower $\sim_\infty^T$, but it is not equal to this tower (to get a contradiction). Write $\sim_m^T = \sim_\infty^T|\mathcal{Z}^m$ and $\sim_m=\sim_\infty|\mathcal{Z}^m$ for each $m\geq 0$. Let E_{n+1}^* denote the critical class of $\sim_\infty^T$. It has (minimal) level $n+1\geq q$. Denote by $E=Q_0(E_{n+1}^*)$ the critical value class of $\sim_\infty^T$ it has level n. Then $\sim_m=\sim_m^T$ for $m\leq n$ and $\sim_n=\sim_n^T$ is a fertile tower whose critical value gap (of level $n-1$) contains E. Since $\sim_\infty\neq\sim_\infty^\bigstar$ there exists some minimal level $l, q-1\leq l\leq n$, and a class E' of $\sim_l$ separating the critical value gap G'_m of $\sim_m$, $m>l$ from the Q_0-invariant class E_0 of $\sim_0$.

For each $m\geq 0$ and $1<j\leq q$ let $\widehat{G}_m^j$ denote the gap of $\sim_m^T$ adjacent to E_0 and with $Q_0^{q-j}(\widehat{G}_m^j)$ contained in the critical gap of $\sim_{j-q}$. Then $\widehat{\Delta}_m := H(E_0)\cup\cup_{j=1}^q \widehat{G}_m^j$ is a neighborhood of $H(E_0)$ for each m satisfying,

$$Q_0(\delta(\widehat{\Delta}_{m+1})) = \delta(\widehat{\Delta}_m) \qquad \text{and} \qquad \overline{\delta(\widehat{\Delta}_{m+1})}\subset\delta(\widehat{\Delta}_m).$$

For $m>l$ the restriction of Q_0 to $\delta(\widehat{\Delta}_m)$ is injective. It follows that:

i) For any $\sim'_m$, which is an extension of $\sim_l$, but not of $\sim_{l+1}^\bigstar$, (e.g. $\sim_m$ and $\sim_m^T$), the gaps of $\sim'_m$ adjacent to E_0 are the gaps $\widehat{G}_m^j$;

ii) $H(E_0)=\cap_m\widehat{\Delta}_m$ and thus $\cap_m(-\widehat{\Delta}_m)=H(-E_0)$.

Let the q gaps G_m^j of $\sim_m^T$ adjacent to E be numbered so that $Q_0^n(G_m^j) = -\widehat{G}_{m-n}^j$ and write $\Delta_m = H(E) \cup \cup_{j=1}^m G_m^j$. Then

iii) For any $m > n + l$ and any $\sim_m'$ which is a descendent of $\sim_n^T = \sim_n$, but not a descendent of $\sim_{n+1}^\bigstar$, the q gaps of $\sim_m'$ adjacent to E coincide with the q gaps G_m^j;

iv) $\cap_m \Delta_m = H(E)$ and

v) $\Delta_m \cap \widehat{\Delta}_m = \emptyset$ for $m > l + n$.

Returning to the parabolic map g and its parabolic puzzle $\mathcal{P} = \mathcal{P}_g$, we assume $\sim_\infty = h_2^{-1}(\sim_\infty^g)$ is adjacent to $\sim_\infty^T$, but is different from $\sim_\infty^\bigstar$. Denote by $\Omega_m = \Omega_m^g = \cup_{j=1}^q P(h_2(G_m^j))$ and by $\widehat{\Omega}_m = \widehat{\Omega}_m^g = \cup_{j=1}^q P(h_2(\widehat{G}_m^j))$, where $P(h_2(G))$ denotes the parabolic puzzle piece corresponding to the gap G. Then each $\widehat{\Omega}_m$ is a disk neighborhood of α and Ω_m is a disk neighborhood of a preimage α' of α, where the parabolic rays corresponding to $h_2(E)$ land and with $g^{n+1}(\Omega_m) = \widehat{\Omega}_{m-n-1}$ for $m > l+n+1$. Moreover, the critical value v, which does not escape within Λ, belongs to $P(h_2(G_m^j)) \subset \Omega_m$ for some j for every $m > n$ (by hypothesis) . As $\Omega_m \cap \widehat{\Omega}_m = \emptyset$ for $m > l + n$ (by v) above) the restriction $g| : \widehat{\Omega}_m \longrightarrow \underline{\widehat{\Omega}_{m-1}}$ is biholomorphic for $m > l + n$. Thus $\cap_m \widehat{\Omega}_m = \alpha$ since α is repelling and $\overline{\widehat{\Omega}_m} \subset \widehat{\Omega}_{m-1}$. Hence $v \in \cap_m \Omega_m = \alpha'$ and thus $\sim_\infty = \sim_\infty^T$. $\square$

<u>*Proof of Theorem 1.1.*</u>

Let $g \in \mathbf{M_1}$. If $|\sigma_g| \leq 1$, define $\Psi^1(g) = \Xi^{-1}(\sigma_g)$ *i.e.,* Q_c has a fixed point of multiplier σ_g.

Now we focus on the case where $|\sigma_g| > 1$. The parabolic puzzle $\mathcal{P}_g$ is well defined (see section 5.1). Let $\sim_\infty^g$ denote the equivalence relation induced on $h_2(\mathcal{Z})$ by $\mathcal{P}_g$. The infinite tower of equivalence relations $\sim_\infty = h_2^{-1}(\sim_\infty^g)$ on $\mathcal{Z}$ defines a p/q-equivalence relation $F = F(\sim_\infty)$. We distinguish four cases (see also Proposition 5.4):

1) If $F \neq F_\bigstar$ contains a terminal tower $\sim_\infty^T$ then, by Lemma 5.7, $\sim_\infty = \sim_\infty^T$ and $\sim_\infty$ has combinatorics $\sim_\infty^T |_{\mathcal{Z}^N}$, where n is the level of the critical class. By Theorem 5.6, there exists a unique $c \in \mathbf{M}$ with tower $\sim_\infty^c = \sim_\infty^T$. Define $\Psi^1(g) = c$.

2) If F is a non-renormalizable p/q-equivalence that does not contain a terminal tower, then $F = \sim_\infty$ and, by Theorem 5.6, there exists a unique $c \in \mathbf{M}$ with $\Xi(c) = F = \sim_\infty$. Define $\Psi^1(g) = c$.

3) Suppose next that $\sim_\infty$ is renormalizable with combinatorics $\sim_N = \sim_\infty |_{\mathcal{Z}^N}$ and critical period k, where $N \geq 0$ and $k \geq q$. If $F(\sim_\infty) \neq F_\bigstar$ then $N, k > q$. By Theorem 5.6, there exists $c_0 \in \mathbf{M}$ such that $\Xi(c_0) = (F(\sim_\infty), 0) \in \mathcal{C}_{p/q}$. Then, for this parameter c_0, $\sim_\infty^{c_0} |_{\mathcal{Z}^N} = \sim_\infty |_{\mathcal{Z}^N}$ and, because of the periodicity of the tower, $\sim_\infty^{c_0} = \sim_\infty$. Therefore, the restriction $Q_{c_0}^k : Y_{c_0}^{N+k} \longrightarrow Y_{c_0}^N$ is proper of degree 2. Hence similarly the restriction $g^k : P_g^{N+k} \longrightarrow P_g^N$ is proper of degree 2. Here we have, in order to shorten notation, used the symbols $Y_{c_0}^k$ and P_g^k for the critical level k Yoccoz and parabolic Yoccoz puzzle pieces of Q_{c_0} and g respectively. The orbit of the critical point of either map $Q_{c_0}^k$ and g^k is contained in $Y_{c_0}^{N+k}$ respectively in P_g^{N+k}, because $\sim_\infty = \sim_\infty^{c_0}$ (Lemma 5.1). Moreover since $k > q$ the annulus $Y_{c_0}^N \backslash \mathring{Y}_{c_0}^{N+k}$ is non degenerate, so that the above restriction of $Q_{c_0}^k$ is quadratic-like with connected filled-in Julia set. But then also $P_g^N \backslash \mathring{P}_g^{N+k}$ is non degenerate and the above restriction of g^k is quadratic-like with connected filled-in Julia set. In order to apply Theorem 4.6 here note that the argument, which shows that $Y_{c_0}^{N+k} \subset\subset Y_{c_0}^N$ actually shows that at least one of the intermediate annuli $Y_{c_0}^l \backslash \mathring{Y}_{c_0}^{l+1}$, $N \leq l < N + k$ is non degenerate. Denote by $c' = \chi(g^k) \in \mathbf{M}$ the Douady-Hubbard straightening parameter for the quadratic like restriction $g^k : P_g^{N+k} \longrightarrow P_g^N$ above.

By Theorem 5.6 there exists a unique parameter $c \in \mathbf{M}$ with $\Xi(c) = (F(\sim_\infty), c')$. We define $\Psi^1(g) = c$.

4) Finally suppose $F(\sim_\infty) = F_\bigstar$ and suppose g does not have a parabolic fixed point with multiplier $\exp(i2\pi p/q)$. Then $k = q$ and $N = 0$. In this case as in the polynomial case we may, since α is repelling, produce a quadratic-like restriction of g^q with connected filled-in Julia sets by slightly thickening the critical puzzle piece P_g^0 along the boundary rays and pulling it back to thickenings of P_g^q. Hence g is q renormalizable. As above let $c' \in \mathbf{M}$ denote the corresponding Douady-Hubbard straightening parameter. By Theorem 5.6 there exists a unique parameter $c \in \mathbf{M}$ with $\Xi(c) = (F(\sim_\infty), c')$. We define $\Psi^1(g) = c$. The details are left to the reader.

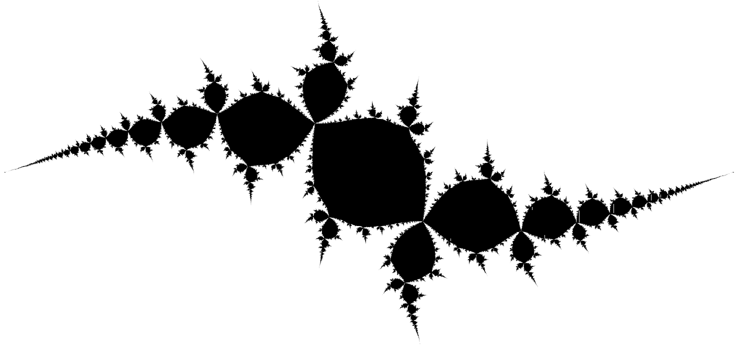

Figure 4. A parabolic Douady rabbit with an attracting cycle of period 3 attracting 0. The last fixed point is repelling and has combinatorial rotation number 1/3.

5.5 *Proof of Theorem 1.2*

Proof. Let $g \in \mathbf{M_1}$ be non-renormalizable. Define its parabolic graph $\mathcal{GP}_g$ as in the previous section. The local connectivity of J_g follows from the fact that the puzzle pieces intersect J_g in a connected set and they form a convergent nest for each point of J_g. Let $c \in \mathbf{M}$ be the parameter such that $\psi^1(g) = c$. We distinguish different cases (as previously).

(1) Let $\{(P^n)\}_{n\geq 0}$ denote the unique fixed nest of the parabolic p/q-Yoccoz puzzle for g. It is proved in the next section (Proposition 5.10) that though this nest is not dynamic ($g(P^n) \neq P^{n-1}$), it is convergent by construction (the sum of moduli is bounded from below by the harmonic series).

(2) If $\sim_\infty$ is terminal then arguing as for Q_c any nest is convergent. In this case the critical points in the Julia sets of Q_c and g both iterate to the α fixed point and in particular are not recurrent. Convergence of almost any nest then follows from a standard Poincaré metric argument. The only exceptions are the nest $\{(P^n)\}_{n\geq 0}$ containing the parabolic fixed point β and its iterated preimages (which were treated above).

(3) If $\sim_\infty$ is not terminal and not renormalizable, then Q_c is not renormalizable and $\mathcal{Y}_c$ has at least one non degenerate critical annulus. Hence by Theo-

rem 4.6 so does $\mathcal{P}_g$. The same tableaux argument, which then shows that the critical nest of $\mathcal{Y}_c$ is convergent, shows that the critical nest of $\mathcal{P}_g$ is convergent since the critical puzzle pieces correspond and the combinatorics defined by the towers are the same. Then arguing as for Q_c any other nest is also convergent, the only exception being again the nest containing the parabolic fixed point β and its iterated preimages. And these are handled exactly as above.

(4) Suppose next that $\sim_\infty$ is renormalizable with combinatorics $\sim_N = \sim_\infty |_{\mathcal{Z}^N}$ and critical period $k \geq q$. Then we have shown in the proof of Theorem 1.1 that g^k is quadratic-like with connected filled-in Julia set *i.e.*, that g is renormalizable.

This proves both local connectivity of the Julia set J_g and that the two maps g and Q_c are topologically conjugate on their respective Julia sets. Because all nests are convergent and because defining the same infinite tower of $\sim_\infty$ implies that puzzles and hence nests have the same combinatorial structure. □

5.6 *Convergence of the fixed nest.*

Recall the definition of the modulus of quadrilaterals and annuli (see also [L-V] for a more complete discussion of annuli, quadrilaterals and their moduli).

Let $Q = Q(a, a') \approx \mathbb{D}$, resp. $A = A(c, c') \approx \{z \in \mathbb{C} \mid r < |z| < R\}$, denote a quadrilateral, resp. an annulus, with sides a and a', resp. with outer boundary c and inner boundary c', where $Q \subset \mathbb{C}$, resp. $A \subset \mathbb{C}$.

A conformal metric ρ (precisely $\rho(z)|dz|$) on Q, resp. A, is called admissible (for $Q(a, a')$, resp. $A(c, c')$), if and only if for any path $\gamma : [0, 1] \longrightarrow \overline{Q}$, resp. $\overline{A}$, connecting a and a', resp. c and c', the ρ-length of γ satisfies: $\mathrm{l}_\rho(\gamma) \geq 1$. The *modulus* of $Q(a, a')$, resp. of $A(c, c')$, is the non negative number:

$$\mathrm{mod}(Q(a, a')) := \inf_{\rho \text{ admissible}} \mathrm{Area}(Q, \rho), \quad \text{resp.} \quad \mathrm{mod}(A) := \sup_{\rho \text{ admissible}} \frac{1}{\mathrm{Area}(A, \rho)}.$$

Then $Q(a, a')$, resp. $A(c, c')$, is said to be non-degenerate if the modulus is neither 0 nor ∞. Any non-degenerate quadrilateral or annulus has a unique extremal conformal metric. That is a unique admissible metric ρ for which the infimum or the supremum (in the definition) is attained.

Definition 5.8. Two non degenerate quadrilaterals $Q_1(a_1, a_1')$ and $Q_2(a_2, a_2')$ are said to *be in annular position* if and only if $Q_1 \cup Q_2$ contains an annulus $A = A(c, c')$ such that any path $\gamma : [0, 1] \longrightarrow \overline{A}$ connecting c and c' has a subarc (proper or not) which is contained in $\overline{Q_i}$ and connects the sides a_i and a_i' for $i = 1$ or $i = 2$. For a pair of quadrilaterals in annular position as above, any annulus A as above is called a spanning annulus for the pair.

Lemma 5.9 Cui *Suppose the quadrilaterals $Q_i(a_i, a_i')$, $i = 1, 2$ are in annular position and $A = A(c, c')$ is any spanning annulus. Then*

$$\mathrm{mod}(A) \geq \frac{1}{\mathrm{mod}(Q_1(a_1, a_1')) + \mathrm{mod}(Q_2(a_2, a_2'))}.$$

Proof. Let ρ_i be the extremal conformal metric on $Q_i(a_i, a_i')$ for $i = 1, 2$ extended by 0 to $\overline{\mathbb{C}} \setminus Q_i$. Define $\rho(z) = \max\{\rho_1(z)), \rho_2(z)\}$. Then $\rho_{|A}$ is an admissible conformal metric on A and

$$\operatorname{mod}(A) \geq \frac{1}{\operatorname{Area}(A,\rho)} \geq \frac{1}{\operatorname{Area}(Q_1,\rho_1)+\operatorname{Area}(Q_2,\rho_2)}$$
$$= \frac{1}{\operatorname{mod}(Q_1(a_1,a_1'))+\operatorname{mod}(Q_2(a_2,a_2'))}.$$

□

Let g be as in Theorem 1.2 and let $\{(P^n)\}_{n\geq 0}$ denote the unique fixed nest of the parabolic p/q-Yoccoz puzzle for g. Define the annuli $A_n := \overline{P^n \backslash P^{n+1}}$ with outer boundary $c_n = \partial P^n$ and inner boundary $c_n' = \partial P^{n+1}$.

PROPOSITION 5.10. *There exists $k_1, k_2 > 0$ such that*

$$\forall\, n \geq 0: \qquad \operatorname{mod}(A_n) \geq \frac{1}{k_1 + k_2 n}.$$

In particular

$$\sum_{n=0}^{\infty} \operatorname{mod}(A_n) = \infty.$$

Proof. The idea of the proof is to define pairs of quadrilaterals $Q_{n,0}(a_{n,0}, a_{n,0}')$, $Q_{n,1}(a_{n,1}, a_{n,1}')$ in annular postion such that

(1) A_n is a spanning annulus,
(2) any two quadrilaterals $Q_{n,1}(a_{n,1}, a_{n,1}')$ and $Q_{m,1}(a_{m,1}, a_{m,1}')$ are conformally equivalent and thus have the same modulus,
(3) the quadrilaterals $Q_{n,0}(a_{n,0}, a_{n,0}')$ are contained in the parabolic basin, have universal models and the moduli are bounded above by an affine function of n.

Applying Lemma 5.9 then yields the proposition. Hence the main ingredient in the proof is the definition of the quadrilaterals.

Let $\Phi = \Phi_g = \Phi_+ \circ \eta : \Lambda_g(\beta) \longrightarrow \mathbb{C}$ denote the attracting Fatou coordinate for g normalized by mapping the unique critical point in $\Lambda_g(\beta)$ onto 0. Let $D_0^g \subset \Lambda_g(\beta)$ denote the disk bounded by the two fixed parabolic rays of g, i.e. $\eta(D_0^g) = D_0$, where the domain D_0 was defined in the subsection 4.1. We shall similarly for any dyadic $n/2^m$ write $D_{n/2^m}^g$ for the disk $\eta^{-1}(D_{n/2^m})$, where $D_{n/2^m} \subset \mathbb{D}$ is the domain mapped univalently onto D_0 by P_2^m and characterized by $\overline{D}_{n/2^m} \cap \mathbb{S}^1 = h_2(e^{2i\pi n/2^m})$.

Define $\widehat{Q}_{n,0} := A_n \cap \overline{D}_0^g$ and $\widehat{Q}_{n,1} := A_n \backslash D_0^g$. Then $\widehat{Q}_{n,0} \cap \widehat{Q}_{n,1}$ consists of two arcs $b_n^0, b_n^1 \subset \partial D_0^g$, for which $\Phi(b_n^i) = [-(n+n_i+1), -(n+n_i)]$ homeomorphically, where $n_i \geq 2$ depends on p/q and are defined in the subsection 4.1 for $i \in \{0,1\}$. We shall extend $\widehat{Q}_{n,0}$ and $\widehat{Q}_{n,1}$ by 'caps' (biholomorphic to half or slit disks under Φ) at either end in order to obtain $Q_{n,0}$ and $Q_{n,1}$. For this we define

$$\Delta_k := \{z \mid |z+k| < 1\} \backslash [-(k+1), -k], \qquad \Delta_k^{\pm} := \{z = x+iy \mid \pm y > 0, |z+k| < 1\}$$

and

$$S_k := \{z \mid |z+k| = 1\}, \qquad S_k\pm := \{z = x+iy \mid \pm y \geq 0, |z+k| = 1\}.$$

We must distinguish the two cases $n_0 > n_1$ and $n_0 < n_1$, whereas in the remaining case ($n_0 = n_1$) we can use either of the following (see Fig. 5) : For $n_0 > n_1$ define

$$\begin{aligned} Q_{n,0} &= (\Phi^{-1}(\Delta^-_{n+n_0+1}) \cap D^g_{1/2^{(n+n_0)}}) \cup \widehat{Q}_{n,0} \cup (\Phi^{-1}(\Delta_{n+n_1}) \cap D^g_{1-1/2^{n+n_1}}), \\ a_{n,0} &= (\partial P^n \cap \overline{D}_0) \cup (\Phi^{-1}(S^-_{n+n_0+1}) \cap \overline{D}_{1/2^{n+n_0}}) \qquad \text{and} \\ a'_{n,0} &= (\Phi^{-1}(S_{n+n_1}) \cap \overline{D}_{1-1/2^{n+n_1}}) \cup (\partial P^{n+1} \cap \overline{D}_0). \end{aligned}$$

So that $Q_{n,0} \subset A_n$ and the two boundary components of $Q_{n,0}$ are separated by A_n. Similarly define

$$\begin{aligned} Q_{n,1} &= (\Phi^{-1}(\Delta^+_{n+n_0+1}) \cap D^g_0) \cup \widehat{Q}_{n,1} \cup (\Phi^{-1}(\Delta^-_{n+n_1}) \cap D^g_0), \\ a_{n,1} &= (\Phi^{-1}(S^+_{n+n_0+1}) \cap \overline{D}_0) \cup (\partial P^n \backslash D^g_0) \qquad \text{and} \\ a'_{n,1} &= (\partial P_{n+1} \backslash D^g_0) \cup (\Phi^{-1}(S^-_{n+n_1}) \cap D^g_0). \end{aligned}$$

Finally for $n_0 < n_1$ we match reversely

$$\begin{aligned} Q_{n,0} &= (\Phi^{-1}(\Delta_{n+n_0}) \cap D^g_{1/2^{n+n_0}}) \cup \widehat{Q}_{n,0} \cup (\Phi^{-1}(\Delta^+_{n+n_1+1}) \cap D^g_{1-1/2^{n+n_1}}), \\ a_{n,0} &= (\partial P^n \cap \overline{D}_0) \cup (\Phi^{-1}(S^+_{n+n_1+1}) \cap \overline{D}_{1-1/2^{n+n_1}}) \qquad \text{and} \\ a'_{n,0} &= (\Phi^{-1}(S_{n+n_0+1}) \cap \overline{D}_{1/2^{n+n_0}}) \cup (\partial P^{n+1} \cap \overline{D}_0), \end{aligned}$$

and

$$\begin{aligned} Q_{n,1} &= (\Phi^{-1}(\Delta^+_{n+n_0}) \cap D^g_0) \cup \widehat{Q}_{n,1} \cup (\Phi^{-1}(\Delta^-_{n+n_1+1}) \cap D^g_0), \\ a_{n,1} &= (\Phi^{-1}(S^+_{n+n_0}) \cap \overline{D}_0) \cup (\partial P^n \backslash D^g_0) \qquad \text{and} \\ a'_{n,1} &= (\partial P_{n+1} \backslash D^g_0) \cup (\Phi^{-1}(S^-_{n+n_1+1}) \cap D^g_0). \end{aligned}$$

Then in either case $A_n \subset Q_{n,0} \cup Q_{n,1}$ and A_n separates the two complementary sides (connected components) of $\partial Q_{n,0} \backslash (a_{n,0} \cup a'_{n,0})$ and of $\partial Q_{n,1} \backslash (a_{n,1} \cup a'_{n,1})$ in the two quadrilaterals.

Then by construction $Q_{n,0}(a_{n,0}, a'_{n,0})$ and $Q_{n,1}(a_{n,1}, a'_{n,1})$ are in annular position with A_n as a spanning annulus for every $n \geq 0$. Also g maps $Q_{n+1,1}(a_{n+1,1}, a'_{n+1,1})$ biholomorphically onto $Q_{n,1}(a_{n,1}, a'_{n,1})$ for every $n \geq 0$ (by construction). Hence there exists a constant $K, 0 < K < \infty$ such that $\mathrm{mod}(Q_{n,1}(a_{n,1}, a'_{n,1})) = K$ for every $n \geq 0$. For the modulus of $Q_{n,0}(a_{n,0}, a'_{n,0})$ we consider the conformal metric $\rho(z) = |\Phi'(z)|$ restricted to $Q_{n,0}$. An easy computation which we leave to the reader shows that

$$0 < \inf\{\mathrm{l}_\rho(\gamma) | \gamma : [0,1] \longrightarrow Q_{n,0} \text{ connects } a_{n,0} \text{ and } a'_{n,0}\} \underset{n\to\infty}{\longrightarrow} 1$$

and

$$\mathrm{Area}(Q_{n,0}) - n \cdot 2\pi - \frac{3}{2}\pi \underset{n\to\infty}{\longrightarrow} 0.$$

From this and Lemma 5.9 the propostion follows with $k_2 = 2\pi$ and $k_1 > K$ for some constant K. $\square$

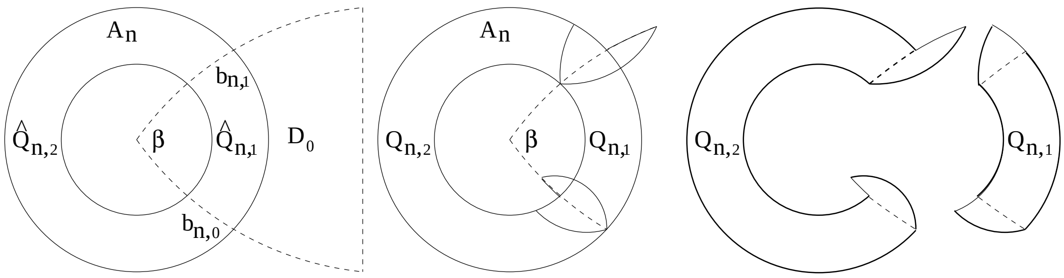

Figure 5. Symbolic drawing of the spanning annulus in the proof above.

Remark 1. The same methods certainly prove local connectivity of J_g in all cases, where $J(Q_c)$ is proved to be locally connected by a proof which uses the combinatorics of puzzles pieces (where $c = \Psi^1(g)$), *i.e.* for finitely renormalizable polynomials or infinitely renormaizable of bounded type.

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