Rapid course on geometric measure theory and the theory of varifolds

I will present an excerpt of classical results in geometric measure theory and the theory of varifolds as developed in [Fed69] and [All72]. Alternate sources for fragments of the theory are [Sim83], [EG92], [KP08], [KP99], [Zie89], [AFP00].

Rectifiable sets and the area and coarea formulas

(1) Area and coarea for maps between Euclidean spaces; see [Fed69, 3.2.3, 3.2.10] or [KP08, 5.1, 5.2]

(2) Area and coarea formula for maps between rectifiable sets; see [Fed69, 3.2.20, 3.2.22] or [KP08, 5.4.7, 5.4.8]

(3) Approximate tangent and normal vectors and approximate differentiation with respect to a measure and dimension; see [Fed69, 3.2.16]

(4) Approximate tangents and densities for Hausdorff measure restricted to an $(\mathscr{H}^m, m)$ rectifiable set; see [Fed69, 3.2.19]

(5) Characterization of countably $(\mathscr{H}^m, m)$-rectifiable sets in terms of covering by submanifolds of class $\mathscr{C}^1$; see [Fed69, 3.2.29] or [KP08, 5.4.3]

Sets of finite perimeter

(1) Gauss-Green theorem for sets of locally finite perimeter; see [Fed69, 4.5.6] or [EG92, 5.7–5.8]

(2) Coarea formula for real valued functions of locally bounded variation; see [Fed69, 4.5.9(12)(13)] or [EG92, 5.5 Theorem 1]

(3) Differentiability of approximate and integral type for real valued functions of locally bounded variation; see [Fed69, 4.5.9(26)] or [EG92, 6.1 Theorems 1 and 4]

(4) Characterization of sets of locally finite perimeter in terms of the Hausdorff measure of the measure theoretic boundary; see [Fed69, 4.5.11] or [EG92, 5.11 Theorem 1]

Varifolds

(1) Notation and basic facts for submanifolds of Euclidean space, see [All72, 2.5] or [Sim83, §7]

(2) Definitions and notation for varifolds, push forward by a smooth map, varifold disintegration; see [All72, 3.1–3.3] or [Sim83, pp. 227–233]

(3) Varifold tangents; see [All72, 3.4] or [Sim83, §42]

(4) Basic facts on rectifiable and integral varifolds; see [All72, 3.5] or [Sim83, §11, §15, §38]

Locally Lipschitzian functions on rectifiable varifolds I will present the contents of [Men15, Section 3]. The result allows to approximate Lipschitzian functions on varifolds by functions of class $\mathscr{C}^1$.

First variation of a varifold

(1) First variation of a varifold; see [All72, 4.2] or [Sim83, 39.1–39.3]

(2) Varifolds whose first variation is representable by integration; see [All72, 4.3] or [Sim83, 39.3, 39.4]

(3) Varifolds contained in submanifolds mostly of the same dimension; see [All72, 4.5–4.7] or [Sim83, §41] for some parts

(4) Relation of varifolds to area minimising currents; see [All72, 4.8] or [Sim83, p. 227]

(5) Cutting a varifold by a smooth functions; see [All72, 4.10]

Prerequisites

(1) Basic measure theory and the theory of Radon measures; see [Fed69, 2.1, 2.2] or [EG92, 1.1, 1.9]

(2) Lusin's and Egoroff's theorems; see [Fed69, 2.3.5–2.3.7] or [EG92, 1.2]

(3) Theory of Lebesgue integration; measurable sets and functions; see [Fed69, 2.4] or [EG92, 1.3]

(4) Riesz's representation theorem; characterization of dual spaces to Lebesgue spaces; see [Fed69, 2.5.1–2.5.15] or [EG92, 1.8] Product measures and Fubini's theorem; see [Fed69, 2.6.1–2.6.5] or [EG92, 1.4]

(5) Covering theorems (Vitali and Besicovitch); see [Fed69, 2.8] or [EG92, 1.5]

(6) Differentiation of a Radon measure with respect to another Radon measure; Radon-Nikodym theorem; Lebesgue points; see [Fed69, 2.9] or [EG92, 1.6, 1.7]

(7) Construction and basic properties of Hausdorff measure; see [Fed69, 2.10.1–2.10.6] or [EG92, Chapter 2]

(8) Arzélà–Ascoli theorem; see [Fed69, 2.10.21] or [Rud91, Appendix A5]

(9) Extension of Lipschitzian maps; Kirszbraun's theorem; see [Fed69, 2.10.43]

(10) Whitney's extension theorem; see [Fed69, 3.1.13] or [KP99, 5.3]

(11) Different equivalent definitions of submanifolds of $\mathbf{R}^n$; see [Fed69, 3.1.19] or [Sim83, §7]

(12) Basic theory of distributions; spaces of smooth test functions and distributions, smoothing distributions, representation by integration; see [Fed69, 4.1.1–4.1.7] or [Rud91, Chapter 6]

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