Weakly differentiable functions and Sobolev functions on varifolds

Summary In geometric analysis both Sobolev functions on smooth Riemannian manifolds and models of possibly singular surfaces, such as varifolds and currents which generalise the concept of submanifold, are tools of basic importance. In [Men15a, Men15b], the author constructed a theory allowing to combine these two tools into a coherent theory of Sobolev functions on varifolds. The present lecture series will present a detailed overview of this theory. The participants are expected to possess a good knowledge of basic geometric measure theory and a familiarity with basic definitions of varifolds such as may be obtained by attending the lecture series of Sławomir Kolasiński.

Monotonicity identity In this part the central monotonicity identity for varifolds and those consequences most relevant for this lecture series will be summarised.

(1) Monotonicity identity, see Allard [All72, 5.1], Simon [Sim83, 17.3, 17.4], and [Men15a, 4.2, 4.5, 4.6].

(2) Stationary cones, see Allard [All72, 5.2, 5.3].

(3) Upper semicontinuity properties of the density, see Allard [All72, 5.4, 8.6] or Simon [Sim83, 17.8, 40.6].

(4) Rectifiability theorem for varifolds, see Allard [All72, 5.5 (1)] or Simon [Sim83, 42.4].

(5) Compactness of rectifiable varifolds, see Allard [All72, 5.6] or Simon [Sim83, 42.7].

(6) Isoperimetric inequality, see Allard [All72, 7.1] or Simon [Sim83, §18].

(7) Isoperimetric lower density ratio bounds, see Allard [All72, 8.3] and [Men09, 2.5].

(8) An example concerning subcritical mean curvature, see [Men09, 1.2].

(9) An example concerning higher multiplicity, see Brakke [Bra78, 6.1] and Kolasiński and the author [KM15, 10.3, 10.8].

Distributional boundary of a set with respect to varifold In this part the distributional boundary of a set with respect to a rectifiable varifold with locally bounded first variation is introduced, see [Men15a, 5.1]. This allows to define a notion of decomposition for such a varifold, see [Men15a, 6.9], and provides the basis for the study of weakly differentiable functions on varifolds.

(1) Existence of a (nonunique) decomposition, see [Men15a, 6.12, 6.13].

(2) A relative isoperimetric inequality which is effective near almost all points at small scales, see [Men15a, 7.8, 7.9, 7.11].

Weakly differentiable functions I In this part a concept of weakly differentiable function on a rectifiable varifold with locally bounded first variation is introduced, see [Men15a, 8.3]. The guiding principle for its definition is to define (without reference to an approximation by smooth functions) a class as large as possible so as to still allow for substantial positive results.

(1) Closure properties under addition/multiplication, see [Men15a, 8.20, 8.25].

(2) A coarea formula, see [Men15a, 8.29].

(3) A constancy theorem, see [Men15a, 8.34].

(4) A concept of zero boundary values, see [Men15a, §9].

(5) Several Sobolev Poincaré inequalities, see [Men15a, 10.1, 10.7].

Weakly differentiable functions II Here the more advanced parts of theory are outlined which rest on the Sobolev Poincaré inequalities.

(1) Approximate differentiability, see [Men15a, 11.2].

(2) Differentiability in Lebesgue spaces, see [Men15a, 11.4].

(3) Rectifiability of the distributional boundary of almost all superlevel sets, see [Men15a, 12.2].

(4) Embedding into continuous functions, see [Men15a, 13.1].

(5) Embedding into Hölder continuous functions, see [Men15a, 13.3].

(6) Rellich type embedding theorem, see [Men15b, 4.8].

(7) Closedness under weak convergence of the class weakly differentiable functions, see [Men15b, 4.9, 4.10].

Possibly also applications to curvature varifolds may be discussed, see [Men15a, §15].

Sobolev functions In this part Sobolev spaces are introduced, see [Men15b, 5.1, 5.7, 5.11, 5.14, 5.18], as subclasses of the in general nonlinear space of weakly differentiable functions. They are defined as suitable completion of locally Lipschitzian functions.

(1) Completeness, see [Men15b, 5.13, 5.17, 5.26].

(2) Zero boundary values, see [Men15b, 5.27].

(3) Geodesic distance on the support of the weight measure as Sobolev function, see [Men15a, 14.2] and [Men15b, 6.8].

(4) A non Hölder continuous Sobolev function with bounded weak derivative, see [Men15b, 6.10].

(5) An alternate description of the topology on local Sobolev spaces, see [Men15b, 7.7].

(6) Embedding theorems for Sobolev functions into Lebesgue spaces and continuous functions, see [Men15b, 7.9, 7.11, 7.12, 7.18, 7.21].

Concluding remark Each of the preceding five topics corresponds to one 90 min lecture, except the first which is longer and the second which is correspondingly shorter.

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