Extremal Process of Last Progeny Modified Branching Random Walks

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Abstract

We consider a last progeny modified branching random walk, in which the position of each particle at the last generation n is modified by an i.i.d. copy of a random variable Y. Depending on the asymptotic properties of the tail of Y, we describe the asymptotic behaviour of the extremal process of this model as $n \to \infty$.

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1 Introduction

Branching random walk (or BRW for short) is a particle system on the real line constructed as follows. It starts from a single particle at position 0 forming the initial generation 0 of the process. Each particle reproduces independently of all others, by creating an identically distributed point process of children around its position. We denote by $\mathcal{U}$ the set of particles in the branching random walk. For all $u \in \mathcal{U}$, we write S_u for the position of particle u, $|u|$ for the generation to which it belongs and u_k for the ancestor of u alive at generation $k \leq |u|$. We only consider in the present article supercritical branching random walks, satisfying the assumption

$$\mathbf{E}\left(\#\{u \in \mathcal{U} : |u| = 1\}\right) > 1. \tag{1.1}$$

Note that we do not make any assumption on the finiteness of $\#\{|u| = 1\}$.

For all $\theta > 0$, we denote by $\kappa(\theta) = \log \mathbf{E}\left(\sum_{|u|=1} e^{\theta S_u}\right) \in (-\infty, \infty]$ the log-Laplace transform of (the intensity measure of the reproduction law of) the BRW. Assuming that $\kappa(\theta) < \infty$, we write

$$\kappa'(\theta) = \mathbf{E}\left(\sum_{|u|=1} S_u e^{\theta S_u - \kappa(\theta)}\right)$$

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whenever this integral is well-defined, irrespectively of the well-definition of κ in a neighbourhood of θ. If $\kappa'(\theta)$ is well-defined, we also write

$$\kappa''(\theta) = \mathbf{E}\left(\sum_{|u|=1} (S_u - \kappa'(\theta))^2 e^{\theta S_u - \kappa(\theta)} \right) \in [0, \infty].$$

Observe that by Lebesgue's dominated convergence theorem, it is straightforward to verify that $\kappa'(\theta)$ and $\kappa''(\theta)$ indeed correspond to the first and second derivative of κ whenever this function is finite in a neighbourhood of θ. We also remark that $\kappa''(\theta) = 0$ implies that almost surely, all particles in the first generation have the same position. We bar this degenerate situation from consideration, by always assuming in this article that

$$\forall x \in \mathbb{R}, \quad \mathbf{P}(\exists |u| = 1 : S_u \neq x) > 0. \tag{1.2}$$

Branching random walks have been the subject of a large and still expanding literature. One of the most studied features of this model is the asymptotic behaviour of the position of particles at the right tip of the BRW. Biggins [7] proved that the *maximal displacement* $M_n := \max_{|u|=n} S_u$ satisfies

$$\lim_{n\to\infty} \frac{M_n}{n} = v := \inf_{\theta>0} \frac{\kappa(\theta)}{\theta} \quad \text{a.s. on the survival event of the BRW,}$$

as long as there exists $\theta > 0$ such that $\kappa(\theta) < \infty$. Addario-Berry and Reed [1] (see also Bramson and Zeitouni [10]) showed that if there exists $\theta_0 > 0$ such that

$$\theta_0 \kappa'(\theta_0) - \kappa(\theta_0) = 0, \tag{1.3}$$

and additional integrability conditions hold, then setting $m_n := nv - \frac{3}{2\theta_0} \log n$, the sequence $(M_n - m_n)$ is tight, although Hu and Shi [13] proved that this sequence exhibits almost sure fluctuations on a logarithmic scale. Aïdékon [2] proved the convergence in distribution of the centred maximal displacement under close to optimal integrability conditions for a BRW satisfying (1.3).

In order to study the joint convergence in distribution of the first few rightmost particles, one can consider the so-called *extremal process* of the BRW, defined as

$$\mathcal{Z}_n := \tau_{-m_n} \mathcal{X}_n,$$

where $\mathcal{X}_n = \sum_{|u|=n} \delta_{S_u}$ is the empirical measure of the BRW and τ_x is the operator corresponding to a shift by x of the point measure. Madaule [17] proved that under mild conditions, the extremal process $\mathcal{Z}_n$ of the BRW converges in law for the topology of vague convergence of point measure to a limiting decorated Cox process $\mathcal{Z}_\infty$. We refer to [20] for an overview of branching random walks, and [18, 21] for background on decorated Cox processes and their connections with branching particle systems.

We take interest in this article to the *last progeny modified BRW*, introduced by Bandyopadhyay and Ghosh in [6], that can be constructed as follows: Let ν be a probability measure on $\mathbb{R}$ and $(Y_u, u \in \mathcal{U})$ be a collection of i.i.d. random variables of law ν, which are independent of the BRW. This model is the family of point measures defined for $n \geq 0$ by

$$\mathcal{E}_n = \sum_{|u|=n} \delta_{S_u + Y_u}.$$

In other words, in the process $\mathcal{E}_n$, the position of particles at the last step n in the BRW are modified by the i.i.d. random variables $(Y_u, |u| = n)$.

The asymptotic behaviour of $\mathcal{E}_n$ can be described using the classical additive martingales of the branching random walk. Let $\theta > 0$ such that $\kappa(\theta) < \infty$, the process defined for $n \geq 0$ by

$$W_n(\theta) = \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}$$

is a non-negative martingale called the *additive martingale* of the BRW. Assuming that $\kappa'(\theta)$ is finite, Biggins [8] obtained a necessary and sufficient condition for the non-degeneracy of its almost sure limit $W_\infty(\theta)$ (with an alternative proof by Lyons [15] based on a spine decomposition argument). More precisely, the martingale $(W_n(\theta))$ is uniformly integrable if and only if

$$\theta\kappa'(\theta) - \kappa(\theta) < 0 \text{ and } \mathbf{E}(W_1(\theta) \log_+ W_1(\theta)) < \infty, \tag{1.4}$$

where $\log_+(x) = \log \max(x, 1)$. If (1.4) does not hold, then $W_\infty(\theta) = 0$ a.s. Alsmeyer and Iksanov [5] obtained a more general necessary and sufficient condition for the non-degeneracy of $W_\infty(\theta)$ that does not depend on the well-definition of $\kappa'(\theta)$.

Let us observe that, by assumption (1.2), the function $\theta \mapsto \kappa(\theta)$ is a strictly convex function, therefore $\theta \mapsto \theta\kappa'(\theta) - \kappa(\theta)$ is increasing on its interval of definition. Consequently (1.4) implies that $\theta < \theta_0$ whenever this quantity is well-defined.

Assume that the constant θ_0 defined in (1.3) exists. In this case, as mentioned above, the additive martingale $W_n(\theta_0)$ converges to 0 a.s. However, the so-called *derivative martingale* defined for $n \geq 0$ by

$$Z_n := \sum_{|u|=n} (n\kappa'(\theta_0) - S_u) e^{\theta_0 S_u - n\kappa(\theta_0)}$$

plays an important role for the study of extremes of the BRW. Assuming that $\kappa''(\theta) < \infty$, Aïdékon [2] obtained necessary condition that Chen [11] proved to be sufficient for the convergence of the (signed, non-uniformly integrable) martingale (Z_n) to an a.s. positive limit Z_∞. This necessary and sufficient conditions is

$$\mathbf{E}\left(W_1(\theta_0)(\log_+ W_1(\theta_0))^2\right) + \mathbf{E}(\overline{W}_1 \log_+(\overline{W}_1)) < \infty, \tag{1.5}$$

where $\overline{W}_1 = \sum_{|u|=1} (\kappa'(\theta_0) - S_u)_+ e^{\theta_0 S_u - \kappa(\theta_0)}$ and $x_+ = \max(x, 0)$.

Under some integrability assumptions on the reproduction law of the BRW, Bandyopadhyay and Ghosh considered in [6] a last progeny modified BRW with perturbation law ν given by the Gumbel law of parameter $1/\theta$. Assuming that $\theta\kappa'(\theta) - \kappa(\theta) \leq 0$, they showed that

$$\lim_{n\to\infty} \tau_{-m_n^{(\theta)}} \mathcal{E}_n = \mathcal{Y}_\theta \quad \text{in law for the topology of vague convergence,} \tag{1.6}$$

with $\mathcal{Y}_\theta$ a random point measure. On the one hand, if $\theta\kappa'(\theta) - \kappa(\theta) < 0$, then $m_n^{(\theta)} = n\frac{\kappa(\theta)}{\theta}$ and the limiting point measure $\mathcal{Y}_\theta$ is a Cox process (i.e. a Poisson point process with a random intensity measure) with intensity $\theta W_\infty(\theta) e^{-\theta x}\,\mathrm{d}x$.

On the other hand, if $\theta\kappa'(\theta)-\kappa(\theta)=0$, (1.6) holds with $m_n^{(\theta)}=n\kappa'(\theta)-\frac{1}{2\theta}\log n$ and $\mathcal{Y}_\theta$ a Cox process with intensity $\sqrt{\frac{2}{\pi\kappa''(\theta)}}\theta Z_\infty(\theta)e^{-\theta x}\,\mathrm{d}x$.

In [14], Kowalski obtained refined estimates on the position of the largest atom in the last progeny modified BRW for a more general class of distributions for ν. The objective of the present paper is to recover the results of Bandyopadhyay and Ghosh under close to minimal conditions for the reproduction law of the BRW or for the law ν. More precisely, we prove that the convergence (1.6) holds in a variety of cases for a well-chosen centring sequence (m_n^θ).

We endow the set of point measures on $\mathbb{R}$ with the topology of vague convergence. For μ a Radon measure on $\mathbb{R}$ and Q a non-negative random variable, we denote by $\mathrm{PPP}(Q\mu)$ a Cox process with intensity $Q\mu$ on $\mathbb{R}$, i.e. conditionally on Q a Poisson point process with intensity $Q\mu$.

We first consider a BRW satisfying the assumptions of [5, Theorem 1.3], and prove the following result.

Theorem 1.1. *Let $\theta>0$ such that $\kappa(\theta)<\infty$. We assume that $(W_n(\theta), n\geq 1)$ is uniformly integrable. Let ν be a probability distribution on $\mathbb{R}$ such that there exists a constant $L\in(0,\infty)$ satisfying*

$$\nu([x,\infty))\sim Le^{-\theta x} \quad \text{as } x\to\infty. \tag{1.7}$$

Then, writing $m_n=\frac{1}{\theta}\left(n\kappa(\theta)+\log L\right)$, the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a PPP($\theta W_\infty(\theta)e^{-\theta x}\,\mathrm{d}x$).

Remark 1.2. For the conclusion of Theorem 1.1 to hold, it is not necessary to assume that κ is finite at any point besides θ, or that $\kappa'(\theta)>-\infty$. Integral conditions in [5, Theorem 1.3] are a necessary and sufficient condition for the convergence in distribution of $\tau_{-m_n}\mathcal{E}_n$ to hold, assuming that (1.7) is satisfied.

Adding some extra integrability conditions on the reproduction law of the BRW, we are also able to describe the asymptotic behaviour of the last progeny modified BRW for any distribution ν satisfying

$$\nu([x,\infty))\sim L(x)e^{-\theta x} \quad \text{as } x\to\infty, \tag{1.8}$$

where L is a positive regularly varying function at ∞. Let us recall that a function L is called *regularly varying at* ∞ with parameter $\alpha\in\mathbb{R}$ if for all $\lambda>0$, we have

$$\lim_{x\to\infty}\frac{L(\lambda x)}{L(x)}=\lambda^\alpha.$$

A function is called *regularly varying at* ∞ if there exists such a parameter α. We refer to the book of Bingham, Goldie and Teugels [9] for background on regularly varying functions.

Theorem 1.3. *Let $\theta>0$ such that $\kappa(\theta)<\infty$. We assume that (1.4) holds, and that there exists $\delta>0$ such that $\kappa(\theta+\delta)+\kappa(\theta-\delta)<\infty$. Let ν be a probability distribution on $\mathbb{R}$ such that there exists a regularly varying function L at ∞ with index α satisfying (1.8). Then, writing*

$$m_n=n\frac{\kappa(\theta)}{\theta}+\frac{1}{\theta}\log L(n) \quad \text{and} \quad c_1=\left(\frac{\kappa(\theta)}{\theta}-\kappa'(\theta)\right)^\alpha,$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a PPP($c_1\theta W_\infty(\theta)e^{-\theta x}\,\mathrm{d}x$).

We now consider a situation in which ν satisfies (1.8) with the parameter θ_0. In this situation, a similar result can be obtained, assuming a stronger condition on the function L and using a modified centring term for the extremal process.

Theorem 1.4. *Let $\theta > 0$ such that $\kappa(\theta) < \infty$ and $\kappa''(\theta) < \infty$. We assume that*

$$\theta\kappa'(\theta) - \kappa(\theta) = 0,$$

i.e. $\theta = \theta_0$ and that (1.5) holds. Let ν be a probability distribution on $\mathbb{R}$ such that there exists a regularly varying function L at ∞ with index $\alpha \in (-2, 0)$ satisfying (1.8). Then, writing

$$m_n = n\frac{\kappa(\theta)}{\theta} + \frac{1}{\theta}\log L\left(\sqrt{n}\right) - \frac{1}{2\theta}\log n$$

$$\text{and} \quad c_2 = \sqrt{\frac{2}{\pi\kappa''(\theta)}}\left(2\kappa''(\theta)\right)^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}+1\right),$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to a $PPP(c_2\theta Z_\infty e^{-\theta x}\,\mathrm{d}x)$.

Remark 1.5. Theorem 1.4 also holds assuming that ν satisfies (1.8) with L a positive constant, using the same arguments as the one we use below. This result therefore extends the previous estimate of Bandyopadhyay and Ghosh [6] for ν a Gumbel distribution with parameter $\frac{1}{\theta_0}$.

Remark 1.6. We believe that up to adding a stronger integrability condition on the reproduction law of the BRW, one could prove a result similar to Theorem 1.4 assuming that ν satisfies (1.8) with L a regularly varying function with parameter $\alpha \geq 0$. However, we predict a sharp phase correction around $\alpha = -2$, and that the limiting extremal process should resemble the one obtained in forthcoming Theorem 1.7 if $\alpha < -2$.

Finally, if ν satisfies (1.8) with $\theta > \theta_0$, then the asymptotic behaviour of $\mathcal{E}_n$ can be deduced from the asymptotic behaviour of $\mathcal{Z}_n$, the extremal process of the BRW.

Theorem 1.7. *We assume that the reproduction law of the BRW is non-lattice, and there exists $\theta_0 > 0$ satisfying (1.3). We assume that $\kappa''(\theta_0) < \infty$ and that (1.5) holds. Let ν be a probability measure such that there exist $C > 0$ and $\theta > \theta_0$ verifying*

$$\nu([x,\infty)) \leq Ce^{-\theta x} \text{ for } x \in \mathbb{R}. \tag{1.9}$$

Then, writing

$$m_n = n\kappa'(\theta_0) - \frac{3}{2\theta_0}\log n,$$

the extremal process $\tau_{-m_n}\mathcal{E}_n$ converges in law to

$$\sum_{i\in\mathbb{N}}\delta_{z_i+Y_i}, \tag{1.10}$$

where $(z_i, i \in \mathbb{N})$ are the atoms of the limiting extremal process $\mathcal{Z}_\infty$ of the BRW and (Y_i) is an independent sequence of i.i.d. variables with law ν.

Theorem 1.7 follows as a direct consequence of the convergence in distribution of the extremal process of the branching random walk of Madaule [17]. The condition (1.9) is used to ensure that the point process defined in (1.10) is well-defined.

The rest of the article is organized as follows. We introduce in the next section some estimates that allow us to study the Laplace transform of $\mathcal{E}_n$. We then use the convergence of this Laplace transform to prove our main theorems in Section 3.

2 Laplace transform of the last progeny modified branching random walk

To prove the convergence in distribution of $\tau_{-m_n}\mathcal{E}_n$ to a limiting point measure $\mathcal{Z}$ in law for the topology of vague convergence, it is sufficient to show that for all continuous compactly supported function φ, we have

$$\lim_{n\to\infty} \langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle = \langle \mathcal{Z}, \varphi\rangle \quad \text{in law},$$

where we write $\langle \mathcal{X}, \varphi\rangle = \int \varphi \, \mathrm{d}\mathcal{X}$. As a result, it is sufficient to prove the convergence of the Laplace transform of $\tau_{-m_n}\mathcal{E}_n$, defined as

$$\varphi \in \mathcal{T} \mapsto \mathbf{E}\left(\exp\left(-\langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle\right)\right),$$

where $\mathcal{T}$ is the set of all non-negative continuous compactly supported functions, i.e., to show that for each $\varphi \in \mathcal{T}$, we have

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\left(-\langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle\right)\right) = \mathbf{E}\left(\exp\left(-\langle \mathcal{Z}, \varphi\rangle\right)\right). \tag{2.1}$$

In the rest of the section, we introduce some methods and results allowing to study this Laplace transform.

Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Using the independence between $(Y_u, u \in \mathcal{U})$ and the BRW, we observe that

$$\begin{aligned} \mathbf{E}\left(\exp\left(-\langle \tau_x\mathcal{E}_n, \varphi\rangle\right)\right) &= \mathbf{E}\left(\exp\left(-\sum_{|u|=n} g_\varphi(x+S_u)\right)\right) \\ &= \mathbf{E}\left(\exp\left(-\langle \tau_x\mathcal{X}_n, g_\varphi\rangle\right)\right) \end{aligned} \tag{2.2}$$

where we have set

$$g_\varphi : x \mapsto -\log \int e^{-\varphi(x+y)}\, \nu(\mathrm{d}y),$$

and we recall that $\mathcal{X}_n$ is the counting measure of the BRW at time n.

As mentioned above, applying Madaule's theorem [17] to (2.2) immediately yields Theorem 1.7. To prove Theorems 1.1, 1.3 and 1.4, we first show that $\max_{|u|=n} S_u - m_n$ converges to $-\infty$ in probability (for m_n the process defined in each of these theorems), then study the asymptotic behaviour of $g_\varphi(x)$ as $x \to -\infty$. Let us first state the following generic lemma for BRW.

Lemma 2.1. *If $\kappa(\theta) < \infty$, then $\lim_{n\to\infty} \max_{|u|=n} \theta S_u - n\kappa(\theta) = -\infty$ a.s.*

Remark 2.2. Recall that $W_n(\theta) = \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}$ is a non-negative martingale, hence converges almost surely. As a result, using the bound

$$W_n(\theta) \geq e^{\theta M_n - n\kappa(\theta)},$$

it is immediate that $\sup_{n\in\mathbb{N}} \theta M_n - n\kappa(\theta) < \infty$ a.s.

Proof. To prove that $\theta M_n - n\kappa(\theta) \to -\infty$, we strengthen the bound of Remark 2.2 with the following observation. The martingale $(W(\theta))$ converges almost surely, therefore $(W_n(\theta))$ is almost surely a Cauchy sequence.

Let $x \in \mathbb{R}$, we set $A_x := \{\limsup_{n\to\infty} \theta M_n - n\kappa(\theta) > x\}$. We define a sequence (T_n) of stopping times by $T_0 = 0$ and

$$T_{n+1} = \inf\{k > T_n : \theta M_k - k\kappa(\theta) > x\}.$$

By definition, T_n is finite for all n on A_x. We define $A_{x,n} := \{T_n < \infty\}$. Note that $A_{x,n} \in \mathcal{F}_{T_n}$ and $A_{x,n+1} \subseteq A_{x,n}$ for all n, and $A_x = \bigcap_{n\in\mathbb{N}} A_{x,n}$, where $(\mathcal{F}_n)$ is the usual filtration of the BRW. We write r_n for the label of one individual at generation T_n such that $S_{r_n} = M_{T_n}$, and we define

$$\overline{W}_{T_n+1}(\theta) = \sum_{|u|=T_n+1} e^{\theta S_u - (T_n+1)\kappa(\theta)} \mathbb{1}_{\{u \not\succ r_n\}} + e^{\theta S_{r_n} - T_n\kappa(\theta)} \sum_{j\in\mathbb{N}} e^{\theta X_j - \kappa(\theta)},$$

where $u \succ v$ means that u is a descendant of v, and $(X_j, j \in \mathbb{N})$ is a random sequence with same law as $(S_u, |u| = 1)$, independent of the BRW. In words, we obtain $\overline{W}_{T_n+1}(\theta)$ by replacing the offspring of an individual alive at position M_{T_n} by an independent copy of that point process. It is therefore apparent that $\overline{W}_{T_n+1}(\theta)$ has, conditionally on $\mathcal{F}_{T_n}$, the same law as $W_{T_n+1}(\theta)$. Moreover, we have that on $A_{x,n}$,

$$\begin{aligned} \left|W_{T_n+1}(\theta) - \overline{W}_{T_n+1}(\theta)\right| &= e^{\theta S_{r_n} - (T_n+1)\kappa(\theta)} \left| \sum_{j\in\mathbb{N}} e^{\theta(S_{r_n j} - S_{r_n})} - e^{\theta X_j} \right| \\ &\geq e^{x-\kappa(\theta)} \left| \sum_{j\in\mathbb{N}} e^{\theta(S_{r_n j} - S_{r_n})} - e^{\theta X_j} \right|, \end{aligned}$$

with $r_n j$ being the jth descendant of individual r_n. In particular, there exists $\delta > 0$ such that

$$\mathbf{E}\left(\mathbb{1}_{A_{x,n}} \left(\left|W_{T_n+1}(\theta) - \overline{W}_{T_n+1}(\theta)\right| \wedge 1\right)\right) \geq \delta\, \mathbf{P}(A_{x,n}) \geq \delta\, \mathbf{P}(A_x).$$

On the other hand, we have

$$\mathbf{E}\left(\mathbb{1}_{A_{x,n}} \left(|W_{T_n+1}(\theta) - W_{T_n}(\theta)| \wedge 1\right)\right) = \mathbf{E}\left(\mathbb{1}_{A_{x,n}} \left|\overline{W}_{T_n+1}(\theta) - W_{T_n}(\theta)\right| \wedge 1\right).$$

Therefore, we have

$$\mathbf{E}\left(\mathbb{1}_{A_{x,n}} \left(|W_{T_n+1}(\theta) - W_{T_n}(\theta)| \wedge 1\right)\right) \geq \delta\, \mathbf{P}(A_x)/2$$

for all $n \in \mathbb{N}$. As $(W_n(\theta))$ converges almost surely as $n \to \infty$, we conclude by dominated convergence theorem that $\mathbf{P}(A_x) = 0$.

To complete the proof, we observe that

$$\mathbf{P}(\limsup_{n\to\infty} \theta M_n - n\kappa(\theta) > -\infty) = \mathbf{P}(\cup_{k\in\mathbb{N}} \mathbf{P}(A_{-k})) = 0. \qquad \square$$

Lemma 2.1 is enough to conclude that $\max_{|u|=n} S_u - m_n \to -\infty$ under the assumptions of Theorems 1.1 and 1.3. To obtain a similar result under the assumptions of Theorem 1.4, we will use the tightness of

$$M_n - n\kappa'(\theta) + \frac{3}{2\theta}\log n,$$

proved under the assumptions[1] of Theorem 1.4 in [19]. Using the above observations, the asymptotic behaviour of the Laplace transform of $\mathcal{E}_n$ as $n \to \infty$ can be obtained by studying the asymptotic behaviour of g_φ under assumption (1.8) as $x \to -\infty$.

Lemma 2.3. *Let ν be a probability distribution satisfying* (1.8)*. For all $\varphi \in \mathcal{T}$, we have*

$$\lim_{x\to-\infty} \frac{e^{-\theta x}}{L(-x)} g_\varphi(x) = \int \theta e^{-\theta z}(1 - e^{-\varphi(z)})\,\mathrm{d}z.$$

Proof. Let $\varphi \in \mathcal{T}$. We observe that we can rewrite

$$g_\varphi(x) = -\log\left(1 - \int 1 - e^{-\varphi(z+x)}\,\nu(\mathrm{d}z)\right),$$

and that $y \mapsto 1 - e^{-\varphi(y)}$ is continuous and compactly supported, i.e. an element of $\mathcal{T}$. By the dominated convergence theorem, we have that

$$\lim_{x\to-\infty} \int 1 - e^{-\varphi(z+x)}\,\nu(\mathrm{d}z) = 0,$$

which implies

$$g_\varphi(x) \sim \int 1 - e^{-\varphi(z+x)}\,\nu(\mathrm{d}z) \quad \text{as } x \to -\infty.$$

Therefore, it is enough to show that

$$\lim_{x\to-\infty} \int \frac{e^{-\theta x}}{L(-x)} \psi(x+z)\,\nu(\mathrm{d}z) = \int \theta e^{-\theta z}\psi(z)\,\mathrm{d}z \tag{2.3}$$

for all $\psi \in \mathcal{T}$ to prove Lemma 2.3. We observe that for all $a < b$,

$$\lim_{x\to-\infty} \frac{e^{-\theta x}}{L(-x)} \nu([a-x, b-x)) = e^{-\theta a} - e^{-\theta b} = \int_a^b \theta e^{-\theta z}\,\mathrm{d}z,$$

from which we deduce (2.3) by approximations, which completes the proof. □

To simplify the notation, in the rest of the article, we write for all $\varphi \in \mathcal{T}$ and $\theta > 0$

$$c_\varphi(\theta) = \int \theta e^{-\theta z}(1 - e^{-\varphi(z)})\,\mathrm{d}z.$$

We can then restate Lemma 2.3 as $g_\varphi(x) \sim c_\varphi(\theta) L(-x) e^{\theta x}$ as $x \to -\infty$ under assumption (1.8).

Similarly to Lemma 2.3, we observe that an upper bound for the right tail of ν implies a similar upper bound for g_φ as $x \to -\infty$.

[1] By [2, Equation (B1)], if $\kappa''(\theta_0) < \infty$ then (1.5) implies [19, Equation (1.4)].

Lemma 2.4. *Let ν be a probability distribution satisfying (1.9). For all $\varphi \in \mathcal{T}$, there exists $C' > 0$ such that for all $x \in \mathbb{R}$,*

$$g_\varphi(x) \leq C' e^{\theta x}.$$

Proof. Let $\varphi \in \mathcal{T}$, as φ is compactly supported, there exists $B > 0$ such that $\varphi(z) = 0$ for all $z < -B$. Therefore,

$$e^{-g_\varphi(x)} = \int e^{-\varphi(x+z)} \nu(\mathrm{d}z) \geq \nu((-\infty, -x-B)) \geq 1 - Ce^{\theta(x+B)}.$$

As a result, we deduce that

$$\limsup_{x \to -\infty} e^{-\theta x} g_\varphi(x) \leq \lim_{x \to -\infty} -e^{-\theta x} \log\left(1 - Ce^{\theta(x+B)}\right) = Ce^{\theta B}.$$

Using that g_φ is bounded, the proof is now complete. □

To complete the proofs of Theorems 1.1, 1.3 and 1.4, it will be enough to show that

$$\sum_{|u|=n} g_\varphi(S_u - m_n) \text{ converges in probability,}$$

and to identify its limit. This is mainly done using the so-called many-to-one lemma (see [20, Theorem 1.1]), that we now state.

Lemma 2.5. *Let $\theta > 0$ such that $\kappa(\theta) < \infty$. There exists a random walk $(T_n, n \geq 0)$ such that for all measurable non-negative function f, we have*

$$\mathbf{E}\left(\sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)} f(S_{u_1}, \ldots, S_{u_n}) \right) = \mathbf{E}\left(f(T_1, \ldots, T_n) \right).$$

Moreover, $\mathbf{E}(T_1) = \kappa'(\theta)$ whenever this quantity is well-defined.

3 Proof of the theorems

We prove in this section our main theorems. We first consider the asymptotic behaviour of $\mathcal{E}_n$ below the boundary case, i.e. when $\theta < \theta_0$, assuming that this quantity is well-defined. We then turn to the proof of Theorem 1.4, i.e. assuming that $\theta = \theta_0$. Finally, we prove Theorem 1.7 in Section 3.3.

3.1 BRW below the boundary case

In this section, θ is a fixed positive constant, and we assume that $\kappa(\theta) < \infty$ and $(W_n(\theta), n \geq 0)$ is uniformly integrable. We denote by $W_\infty(\theta) = \lim_{n \to \infty} W_n(\theta)$ the almost sure limit of this martingale. We start by proving Theorem 1.1.

Proof of Theorem 1.1. Let $\varphi \in \mathcal{T}$, using Lemma 2.3, under assumption (1.7), for all $\varepsilon > 0$, there exists $A > 0$ such that for all $x \leq -A$, we have

$$\left| g_\varphi(x) - c_\varphi(\theta) L e^{\theta x} \right| \leq \varepsilon e^{\theta x}. \tag{3.1}$$

Observe that for all $a, b > 0$, we have $|e^{-a} - e^{-b}| \leq |a - b| \wedge 1$ using that exp is 1-Lipschitz on $\mathbb{R}_-$. Therefore,

$$\mathbf{E}\left(\left|e^{-\langle \tau_{-m_n}\mathcal{X}_n, g_\varphi\rangle} - e^{-c_\varphi(\theta)W_n(\theta)}\right|\right) \leq \mathbf{E}\left(|\langle \tau_{-m_n}\mathcal{X}_n, g_\varphi\rangle - c_\varphi(\theta)W_n(\theta)| \wedge 1\right)$$
$$\leq \mathbf{E}\left(\left|\sum_{|u|=n} h_\varphi(S_u - m_n)\right| \wedge 1\right),$$

where $h_\varphi(x) = g_\varphi(x) - c_\varphi(\theta)Le^{\theta x}$, using that $e^{-\theta m_n} = L^{-1}e^{-n\kappa(\theta)}$. By (3.1), we have

$$\mathbf{E}\left(\mathbb{1}_{\{M_n \leq m_n - A\}}\left|\sum_{|u|=n} h_\varphi(S_u - m_n)\right| \wedge 1\right) \leq \varepsilon L^{-1}\,\mathbf{E}(W_n(\theta)),$$

hence

$$\mathbf{E}\left(\left|e^{-\langle \tau_{-m_n}\mathcal{X}_n, g_\varphi\rangle} - e^{-c_\varphi(\theta)W_n(\theta)}\right|\right) \leq \varepsilon L^{-1} + \mathbf{P}(M_n \geq m_n - A).$$

Letting $n \to \infty$, then $\varepsilon \to 0$, we conclude, by Lebesgue's dominated convergence theorem and (2.2), that

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle}\right) = \lim_{n\to\infty} \mathbf{E}\left(e^{-c_\varphi(\theta)\sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}}\right)$$
$$= \mathbf{E}\left(\exp\left(-W_\infty(\theta)\int \theta e^{-\theta z}(1 - e^{-\varphi(z)})\,\mathrm{d}z\right)\right).$$

To complete the proof, it is then enough to observe that this limit is the Laplace transform of the $\mathrm{PPP}(\theta W_\infty(\theta)e^{-\theta z}\,\mathrm{d}z)$. □

Theorem 1.3 follows from very similar computations. We use the extra integrability condition $\kappa(\theta+\delta) < \infty$ to guarantee that under our stated assumptions, $M_n - m_n$ almost surely decays linearly, and the condition $\kappa(\theta-\delta) < \infty$ to control the contributions of particles far from position m_n.

Proof of Theorem 1.3. We compute the asymptotic behaviour of the Laplace transform of $\tau_{-m_n}\mathcal{E}_n$ using similar computations as for the proof of Theorem 1.1. We recall that for all $\varphi \in \mathcal{T}$, we have

$$\mathbf{E}\left(\exp\left(-\langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle\right)\right) = \mathbf{E}\left(\exp\left(-\sum_{|u|=n} g_\varphi(S_u - m_n)\right)\right),$$

and that $c_1 = \left(\frac{\kappa(\theta)}{\theta} - \kappa'(\theta)\right)^\alpha$. Let $\eta > 0$ such that $4\eta < \frac{\kappa(\theta)}{\theta} - \kappa'(\theta)$. Using Lemma 2.3 and the regular variations of L at ∞, we deduce that almost surely, for all n large enough,

$$(1 - \tfrac{2}{c_1^{1/\alpha}}\eta)^{|\alpha|} c_1 c_\varphi(\theta) \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}\mathbb{1}_{\{|S_u - n\kappa'(\theta)| \leq n\eta\}}$$
$$\leq \sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{\{|S_u - n\kappa'(\theta)| \leq n\eta\}}$$
$$\leq (1 - \tfrac{2}{c_1^{1/\alpha}}\eta)^{-|\alpha|} c_1 c_\varphi(\theta) \sum_{|u|=n} e^{\theta S_u - n\kappa(\theta)}\mathbb{1}_{\{|S_u - n\kappa'(\theta)| \leq n\eta\}}.$$

Here we use the Uniform Convergence Theorem [9, Theorem 1.5.2], which states that $\lim_{z\to\infty}\frac{L(\lambda z)}{L(z)}=\lambda^\alpha$ holds uniformly on compact subsets of $(0,\infty)$. Note that there exists $r>0$ so that

$$[(1-\tfrac{3}{c_1^{1/\alpha}}\eta)^{|\alpha|},(1-\tfrac{3}{c_1^{1/\alpha}}\eta)^{-|\alpha|}]\subset[1-r\eta,1+r\eta].$$

Using Lemma 2.5, together with the law of large numbers, we observe that

$$\lim_{n\to\infty}\mathbf{E}\left(\sum_{|u|=n}e^{\theta S_u-n\kappa(\theta)}\mathbb{1}_{\{|S_u-n\kappa'(\theta)|>n\eta\}}\right)=\lim_{n\to\infty}\mathbf{P}(|T_n-\mathbf{E}(T_n)|>n\eta)=0. \tag{3.2}$$

Consequently, using the a.s. convergence of $W_n(\theta)$ to $W_\infty(\theta)$, we deduce that

$$\lim_{n\to\infty}\sum_{|u|=n}e^{\theta S_u-n\kappa(\theta)}\mathbb{1}_{\{|S_u-n\kappa'(\theta)|\le n\eta\}}=W_\infty(\theta)\quad\text{in probability.}$$

Hence, we conclude that

$$\mathbf{P}\left(\left|\sum_{|u|=n}g_\varphi(S_u-m_n)\mathbb{1}_{\{|S_u-n\kappa'(\theta)|\le n\eta\}}-c_1c_\varphi(\theta)W_\infty(\theta)\right|>r\eta c_1c_\varphi(\theta)W_\infty(\theta)\right)$$

converges to 0 as $n\to\infty$. In order to show that $\sum_{|u|=n}g_\varphi(S_u-m_n)$ converges to $c_1c_\varphi(\theta)W_\infty(\theta)$ in probability and complete the proof of Theorem 1.3, we now show that

$$\lim_{n\to\infty}\sum_{|u|=n}g_\varphi(S_u-m_n)\mathbb{1}_{\{|S_u-n\kappa'(\theta)|>n\eta\}}=0\quad\text{in probability.} \tag{3.3}$$

On the one hand, using that $\theta\mapsto\kappa(\theta)$ is $\mathcal{C}^2$ on $(\theta-\delta,\theta+\delta)$, and that $\theta\kappa'(\theta)-\kappa(\theta)<0$, we observe that $\vartheta\mapsto\kappa(\vartheta)/\vartheta$ is decreasing on $[\theta,\theta+\delta]$ for δ small enough. Thus, by Lemma 2.1, there exists $\varepsilon>0$ such that

$$\lim_{n\to\infty}M_n-n(\kappa(\theta)/\theta-\varepsilon)=-\infty\quad\text{a.s.}$$

As $\log L(n)/n\to0$ as $n\to\infty$, we conclude that almost surely, for all n large enough, $M_n-m_n\le-\varepsilon n/2$. Therefore, almost surely, for n large enough

$$\begin{aligned}&\sum_{|u|=n}g_\varphi(S_u-m_n)\mathbb{1}_{\{S_u-n\kappa'(\theta)>n\eta\}}\\&\quad=\sum_{|u|=n}g_\varphi(S_u-m_n)\mathbb{1}_{\{S_u-n\kappa'(\theta)>n\eta,S_u-m_n<-n\varepsilon/2\}}\\&\quad\le2c_\varphi(\theta)\sum_{|u|=n}L(m_n-S_u)\frac{e^{\theta S_u-n\kappa(\theta)}}{L(n)}\mathbb{1}_{\{S_u-n\kappa'(\theta)>n\eta,S_u-m_n<-n\varepsilon/2\}}.\end{aligned}$$

Therefore, using again [9, Theorem 1.5.2], there exists a constant C depending on ε and η such that almost surely, for all n large enough

$$\sum_{|u|=n}g_\varphi(S_u-m_n)\mathbb{1}_{\{S_u-n\kappa'(\theta)>n\eta\}}\le C\sum_{|u|=n}e^{\theta S_u-n\kappa(\theta)}\mathbb{1}_{\{S_u-n\kappa'(\theta)>n\eta\}}.$$

This, together with (3.2), implies that

$$\lim_{n\to\infty} \sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - n\kappa'(\theta) > n\eta\}} = 0 \quad \text{in probability.} \tag{3.4}$$

With similar computations, we observe that for all $B > 0$, we have

$$\lim_{n\to\infty} \sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - n\kappa'(\theta) < -n\eta\}} \mathbb{1}_{\{S_u - m_n > -nB\}} = 0 \quad \text{in probability.} \tag{3.5}$$

On the other hand, using that $\lim_{x\to\infty} x^{-1} \log L(x) = 0$, we observe that for all x large enough, we have

$$g_\varphi(-x) \leq 2c_\varphi(\theta) L(x) e^{-\theta x} \leq 4c_\varphi(\theta) e^{-(\theta-\delta/2)x}.$$

Therefore, for all n large enough, we have

$$\begin{aligned}\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n < -nB\}} &\leq 4c_\varphi(\theta) \sum_{|u|=n} e^{(\theta-\delta/2)(S_u - m_n)} \mathbb{1}_{\{S_u - m_n < -nB\}} \\ &\leq 4c_\varphi(\theta) W_n(\theta - \delta) e^{n\kappa(\theta-\delta) - (\theta-\delta)m_n - \delta n B/2}.\end{aligned}$$

Using that $W_n(\theta - \delta)$ converges almost surely, we observe that for B large enough, we have

$$\lim_{n\to\infty} \sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n < -nB\}} = 0 \quad \text{in probability.} \tag{3.6}$$

Consequently, combining (3.4), (3.5) and (3.6), we get (3.3), which implies

$$\lim_{n\to\infty} \sum_{|u|=n} g_\varphi(S_u - m_n) = c_1 c_\varphi(\theta) W_\infty(\theta) \quad \text{in probability.}$$

Using the dominated convergence theorem together with (2.2), we conclude that for all for all $\varphi \in \mathcal{T}$,

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \tau_{-m_n} \mathcal{E}_n, \varphi \rangle} \right) = \mathbf{E}\left(\exp\left(-c_1 W_\infty(\theta) c_\varphi(\theta) \right) \right).$$

We can now complete the proof, observing that the right-hand side is the Laplace transform of the PPP$(c_1 \theta W_\infty(\theta) e^{-\theta x} \mathrm{d}x)$. □

3.2 BRW in the boundary case

We turn in this section to the proof of Theorem 1.4. The proof follows a similar scheme as the one used above, with the added introduction of a shaving procedure. More precisely, for all $A > 0$, we set

$$\mathcal{G}_n(A) = \{|u| = n : S_{u_k} \leq k\kappa'(\theta) + A, k \leq n\},$$

the set of particles that stayed at all times below the line $x \mapsto x\kappa'(\theta) + A$. Using that $\lim_{n\to\infty} M_n - n\kappa'(\theta) = -\infty$ a.s. we observe that almost surely, for A large enough, we have $\mathcal{G}_n(A) = \{|u| = n\}$. In other words, writing

$$\mathcal{S}_A := \{\sup_{n\in\mathbb{N}} M_n - n\kappa'(\theta) < A\},$$

we observe that on the event $\mathcal{S}_A$, we have $\mathcal{G}_n(A) = \{|u| = n\}$ for all $n \in \mathbb{N}$, and that $\lim_{A\to\infty} \mathbf{P}(\mathcal{S}_A) = 1$.

As a first step towards the proof of Theorem 1.4, we show that no particle in $\mathcal{G}_n(A)$ above position $m_n - \varepsilon n^{1/2}$ contribute to the extremal process with high probability, as soon as $\varepsilon > 0$ is small enough.

Lemma 3.1. *Under the conditions and notation of Theorem 1.4, for all $B > 0$, we have*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{P}(\exists |u| = n : S_u \geq m_n - \varepsilon n^{1/2}, S_u + Y_u \geq m_n - B) = 0.$$

Proof. We first recall from [19, Theorem 1.1], that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{P}(M_n \geq n\kappa'(\theta) - \frac{3}{2\theta} \log n + \varepsilon^{-1}) = 0.$$

Letting $a_n = \frac{1}{\theta} (\log n + \log L(\sqrt{n}))$, we can rewrite this as

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{P}(M_n \geq m_n - a_n + \varepsilon^{-1}) = 0.$$

Moreover, we have $\lim_{A\to\infty} \mathbf{P}(\mathcal{S}_A) = 1$. Therefore it is enough to prove that for all $A > 0$ large enough,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{P}\left(\exists u \in \mathcal{G}_n(A) : \begin{array}{l} m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], \\ S_u + Y_u \geq m_n - B \end{array}\right) = 0. \tag{3.7}$$

Using the Markov inequality, we have

$$\begin{aligned}
&\mathbf{P}\left(\exists u \in \mathcal{G}_n(A) : m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], S_u + Y_u \geq m_n - B\right) \\
&\leq \mathbf{E}\left(\sum_{u\in\mathcal{G}_n(A)} \mathbb{1}_{\left\{m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}]\right\}} \nu([m_n - B - S_u, \infty))\right) \\
&\leq 2\,\mathbf{E}\left(\sum_{u\in\mathcal{G}_n(A)} \mathbb{1}_{\left\{m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}]\right\}} L(m_n - S_u) e^{\theta(S_u + B - m_n)}\right)
\end{aligned}$$

for all n large enough, where we used the independence between Y and S, and the fact that $a_n \to \infty$, therefore we can apply (1.8) to $m_n - S_u$.

We then use the formula of m_n and the many-to-one lemma to compute

$$\begin{aligned}
&\mathbf{E}\left(\sum_{u\in\mathcal{G}_n(A)} \mathbb{1}_{\left\{m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}]\right\}} L(m_n - S_u) e^{\theta(S_u - m_n)}\right) \\
&= n^{1/2}\,\mathbf{E}\left(\sum_{u\in\mathcal{G}_n(A)} \frac{L(m_n - S_u)}{L(n^{1/2})} e^{\theta S_u - n\kappa(\theta)} \mathbb{1}_{\left\{m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}]\right\}}\right) \\
&= n^{1/2}\,\mathbf{E}\left(\frac{L(\widehat{m}_n - \widehat{T}_n)}{L(n^{1/2})} \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right),
\end{aligned}$$

where $\widehat{T}_k = T_k - k\kappa'(\theta)$ and $\widehat{m}_n = m_n - n\kappa'(\theta)$. Let $\rho \in (0, \alpha + 2)$, we define $\mathfrak{L} : x \mapsto x^{\rho-\alpha} L(x)$. Observe that $\mathfrak{L}$ is a regularly varying function at ∞ with index $\rho > 0$. Therefore, by [9, Theorem 1.5.2], for all $\delta > 0$, there exists N_δ such that for all $x > N_\delta$,

$$\left|\frac{L(\lambda x)}{L(x)} - \lambda^\alpha\right| = \lambda^{\alpha-\rho}\left|\frac{\mathfrak{L}(\lambda x)}{\mathfrak{L}(x)} - \lambda^\rho\right| < \delta\lambda^{\alpha-\rho} \quad \text{for all } \lambda \in (0, 2\varepsilon].$$

As a result, for all n large enough, we have

$$\begin{aligned}
&\mathbf{E}\left(\frac{L(\widehat{m}_n - \widehat{T}_n)}{L(n^{1/2})}\mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right) \\
&\qquad \leq \mathbf{E}\left(\left(\frac{\widehat{m}_n - \widehat{T}_n}{n^{1/2}}\right)^\alpha \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right) \\
&\qquad\quad + \delta\,\mathbf{E}\left(\left(\frac{\widehat{m}_n - \widehat{T}_n}{n^{1/2}}\right)^{\alpha-\rho} \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right).
\end{aligned}$$

We now compute this quantity using the ballot theorem to the centred random walk $(-\widehat{T}_n)$ with finite variance.

Using e.g. [3, Lemma 4.1], there exist $C > 0$ and $h > 0$ such that for all $n \in \mathbb{N}$, $a \geq 0$ and $b \geq -a$, we have

$$\mathbf{P}(-\widehat{T}_n \in [b, b+h], -\widehat{T}_j \geq -a, j \leq n) \leq C\frac{((a+1) \wedge n^{1/2})((a+b+1) \wedge n^{1/2})}{n^{3/2}}.$$

Let $\gamma \in (-2, 0)$. For all n large enough, we have

$$\begin{aligned}
&\mathbf{E}\left(\left(\widehat{m}_n - \widehat{T}_n\right)^\gamma \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right) \\
&\leq \sum_{k=1}^{\lceil \varepsilon n^{1/2}/h \rceil} (kh)^\gamma\, \mathbf{P}(\widehat{m}_n - \widehat{T}_n \in [kh, (k+1)h], -\widehat{T}_j \geq -A, j \leq n) \\
&\leq \frac{Ch^\gamma(A+1)}{n^{3/2}} \sum_{k=1}^{\lceil \varepsilon n^{1/2}/h \rceil} k^\gamma(A + kh + C'\log n + 1) \\
&\leq \frac{2Ch^\gamma(A+1)^2}{n^{3/2}} \sum_{k=1}^{\lceil \varepsilon n^{1/2}/h \rceil} k^\gamma(kh + C'\log n),
\end{aligned}$$

where C' is chosen so that $|\widehat{m}_n| \leq C'\log n$ for all n large enough. Decomposing this sum for $k \leq \log n$ and $k \geq \log n$, we obtain, for n large enough

$$\begin{aligned}
&\mathbf{E}\left((\widehat{m}_n - \widehat{T}_n)^\gamma \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right) \\
&\qquad \leq \frac{2Ch^\gamma(A+1)^2(C'+h)}{n^{3/2}}\left((\log n)^{\gamma+2} + \sum_{k=\lceil \log n \rceil}^{\lceil \varepsilon n^{1/2}/h \rceil} k^{\gamma+1}\right) \\
&\qquad\qquad \leq \frac{K_\gamma}{n^{3/2}}(\log n)^{\gamma+2} + \frac{K'_\gamma}{n^{3/2}}\varepsilon^{\gamma+2} n^{\gamma/2+1},
\end{aligned}$$

for some $K_\gamma, K'_\gamma > 0$ since $\gamma + 1 > -1$. As a result, letting $n \to \infty$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbf{P}\left(\exists u \in \mathcal{G}_n(A) : \begin{array}{l} m_n - S_u \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], \\ S_u + Y_u \geq m_n - B \end{array}\right) \\
&\leq 2e^{\theta B} \limsup_{n\to\infty} n^{1/2}\, \mathbf{E}\left(\frac{L(\widehat{m}_n - \widehat{T}_n)}{L(n^{1/2})} \mathbb{1}_{\left\{\widehat{m}_n - \widehat{T}_n \in [a_n - \varepsilon^{-1}, \varepsilon n^{1/2}], -\widehat{T}_j \geq -A, j \leq n\right\}}\right) \\
&\leq 2e^{\theta B} K'_\alpha \varepsilon^{\alpha+2} + 2e^{\theta B} \delta K'_{\alpha-\rho} \varepsilon^{\alpha-\rho+2}.
\end{aligned}$$

Using that $\alpha > \alpha - \rho > -2$, we conclude that (3.7) holds, which completes the proof. □

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. For any $n \in \mathbb{N}$ and $\varepsilon > 0$, we write

$$\widetilde{\mathcal{E}}_n^{(\varepsilon)} = \sum_{|u|=n} \delta_{S_u + Y_u} \mathbb{1}_{\left\{S_u \leq m_n - \varepsilon n^{1/2}\right\}}.$$

Using Lemma 3.1, together with the inequality $|e^{-a} - e^{-b}| \leq |a - b| \wedge 1$, we observe that for all $\varphi \in \mathcal{T}$,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \left| \mathbf{E}\left(e^{-\langle \tau_{-m_n} \mathcal{E}_n, \varphi\rangle}\right) - \mathbf{E}\left(e^{-\langle \tau_{-m_n} \widetilde{\mathcal{E}}_n^{(\varepsilon)}, \varphi\rangle}\right)\right| \\
&\leq \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbf{E}\left(\left(\sum_{|u|=n} \varphi(S_u + Y_u - m_n) \mathbb{1}_{\left\{S_u \geq m_n - \varepsilon n^{1/2}\right\}}\right) \wedge 1\right) = 0. \qquad (3.8)
\end{aligned}$$

It is therefore enough to study the asymptotic behaviour of $\mathbf{E}\left(e^{-\langle \tau_{-m_n} \widetilde{\mathcal{E}}_n^{(\varepsilon)}, \varphi\rangle}\right)$ to identify the limiting distribution of $\tau_{-m_n} \mathcal{E}_n$.

Let $\varphi \in \mathcal{T}$. Using the same computations as in (2.2), we have

$$\begin{aligned}
&\mathbf{E}\left(\exp\left(-\langle \tau_{-m_n} \widetilde{\mathcal{E}}_n^{(\varepsilon)}, \varphi\rangle\right)\right) \\
&\qquad\qquad = \mathbf{E}\left(\exp\left(-\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\left\{S_u \leq m_n - \varepsilon n^{1/2}\right\}}\right)\right).
\end{aligned}$$

We study the asymptotic behaviour of $\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\left\{S_u \leq m_n - \varepsilon n^{1/2}\right\}}$ as $n \to \infty$ then $\varepsilon \to 0$. By Lemma 2.3, we get that for all $\delta > 0$, for all large enough n,

$$\begin{aligned}
&(1 - \delta) c_\varphi(\theta) n^{1/2} \sum_{|u|=n} \frac{L(m_n - S_u)}{L(n^{1/2})} e^{\theta S_u - n\kappa(\theta)} \mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}} \\
&\quad \leq \sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}} \\
&\qquad \leq (1 + \delta) c_\varphi(\theta) n^{1/2} \sum_{|u|=n} \frac{L(m_n - S_u)}{L(n^{1/2})} e^{\theta S_u - n\kappa(\theta)} \mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}}.
\end{aligned}$$

Recall that L is regularly varying with index $\alpha < 0$. We use again [9, Theorem 1.5.2], yielding

$$\lim_{x\to\infty} \sup_{\lambda > \varepsilon/2} \left| \frac{L(\lambda x)}{L(x)} - \lambda^\alpha \right| = 0.$$

As a result, for all $0 < \delta < \varepsilon$, for all n large enough, we have

$$\begin{aligned}(1-\delta)c_\varphi(\theta)n^{1/2} &\sum_{|u|=n} \underline{h}_{\varepsilon,\delta}((m_n - S_u)/n^{1/2})e^{\theta S_u - n\kappa(\theta)} \\ &\leq \sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}} \\ &\qquad \leq (1+\delta)c_\varphi(\theta)n^{1/2} \sum_{|u|=n} \overline{h}_{\varepsilon,\delta}((m_n - S_u)/n^{1/2})e^{\theta S_u - n\kappa(\theta)},\end{aligned}$$

where $\underline{h}_{\varepsilon,\delta}$ and $\overline{h}_{\varepsilon,\delta}$ are continuous functions satisfying for all $x \in \mathbb{R}$

$$\begin{aligned}(x^\alpha - \delta)\mathbb{1}_{\{[\varepsilon+\delta,\infty)\}} \leq \underline{h}_{\varepsilon,\delta}(x) \leq (x^\alpha - \delta)\mathbb{1}_{\{[\varepsilon,\infty)\}} \\ \text{and } (x^\alpha + \delta)\mathbb{1}_{\{[\varepsilon,\infty)\}} \leq \overline{h}_{\varepsilon,\delta}(x) \leq (x^\alpha + \delta)\mathbb{1}_{\{[\varepsilon-\delta,\infty)\}}.\end{aligned}$$

As a result, using a combination of [16, Theorem 1.2] and [4, Theorem 1.1], then letting $\delta \downarrow 0$ we conclude that

$$\begin{aligned}\lim_{n\to\infty} &\sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}} \\ &= \sqrt{\frac{2}{\pi\kappa''(\theta)}} c_\varphi(\theta) Z_\infty \, \mathbf{E}\left(\left(\sqrt{\kappa''(\theta)}R_1\right)^\alpha \mathbb{1}_{\left\{\sqrt{\kappa''(\theta)}R_1 \geq \varepsilon\right\}} \right) \quad \text{in probability,}\end{aligned}$$

where $(R_t, t \in [0,1])$ is a Brownian meander. Since R_1 has Rayleigh distribution, we observe that $\mathbf{E}\left((R_1)^\alpha\right) < \infty$. Now, letting $\varepsilon \downarrow 0$, we obtain by monotonicity that

$$\begin{aligned}\lim_{\varepsilon\to 0} \lim_{n\to\infty} &\sum_{|u|=n} g_\varphi(S_u - m_n)\mathbb{1}_{\left\{m_n - S_u \geq \varepsilon n^{1/2}\right\}} \\ &= \sqrt{\frac{2}{\pi\kappa''(\theta)}} c_\varphi(\theta) Z_\infty \, \mathbf{E}\left(\left(\sqrt{\kappa''(\theta)}R_1\right)^\alpha \right) \quad \text{in probability.}\end{aligned}$$

Finally, we set

$$c_2 = \sqrt{\frac{2}{\pi\kappa''(\theta)}} \mathbf{E}\left(\left(\sqrt{\kappa''(\theta)}R_1\right)^\alpha \right) = \sqrt{\frac{2}{\pi\kappa''(\theta)}} \left(2\kappa''(\theta)\right)^{\frac{\alpha}{2}} \Gamma\left(\frac{\alpha}{2} + 1\right).$$

We now conclude, by equation (3.8) that

$$\begin{aligned}\mathbf{E}(e^{-c_2 c_\varphi(\theta) Z_\infty}) &= \lim_{\varepsilon\to 0} \lim_{n\to\infty} \mathbf{E}(e^{-\langle \tau_{-m_n} \widetilde{\mathcal{E}}_n^{(\varepsilon)}, \varphi\rangle}) \\ &= \lim_{n\to\infty} \mathbf{E}(e^{-\langle \tau_{-m_n} \mathcal{E}_n, \varphi\rangle}).\end{aligned}$$

Identifying the Laplace transform of the PPP$(c_2\theta Z_\infty e^{-\theta x}\,\mathrm{d}x)$, the proof of Theorem 1.4 is now complete. □

3.3 BRW above the boundary case

We prove in this section Theorem 1.7 as a consequence of the convergence of the extremal process of the BRW observed by Madaule [17]. Recall that by [17, Theorem 1.1], for all $\varphi \in \mathcal{T}$, we have

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \mathcal{Z}_n, \varphi\rangle}\right) = \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, \varphi\rangle}\right),$$

where $\mathcal{Z}_\infty$ is the limiting extremal process of the BRW, a decorated, randomly shifted Poisson point process with exponential intensity. Moreover, [17, Proposition 1.3] shows that for all $\theta > \theta_0$, we have

$$\lim_{n\to\infty} \mathbf{E}\left(\exp\left(-\sum_{|u|=n} e^{\theta(S_u - m_n)}\right)\right) = \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, \mathrm{e}_\theta\rangle}\right),$$

where $\mathrm{e}_\theta : x \mapsto e^{\theta x}$. Using these two results and assuming (1.9), we show that for all $\varphi \in \mathcal{T}$,

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \mathcal{Z}_n, g_\varphi\rangle}\right) = \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, g_\varphi\rangle}\right),$$

which implies the convergence of the extremal process of the last progeny modified BRW.

Proof of Theorem 1.7. Let $K > 0$, we write χ_K a function in $\mathcal{T}$ such that

$$\mathbb{1}_{[-K,K]} \leq \chi_K \leq \mathbb{1}_{[-2K,2K]}.$$

Let $\varphi \in \mathcal{T}$, using (2.2), we can write for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\left|\mathbf{E}\left(e^{-\langle \tau_{-m_n}\mathcal{E}_n, \varphi\rangle}\right) - \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, g_\varphi\rangle}\right)\right| \\
&= \left|\mathbf{E}\left(e^{-\sum_{|u|=n} g_\varphi(S_u - m_n)}\right) - \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, g_\varphi\rangle}\right)\right| \\
&\leq I_1(n,K) + I_2(n,K) + I_3(K),
\end{aligned}$$

where we have set

$$\begin{aligned}
I_1(n,K) &= \left|\mathbf{E}\left(e^{-\sum_{|u|=n} g_\varphi(S_u - m_n)}\right) - \mathbf{E}\left(e^{-\sum_{|u|=n} \chi_K g_\varphi(S_u - m_n)}\right)\right| \\
I_2(n,K) &= \left|\mathbf{E}\left(e^{-\sum_{|u|=n} \chi_K g_\varphi(S_u - m_n)}\right) - \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, \chi_K g_\varphi\rangle}\right)\right| \\
I_3(K) &= \left|\mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, \chi_K g_\varphi\rangle}\right) - \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, g_\varphi\rangle}\right)\right|.
\end{aligned}$$

Observe that $\chi_K g_\varphi \in \mathcal{T}$. Therefore, by [17, Theorem 1.1], we know that for all $K > 0$, $\lim_{n\to\infty} I_2(n,K) = 0$. Additionally, $\lim_{K\to\infty} I_3(K) = 0$ by Lebesgue's dominated convergence theorem. To complete the proof of Theorem 1.7, we now bound $I_1(n,K)$ in n.

Let $\theta_1 \in (\theta_0, \theta)$, using the inequality $|e^{-a} - e^{-b}| \leq |a - b| \wedge 1$, we have

$$\begin{aligned} I_1(n,K) &\leq \mathbf{E}\left(\left(\sum_{|u|=n} (1-\chi_K) g_\varphi(S_u - m_n)\right) \wedge 1\right) \\ &\leq \mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{|S_u - m_n| \geq K\}}\right) \wedge 1\right) \\ &\leq \mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n \leq -K\}}\right) \wedge 1\right) \\ &\qquad + \mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n \geq K\}}\right) \wedge 1\right), \end{aligned}$$

by sub-additivity of $x \mapsto x \wedge 1$. We first observe that

$$\mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n \geq K\}}\right) \wedge 1\right) \leq \mathbf{P}(M_n - m_n \geq K),$$

which converges to 0 uniformly in n as $K \to \infty$, by tightness of $(M_n - m_n)$. Moreover, using Lemma 2.4, we have

$$\begin{aligned} &\mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n \leq -K\}}\right) \wedge 1\right) \\ \leq\ &\mathbf{E}\left(\left(C' \sum_{|u|=n} e^{\theta(S_u - m_n)} \mathbb{1}_{\{S_u - m_n \leq -K\}}\right) \wedge 1\right) \\ \leq\ &C' \,\mathbf{E}\left(\mathbb{1}_{\left\{\sum_{|u|=n} e^{\theta_1(S_u - m_n)} \leq K\right\}} \sum_{|u|=n} e^{\theta(S_u - m_n)} \mathbb{1}_{\{S_u - m_n \leq -K\}}\right) \\ &\qquad + \mathbf{P}\left(\sum_{|u|=n} e^{\theta_1(S_u - m_n)} \geq K\right), \end{aligned}$$

for an arbitrary $\theta_1 \in (\theta_0, \theta)$. As a result

$$\begin{aligned} &\mathbf{E}\left(\left(\sum_{|u|=n} g_\varphi(S_u - m_n) \mathbb{1}_{\{S_u - m_n \leq -K\}}\right) \wedge 1\right) \\ &\qquad\qquad \leq C' K e^{-K(\theta - \theta_1)} + \mathbf{P}\left(\sum_{|u|=n} e^{\theta_1(S_u - m_n)} \geq K\right). \end{aligned}$$

Using [17, Proposition 1.3], we observe that $\sum_{|u|=n} e^{\theta_1(S_u - m_n)}$ converges in distribution, thus is tight. As a result, we conclude that

$$\lim_{K \to \infty} \sup_{n \in \mathbb{N}} I_1(n,K) = 0.$$

We conclude that

$$\lim_{n\to\infty} \mathbf{E}\left(e^{-\langle \tau_{-m_n}\mathcal{E}_n,\varphi\rangle}\right) = \mathbf{E}\left(e^{-\langle \mathcal{Z}_\infty, g_\varphi\rangle}\right) = \mathbf{E}\left(\exp\left(-\sum_{i\in\mathbb{N}}\varphi(z_i+Y_i)\right)\right)$$

which completes the proof. □

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