Bounded type Siegel disks of finite type maps with few singular values

Arnaud Chéritat[1]
joint with
Adam Epstein[2]

(1): CNRS & Institut de Mathématiques de Toulouse
(2): Warwick University Mathematics Institute

Published in Science China Mathematics
Vol. 61, No. 12, pp. 2139–2156 (Nov. 2018)

Będlewo, September 2021

Terminology

In 1D holomorphic dynamics Siegel disks are maximal domains of conjugacy to a rotation.

Terminology

In 1D holomorphic dynamics Siegel disks are maximal domains of conjugacy to a rotation.

Relatively compact Siegel disks are those which are relatively compact subset of the domain of f.

Terminology

In 1D holomorphic dynamics Siegel disks are maximal domains of conjugacy to a rotation.

Relatively compact Siegel disks are those which are relatively compact subset of the domain of f.

If the angle of the rotation is $2\pi\theta$ radians then $\theta \in \mathbb{R}/\mathbb{Z}$ is called the rotation number.

Terminology

In 1D holomorphic dynamics Siegel disks are maximal domains of conjugacy to a rotation.

Relatively compact Siegel disks are those which are relatively compact subset of the domain of f.

If the angle of the rotation is $2\pi\theta$ radians then $\theta \in \mathbb{R}/\mathbb{Z}$ is called the rotation number.

Expand θ as a continued fraction $1/(a_1 + 1/(a_2 \ldots$, $a_n \in \mathbb{N}^*$. Recall that θ has bounded type means "$\theta \notin \mathbb{Q}$ and (a_n) is bounded".

Finite type maps

Definition (A. Epstein)

Given a Riemann surfaces X, Y with Y *compact*, a finite-type map is a holomorphic map $f : X \to Y$ which is open, whose set of singular values $\operatorname{Sing} f$ is finite, and which has no removable isolated singularity.

Finite type maps

Definition (A. Epstein)

Given a Riemann surfaces X, Y with Y *compact*, a finite-type map is a holomorphic map $f : X \to Y$ which is open, whose set of singular values $\operatorname{Sing} f$ is finite, and which has no removable isolated singularity.

When $X \subset Y$ we can iterate and f^n is also a finite type map.

Finite type maps

Definition (A. Epstein)

Given a Riemann surfaces X, Y with Y *compact*, a finite-type map is a holomorphic map $f : X \to Y$ which is open, whose set of singular values $\operatorname{Sing} f$ is finite, and which has no removable isolated singularity.

When $X \subset Y$ we can iterate and f^n is also a finite type map.

Polynomials, rational maps, transcendental entire maps in the Spieser class, are finite type maps.

Finite type maps

Definition (A. Epstein)

Given a Riemann surfaces X, Y with Y *compact*, a finite-type map is a holomorphic map $f : X \to Y$ which is open, whose set of singular values $\operatorname{Sing} f$ is finite, and which has no removable isolated singularity.

When $X \subset Y$ we can iterate and f^n is also a finite type map.

Polynomials, rational maps, transcendental entire maps in the Spieser class, are finite type maps.

The horn maps (a.k.a. parabolic renormalization) of a parabolic point of a finite type map are also finite type maps.

Finite type maps

Definition (A. Epstein)

Given a Riemann surfaces X, Y with $Y compact$, a finite-type map is a holomorphic map $f : X \to Y$ which is open, whose set of singular values $\mathrm{Sing}\, f$ is finite, and which has no removable isolated singularity.

When $X \subset Y$ we can iterate and f^n is also a finite type map.

Polynomials, rational maps, transcendental entire maps in the Spieser class, are finite type maps.

The horn maps (a.k.a. parabolic renormalization) of a parabolic point of a finite type map are also finite type maps.

Many theorems of Fatou, Julia concerning rational maps still hold for finite type maps, and Sullivan's non-wandering theorem too.

Reminder

If a fixed point of a holomorphic map has multiplier $e^{2\pi i\theta}$ with θ of bounded type then the fixed point has a Siegel disk.

Reminder

If a fixed point of a holomorphic map has multiplier $e^{2\pi i\theta}$ with θ of bounded type then the fixed point has a Siegel disk.

In fact it is true when θ is in the bigger class of *Brjuno numbers*.

Main theorem

Theorem (Chéritat, Epstein)

Let $Y = \widehat{\mathbb{C}}$, $X = U \subset \hat{\mathbb{C}}$, $f : U \to \widehat{\mathbb{C}}$ a finite-type map with

a. $\operatorname{Sing} f \subset \{a, b, c\}$ *for some distinct* $a, b, c \in \widehat{\mathbb{C}}$

b. $a \in U$, *is fixed with multiplier* $e^{2\pi i\theta}$ *and* θ *has bounded type*

c. *either* $c \in \widehat{\mathbb{C}} - U$ *or* $f(c) = c$.

Consider an injective path γ *from* a *to* b *in* $\widehat{\mathbb{C}} - \{a, b, c\}$. *Consider its unique lift* $\tilde{\gamma}$ *by* f *starting from* a. *Then either*

1. $\tilde{\gamma}$ *ends on a non-critical point in* U *and then* $U = \widehat{\mathbb{C}}$ *and* f *is a homography,*
2. $\tilde{\gamma}$ *ends on a critical point in* U *and then* Δ *is a quasidisk whose boundary contains this critical point and no other critical point,*
3. $\tilde{\gamma}$ *leaves every compact of* U *and then* Δ *does not have compact closure in* U.

Context

Boundaries of relatively compact Siegel disks come in many flavours.

1. Jordan curve without a critical point (Ghys, Herman)
2. Smooth (Pérez Marco, Buff-Chéritat, Ávila)
3. Jordan curve with a critical point (Douady, Herman, Świątek)
4. Jordan with any kind of regularity (Buff-Chéritat)
5. Pseudocircles (Herman, Chéritat)

Context

Boundaries of relatively compact Siegel disks come in many flavours.

1. Jordan curve without a critical point (Ghys, Herman)
2. Smooth (Pérez Marco, Buff-Chéritat, Ávila)
3. Jordan curve with a critical point (Douady, Herman, Świątek)
4. Jordan with any kind of regularity (Buff-Chéritat)
5. Pseudocircles (Herman, Chéritat)

1. and 3. thanks to the Ghys *surgery procedure*. (see further slides)

Context

Boundaries of relatively compact Siegel disks come in many flavours.

1. Jordan curve without a critical point (Ghys, Herman)
2. Smooth (Pérez Marco, Buff-Chéritat, Ávila)
3. Jordan curve with a critical point (Herman, Świątek)
4. Jordan with any kind of regularity (Buff-Chéritat)
5. Pseudocircles (Herman, Chéritat)

1. and 3. thanks to the Ghys *surgery procedure*. (see further slides)

Acknlowledgements

These works have been further extended by Shishikura, Petersen, Zakeri, Zhang Gaofei, Roesch, Graczyk, Jones and several other authors.

Works by Buff and Chéritat used Yoccoz renormalization.

Using other kinds of renormalization, there are works by Yamplosky, Shishikura, Inou, Cheraghi and several other authors. New developments with Dzimitri Dudko.

Relatively compact Siegel disk with a critical point on the boundary

Theorem (Graczyk, Świątek 2003)

A relatively compact Siegel disk with bounded type rotation number necessarily has a critical point on its boundary.

The proof uses the normalized Schwarzian derivative of the conjugacy from the rotation to f, area estimates, and properties of univalent functions.

Relatively compact Siegel disk with a critical point on the boundary

Theorem (Graczyk, Świątek 2003)

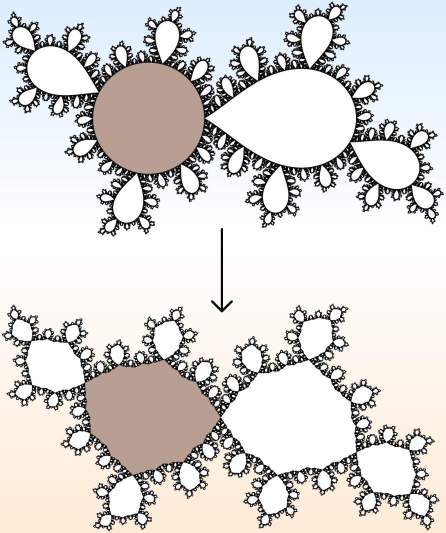
A relatively compact Siegel disk with bounded type rotation number necessarily has a critical point on its boundary.

The proof uses the normalized Schwarzian derivative of the conjugacy from the rotation to f, area estimates, and properties of univalent functions.

Theorem (Douady, Ghys, Herman, Świątek late 1980's)

If $\theta \in \mathbb{R} - \mathbb{Q}$ has bounded type then $z \mapsto e^{2\pi i\theta} z + z^2$ has a Siegel disk whose boundary is a quasicircle going through the critical point.

The Douady-Ghys surgery

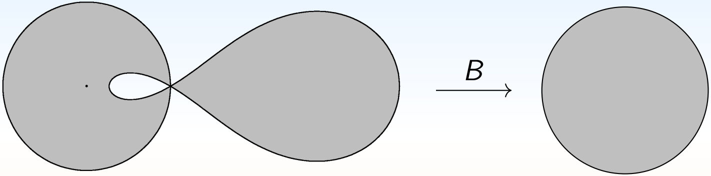


Description

A specific Blaschke *fraction*: $B(z) = z^2 \dfrac{1 - \bar{a}z}{z - a}$, $a = 1/3$.

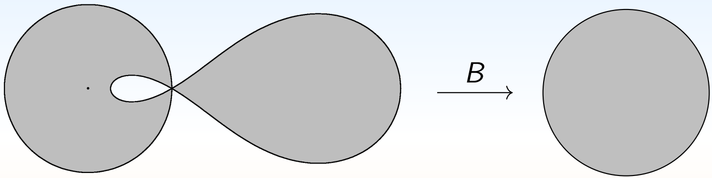
Description

A specific Blaschke *fraction*: $B(z) = z^2 \dfrac{1 - \bar{a}z}{z - a}$, $a = 1/3$.



Description

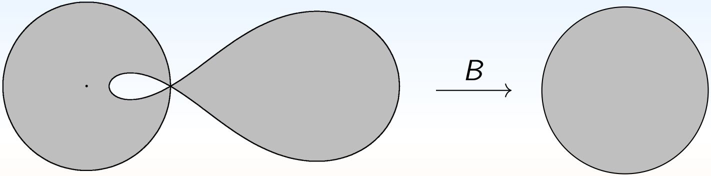
A specific Blaschke *fraction*: $B(z) = z^2 \dfrac{1-\bar{a}z}{z-a}$, $a = 1/3$.



preserves S^1, but *not* $\mathbb{D}$.

Description

A specific Blaschke *fraction*: $B(z) = z^2 \dfrac{1 - \bar{a}z}{z - a}$, $a = 1/3$.

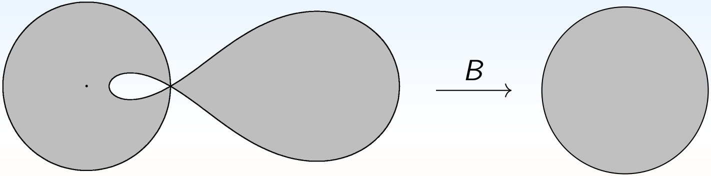


preserves S^1, but *not* $\mathbb{D}$.

On S^1, B is a homeomorphism, but *not* a diffeomorphism: $z = 1$ is a critical point of local degree 3.

Description

A specific Blaschke *fraction*: $B(z) = z^2 \dfrac{1 - \bar{a}z}{z - a}$, $a = 1/3$.



preserves S^1, but *not* $\mathbb{D}$.

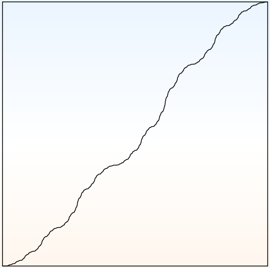
On S^1, B is a homeomorphism, but *not* a diffeomorphism: $z = 1$ is a critical point of local degree 3.

For all $\theta \in \mathbb{R} - \mathbb{Q}$, there exists a unique $\rho \in \partial\mathbb{D}$ such that $\rho.B|_{S_1}$ has rotation number θ mod $\mathbb{Z}$.

For all $\theta \in \mathbb{R} - \mathbb{Q}$, there exists a unique $\rho \in \partial\mathbb{D}$ such that $\rho.B|_{S_1}$ has rotation number θ mod $\mathbb{Z}$.

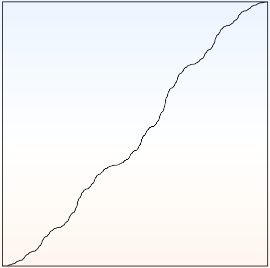
For all $\theta \in \mathbb{R} - \mathbb{Q}$, there exists a unique $\rho \in \partial\mathbb{D}$ such that $\rho.B|_{S_1}$ has rotation number θ mod $\mathbb{Z}$.

(Yoccoz) There exists a *Poincaré conjugacy h* to the rotation $x \mapsto x + \theta$ on $\mathbb{R}/\mathbb{Z}$.



For all $\theta \in \mathbb{R} - \mathbb{Q}$, there exists a unique $\rho \in \partial\mathbb{D}$ such that $\rho.B|_{S_1}$ has rotation number θ mod $\mathbb{Z}$.

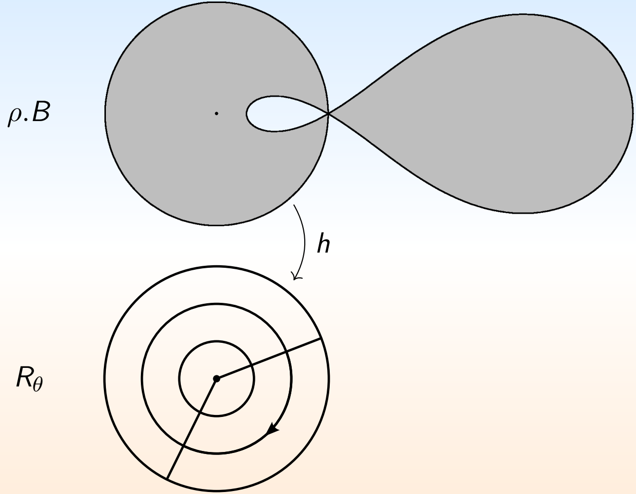
(Yoccoz) There exists a *Poincaré conjugacy h* to the rotation $x \mapsto x + \theta$ on $\mathbb{R}/\mathbb{Z}$.



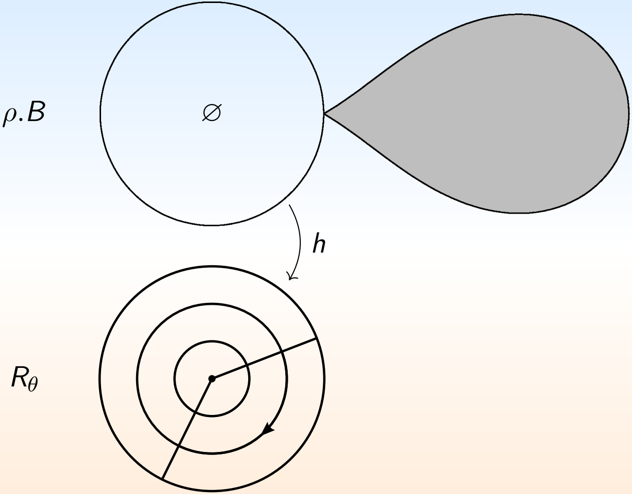
Theorem (Herman, Świątek)

If θ has bounded type, then h is quasisymmetric.

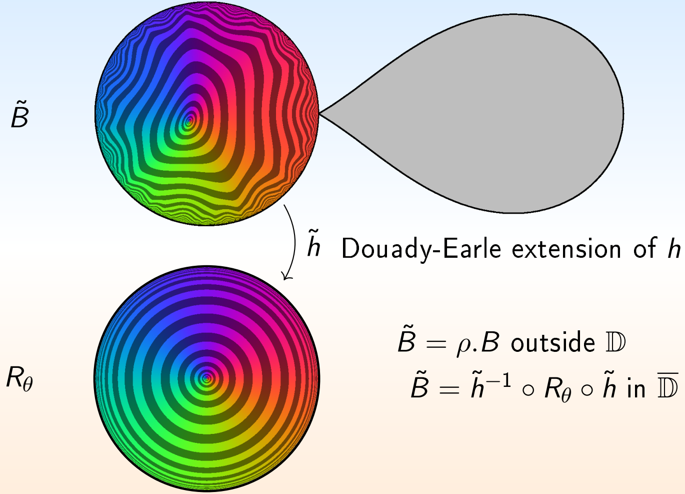
(Ahlfors, Beurling, Douady, Earle) Quasisymmetric orientation preserving maps $h : \partial\mathbb{D} \to \partial\mathbb{D}$ have quasiconformal extensions $\tilde{h}$ to $\mathbb{D}$.



(Ahlfors, Beurling, Douady, Earle) Quasisymmetric orientation preserving maps $h : \partial\mathbb{D} \to \partial\mathbb{D}$ have quasiconformal extensions $\tilde{h}$ to $\mathbb{D}$.



(Ahlfors, Beurling, Douady, Earle) Quasisymmetric orientation preserving maps $h : \partial\mathbb{D} \to \partial\mathbb{D}$ have quasiconformal extensions $\tilde{h}$ to $\mathbb{D}$.

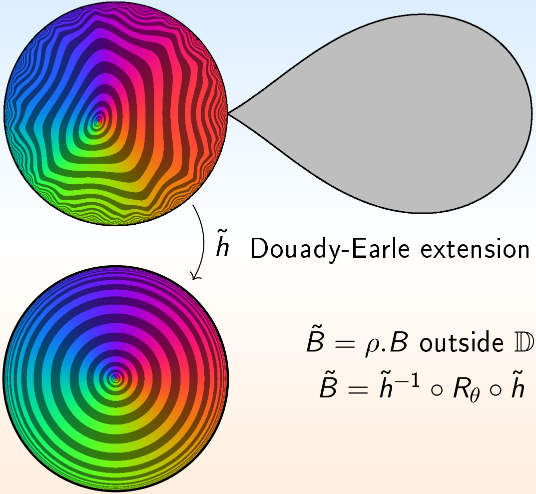


$\tilde{B} = \rho.B$ outside $\mathbb{D}$

$\tilde{B} = \tilde{h}^{-1} \circ R_\theta \circ \tilde{h}$ in $\overline{\mathbb{D}}$

(Ahlfors, Beurling, Douady, Earle) Quasisymmetric orientation preserving maps $h : \partial\mathbb{D} \to \partial\mathbb{D}$ have quasiconformal extensions $\tilde{h}$ to $\mathbb{D}$.

$\tilde{B}$



$\tilde{h}$ Douady-Earle extension of h

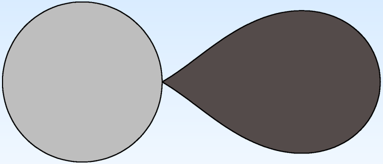
R_θ

$\tilde{B} = \rho.B$ outside $\mathbb{D}$

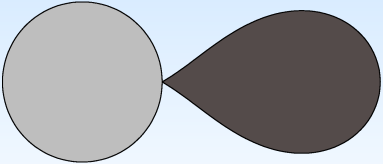
$\tilde{B} = \tilde{h}^{-1} \circ R_\theta \circ \tilde{h}$ in $\overline{\mathbb{D}}$

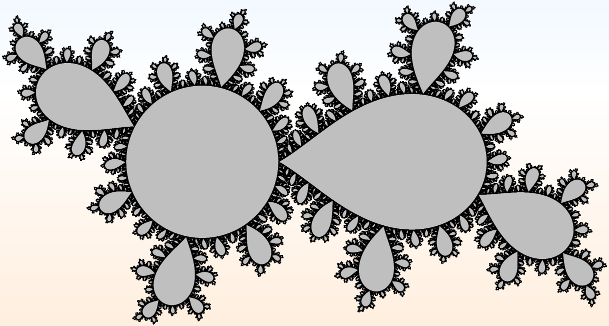
Invariant Beltrami form $\mu = \tilde{h}^*(0)$ in $\mathbb{D}$.

$\tilde{B}$-invariant Beltrami form $\mu = \tilde{h}^*(0)$ on $\mathbb{D}$.



$\tilde{B}$-invariant Beltrami form $\mu = \tilde{h}^*(0)$ on $\mathbb{D}$.





pulled-back by $\tilde{B}$ into a $\tilde{B}$-invariant μ on $\mathbb{C}$.

By Morrey's theorem, there exists a quasiconformal homeomorphism $S : \mathbb{C} \to \mathbb{C}$ straightening μ, i.e. $S_*\mu = 0$.

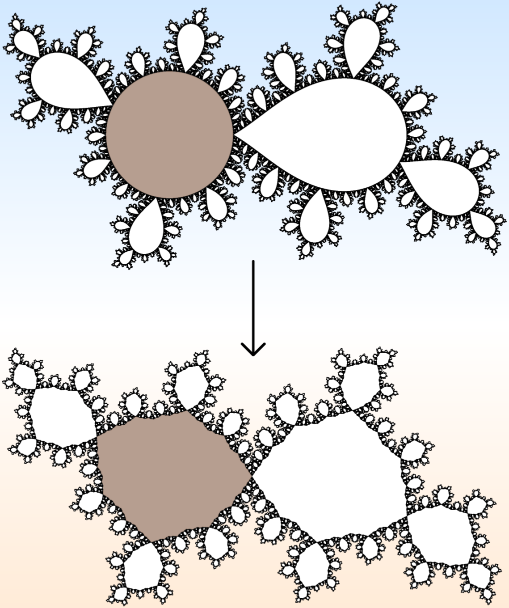
By Morrey's theorem, there exists a quasiconformal homeomorphism $S : \mathbb{C} \to \mathbb{C}$ straightening μ, i.e. $S_*\mu = 0$.

The map $S \circ \tilde{B} \circ S^{-1}$ sends the null Beltrami form to itself: it is thus *holomorphic*.

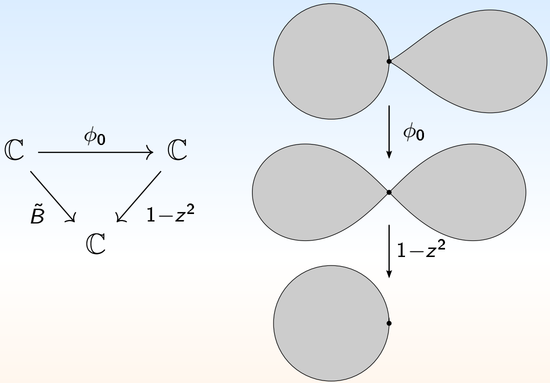
By Morrey's theorem, there exists a quasiconformal homeomorphism $S : \mathbb{C} \to \mathbb{C}$ straightening μ, i.e. $S_* \mu = 0$.

The map $S \circ \tilde{B} \circ S^{-1}$ sends the null Beltrami form to itself: it is thus *holomorphic*.

It can be nothing but $z \mapsto e^{2i\pi\theta} z + z^2$.

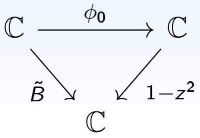


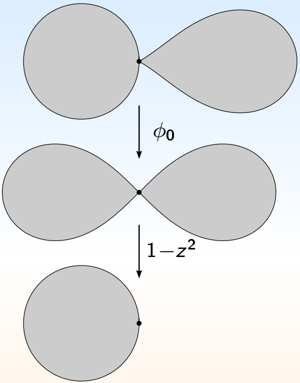
In the Douady-Ghys Surgery, $\tilde{B}$ is cover-equivalent to $z \mapsto z^2$ (eq. $1 - z^2$):



and the q.c. homeo. ϕ_0 is *analytic* outside $\mathbb{D}$.

In the Douady-Ghys Surgery, $\tilde{B}$ is cover-equivalent to $z \mapsto z^2$ (eq. $1 - z^2$):





and the q.c. homeo. ϕ_0 is *analytic* outside $\mathbb{D}$.

B can be recovered from the restriction to $\mathbb{C} - \mathbb{D}$ of $(z \mapsto 1 - z^2) \circ \phi_0$ by *Schwarz reflection*.

Developments

In my thesis (2001): I found a trick to adapt this to horn maps of quadratic polynomials.

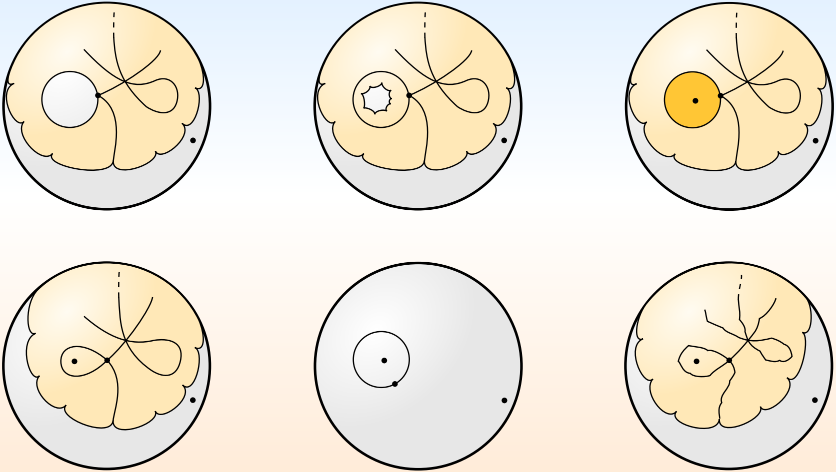
In “Ghys-like model providing trick for Lavaurs and simple entire maps” [Chéritat 2006]: the trick works for a sub class of the set of entire functions with at most two singular values (determined thanks to a discussion with Dierk Schleicher in 2002). Uncountably many examples.

In 2006 I had the honor of having A. Epstein present my proof in Douady’s 70th birthday’s conference. At the end of his lecture, Adam noted that we could simplify the argument. As a consequence: generalization of the above two results to a much bigger class of finite type maps with few singular values.

Philosophy of the method

$$\text{map} \xrightarrow[\text{unif.+refl.}]{} \text{premodel} \xrightarrow[\text{qc rot. in } \mathbb{D}]{} \text{model} \xrightarrow[\text{straighten}]{} \text{(new?) map}$$

Philosophy of the method



The main theorem, again

Theorem (Chéritat, Epstein)

Let $f : U \to \widehat{\mathbb{C}}$ be a finite-type map with $U \subset \hat{\mathbb{C}}$ and

a. $\operatorname{Sing} f \subset \{a, b, c\}$ *for some distinct* $a, b, c \in \widehat{\mathbb{C}}$

b. $a \in U$*, is fixed with multiplier* $e^{2\pi i\theta}$ *and* θ *has bounded type*

c. *either* $c \in \widehat{\mathbb{C}} - U$ *or* $f(c) = c$.

Consider an injective path γ *from* a *to* b *in* $\widehat{\mathbb{C}} - \{a, b, c\}$. *Consider its unique lift* $\tilde{\gamma}$ *by* f *starting from* a. *Then either*

1. $\tilde{\gamma}$ *ends on a non-critical point in* U *and then* $U = \widehat{\mathbb{C}}$ *and* f *is a homography,*
2. $\tilde{\gamma}$ *ends on a critical point in* U *and then* Δ *is a quasidisk whose boundary contains this critical point and no other critical point,*
3. $\tilde{\gamma}$ *leaves every compact of* U *and then* Δ *does not have compact closure in* U.

Note

In the theorem: the injective path γ goes from a to b in $\widehat{\mathbb{C}} - \{a, b, c\}$ and $\tilde{\gamma}$ is its unique lift by f starting from a.

The endpoint of $\tilde{\gamma}$ in the Alexandrov compactification of U (a point of U or the leave-every-compact-point) is independent of the choice of γ.

Proof of the main theorem

w.l.o.g. $a = 0$, $b = 1$, $c = \infty$

Proof of the main theorem

w.l.o.g. $a = 0$, $b = 1$, $c = \infty$

$\mathbb{D}$ has a unique pull-back D_0 that contains 0 and $f : D_0 \to \mathbb{D}$ is a bijection

Proof of the main theorem

w.l.o.g. $a = 0$, $b = 1$, $c = \infty$

$\mathbb{D}$ has a unique pull-back D_0 that contains 0 and $f : D_0 \to \mathbb{D}$ is a bijection

Case 1. $\tilde{\gamma}$ ends on a non-critical point in U. [...]

Proof of the main theorem

w.l.o.g. $a = 0$, $b = 1$, $c = \infty$

$\mathbb{D}$ has a unique pull-back D_0 that contains 0 and $f : D_0 \to \mathbb{D}$ is a bijection

Case 1. $\tilde{\gamma}$ ends on a non-critical point in U. [...]

Case 2. $\tilde{\gamma}$ ends on a critical point in U.

Proof of the main theorem

w.l.o.g. $a = 0$, $b = 1$, $c = \infty$

$\mathbb{D}$ has a unique pull-back D_0 that contains 0 and $f : D_0 \to \mathbb{D}$ is a bijection

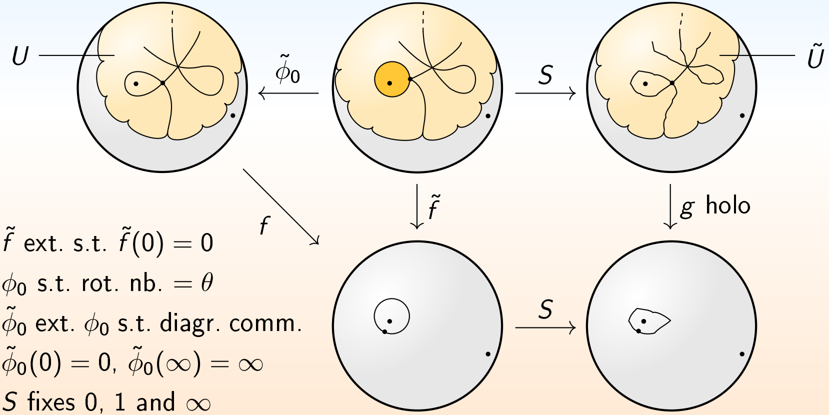
Case 1. $\tilde{\gamma}$ ends on a non-critical point in U. [...]

Case 2. $\tilde{\gamma}$ ends on a critical point in U. Then D_0 is a tear.

Case 2.

Perform the surgery and get a model map $\tilde{f}$ with an invariant Beltrami form $\tilde{\mu}$ defined on $\hat{\mathbb{C}}$. Let $\tilde{U} = \operatorname{dom} \tilde{f}$.

Objective: prove that the domain of the resulting map is the same, i.e. $\tilde{U} = U$.



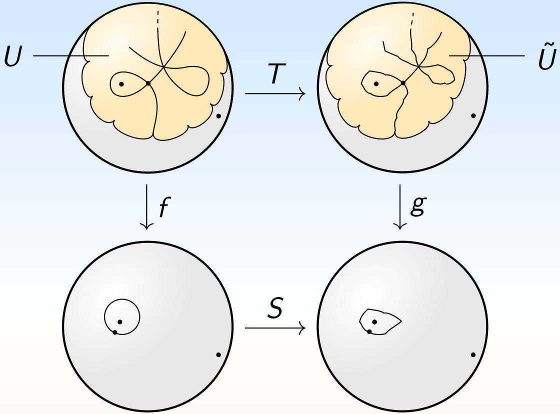
$\tilde{f}$ ext. s.t. $\tilde{f}(0) = 0$

ϕ_0 s.t. rot. nb. $= \theta$

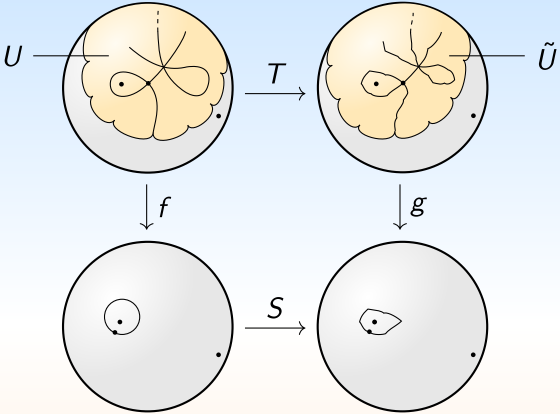
$\tilde{\phi}_0$ ext. ϕ_0 s.t. diagr. comm.

$\tilde{\phi}_0(0) = 0$, $\tilde{\phi}_0(\infty) = \infty$

S fixes 0, 1 and ∞



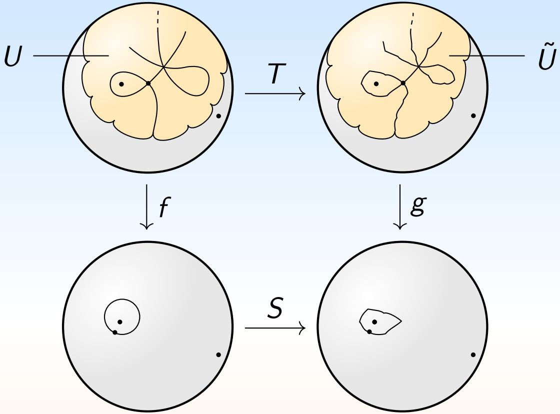
with $T = S \circ \tilde{\phi}_0^{-1}$.



with $T = S \circ \tilde{\phi}_0^{-1}$.

$\bar{\partial} S = 0$ on $\hat{\mathbb{C}} - \tilde{U}$ and $\tilde{\phi}_0$ is conformal outside $\mathbb{D}$ hence

$$\bar{\partial} T = 0 \text{ on } \hat{\mathbb{C}} - U$$



with $T = S \circ \tilde{\phi}_0^{-1}$.

$\bar{\partial} S = 0$ on $\hat{\mathbb{C}} - \tilde{U}$ and $\tilde{\phi}_0$ is conformal outside $\mathbb{D}$ hence

$$\bar{\partial} T = 0 \text{ on } \hat{\mathbb{C}} - U$$

$T(0) = 0, \; T(\infty) = \infty$

T is a q.c. homeo of $\hat{\mathbb{C}}$, its restriction from U to $\tilde{U}$ is a lift of S.

$$\begin{array}{ccc} U & \xrightarrow{\ T\ } & \tilde{U} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[\ S\ ]{} & \hat{\mathbb{C}} \end{array}$$

T is a q.c. homeo of $\hat{\mathbb{C}}$, its restriction from U to $\tilde{U}$ is a lift of S.

$$\begin{array}{ccc} U & \xrightarrow{T} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S]{} & \hat{\mathbb{C}} \end{array}$$

Let $\mu_t = t.\mu$. Straighten μ_t by the unique q.c. isomorphism S_t of $\hat{\mathbb{C}}$ fixing a, b and c. Note that $t \mapsto S_t$ is an isotopy and $S_0 = \mathrm{id}$. Lift this isotopy starting from $t = 1$ down to $t = 0$ as a family of homeomorphisms $T_t : U \to \tilde{U}$ starting from $T_1 = T$:

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

Note that $t \mapsto T_t$ is the restriction to $t \in [0, 1]$ of a holomorphic motion. Also, T_t maps $f^{-1}\{a, b, c\}$ to $g^{-1}\{a, b, c\}$ and is immobile on this set. In particular $T_t(0) = 0$.

Specializing to $t = 0$, using $S_0 = \mathrm{id}$:

$$\begin{array}{ccc} U & \xrightarrow{T_0} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[\mathrm{id}]{} & \hat{\mathbb{C}} \end{array}$$

in particular T_0 is holomorphic.

$$
\begin{array}{ccccc}
\tilde{U} & \xrightarrow{T^{-1}} & U & \xrightarrow{T_t} & \tilde{U} \\
\big\downarrow g & & f \big\downarrow & & \big\downarrow g \\
\hat{\mathbb{C}} & \xrightarrow[S^{-1}]{} & \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}}
\end{array}
$$

The family $t \mapsto T_t \circ T^{-1} : \tilde{U} \to \tilde{U}$ is also a holomorphic motion and is the identity of $\tilde{U}$ for $t = 1$.

$$\begin{array}{ccccc} \tilde U & \xrightarrow{T^{-1}} & U & \xrightarrow{T_t} & \tilde U \\ \downarrow g & & f \downarrow & & \downarrow g \\ \hat{\mathbb{C}} & \xrightarrow[S^{-1}]{} & \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

The family $t \mapsto T_t \circ T^{-1} : \tilde U \to \tilde U$ is also a holomorphic motion and is the identity of $\tilde U$ for $t = 1$.

Use T to extend T_t to $\tilde T_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\tilde T_t(z) = T(z)$ when $z \notin \tilde U$. Then $\tilde T_t \circ T^{-1}$ is a holomorphic motion of $\hat{\mathbb{C}}$.

$$
\begin{array}{ccccc}
\tilde{U} & \xrightarrow{T^{-1}} & U & \xrightarrow{T_t} & \tilde{U} \\
\downarrow g & & f \downarrow & & \downarrow g \\
\hat{\mathbb{C}} & \xrightarrow[S^{-1}]{} & \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}}
\end{array}
$$

The family $t \mapsto T_t \circ T^{-1} : \tilde{U} \to \tilde{U}$ is also a holomorphic motion and is the identity of $\tilde{U}$ for $t = 1$.

Use T to extend T_t to $\tilde{T}_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by setting $\tilde{T}_t(z) = T(z)$ when $z \notin \tilde{U}$. Then $\tilde{T}_t \circ T^{-1}$ is a holomorphic motion of $\hat{\mathbb{C}}$.

The map $\tilde{T}_0$ is quasiconformal, is holomorphic in U. Moreover on $\hat{\mathbb{C}} - U$ we have $T_0 = T$ and $\bar{\partial} T = 0$. Rickman's lemma implies $\tilde{T}_0$ is holomorphic everywhere, and hence $\tilde{T}_0$ *is a homography*.

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

$\tilde{T}_0$ fixes 0

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

$\tilde{T}_0$ fixes 0

For rotation number reasons, $f'(0) = g'(0)$ hence $\tilde{T}_0'(0) = 1$.

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

$\tilde{T}_0$ fixes 0

For rotation number reasons, $f'(0) = g'(0)$ hence $\tilde{T}_0'(0) = 1$.

- Case 1: $\infty \notin U$. Since $\tilde{T}_0 = T$ on $\hat{\mathbb{C}} - U$, and $T(\infty) = \infty$, we get $\tilde{T}_0(\infty) = \infty$.

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

$\tilde{T}_0$ fixes 0

For rotation number reasons, $f'(0) = g'(0)$ hence $\tilde{T}_0'(0) = 1$.

- Case 1: $\infty \notin U$. Since $\tilde{T}_0 = T$ on $\hat{\mathbb{C}} - U$, and $T(\infty) = \infty$, we get $\tilde{T}_0(\infty) = \infty$.
- Case 2: $f(\infty) = \infty$. Since $T_t(z)$ is independent of t when $z \in f^{-1}\{0, 1, \infty\}$, we get $\tilde{T}_0(\infty) = T_0(\infty) = T_1(\infty) = \infty$.

$$f = g \circ \tilde{T}_0$$

$\tilde{T}_0$ is a homography.

$$\begin{array}{ccc} U & \xrightarrow{T_t} & \tilde{U} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{\mathbb{C}} & \xrightarrow[S_t]{} & \hat{\mathbb{C}} \end{array}$$

$\tilde{T}_0$ fixes 0

For rotation number reasons, $f'(0) = g'(0)$ hence $\tilde{T}_0'(0) = 1$.

- Case 1: $\infty \notin U$. Since $\tilde{T}_0 = T$ on $\hat{\mathbb{C}} - U$, and $T(\infty) = \infty$, we get $\tilde{T}_0(\infty) = \infty$.
- Case 2: $f(\infty) = \infty$. Since $T_t(z)$ is independent of t when $z \in f^{-1}\{0, 1, \infty\}$, we get $\tilde{T}_0(\infty) = T_0(\infty) = T_1(\infty) = \infty$.
- Other cases: not covered by our theorem.

Finally: $\tilde{T}_0 = \mathrm{id}$ i.e. $g = f$. Q.E.D.

Case 3

Case 3. $\tilde{\gamma}$ eventually leaves every compact subset of U.

We proceed by contradiction and assume Δ compactly contained in U.

Case 3

Case 3. $\tilde{\gamma}$ eventually leaves every compact subset of U.

We proceed by contradiction and assume Δ compactly contained in U.

We then use Graczyk-Świątek : there must be a critical point $p \in \partial\Delta$.

Case 3

Case 3. $\tilde{\gamma}$ eventually leaves every compact subset of U.

We proceed by contradiction and assume Δ compactly contained in U.

We then use Graczyk-Świątek : there must be a critical point $p \in \partial\Delta$.

Case 3.1: $c \notin U$.

$f(p) \in \{a, b, c\} \cap \partial\Delta$, hence $f(p) = b$

Case 3

Case 3. $\tilde{\gamma}$ eventually leaves every compact subset of U.

We proceed by contradiction and assume Δ compactly contained in U.

We then use Graczyk-Świątek : there must be a critical point $p \in \partial\Delta$.

Case 3.1: $c \notin U$.

$f(p) \in \{a, b, c\} \cap \partial\Delta$, hence $f(p) = b$

One can then construct an injective path from a to b whose lift ends on p.

Case 3.1: $f(c) = c$.

Case 3.1: $f(c) = c$.

There are finitely many critical points $p_1, p_2, \ldots, p_n$ on $\partial\Delta$.

By an injective path construction as above, $b \notin \partial\Delta$. As a consequence, all critical points map to c in one iteration. The point c cannot be critical otherwise its basin would separate c from U. Similarly, it cannot be attracting or Siegel.

Case 3.1: $f(c) = c$.

There are finitely many critical points $p_1, p_2, \dots, p_n$ on $\partial\Delta$.

By an injective path construction as above, $b \notin \partial\Delta$. As a consequence, all critical points map to c in one iteration. The point c cannot be critical otherwise its basin would separate c from U. Similarly, it cannot be attracting or Siegel.

We then use the technique of Graczyk and Świątek, using an area form that is infinite and cylindrical $\omega = \left(\sum_k \frac{1}{|z-p_k|^2}\right) dx \wedge dy$ in place of the Lebesgue measure.

Case 3.1: $f(c) = c$.

There are finitely many critical points $p_1, p_2, \ldots, p_n$ on $\partial\Delta$.

By an injective path construction as above, $b \notin \partial\Delta$. As a consequence, all critical points map to c in one iteration. The point c cannot be critical otherwise its basin would separate c from U. Similarly, it cannot be attracting or Siegel.

We then use the technique of Graczyk and Świątek, using an area form that is infinite and cylindrical $\omega = \left(\sum_k \frac{1}{|z-p_k|^2}\right) dx \wedge dy$ in place of the Lebesgue measure.

We finally prove that the area of Δ w.r.t. ω is *finite*, in which case the technique of Graczyk and Świątek prove $\partial\Delta$ is locally connected, which prevents the existence of a fixed point thereon, leading to a contradiction with $f(c) = c$.

Case 3.1: $f(c) = c$.

There are finitely many critical points $p_1, p_2, \ldots, p_n$ on $\partial\Delta$.

By an injective path construction as above, $b \notin \partial\Delta$. As a consequence, all critical points map to c in one iteration. The point c cannot be critical otherwise its basin would separate c from U. Similarly, it cannot be attracting or Siegel.

We then use the technique of Graczyk and Świątek, using an area form that is infinite and cylindrical $\omega = \left(\sum_k \frac{1}{|z-p_k|^2}\right) dx \wedge dy$ in place of the Lebesgue measure.

We finally prove that the area of Δ w.r.t. ω is *finite*, in which case the technique of Graczyk and Świątek prove $\partial\Delta$ is locally connected, which prevents the existence of a fixed point thereon, leading to a contradiction with $f(c) = c$.

The proof of finiteness splits in three case, according to the fixed point c being parabolic, repelling or Cremer.

Thanks