Using similarity surfaces to straighten the square and to reprove Ahlfors-Bers

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The Beltrami equation

Problem: Given $\mu \in L^\infty(\mathbb{C})$ complex valued with $\|\mu\|_\infty < 1$, find a homeomorphism of the plane whose distribution derivatives are locally L^2 and such that

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This known nowadays as the *Measurable Riemann Mapping Theorem*.

Ellipse field

The ellipses encoded by some $\mu \in \mathbb{D}$ have equation $|z + \mu \times \bar{z}| = r$. They have ratio

$$\frac{\text{major axis}}{\text{minor axis}} = K = \frac{1+|\mu|}{1-|\mu|}$$

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The Beltrami equation is usually used to change the complex structure of a space or a subset.

Solving

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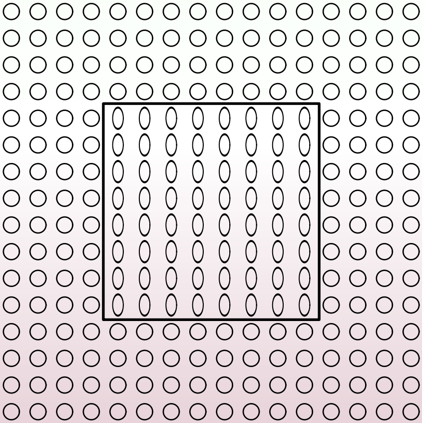
It is quite uncommon that there is a closed formula for a solution, even when μ is simple.

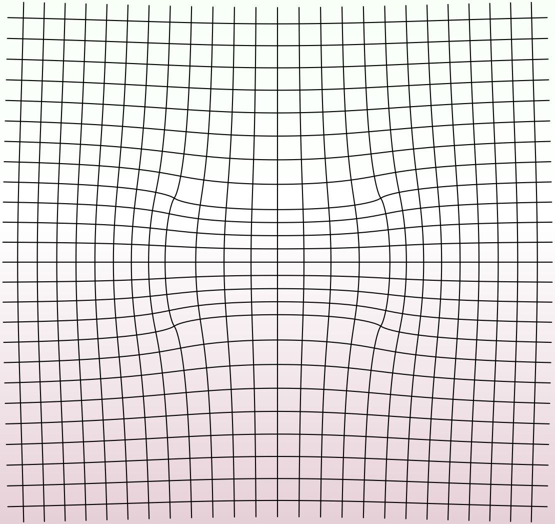
A case study

Setting: an ellipse field on the plane that is constant in a square ($\mu = \frac{K-1}{K+1}$), and circles outside ($\mu = 0$).



Numerically solving a modified Laplacian

$K = 2$



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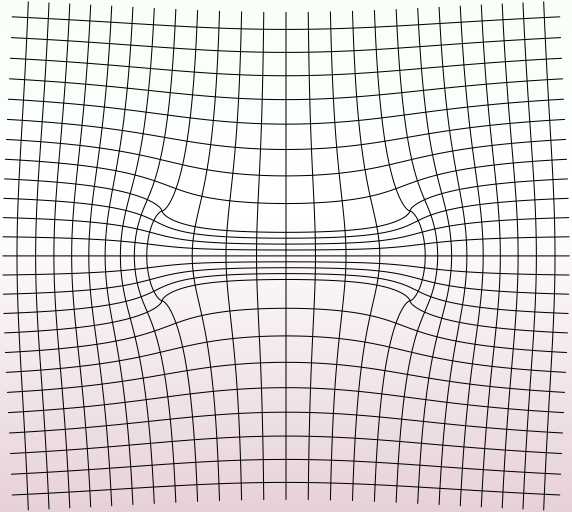

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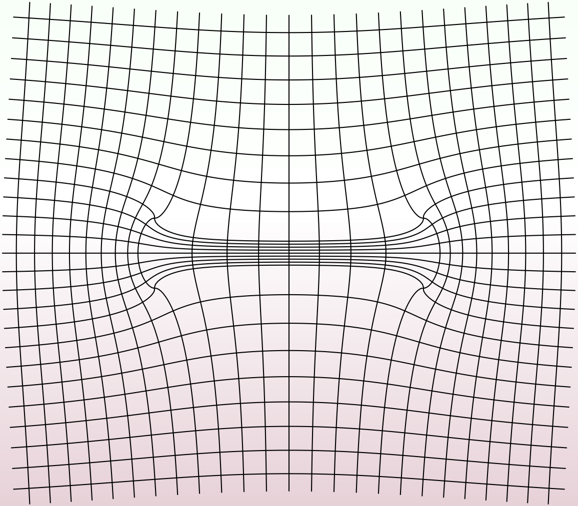
$K = 10$



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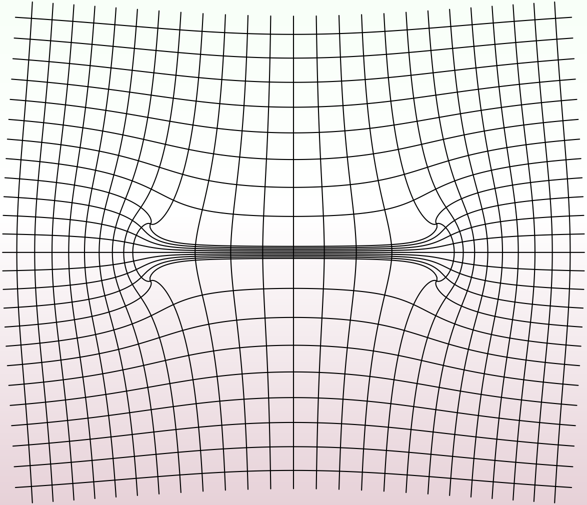
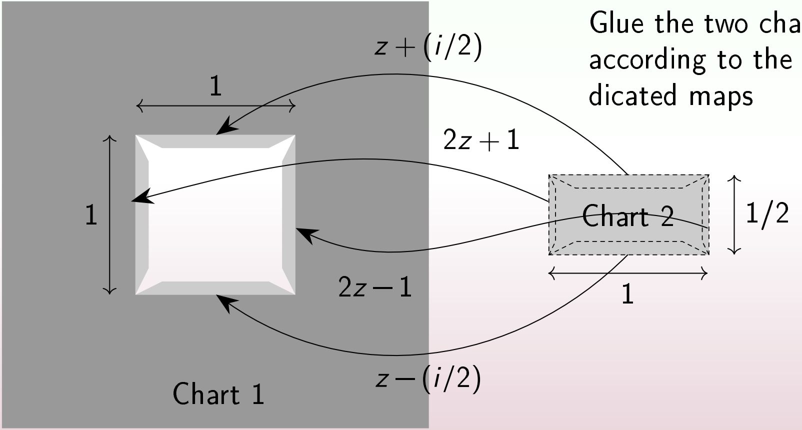


The square

Reformulation into an equivalent problem



Glue the two charts according to the indicated maps

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. . . a locally trivial parallel transport.

`Click here to run applet`

Uniformization theorem

Theorem: (Poincaré, Koebe) *A Riemann surface that is homeomorphic to a sphere is necessarily conformally equivalent* $\hat{\mathbb{C}}$.

In our case, we can complete our gluing by adding 5 points, one at infinity, four at the corners, and 5 Riemann charts near these points.

Completing the Riemann surface

1. Near ∞, the map $z \mapsto 1/z$ gives a local Riemann surface chart (exactly like the *Riemann sphere*).

2. Near a corner, we can glue one side of the rectangle to one side of the square and are left with the following local picture: a slit plane where one side of the slit is glued to the other side by a homothety of ratio K.

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Then the map

$$z \mapsto z^{\alpha}, \qquad \alpha = \frac{2\pi i}{2\pi i \pm \log K}$$

is a local Riemann surface chart: in particular it glues each side of slit exactly according to the required homothety.

`Click here to run applet`

A cultural remark

M.C. Escher's lithography: Print Gallery (1956)

Source: de Smit and Lenstra, Notices of the AMS.

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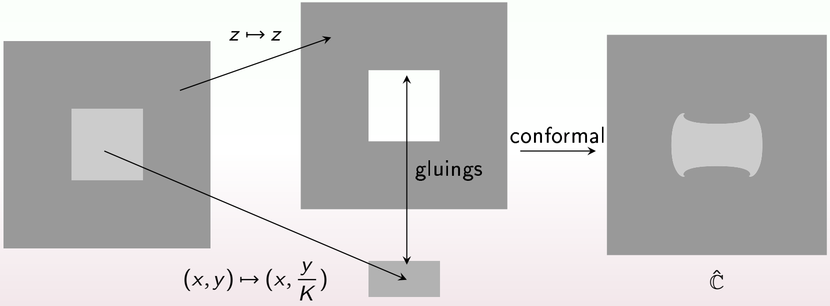

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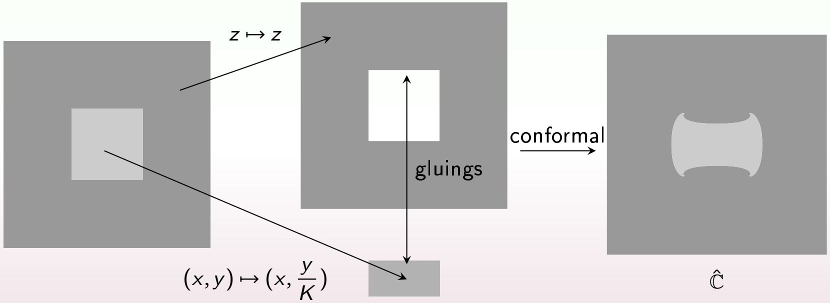

But usually, finding the explicit uniformization of abstract Riemann surfaces is a very hard problem. So why does it help us here?

polygon decomp. of $\mathbb{C}$ $\xrightarrow{\text{similarities}}$ polygons: sim. charts $\xrightarrow{\text{gluing}}$ $\mathscr{S}$ sim. surface (abstract) $\xrightarrow{\text{unif.}^\circ}$ $\mathbb{C} - \{z_1, z_2, z_3, z_4\}$

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We can consider $\mathbb{C}-\{z_1,\ldots,z_4\}$ as a Riemann chart of $\mathscr{S}$ but it is not a sim-chart. The change of coordinates from this chart to the sim-charts are holomorphic functions $\phi : U \to \mathbb{C}$ with $U \subset \mathbb{C}-\{z_1,\ldots,z_4\}$.

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Two such sim-charts, ϕ_1, ϕ_2 satisfy locally on $U_1 \cap U_2$:

$$\phi_1 = a\phi_2 + b$$

for some constants a, b. Hence

$$\boxed{\frac{\phi_2''}{\phi_2'} = \frac{\phi_1''}{\phi_1'}}$$

It follows that there exists a global holomorphic function

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Note: (differential geometry viewpoint) the function η is the expression* of a holomorphic and locally flat conformal *connection*.

(*) a.k.a. a Christoffel symbol.

Example

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Then $\phi''/\phi' = \frac{\frac{1}{\alpha}-1}{z}$ and the Christoffel symbol on the uniformization $\mathbb{C}^*$ has the extremely simple expression

$$\eta(z) = \frac{\log K}{2\pi i} \cdot \frac{1}{z}.$$

Analyzing η at the singularities

Change of variable for the connection: if one expresses η in two Riemann charts C_1 and C_2 with change of coordinates ψ between them, then the expressions η_1 and η_2 in the respective charts are related by:

$$\eta_1 = \psi' \times \eta_2 \circ \psi + \frac{\psi''}{\psi'} \ . \tag{1}$$

(It is almost like a differential form).

Solution

As a consequence:

- η has a simple pole at z_k with residue $\pm\log(K)/2\pi i$.
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$$\eta = \frac{\log K}{2\pi i} \cdot \left(\frac{-1}{z - z_1} + \frac{1}{z - z_2} + \frac{-1}{z - z_3} + \frac{1}{z - z_4} \right)$$

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The conformal map sought for is locally the *inverse mapping* of ϕ (for appropriate choices of the integration constants a, b).

The Schwarz-Christoffel formula

The formula we found

$$a+b\int\left(\frac{z-z_2}{z-z_4}\cdot\frac{z-z_1}{z-z_3}\right)^{\frac{\log K}{2\pi i}}dz$$

is an analogue of the Schwarz-Christoffel formula that gives an expression of the conformal map from the upper half plane to any polygon in the plane: for an n-gon with angles $\alpha_k \in (0,2\pi)$, there exists real numbers $x_1,\ldots,x_n$ such that

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The x_i are mapped to the vertices of the polygon. They can be hard to determine: each depends on all the angles and the length of all sides of the polygon. This is called the *parameter problem*.

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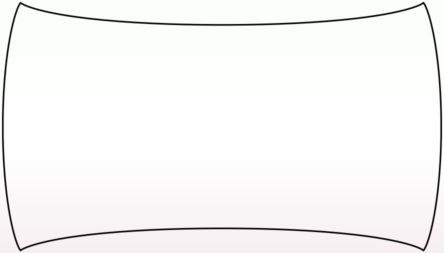

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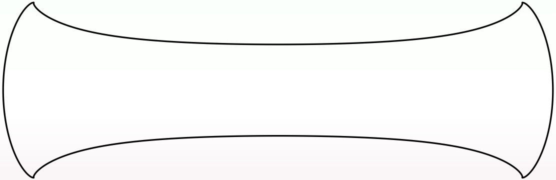
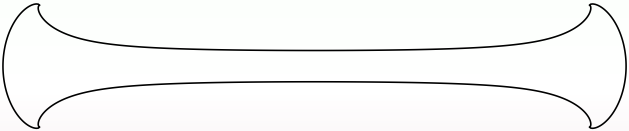
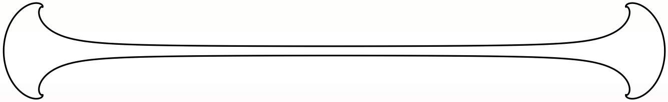


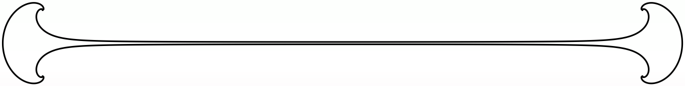
$K = 15$



$K = 50$



$K = 200$



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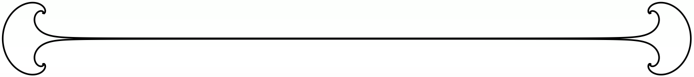
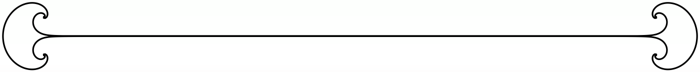


$K = 10^4$



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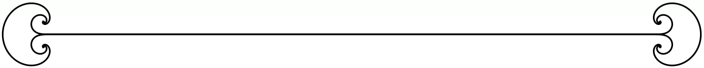
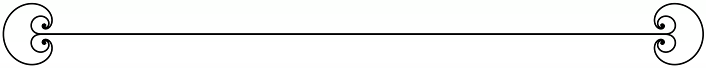


$K = 10^9$



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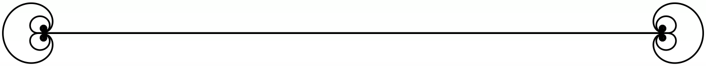

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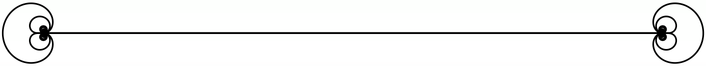

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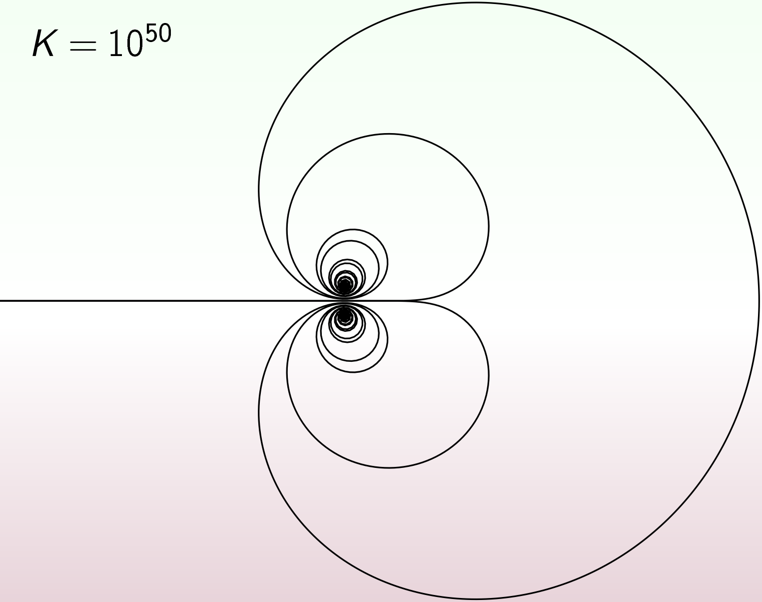
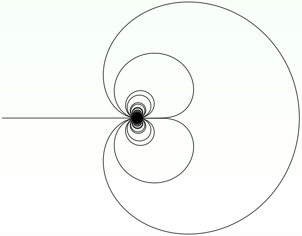


The limit

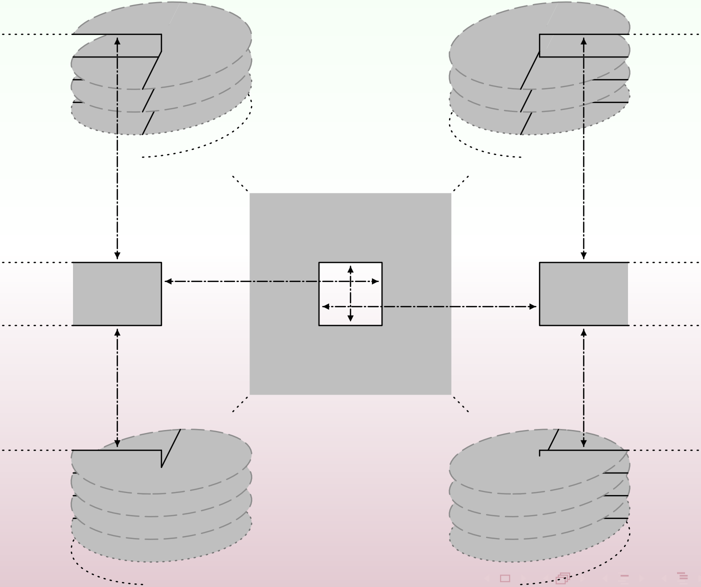


As $K \longrightarrow +\infty$ we see a limit shape and can prove

$$\eta_K \longrightarrow \eta_\infty = \frac{\sigma_0}{(z-x_0)^2} - \frac{\sigma_0}{(z+x_0)^2}$$

This limit shape also has an interpretation in terms of similarity surfaces:

The limit



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Theorem (Ahlfors-Bers)

The normalized solution of the Beltrami equation depends holomorphically on μ.

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Theorem (Ahlfors-Bers)

The normalized solution of the Beltrami equation depends holomorphically on μ.

What is meant: if $\mu[\tau](z)$ depends holomorphically on $\tau \in \mathbb{D}$ for all $z \in \mathbb{C}$ then $f[\tau](z)$ also does.

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The one we propose is based on the Poincaré-Koebe theorem, of which there are softer proofs.

An approach without the AB operator

Inspired by Lavrentiev's

We consider the square $|\operatorname{Im} z| < n$, $|\operatorname{Re} z| < n$ and divide it into litte squares of side $1/n$, totalling $(2n^2)^2$ squares.

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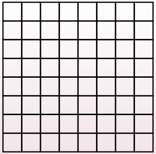

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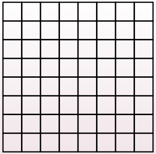

Then $\|\mu_n\|_\infty \leq \|\mu\|_\infty$ and for all $R > 0$, $\int_{B(0,R)} |\mu - \mu_n| \longrightarrow 0$.

Bounds

MRMT: Measurable Riemann Mapping Theorem (Morrey's theorem)

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Except Gauss's, all proofs of the MRMT and its variants use bounds on quasiconformal homeomorphisms or similar bounds. Ours is no exception.

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Except Gauss's, all proofs of the MRMT and its variants use bounds on quasiconformal homeomorphisms or similar bounds. Ours is no exception.

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Bounds: One can also prove without the MRMT that:

- the set of normalized K-quasiconformal homeomorphisms for a fixed K forms a normal family;
- there is a locally uniform bound on the L^2 norm of their first-order distribution partial derivatives.

Recall that:

- K-quasiconformal homeomorphisms form a normal family, and $\int_{B(0,R)} \left|\frac{\partial f}{\partial z}\right|^2 + \left|\frac{\partial f}{\partial \bar{z}}\right|^2 \leq C(K,R)$,
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More generaly:

- To reprove MRMT it is enough to have any dense sub-family of μ for which we know a solution f.
- To reprove Ahlfors-Bers it is enough to have any dense family of holomorphic maps $\tau \in \mathbb{D} \mapsto \mu[\tau]$ for which we know a solution $f[\tau]$ that depends holomorphically on τ.

Consider the following propositions:

A : $\mu[\tau]$ depends holomorphically on the parameter

B : $\mu_n[\tau]$ depends holomorphically on the parameter

C : $f_n[\tau]$ depends holomorphically on the parameter

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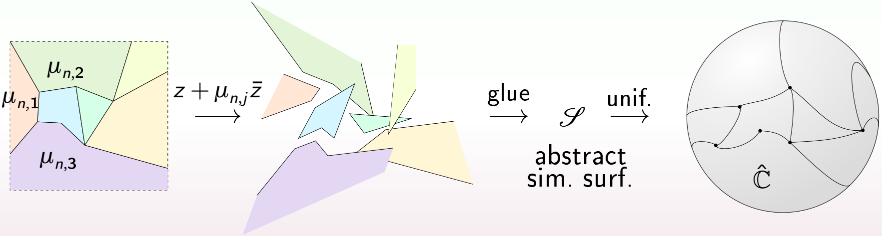
The idea is to use the fact that the Schwarz-Christoffel formula depends holomorphically on the affix and residues of the poles and recompose the values of μ_n on each little square using the formula and use an inversion principle.

A first generalization

Holomorphic dependence of the straightening of Beltrami forms constant on a fixed finite polygonal subdivision of $\mathbb{C}$



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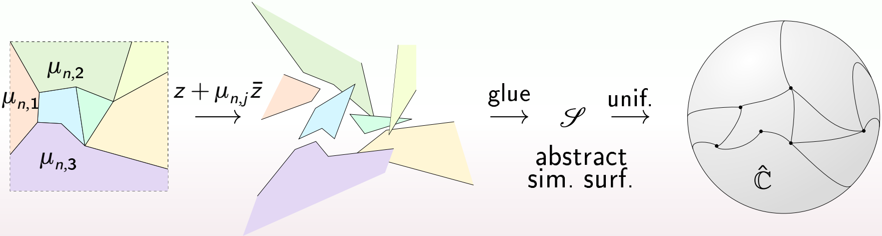
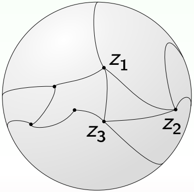


Note that the affix of the vertices of the polygons in the second frame vary holomorphically with μ.

Holo. dep. on the polygons

of the Christoffel symbol on $\mathbb{C}$ given by of a finite polygonal gluing



On the Riemann sphere in the last frame there is also a meromorphic Christoffel symbol, of expression $\zeta = \sum \frac{\mathrm{res}_k}{z - z_k}$ in the canonical chart $\mathbb{C}$, such that the similarity charts are locally the holomorphic solutions ϕ of $\phi''/\phi' = \zeta$.

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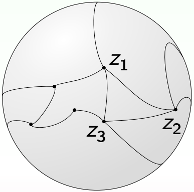

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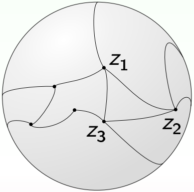

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Once the lemma is proved, the implication $B \implies C$ follows easily, which will complete our proof of the Ahlfors-Bers theorem. But the lemma itself is not that easy to prove.

Generalization of the lemma

We allow ourselves any finite collection of bounded or unbounded polygons with finitely many sides in $\mathbb{C}$, that we glue affinely along a chosen pairing of bounded sides and of unbounded sides. We fix this combinatorics.

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Up to refining we may assume that all polygons are strictly convex, and that unbounded polygons have at least one bounded vertex.

Problem: the affine map gluing unbounded sides is not unique. To recover uniqueness we add *marked points* on the unbounded sides and require the gluing to match them.

Generalization of the lemma

Let z_k be the image in $\hat{\mathbb{C}}$ of the vertices and marked points. We normalize by $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$.

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Again, the point is to prove the lemma without Ahlfors-Bers.

Notations

There are p polygons P_j, each has s_j sides.

Let S-Conf be the set of collections of p strictly convex polygons P_j with s_j sides each considered up to the action of the $\mathbb{C}$-affine group $\operatorname{Aff}\mathbb{C}$.

Let m be the total number of vertices and marked points in the similarity surface $\mathscr{S}$ (after gluing).

To a polygon configuration $\mathscr{P} \in$ S-Conf the construction associates a similarity surface $\mathscr{S}$ and a Christoffel symbol ζ on $\hat{\mathbb{C}}$, normalized so that $z_1 = \infty$, $z_2 = 0$, $z_3 = 1$. The data of ζ is equivalent to the data of the z_k and res_k.

If we decide for a basepoint in $\mathscr{P}_0 \in$ S-Conf, the construction allows to also define a point in the Teichmüller space of $\hat{\mathbb{C}}$ with m marked points.

Let $\mathrm{Conf} = \prod_{j=1}^{p} (\mathbb{C}^{s_j} / \mathrm{Aff}\,\mathbb{C})$.

Let $\mathrm{Conf}^* = \prod_{j=1}^{p} ((\mathbb{C}^{s_j} - \Delta) / \mathrm{Aff}\,\mathbb{C})$ where $\Delta = \left\{ (z, \ldots, z) \,\middle|\, z \in \mathbb{C} \right\}$.

$$\text{S-Conf} \subset \mathrm{Conf}^* \subset \mathrm{Conf}$$

Conf^* is a complex manifold by Conf^* is not.

Let $\mathrm{TR} = \left\{ ([\phi], (\mathrm{res}_k)) \right\} \subset \mathrm{Teich} \times \mathbb{C}^m$ so that the res_k are 0 if z_k corresponds to a marked points, of real part > 1 if z_k corresponds to a bounded vertex, of real part < 1 if z_k corresponds to an unbounded vertex, and so that $\sum \mathrm{res}_k = 2$.

A map $\mathrm{Glu} : \mathscr{P}_0 \in \text{S-Conf} \to \mathrm{TR}$ is thus defined, we want to prove it is holomorphic but it is not even obvious that it is continuous.

A map $\mathrm{Per} : \mathrm{TR} \to \mathrm{Conf}$ can be defined by integration along homotopy classes on the edges of the cell-complex. It is not hard to prove that it is analytic on the preimage of Conf^*.

$\mathrm{Glu} : \text{S-Conf} \to \mathrm{TR}, \qquad \mathrm{Per} : \mathrm{TR} \to \mathrm{Conf}$

We have $\mathrm{Per} \circ \mathrm{Glu}|_{\text{S-Conf}} = \mathrm{Id}_{\text{S-Conf}}$ (almost tautological).

Let $\mathrm{Eff} = \mathrm{Glu}(\text{S-Conf}) \subset \mathrm{TR}$ (the effective Teichmüller-residue pairs). A key step is to prove that every point in Eff has a neighbourhood $W \subset \mathrm{Per}^{-1}(\text{S-Conf})$ on which $\mathrm{Glu} \circ \mathrm{Per}|_W = \mathrm{Id}_W$. This is obtained by following continuously the saddle connections and completing carefully the picture.

Analyticity of Glu then follows from a classical theorem in several variable complex analysis.

Thanks

for listening

and

"Joyeux anniversaire, Mitsu !"