Polynomial Julia sets with positive measure

Xavier Buff & Arnaud Chéritat
Université Paul Sabatier (Toulouse III)

À la mémoire d'Adrien Douady

Challenges

At the end of the 1920's, after the root works of Fatou and Julia on the iteration of rational maps, there remained important open questions. Here is a selection:

Challenges

At the end of the 1920's, after the root works of Fatou and Julia on the iteration of rational maps, there remained important open questions. Here is a selection:

– Can a rational map have a wandering Fatou components?

Challenges

At the end of the 1920's, after the root works of Fatou and Julia on the iteration of rational maps, there remained important open questions. Here is a selection:

– Can a rational map have a wandering Fatou components?

– Examples of rational maps were known, for which the Julia set J is the whole Riemann sphere. The others have a Julia set of empty interior. But what about their measure?

Challenges

At the end of the 1920's, after the root works of Fatou and Julia on the iteration of rational maps, there remained important open questions. Here is a selection:

– Can a rational map have a wandering Fatou components?

– Examples of rational maps were known, for which the Julia set J is the whole Riemann sphere. The others have a Julia set of empty interior. But what about their measure?

– Fatou asked whether, in the set of rational maps of given degree $d \geqslant 2$, those that are hyperbolic form a dense subset[1]. The same question holds in the set of polynomials.

All these questions are already difficult for degree 2 polynomials.

[1] it is known to be an open subset

Rebirth

In the 80s, computers helped revive the subject. Mandelbrot drawed the notion of fractals. J.H. Hubbard investigated Newton's method and got Adrien to enter in the field. This was the birth of the holomorphic Dynamics school.

Rebirth

In the 80s, computers helped revive the subject. Mandelbrot drawed the notion of fractals. J.H. Hubbard investigated Newton's method and got Adrien to enter in the field. This was the birth of the holomorphic Dynamics school.

After having investigated elaborate questions on infinite dimensional Banach algebraic varieties, Adrien told his colleagues he would focus on iterating $z^2 + c$.

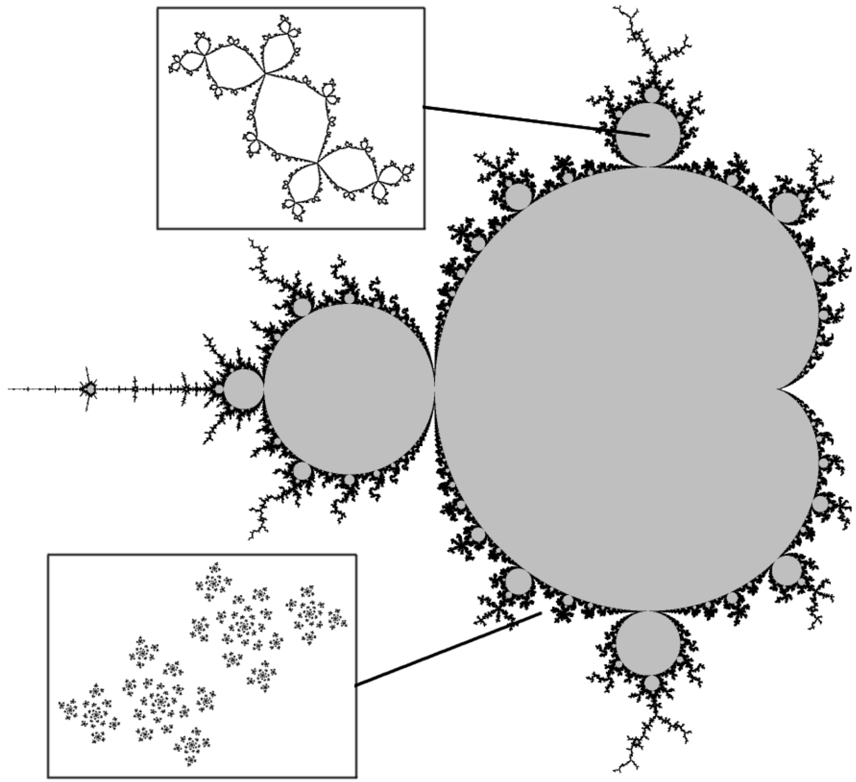
Rebirth

In the 80s, computers helped revive the subject. Mandelbrot drawed the notion of fractals. J.H. Hubbard investigated Newton's method and got Adrien to enter in the field. This was the birth of the holomorphic Dynamics school.

After having investigated elaborate questions on infinite dimensional Banach algebraic varieties, Adrien told his colleagues he would focus on iterating $z^2 + c$.

The family $P_c : z \mapsto z^2 + c$
– looks simple and useless in its aspect
– is very complicated in the facts
– and universal

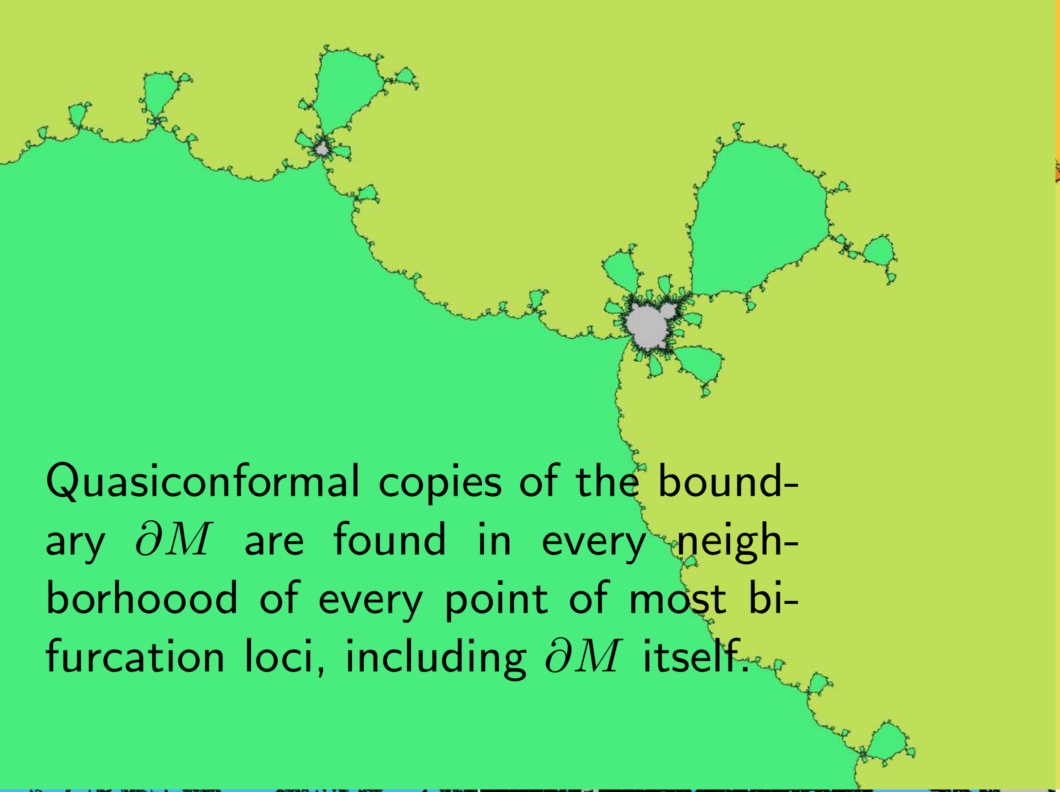
The Mandelbrot set



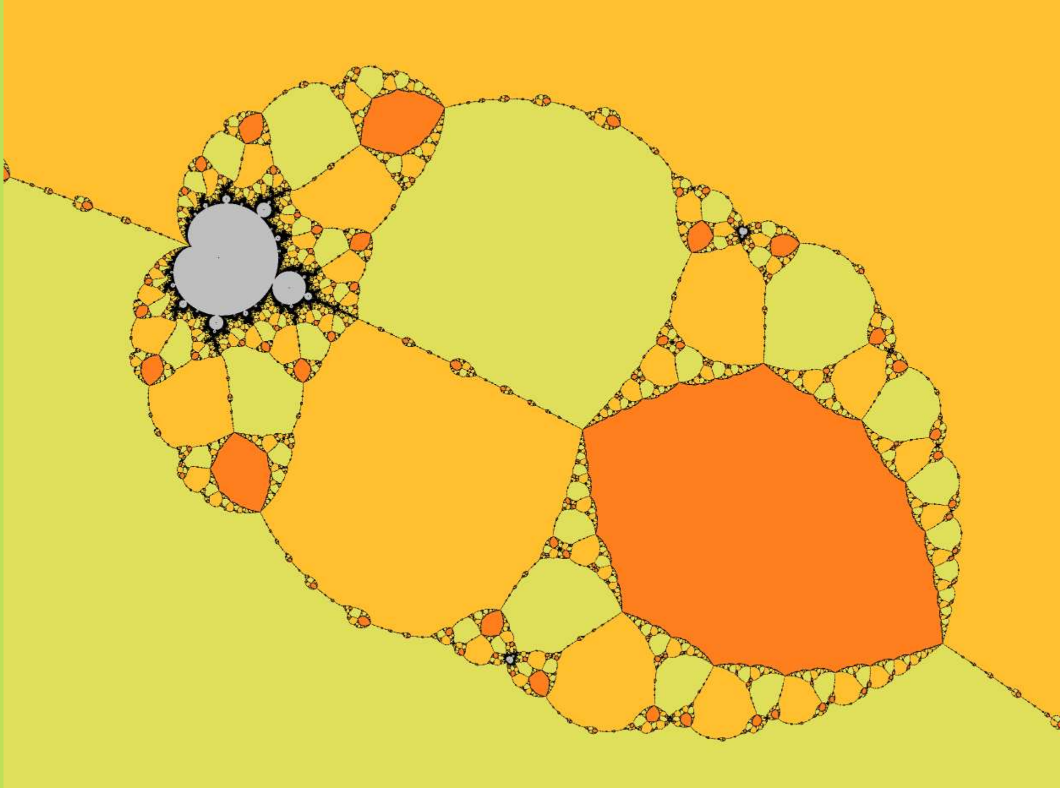
Dichotomy :
–$J(P_c)$ connected $\iff c \in M$
–$J(P_c)$ Cantor oterwise

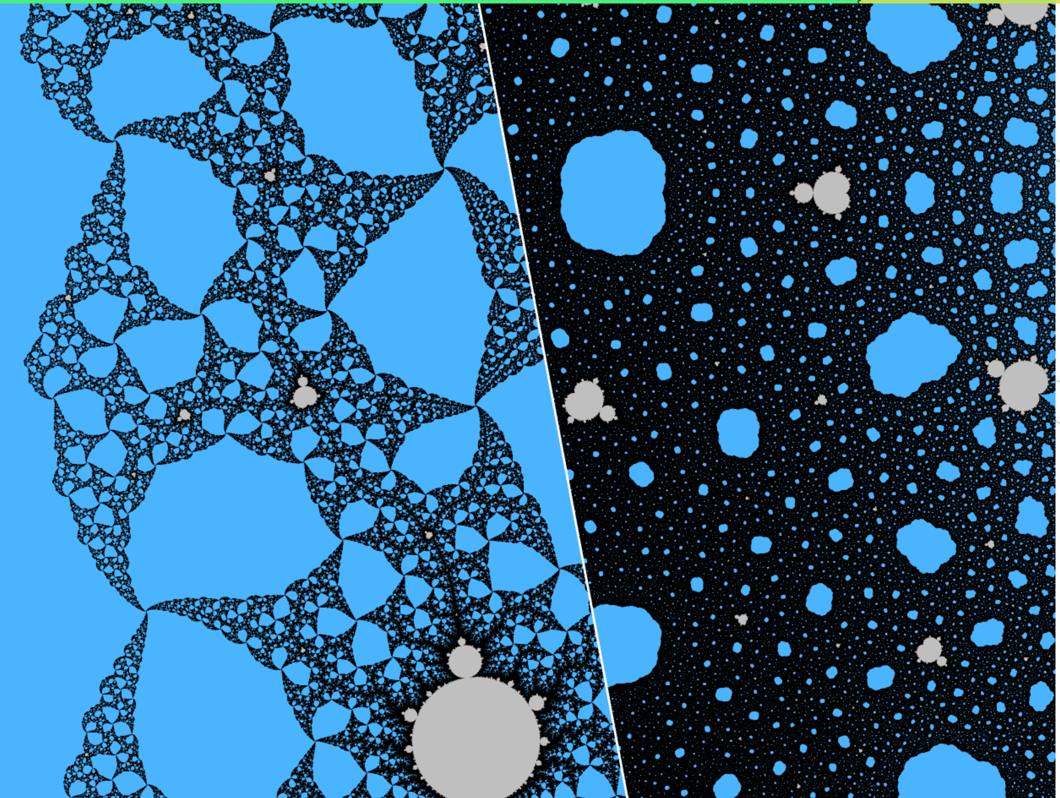
The boundary ∂M is the bifurcation locus of the dynamics, i.e. the set of parameters c where the Julia set do not vary continuously with respect to c.

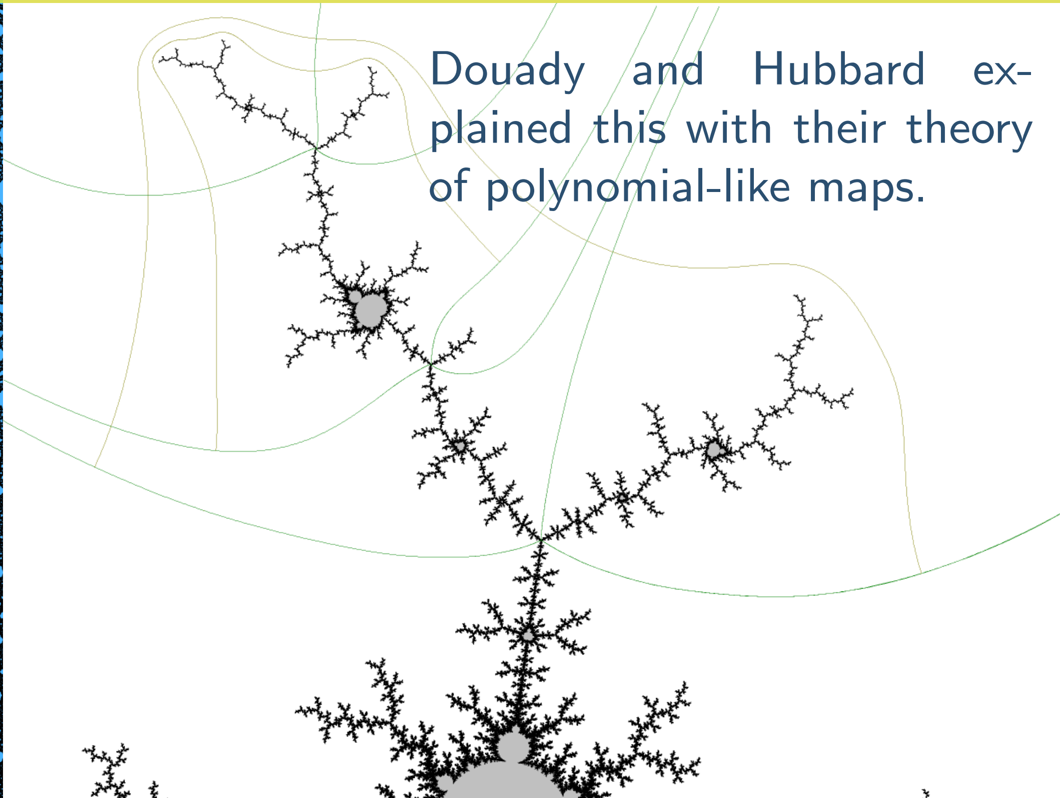
Theorem. MLC $\implies$ Fatou$_2$ (Douady, Hubbard) If the Mandelbrot set is locally connected, then the set of c such that P_c is hyperbolic is dense in $\mathbb{C}$.



Quasiconformal copies of the boundary ∂M are found in every neighborhoood of every point of most bifurcation loci, including ∂M itself.







Douady and Hubbard explained this with their theory of polynomial-like maps.

Why bother?

The Julia set is the place where a given rational map is chaotic, and one may wonder whether there is a non-zero probability that a randomly chosen point may belong to the locus of chaos.

Why bother?

The Julia set is the place where a given rational map is chaotic, and one may wonder whether there is a non-zero probability that a randomly chosen point may belong to the locus of chaos.

Julia sets of positive measure were known in other settings:

- Indeed there are rational maps whose Julia set is the whole Riemann sphere. Mary rees proved there can be lots of them.
- Transcendental entire maps: McMullen proved that in the sine family, the Julia sets always have positive measure, even when their interior is empty.

Quasiconformal methods

At the end of the 80s, quasiconformal method were introduced. These powerful methods allowed new progress, among which:

– The end of the classification of the connected components of the Fatou sets, with Sullivan's proof that *there are no wandering components*.

– Shishikura's optimal sharpening of Fatou's inequality : a degree d rational map has at most $2d-2$ non repelling cycles.

– An equivalent formulation of Fatou's conjecture (Mañe, Sad, Sullivan) **Fatou**$_2$ $\iff$ **NILF**: the density of hyperbolicity for degree 2 polynomials is equivalent to "No degree 2 polynomial has an invariant line field".

No invariant line field

Fatou$_2$ $\iff$ **NILF** (Mañe, Sad, Sullivan): the density of hyperbolicity for degree 2 polynomials is equivalent to "No degree 2 polynomial has an invariant line field".

An invariant line field is an element $\mu \in L^\infty(\mathbb{C})$ with values in $\mathbb{S}^1 \cup \{0\}$ almost everywhere (a.e.), such that

$$\mu(z) = \mu(P(z))\frac{\overline{P'(z)}}{P'(z)} \text{ a.e.,}$$

such that the support of μ is contained in the Julia set, and such that μ is not vanishing a.e.

No invariant line field

Fatou$_2$ $\iff$ **NILF** (Mañe, Sad, Sullivan): the density of hyperbolicity for degree 2 polynomials is equivalent to "No degree 2 polynomial has an invariant line field".

An invariant line field is an element $\mu \in L^\infty(\mathbb{C})$ with values in $\mathbb{S}^1 \cup \{0\}$ almost everywhere (a.e.), such that

$$\mu(z) = \mu(P(z))\frac{\overline{P'(z)}}{P'(z)} \text{ a.e.,}$$

such that the support of μ is contained in the Julia set, and such that μ is not vanishing a.e.

Such a line field requires a Julia set with non zero Lebesgue measure.

The measure zero conjecture

So if the Fatou_2 conjecture fails, then there is a Julia set with positive measure (the converse likely does not hold).

The hope was then that every Julia set of a P_c has Lebesgue measure equal to 0, which would have proved Fatou_2, whence the **measure zero conjecture** and its generalization:

- Every degree 2 polynomial has a Julia set of measure 0.
- (gnrlz.) Every degree $d \geqslant 2$ rational map has a Julia set either equal to the Riemann sphere or of measure 0.

The measure zero conjecture

So if the Fatou$_2$ conjecture fails, then there is a Julia set with positive measure (the converse likely does not hold).

The hope was then that every Julia set of a P_c has Lebesgue measure equal to 0, which would have proved Fatou$_2$, whence the **measure zero conjecture** and its generalization:

- Every degree 2 polynomial has a Julia set of measure 0.
- (gnrlz.) Every degree $d \geqslant 2$ rational map has a Julia set either equal to the Riemann sphere or of measure 0.

Ahlfors had formulated an analog conjecture for Kleinian groups.

The measure zero conjecture

So if the Fatou$_2$ conjecture fails, then there is a Julia set with positive measure (the converse likely does not hold).

The hope was then that every Julia set of a P_c has Lebesgue measure equal to 0, which would have proved Fatou$_2$, whence the **measure zero conjecture** and its generalization:

- Every degree 2 polynomial has a Julia set of measure 0.
- (gnrlz.) Every degree $d \geqslant 2$ rational map has a Julia set either equal to the Riemann sphere or of measure 0.

Ahlfors had formulated an analog conjecture for Kleinian groups.

Ahlfors' conjecture proof has been completed in 2004, at end of a long process.

Case where measure 0 was known to hold

Theorem *Hyperbolic polynomials have a Julia set J with $\operatorname{leb} J = 0$ (Douady-Hubbard) and even better: $\dim_H J < 2$ (Sullivan).*

Theorem (Lyubich, Shishikura): *If P has no indifferent periodic points and is not infinitely renormalizable, then $\operatorname{Leb} J(P) = 0$.*

Theorem: (Fatou, Julia, Douady, Hubbard): *a quadratic polynomial has at most one non repelling cycle.*

We now assume that P is a quadratic polynomial having an indifferent periodic point with multiplier $e^{2i\pi\theta}$.

Theorem (Denker, Urbanski): *If $\theta \in \mathbb{Q}$ then $\dim_H J(P) < 2$.*

Let $\mathrm{PZ} = \left\{\theta = a_0 + 1/(a_1 + \ddots) \,\middle|\, \ln a_n = \mathcal{O}(\sqrt{n})\right\}$.

Theorem (Petersen, Zakeri): *If $\theta \in \mathrm{PZ}$, then $z = 0$ is linearizable and $\operatorname{Leb} J(P) = 0$. Note that PZ has full measure.*

Theorem (Graczyk, Przytycki, Rohde, Swiatek, ...): *For almost every $\theta \in \mathbb{R}$, the external ray of angle θ of the Mand. set lands on a param. c such that the polynomial $P = z^2 + c$ satisfies Collet-Eckmann's condition and in particular $\dim_{\mathrm{H}} J(P) < 2$.*

Encouraging results ?

Theorem (Shishikura):

1. For a Baire generic set of values of $c \in \partial M$, $P = z^2 + c$ has Hausdorff dimension 2 Julia set.

2. For a Baire generic set of values of θ, $P = e^{2i\pi\theta} z + z^2$ has Hausdorff dimension 2 Julia set.

Remark: About case 1, for a Baire generic set of values of $c \in \partial M$, P has no indifferent cycle, and is not renormalizable, and thus by a previously mentioned theorem, $\mathrm{Leb}\, J(P) = 0$.

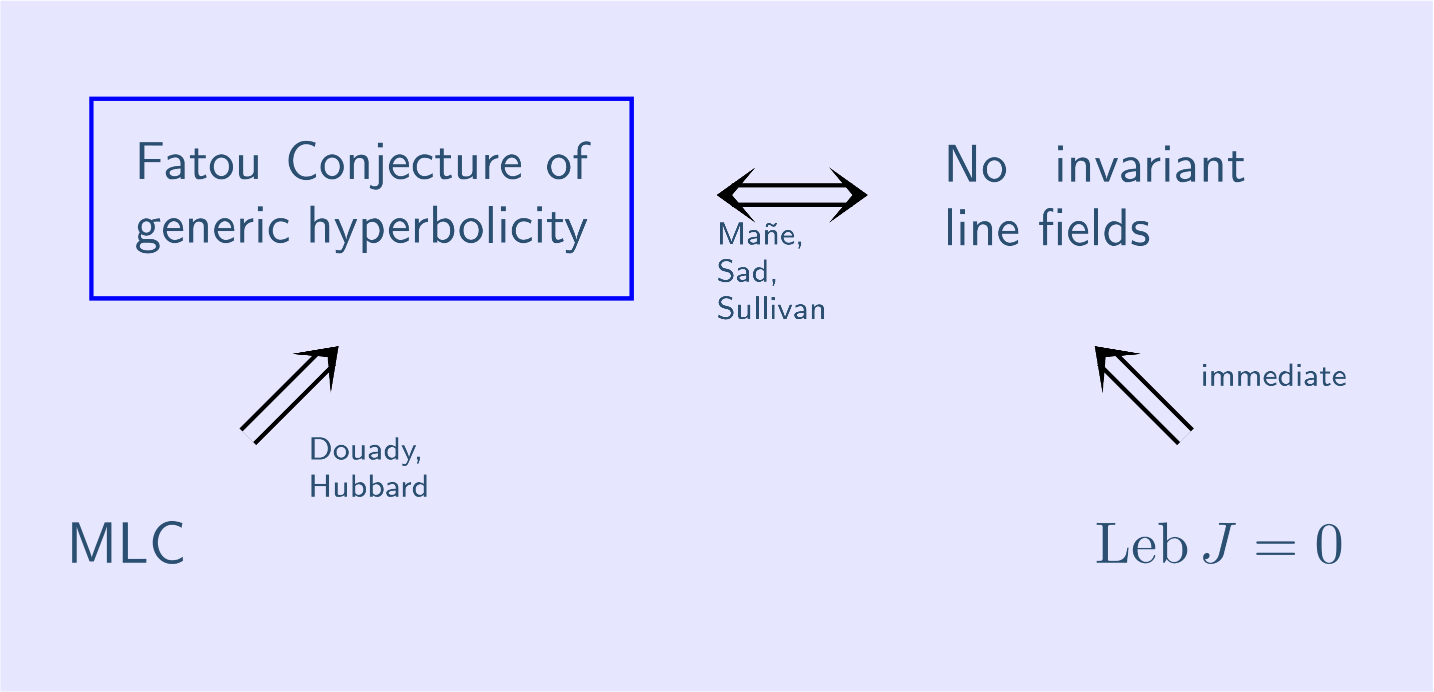
Degree 2 polynomials

Fatou Conjecture of generic hyperbolicity ⟺ (Mañe, Sad, Sullivan) No invariant line fields

⇑ (Douady, Hubbard) ⇑ (immediate)

MLC $\qquad$ $\mathrm{Leb}\, J = 0$

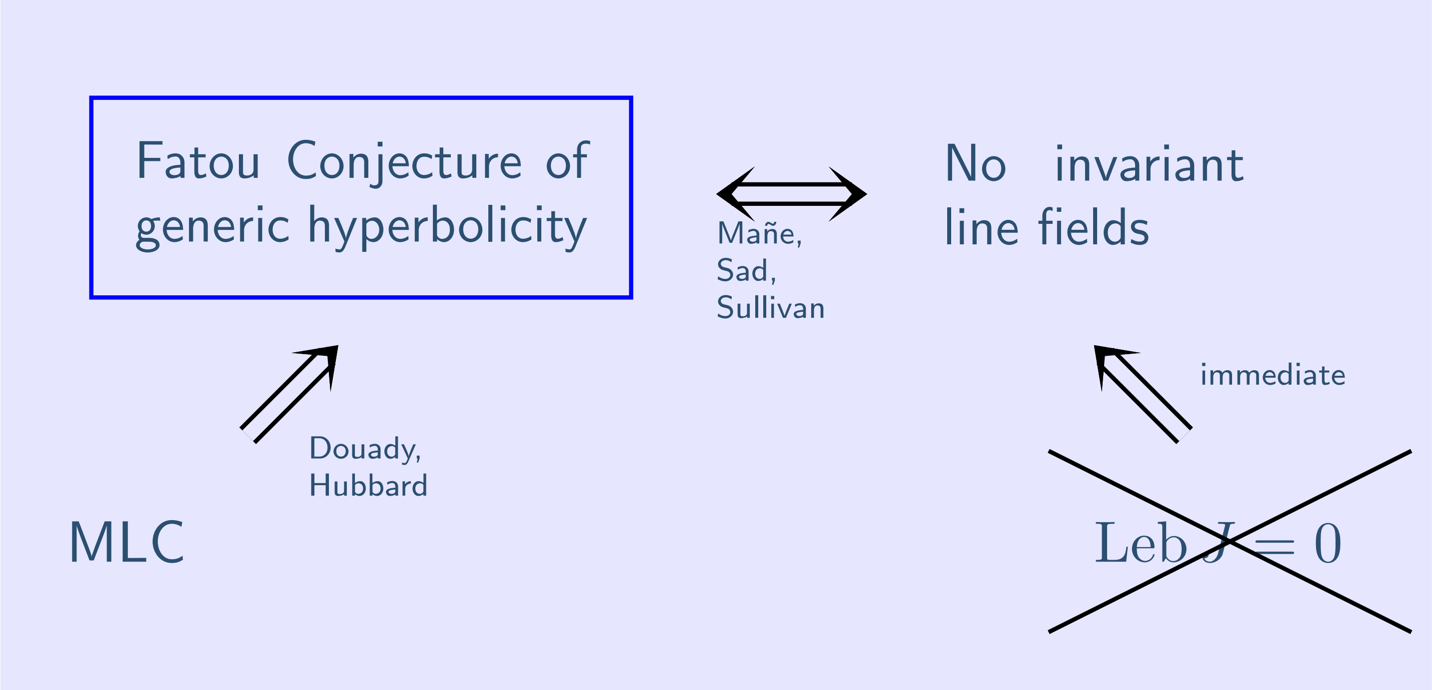
Degree 2 polynomials



Though, in the 90s, Adrien caught a glimpse of an approach, that might lead to a degree 2 polynomial with a Julia set of positive Lebesgue measure.

Adrien's insight

Degree 2 polynomials



Though, in the 90s, Adrien caught a glimpse of an approach, that might lead to a degree 2 polynomial with a Julia set of positive Lebesgue measure.

Adrien's plan for positive measure

Notations: $P_c(z) = z^2 + c$, $J_c = J(P_c)$.

Properties:

- Let K_c be the complement of the basin of ∞: $J_c = \partial K_c$ (Fatou, Julia).
- If $\mathring{K}_c = \varnothing$ then $J_c = K_c$ whence $\mathrm{Leb}(J_c) = \mathrm{Leb}(K_c)$.
- If P_c has a non-linearizable indifferent periodic point (Cremer point) then this is the case.
- If $\mathring{K}_c \neq \varnothing$ then of course $\mathrm{Leb}(K_c) > 0$.
- The map $c \mapsto \mathrm{Leb}\, K_c$ is upper semi-continuous.

Adrien's plan for positive measure

Notations: $P_c(z) = z^2 + c$, $J_c = J(P_c)$.

Properties:

- Let K_c be the complement of the basin of ∞: $J_c = \partial K_c$ (Fatou, Julia).
- If $\mathring{K}_c = \varnothing$ then $J_c = K_c$ whence $\mathrm{Leb}(J_c) = \mathrm{Leb}(K_c)$.
- If P_c has a non-linearizable indifferent periodic point (Cremer point) then this is the case.
- If $\mathring{K}_c \neq \varnothing$ then of course $\mathrm{Leb}(K_c) > 0$.
- The map $c \mapsto \mathrm{Leb}\, K_c$ is upper semi-continuous.

Consequence: if you have a convergent sequence $c_n \longrightarrow b \in \mathbb{C}$ such that P_b has a Cremer point, and $\mathrm{Leb}\, \mathring{K}_{c_n} > \varepsilon$ then $\mathrm{Leb}\, J_b > 0$.

Adrien's plan for positive measure

To define such a sequence and its limit, one would work with quadratic polynomials with an indifferent fixed point. It is equivalent to be working with the family $P_\theta(z) = e^{2\pi i\theta} z + z^2$, $\theta \in \mathbb{R}$, wich is conjugated to $z^2 + c$ with $c = e^{2\pi i\theta}/2 - e^{4\pi i\theta}/4$.

Start from some bounded type irrational θ_0. Then K_{θ_0} has non-empty interior since it contains a Siegel disk.

Then define θ_n by induction, so that θ_{n+1} is close to θ_n and the interior of K_{θ_n} does not lose too much Lebesgue measure.

By requiring $\theta_{n+1} - \theta_n$ very small at each step, it is easy to ensure convergence to a θ satisfying Cremer's condition for non linearizability.

The hard part is to control the loss of measure. It uses the theory of *parabolic implosion*, that Adrien initiated and developped a lot. It also uses the *control on the explosion of parabolic points* (Chéritat), *renormalization* techniques (McMullen, Shishikura, Yoccoz, . . .) and *quasiconformal models* (Ghys, Herman, Swiatek, etc. . .).

Our contribution

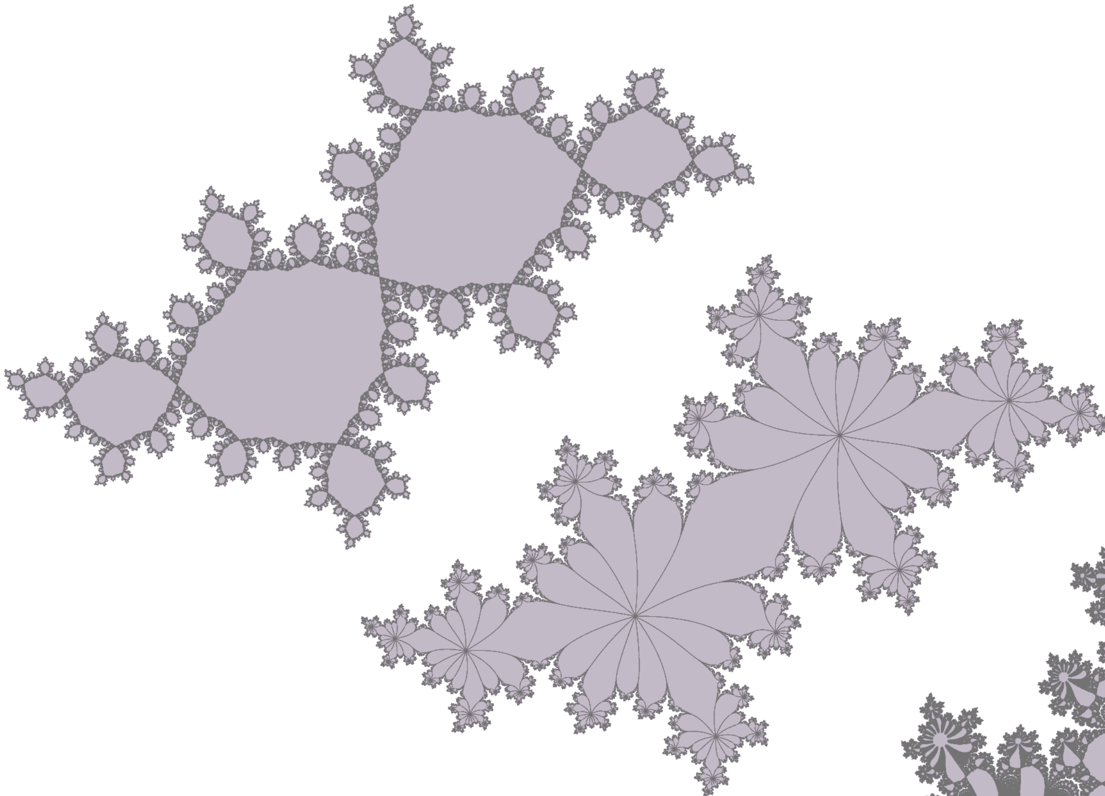
In my PhD, I proved that a measure loss control could be done provided some reasonable conjecture would hold (conjecture analog to things already proved by McMullen and backed by computer experiments). I even managed to convince Adrien that his own plan would work.

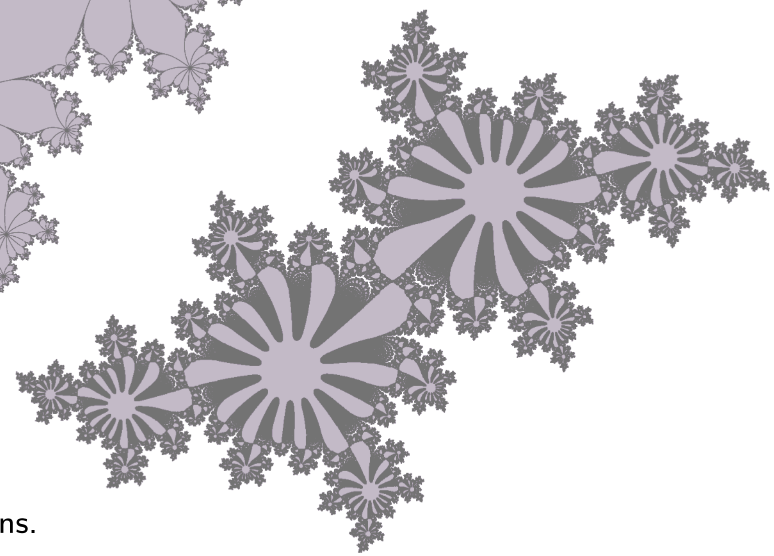
Our contribution

In my PhD, I proved that a measure loss control could be done provided some reasonable conjecture would hold (conjecture analog to things already proved by McMullen and backed by computer experiments). I even managed to convince Adrien that his own plan would work.

Using a breakthrough of Inou and Shishikura on near parabolic renormalization, Xavier Buff and I were able to complete the proof in 2005.

Thanks





To our common thesis advisor, Adrien Douady,
taught us good mathematics and good mood.

To all the people with whom we had fruitful discussions.

Flower picture borrowed on olharfeliz.typepad.com (Lugar do Olhar Feliz).