1/7, 2/7, 4/7

Echos of Teichmüller theory in the dynamics of holomorphic maps on the unit disk

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The space of Blaschke products

$$\mathcal{B}_d = \left\{ f(z) = z \prod_1^{d-1} \left(\frac{z-a_i}{1-\overline{a}_i z} \right) \right\} \cong \Delta^{(d-1)}$$

= {degree d maps $f : \Delta \to \Delta$, with $f|S^1$ expanding}

Marking: $f \circlearrowright S^1 \xrightarrow{\phi} S^1 \circlearrowright z^d$

Length of cycle C for z^d: $L(C,f) = \log |(f^p)'(z)| > 0$

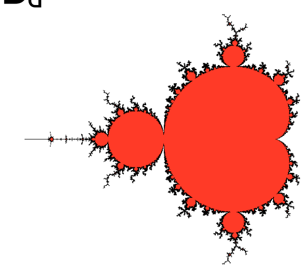
Features of B_d

Complex structure:

coming from $f \to$ mating $f \cup z^d$

Mapping class group:

$\mathrm{Aut}(z^d) = \mathbb{Z}/(d-1)$



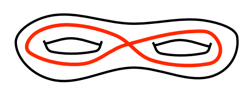
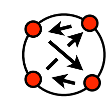
B_2 = main cardioid

Algebraic boundary:

$\partial B_d = \{(F,S) : F : \Delta \to \Delta, S \text{ divisor on } S^1, \deg(F)+\deg(S)=d\}$

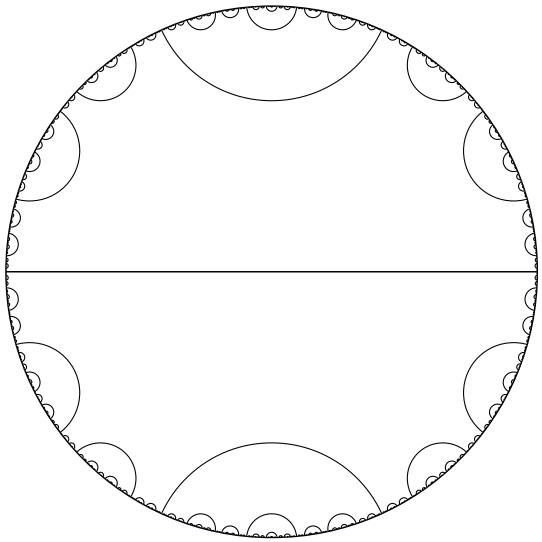
(bubbling)

Dictionary

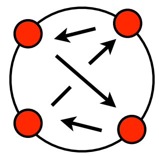
Teichmüller space T_g	Space of Blaschke products B_d
$\Delta/\Gamma = X$ compact	$f:\Delta \to \Delta$
	
closed geodesics γ	cycles C
$L(\gamma,X)$	$L(C,f)$

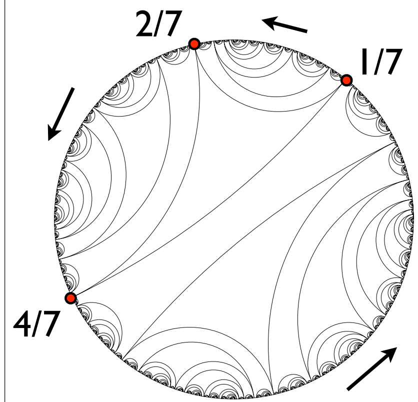
Simple loop? Measured lamination?
$PML_g = \partial T_g$? Ratios of lengths? Metric?

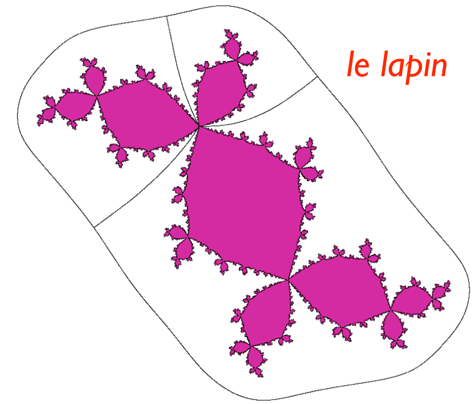
Lifts to Δ of a simple geodesic



not simple







A cycle C for z^d is *simple* if $z^d|C$ extends to a degree one map on S^1

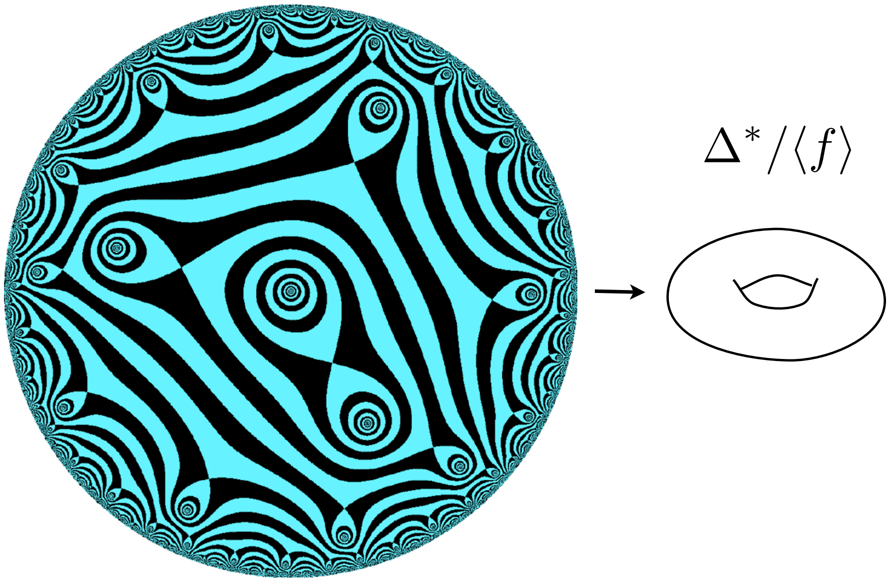
Simple geodesic on Riemann surfaces:

- Any loop with $L(\gamma,X) < \log(3 + 2\sqrt{2})$ is simple.
- There exists a simple loop with $L(\gamma,X) = O(\log g)$.
- The closure of the simple loops has Hausdorff dimension = 1.
- If (γ_i) are *binding,* then $\{X : \Sigma\, L(\gamma_i,X) \leq M\}$ is compact.
- The number of simple loops with $L(\gamma,f) < M$ is $O(M^{6g-6})$. *(polynomial growth)*

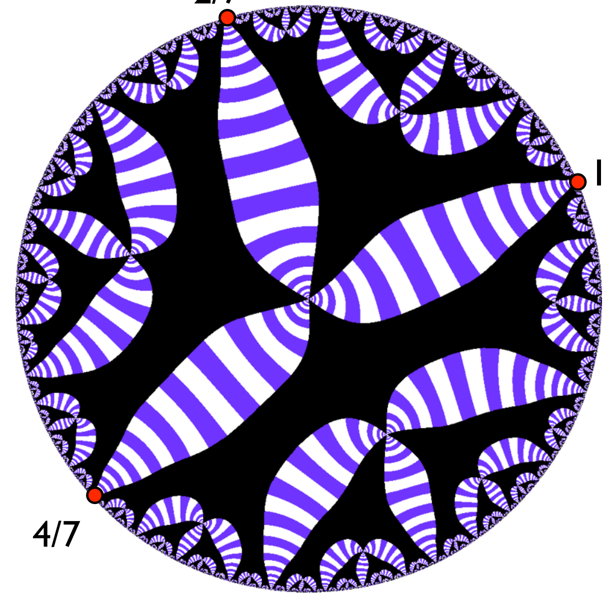
Simple cycles for Blaschke products:

- Any cycle with $L(C,f) < \log 2$ is simple.
- There exists a simple cycle with $L(C,f) = O(d)$.
- The closure of the simple cycles in S^1 has Hausdorff dimension = 0.
- If (C_i) are *binding,* then $\{f : \Sigma\, L(C_i,f) \leq M\}$ is compact.
- The number of simple cycles with $L(C,f) < M$ is $O(M^d)$. *(polynomial growth)*

Degree two dynamics in Δ

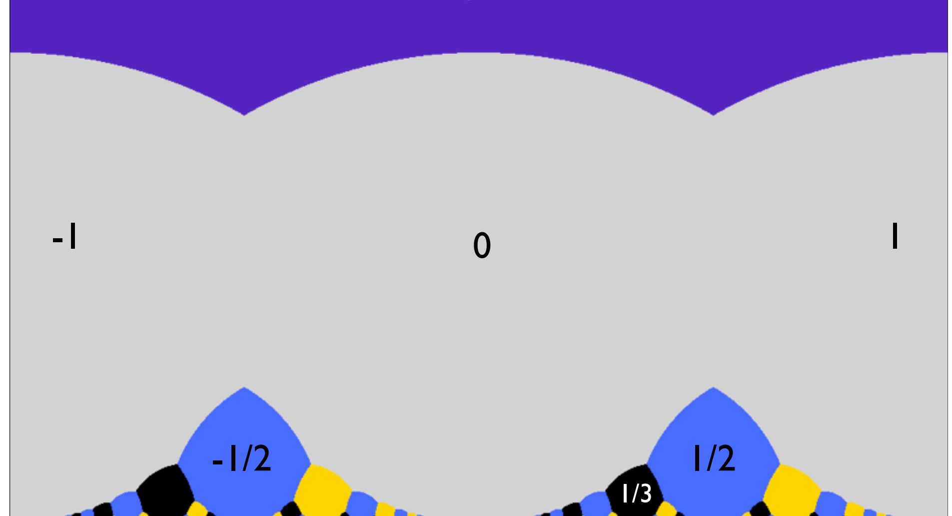


Short cycle with rotation number 1/3

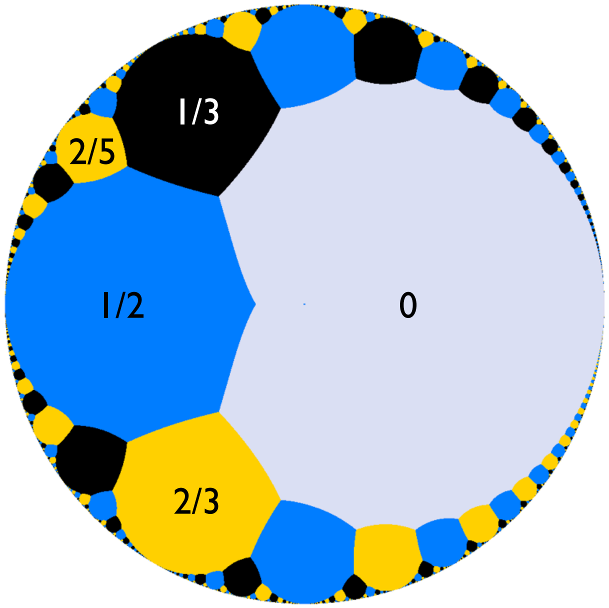


Spiraling 0/1 curves

Tiling of $\mathbb{H}$



Predicting p/q from $f'(0)$



Simple cycles and ∂B_d: measures on S^1

$\nu : B_d \to M_d(S^1) = \{z^d\text{-invariant probability measures on } S^1\}$

$\cup$

$\{\text{simple cycles } C\}$

using marking

$\nu(f) = \varphi_*(\text{Lebesgue measure on } S^1).$

Compactification by measures

- $\nu : B_d \to M_d(S^1)$ is an embedding.
- boundary of $\nu(B_d)$ is a sphere.
- (simple cycles+weights) dense in $(S^1)^{(d-1)}$ = Shilov boundary of $\nu(B_d)$
- *measure boundary ⇔ algebraic boundary*

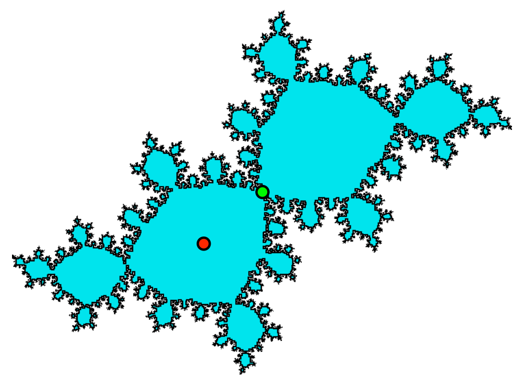
strata

- Hdim($\nu(F,S)$) = log e/log d, e = deg(F)

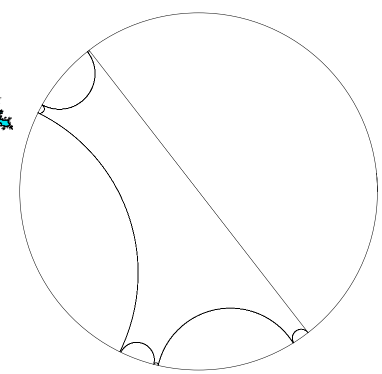
Compare (loops+weights) $\subset PML_g = \partial T_g$

Invariant measures for z^2

$(F,S) = (\lambda z, -\lambda), \quad |\lambda| = 1$



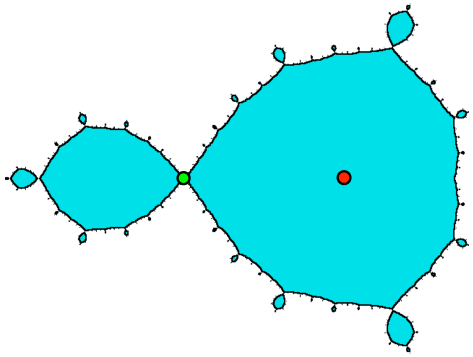
Compare Brownian motion inside/outside

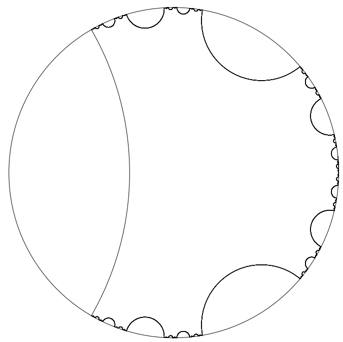


H.dim(Cantor set) = 0

Invariant measures for z^3

$(F,S) = (z^2,-1)$





More generally:

$$\left\{\begin{array}{l}\textit{all deg d expanding}\\ \textit{covering relations}(F,S)\end{array}\right\} \Leftrightarrow M_d(S^1)$$

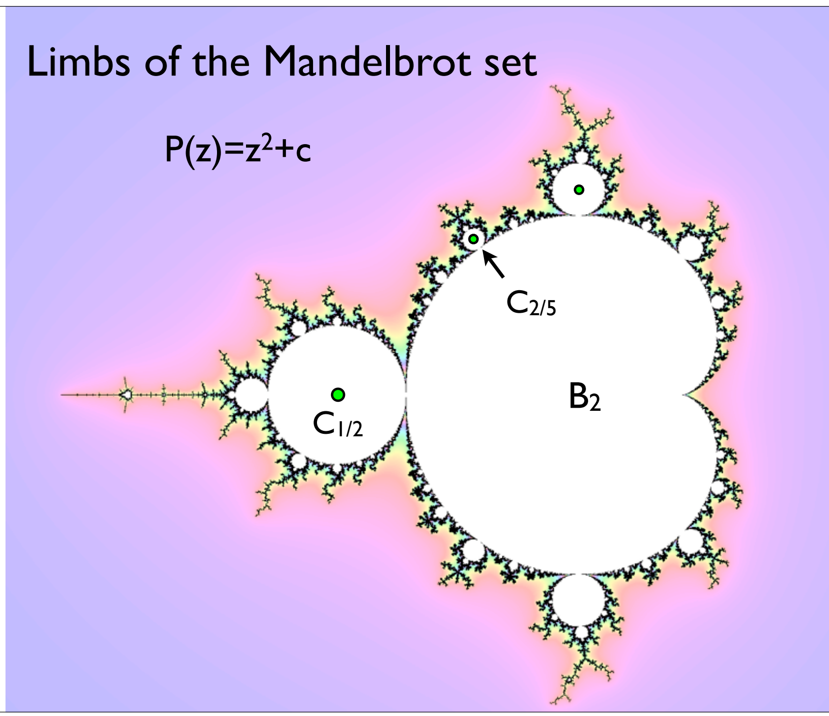
H.dim(Cantor set) = log 2/log 3

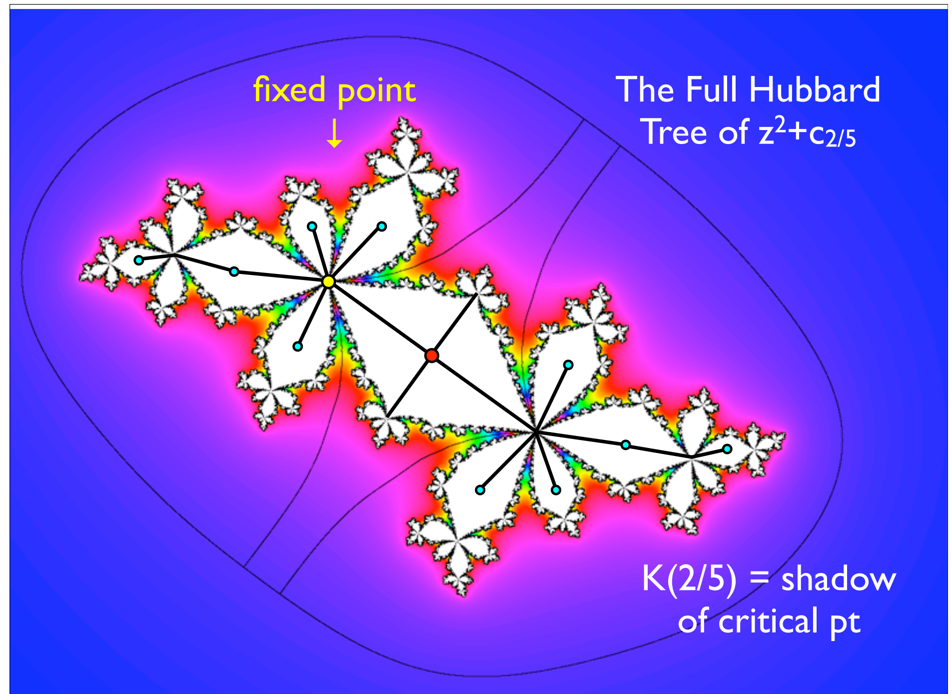
Compactification by marked lengths

- $\{f_n\}$ diverges in $B_d \Rightarrow (f_n : \Delta \rightarrow \Delta)$ converges geometrically to an isometric branched covering $f{:}T \rightarrow T$ of a *ribbon R-tree.*
- $L(C,f_n) \sim L(C,f)$ = translation of periodic end of T
- *T is simplicial, and T/f is a finite tree.*
- $L(C,f)$ span a finite-dimensional vector space / Q

Morgan-Shalen, Bestvina, Paulin... M. Wolff

Limbs of the Mandelbrot set





Quadratic Trees

- $f_n \to$ {(p/q) root} radially$\Rightarrow$ limit is the full Hubbard tree of $z^2 + c_{p/q}$.
- $L(C,f_n) \sim |C \cap K(p/q)|$
- The space of all quadratic trees is $S^1 = R/Z$ with the rational points blown up to intervals.

different from the measure boundary!

(while same in T_g -- because of i(a,b))

Thermodynamic formalism

$$B_d \to \frac{C^{1+\epsilon}(S^1)}{\{\text{coboundaries}\}} \to M_d(S^1)$$

$$f \to Lf = \varphi_*(\log f') \to \nu(f)$$

length function

$$L(C,f) = \Sigma \{Lf(z) : z \in C\}$$

Pressure $P(Lf) = 0$ *convex*

Metric on B_d

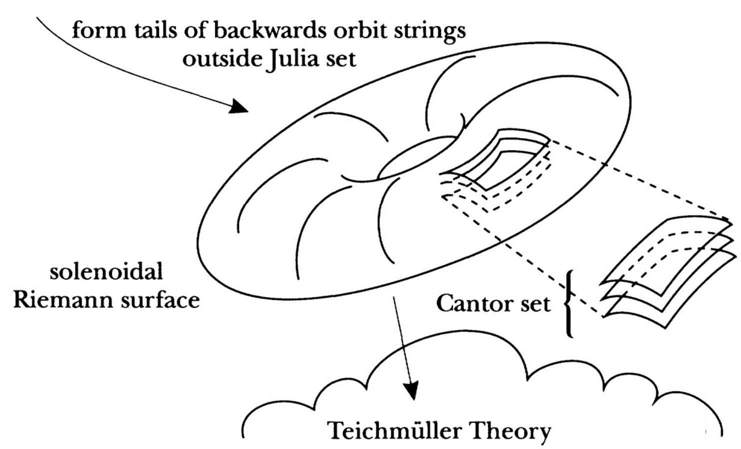
$$\begin{aligned}
\|\dot{f}_0\|^2 &= h(f_0,m)\cdot D^2 P\left(\frac{d}{dt} L f_t\Big|_{t=0}\right) && \textit{pressure} \\
&= \frac{d^2}{dt^2} L(f_t, \textit{random cycle } C \text{ for } f_0)\Big|_{t=0} \\
&= 4\,\frac{d^2}{dt^2}\,\mathrm{H.\,dim}\, J(f_t \cup f_0)\Big|_{t=0} && \textit{mating} \\
&= \frac{16}{3}\int_{\widehat{X}} \rho^{-4} |v'''|^2 \, d\xi && \textit{Schwarzian, quadratic differential}
\end{aligned}$$

Weil-Petersson metric

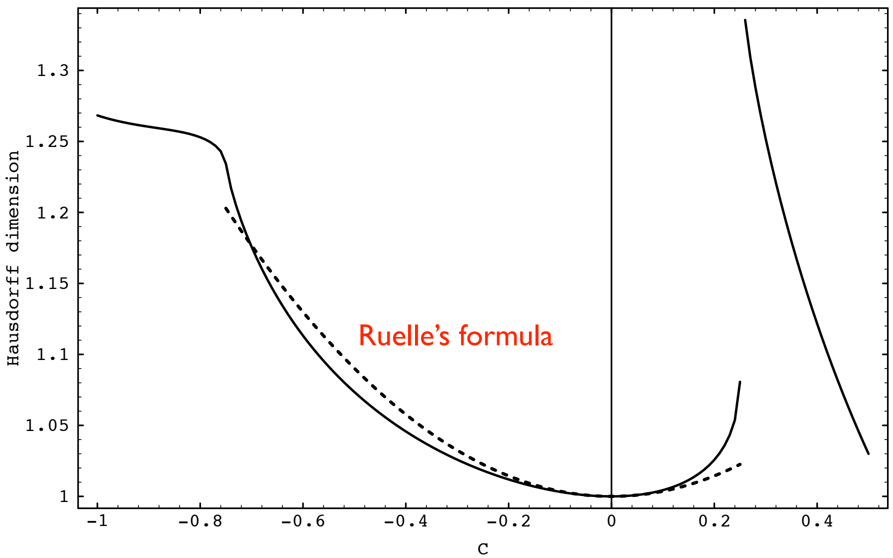
Riemann surface lamination

$$\widehat{\Delta} = \{(z_i) \in \Delta^{\mathbb{Z}} : f(z_i) = z_{i+1} \text{ and } |z_i| \to 1 \text{ as } i \to -\infty\}$$

$$\widehat{X} = \widehat{\Delta}/\langle \widehat{f} \rangle$$ *(Sullivan)*



Dimension of Julia set of $f(z) = z^2 + c$



Dimension Formula

Theorem *For t near zero, the family of polynomials*

$$F_t(z) = z^d + t\left(b_2 z^{d-2} + b_3 z^{d-3} + \cdots + b_d\right)$$

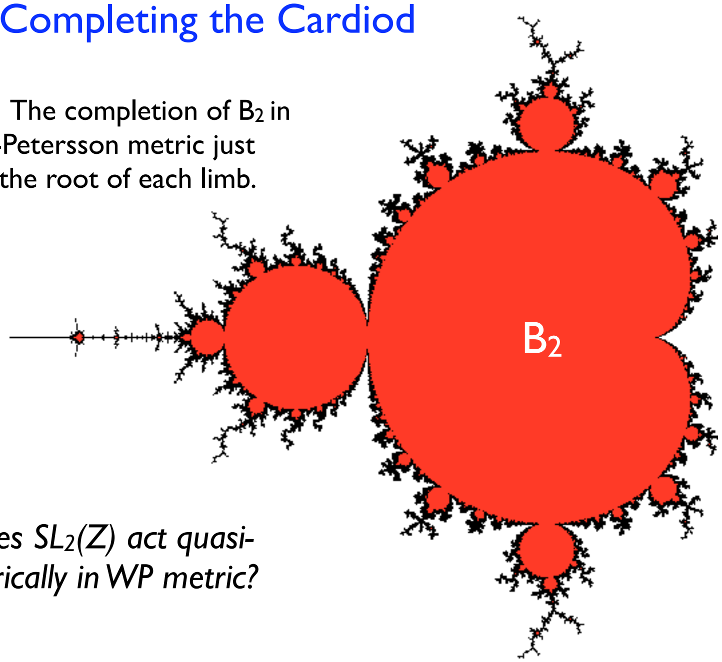
satisfies

$$\mathrm{H.}\dim J(F_t) = 1 + \frac{|t|^2}{4d^2 \log d} \sum k^2 |b_k|^2 + O(|t|^3).$$

Abenda-Moussa-Osbaldestin 1999

Completing the Cardiod

Conjecture: The completion of B_2 in the Weil-Petersson metric just attaches the root of each limb.



Q: Does $SL_2(Z)$ act quasi-isometrically in WP metric?