Impressions of the Mandelbrot set

Celebrating the spirit and ideas of Adrien

Carsten Lunde Petersen

IHP May 30 2008

Prolog

Prolog 1. *We all remember Adrien walking around, talking and singing. I remember one such instance particularly well. We were all installed in offices at some mathematical venue. All but Adrien had we gone to our respective offices. I met him in the corridor and asked: "Don't you want to use your office?" He answered philosophically with a reference to quantum physics: "The wave function of my spirit can not be localized to such a small place! "*
Adriens way of thinking and doing mathematics and life influenced everyone on his way and in this sense his spirit penetrated us all. Thus we become carriers of this same spirit, and can continue the project of spreading it around.

Introduction I

- The work I will be talking about today is mainly due to others. There is a large part due to Adam Epstein and to my ph.d student Eva Uhre.
  There will however also be results of John Milnor, Xavier Buff, Anja Kabelka, Jan Kiwi, Laura de Marco, Mary Rees and presumeably many more to whom I apologize if they are not mentioned.
- When exploring mathematics as so many other aspects of life, we often try to understand things by coordinatizing them or by other means matching them against a well known background or model(space), which we believe to know well.

Introduction II

- One way to do so is to find some properties, which persists in some neighbourhood. One of Adriens many gifts was his ability to find the persisting quantites, which provides links between objects.
- An obvious example is external rays.
- Another important such entity is encoded via the horn maps or equivalently via the Lavours maps of a parabolic point.
- A third one, which was (re)discovered by Milnor and explored by Adam Epstein is the holomorphic fixed point index.
- In this talk we shall make use of variants of all three.

Basic definitions

- Denote by Rat_2 the space of quadratic rational maps $f(z) = \frac{p(z)}{q(z)}$, where p and q are polynomials without common roots and $\deg(f) = \max\{\deg(p), \deg(q)\} = 2$.
- Denote by $\mathcal{M}_2 = \mathrm{Rat}_2/\mathrm{Rat}_1$ the moduli space of quadratic rational maps modulo Möbius conjugacy.
- I will in this talk mainly be focusing on maps with an indifferent fixed point as seen from the set of maps with an attracting fixed point.
- Ultimately we look for new dynamics involving the interplay between two critical points. A simillar study can be done on the space of cubic polynomials and also I presume more generally on spaces of bicritical rational maps and bicritical polynomials.

Coordinatizing I

- Any quadratic rational map f has, counting multiplicity 3 fixed points, whose multipliers we usually denote by λ, μ and γ.
- Milnor showed that the set of fixed point eigenvalues $\{\lambda, \mu, \gamma\}$ uniquely determines $[f] \in \mathcal{M}_2$.
- The holomorphic fixed point theorem gives us (provided $1 \notin \{\lambda, \mu, \gamma\}$) :

$$\frac{1}{1-\lambda} + \frac{1}{1-\mu} + \frac{1}{1-\gamma} = 1.$$

from which it easily follows that for any f

$$\lambda + \mu + \gamma = 2 + \lambda\mu\gamma.$$

Coordinatizing II

In terms of the elementary symmetric functions of the roots:

$$\begin{aligned}\sigma_1 &= \lambda + \mu + \gamma, \\ \sigma_2 &= \lambda\mu + \mu\gamma + \gamma\lambda, \\ \sigma_3 &= \lambda\mu\gamma.\end{aligned}$$

the index formula yields $\sigma_3 = \sigma_1 - 2$. Hence

$$[f] \mapsto (\sigma_1(f), \sigma_2(f)) : \mathcal{M}_2 \to \mathbb{C}^2$$

defines an injective holomorphic mapping. Milnor defined this map and showed that it is also surjective and hence biholomorphic.

The lines $Per_1(\lambda)$.

- Following Milnor we shall fix the eigenvalue λ of one of the fixed points of f generically denoted by a.
-

$$Per_1(\lambda) = \{[f] | f \text{ has a fixed point } a \text{ of multiplier } \lambda\}.$$

- It turns out that each set $Per_1(\lambda)$ is a complex line in the Milnor-coordinates.
- Moreover writing $\sigma = \mu\gamma$ for the product of the remaining two fixed point eigenvalues, the mapping

$$[f] \mapsto \sigma(f) : Per_1(\lambda) \to \mathbb{C}$$

is an isomorphism and thus gives a natural coordinate.

The Fatou relatedness loci.

- Any quadratic rational map f has two distinct critical points c_1, c_2.
  Following Uhre we say that :
  c_1 and c_2 are Fatou related if they belong to the same grand-orbit of Fatou components.
- For each λ we define the Fatou relatedness locus

$$\mathcal{R}^\lambda = \{[f] \in Per_1(\lambda) | c_1, c_2 \text{ are Fatou related}\}$$

The Fatou relatedness locus is a conglomerate of different Rees types of hyperbolic components. However it is the natural entity for the problem.

The line $Per_1(0)$.

- $Per_1(0) = \{[Q_c(z) = z^2 + c] | c \in \mathbb{C}\}$, that is $Per_1(0)$ is parametrized by the normal form Q_c, where $\sigma = 4c$.
- The Mandelbrot set

$$\begin{aligned} M = M^0 &= \{[Q_c] | J_{Q_c} \text{ is connected}\} \\ &= \{[Q_c] | \text{the critical point } 0 \text{ does not escape}\} \\ &= Per_1(0) \setminus \mathcal{R}^0. \end{aligned}$$

- For p/q, $(p,q) \neq 1$ and α_c the least repelling fixed point of Q_c, the p/q-limb of M (M^0) is

$$L_{p/q} = \{[Q_c] | \alpha_c \text{ has combinatorial rotation number } p/q\}$$

The motion M^λ of M.

- For $\lambda \in \mathbb{D}$ any map f with $[f] \in Per_1(\lambda)$ has a quadratic like restriction $f_| : U' \to U$ where U', U are topological disks with $J_f \subset U' \subset\subset U$.
- $$M^\lambda = \{[f] \in Per_1(\lambda) | J_f \text{ is connected}\}$$
$$= Per_1(\lambda) \setminus \mathcal{R}^\lambda.$$
- The inverse of the Douady-Hubbard straightening map $\Psi^\lambda : M^\lambda \to M^0$ defines a holomorphic motion
$$\Phi : \mathbb{D} \times M^0 \to \mathbb{C}$$
of M^0 over $\mathbb{D}$ with base point 0.

Holomorphic motions

Definition 1. *A holomorphic motion of a set* $K \in \mathbb{C}$ *over a Complex analytic manifold* Λ *with base point* $\lambda_0 \in \Lambda$ *is a mapping*

$$\Phi : \Lambda \times K \longrightarrow \mathbb{C}, \qquad \textit{such that:}$$

i) $\forall z \in K$*:* $\lambda \mapsto \Phi(\lambda, z)$ *is holomorphic*
ii) $\forall \lambda \in \Lambda$*:* $z \mapsto \Phi^\lambda(z) := \Phi(\lambda, z)$ *is injective.*
iii) $\Phi^0 = \mathrm{id}$

The amazing λ-lemma states that

Theorem 2 (Mañe-Sad-Sullivan). *Any holomorphic motion has a unique continuous extension to* $\Lambda \times \overline{K}$ *and each time* λ *map* Φ^λ *is quasi-conformal.*

Impressions of M I

- However in general no extension properties what so ever to the motion boundary $\partial\Lambda$ can be infered. This motivates:
- The main question I will investigate in this talk:

Question 1. *What is the impression of the motion* M^λ *on the lines* $Per_1(\omega)$ *when* $|\omega| = 1$*?*

- Using the computer properly it is not difficult to make conjectures.
- I shall focus on the case $\omega = \omega_{p/q} = \mathrm{e}^{i2\pi p/q}$, $(p, q) = 1$, where we actually have proofs of some of these conjectures.
- FILMS F1, F2, F3 AND F4

Impressions of M II.

- To this end I need to be precise on which kind of limits we take
- There are essentially two types of limits, the unrestricted limit from $\mathbb{D}$ and the subtangential limit written respectively as

$$\lambda \to \omega, \qquad \lambda \underset{\text{subtan.}}{\longrightarrow} \omega.$$

Where the latter means that

$$\Re\left(\frac{1}{1-\lambda/\omega}\right) \to \infty.$$

We consider two cases:

Case 1: $\omega = 1$ I

For $\omega = 1$ we still have the connectedness dichotomy:

$$\begin{aligned} M^1 &= \{[f] \in Per_1(1) | J_f \text{ is connected}\} \\ &= Per_1(1) \setminus \mathcal{R}^1 \end{aligned}$$

M^1 is a compact subset of $Per_1(1)$.

Theorem 3 (Roesch, P). *As* $\lambda \underset{\textit{subtan.}}{\longrightarrow} 1$

$$M^\lambda \to M^1, \qquad \textit{Hausdorff}$$

$$\Phi^\lambda \to \Phi^1 \qquad \textit{at least pointwise}$$

where Φ^1 *is a bijection, holomorphic on the interior and preserving dynamics.*

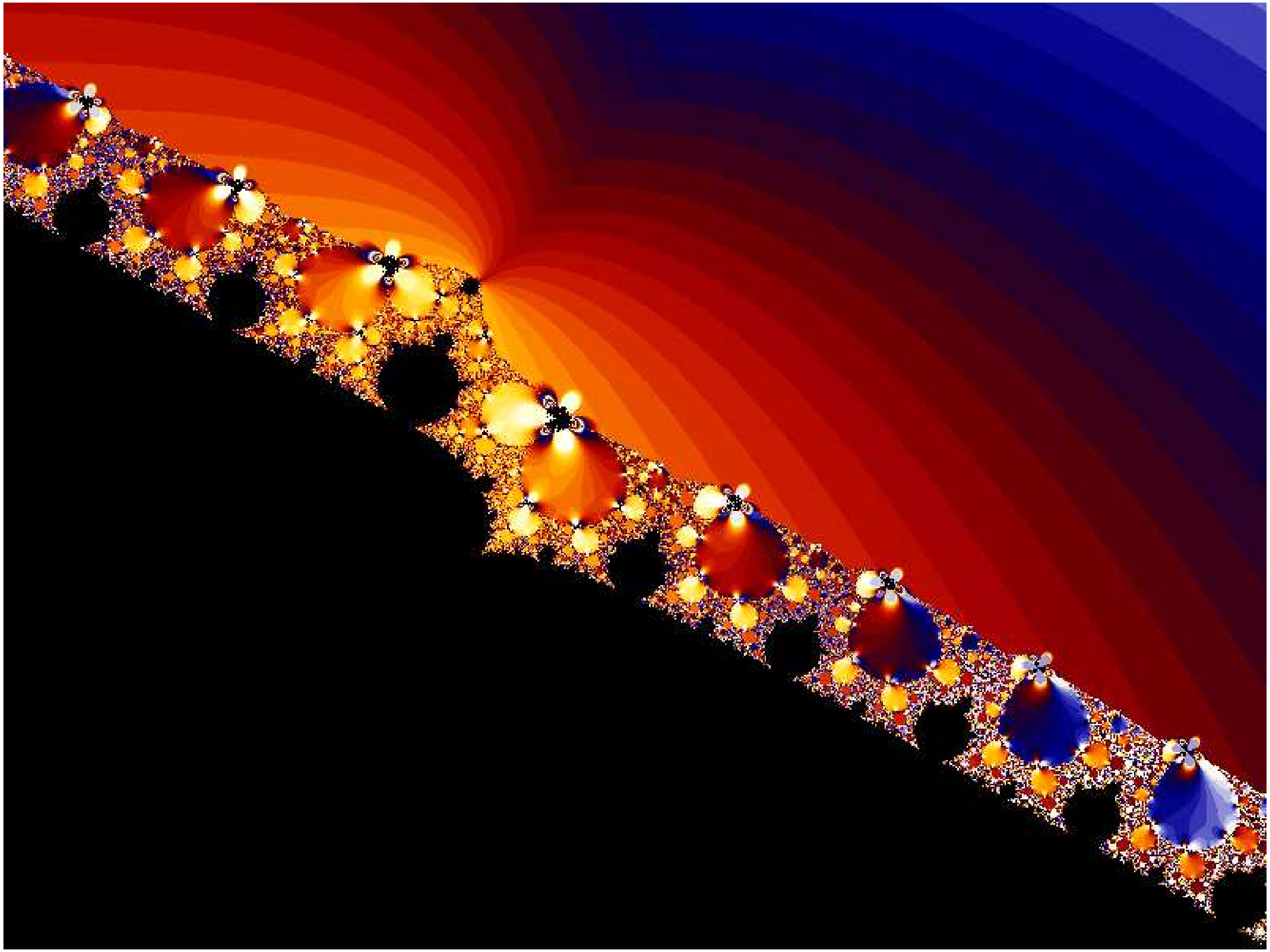
Case 2: $\omega = \omega_{p/q}$, $(p, q) = 1$.

Theorem 4 (P).

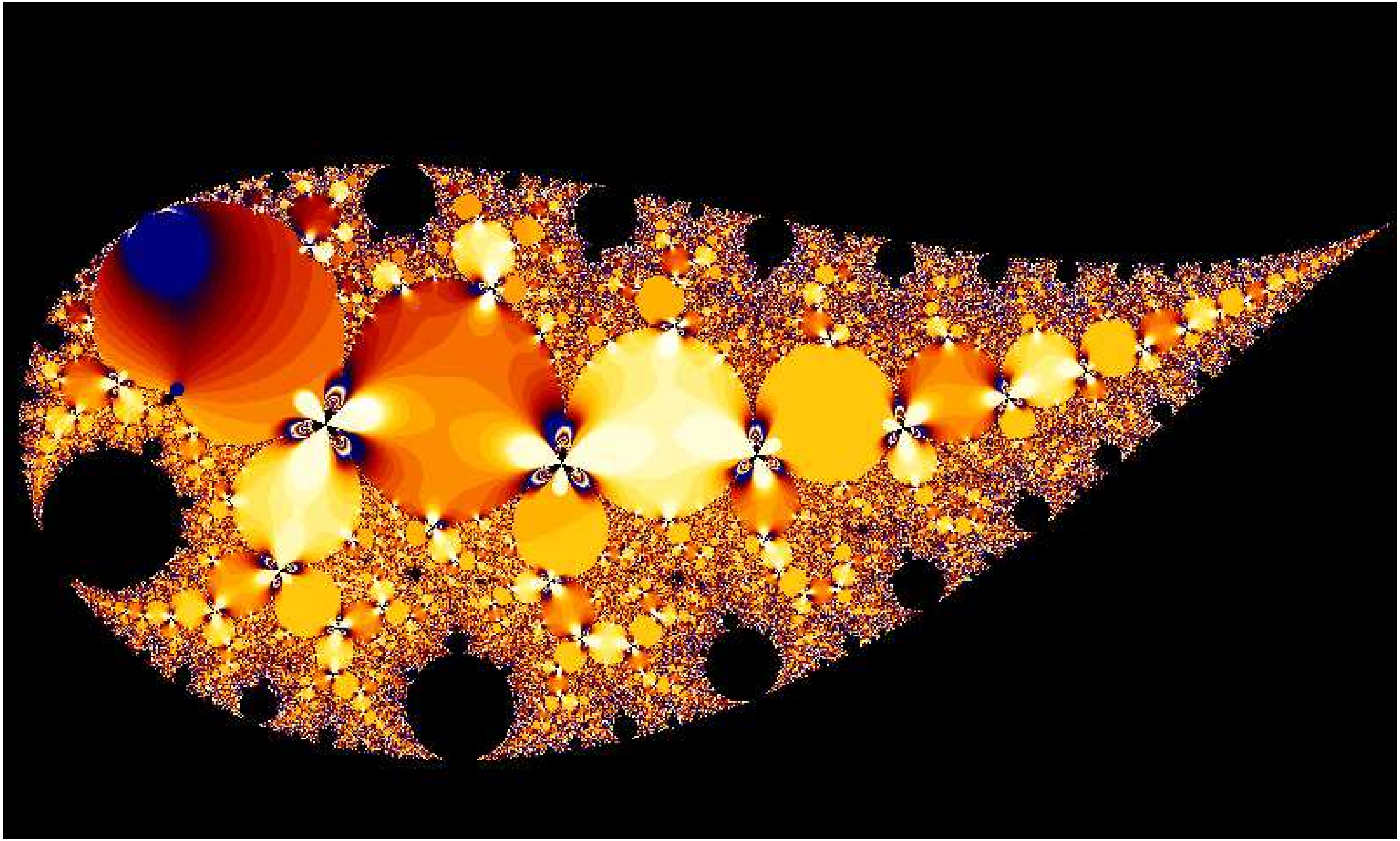
$$L^{\lambda}_{-p/q} = \Phi^{\lambda}(L^{0}_{-p/q}) \underset{\lambda \underset{\textit{subtan.}}{\to} \omega_{p/q}}{\longrightarrow} \infty, \qquad \textit{Hausdorff}.$$

For the following discussion we use the terminology:
A (relatively) hyperbolic component $H \subset Per_1(\omega)$ is a maximal domain on which $[f] \in H$ has an attracting periodic orbit.

$M^{\omega_{1/3}}$ from $\mathbb{C}$



$M^{\omega_{1/3}}$ from ∞.



Subtan. convgce og hyp. comp.

Theorem 5 (Epstein and Uhre). *For any* p/q, $(p,q)=1$, *for any hyperbolic component* $H^0 \subset M^0 \setminus L^0_{-p/q}$ *let*

$$\sigma_H^\lambda = \Phi^\lambda \circ \sigma_H^0 : \overline{\mathbb{D}} \to \overline{H^\lambda} = \Phi^\lambda(H)$$

denote the Douady-Hubbard multiplier parameters. Then

$$\sigma_H^\lambda \underset{\lambda \underset{\text{subtan.}}{\longrightarrow} \omega}{\rightrightarrows} \sigma_H^{\omega_{p/q}} : \overline{\mathbb{D}} \to \overline{H^{\omega_{p/q}}} \subset Per_1(\omega_{p/q}),$$

where $H^{\omega_{p/q}}$ *is some hyperbolic component of* $Per_1(\omega_{p/q})$. *Moreover for any hyperbolic component* $H^{\omega_{p/q}}$ *of* $Per_1(\omega_{p/q})$ *there is a unique hyperbolic component* $H^0 \subset M^0 \setminus L^0_{-p/q}$ *such that the above holds. In particular* $H^{\omega_{p/q}}$ *is relatively compact.*

Preliminaries I

To understand non tangential limits we need to re-introduce the parabolic index of a parabolic fixed point. For a map

$$f(z) = \omega_{p/q} z + \mathcal{O}(z^2)$$

we define the parabolic index as the holomorphic index of f^q:

$$I(f) = \frac{1}{2\pi i} \oint \frac{dz}{z - f^q(z)}$$

The parabolic index $I : Per_1(\omega_{p/q}) \longrightarrow \overline{\mathbb{C}}$ is a rational function. For $q = 1$ it equals $1/\sigma$ and for $q > 1$ it has a pole at ∞.

Preliminaries II

- Recall that the root of any hyperbolic component H of M is the landing point of two periodic (root) rays (except $H = \heartsuit$).
- If H is a satelite then the root rays belongs to the same cycle.
- If H is a primitive component, then they belong to two different cycles.

Definition 6. *A virtual hyperbolic component of $Per_1(\omega_{p/q})$ is a connected component of*

$$I^{-1}(\{x + iy | x > (q+1)/2\}).$$

Any virtual hyperbolic component is rooted at a pole of I.

Unrestricted convgnce of hyp. Comp.

Let θ_1, θ_2 denote the arguments of the root rays of $L^0_{-p/q}$.

Theorem 7 (Epstein and Uhre). *For any hyperbolic component* $H^0 \subset M^0 \setminus L^0_{-p/q}$ *we have two cases:*

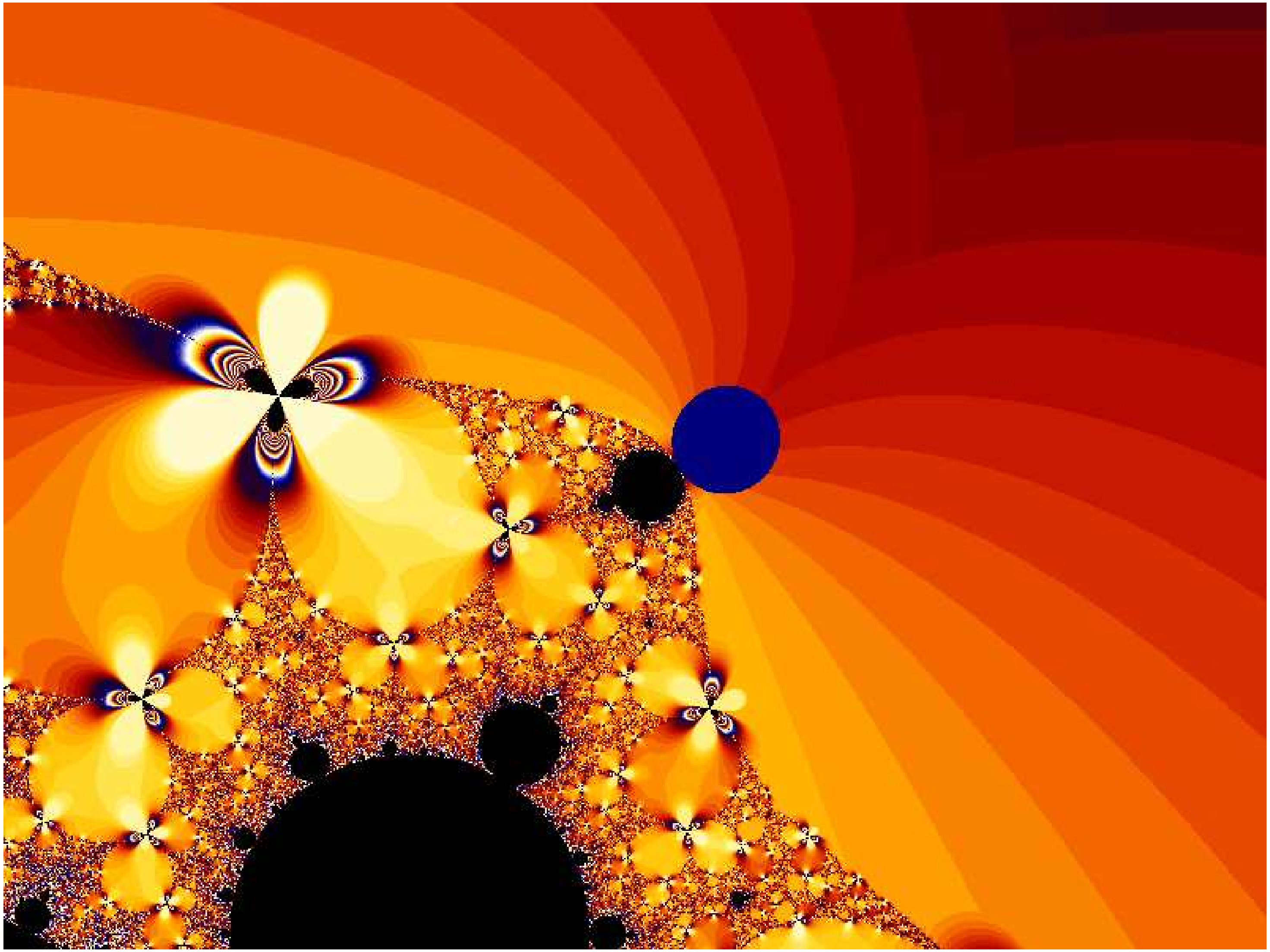
either the argument of one of the rootrays of H^0 *belongs to the cycle of* θ_1, θ_2 *or it does not. If not then*

$$\overline{H^\lambda} = \Phi^\lambda(\overline{H^0}) \underset{\lambda \to \omega}{\longrightarrow} \overline{H^{\omega_{p/q}}}, \qquad \text{Hausdorff.}$$

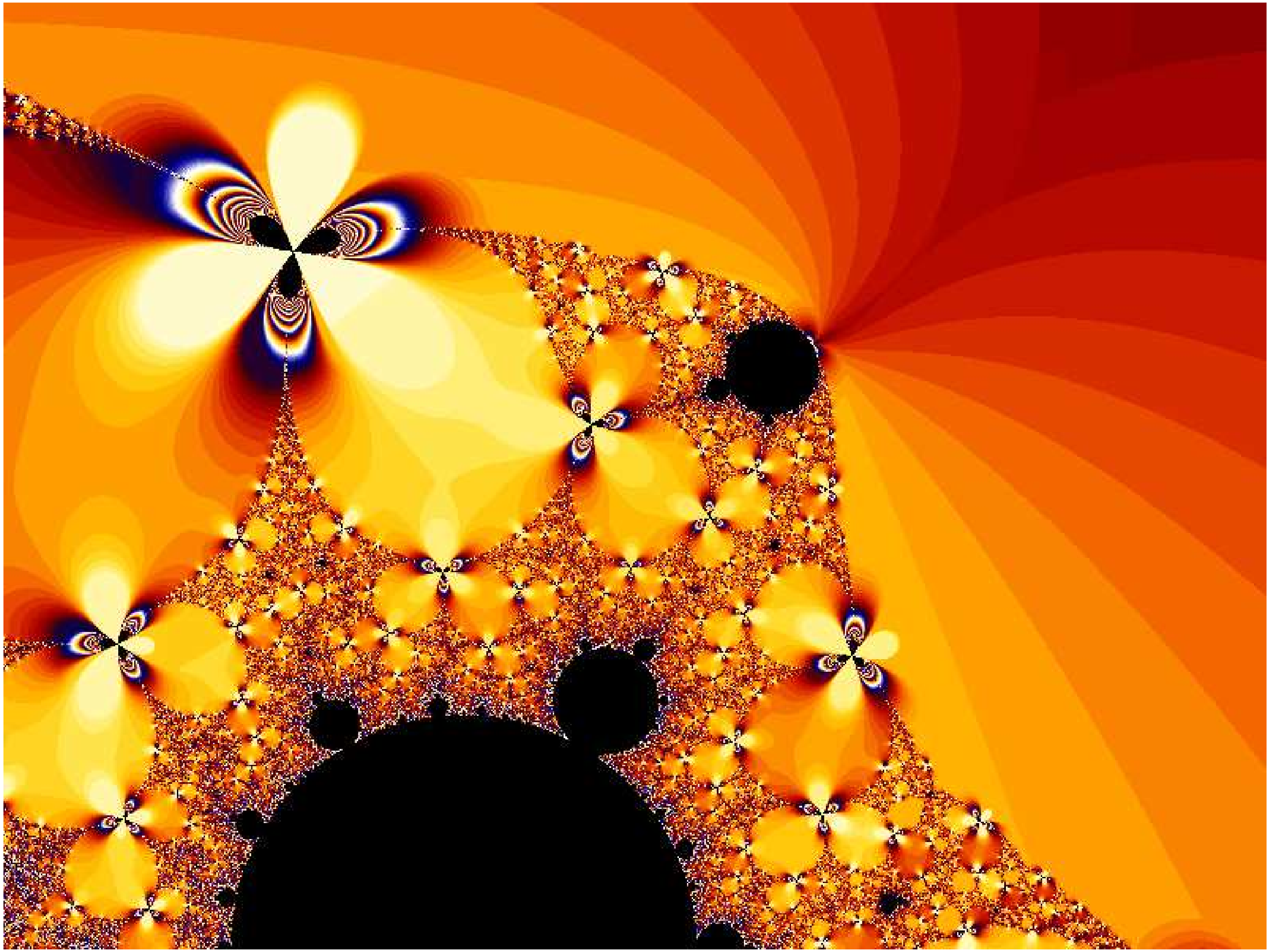
where $\overline{H^{\omega_{p/q}}}$ *is the subtangential limit.*

If yes, then the root r *of* $\overline{H^{\omega_{p/q}}}$ *is a pole of* I *and the unrestricted impression also includes the closure(s) of the virtual hyperbolic component(s) attached at* r*.*

$M^{\omega_{1/3}}$ with virtual hyp. Comp.



$M^{\omega_{1/3}}$ without virtual hyp. Comp.



A model of $\widehat{\mathcal{R}}^{\omega}$

Construction on blackboard. Make your own illustration..

In this version it is replaced by the following slides:

A model of $\widehat{\mathcal{R}}^\omega$ I

Consider the quadratic polynomial $P = P_\omega(z) = \omega z + z^2$.
It has a parabolic fixed point at 0 with a q-cycle of immediate basins $\Lambda_0, \ldots, \Lambda_{q-1}$ numbered counter-clockwise and with the critical point $c = -\omega/2 \in \Lambda_0$ and the critical value $v = -\lambda^2/4 \in \Lambda_p$.
Denote by $\phi_j : \Lambda_j \longrightarrow \mathbb{C}$ the Fatou coordinates for P_ω normalized by

$$\phi_p(v) = 0,$$
$$P \circ \phi_j = 1/q + \phi_{j+p \mod q}.$$

For each j let $\Delta_j \subset \Lambda_j$ denote the petal mapped univalently to the half plane $\{z = x + iy | x > 0\}$

A model of $\widehat{\mathcal{R}}^\omega$ II

Define a new Riemann surface $\mathcal{Y}^\omega$ isomorphic to $\mathbb{C}$ obtained as $(\overline{\mathbb{C}} \setminus \mathbb{D})/\sim^\omega$. Where $\sim^\omega$ is given as follows: Let

$$\chi : \overline{\mathbb{C}} \setminus \mathbb{D} \to \overline{\mathbb{C}} \setminus \bigcup_{j=0}^{q-1} \Delta_j$$

denote the continuously extended uniformizing parameter with $\chi(\infty) = \infty$, $\chi(1) = v$.
Then $z_1 \sim^\omega z_2$ iff

$$\chi(z_1) \in \partial\Delta_j, \quad \chi(z_2) \in \partial\Delta_k, \quad j + k \equiv 2p \mod q, \text{ and}$$
$$\phi_j(\chi(z_1)) + \phi_k(\chi(z_2)) = 0.$$

This relation is extended continuously to the q points $\chi^{-1}(0)$.

A model of $\widehat{\mathcal{R}}^{\omega}$ III

- Let $\Pi^{\omega} : \overline{\mathbb{C}} \setminus \mathbb{D} \longrightarrow \mathcal{Y}^{\omega}$ denote the natural projection. Define $\widehat{\infty} = \Pi^{\omega}(v)$, $\widehat{\delta} = \Pi^{\omega}(\mathbb{S}^1)$ and define $\widetilde{\Omega}^{\omega}$ to be the critical value ear of K_P.
- Define

$$\mathcal{X}^{\omega} = \Pi^{\omega}\left(K_P \setminus \widetilde{\Omega}^{\omega} \bigcup \bigcup_{n=0}^{\infty} P^{-1}(0)\right).$$

- We shall endow $\mathcal{X}^{\omega}$ with a tree $\widehat{\mathcal{T}}^{\omega}$ by which we can parametrize its combinatorial structure. This leads to the notion of bubble rays, an idea which first appears in Josi Lou's thesis.

Bubble rays in $\mathcal{X}^\omega$ I

- We shall extend the Fatou-coordinates ϕ_j to a Fatou coordinate ϕ^ω defined on the entire basin for 0 under P by iteration.
- Define the parabolic ray-tree $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}^\omega$ as

$$\widehat{\mathcal{T}}^\omega = (\phi^\omega)^{-1}(\mathbb{R}) \bigcup \bigcup_{n\geq 1} (P^{-n}(0))$$

- A subset $\widetilde{M} \subset \widehat{\mathcal{T}}^\omega$ is defined to be compact iff it is closed and there exists n such that

$$P^n(M) \subset \bigcup_{j=0}^{q-1} \overline{\Delta_j}.$$

Bubble rays in $\mathcal{X}^\omega$ II

- We project $\widetilde{\mathcal{T}}^\omega$ to a parabolic model ray-tree $\widehat{\mathcal{T}} = \widehat{\mathcal{T}}^\omega$:

$$\widehat{\mathcal{T}}^\omega = \Pi^\omega(\widetilde{\mathcal{T}}^\omega) \cup (\widehat{\delta} \cap \mathcal{X}^\omega).$$

- A subset $\widehat{M} \subset \widehat{\mathcal{T}}^\omega$ is defined to be compact iff there are compact subsets $\widetilde{M} \subset \widehat{\mathcal{T}}^\omega$ and $\delta' \subset \widehat{\delta}$ such that

$$\widehat{M} = \delta' \cup \Pi^\Omega(\widetilde{M}).$$

Bubble rays in $\mathcal{X}^\omega$ III

Definition 8. *A bubble ray is any infinite path in* $\widehat{\mathcal{T}}^\omega$ *starting at* $\widehat{\infty}$*. Any Bubble ray is naturally dynamically marked, e.g. preperiodic, periodic, ...*

Because the Julia set of P is locally connected we have.

Lemma 9. *Any Bubble ray converges to a unique point on* $\partial\mathcal{X}^\omega$*. And the set of Bubble rays parametrizes (non injectively) the boundary of* $\mathcal{X}^\omega$*. More precisely the projection of any strict iterated preimage of* 0 *has both a finte address and is the landing point of* $2q$ *rays. Any other point is the landing point of precisely one Bubble ray.*

Parametrizing $\mathcal{R}^\lambda$.

Theorem 10 (Uhre, work in progress). *There is a natural dynamically defined injection*

$$\eta^\omega : \mathcal{X}^\omega \to \mathcal{R}^\omega,$$

holomorphic in the interior and continuous on any compact subset of $\widehat{\mathcal{T}}^\omega$.

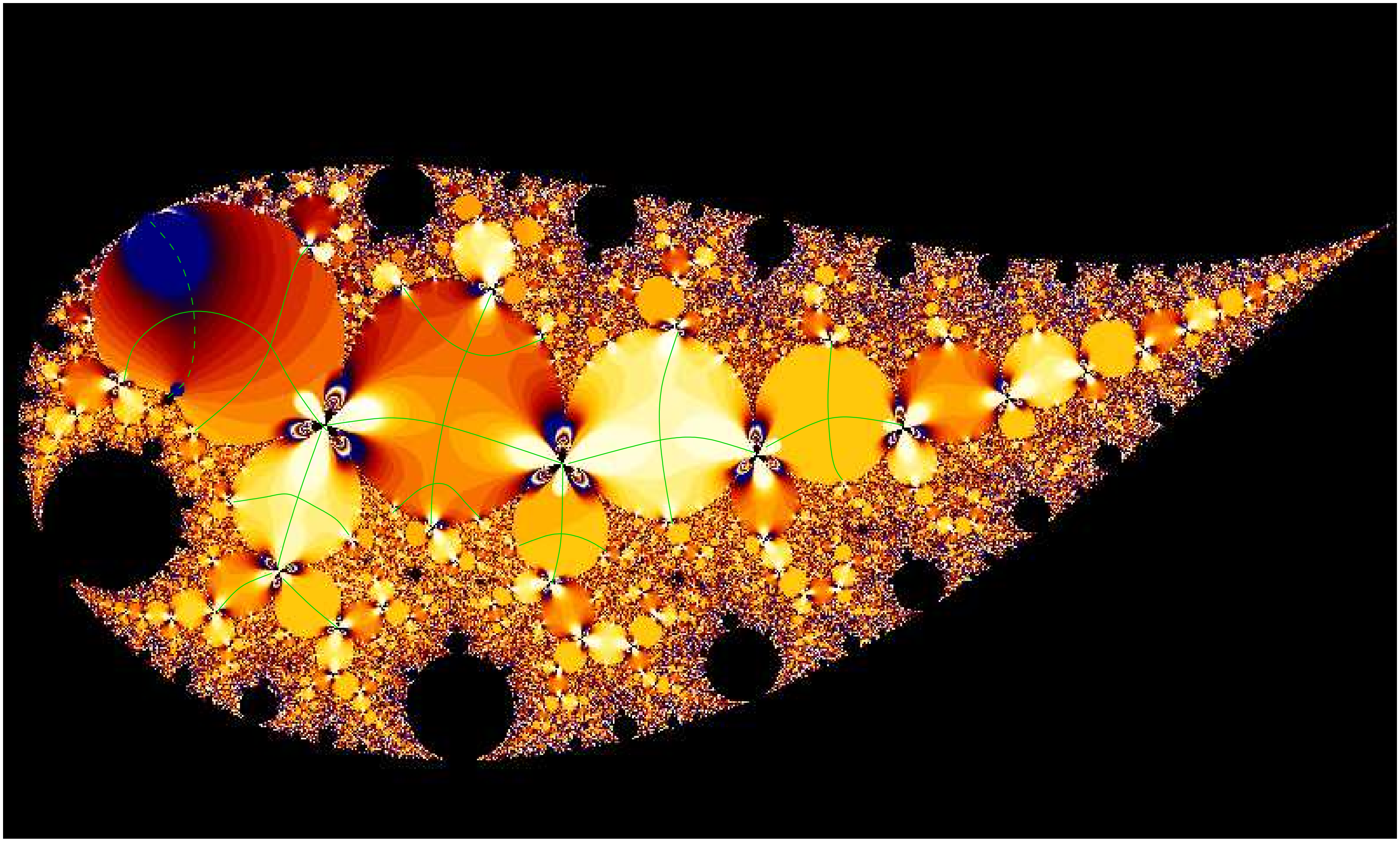
Definition 11. *Define the Parameter Bubble tree*

$$\mathcal{T}^\omega = \eta^\omega(\widehat{\mathcal{T}}^\omega)$$

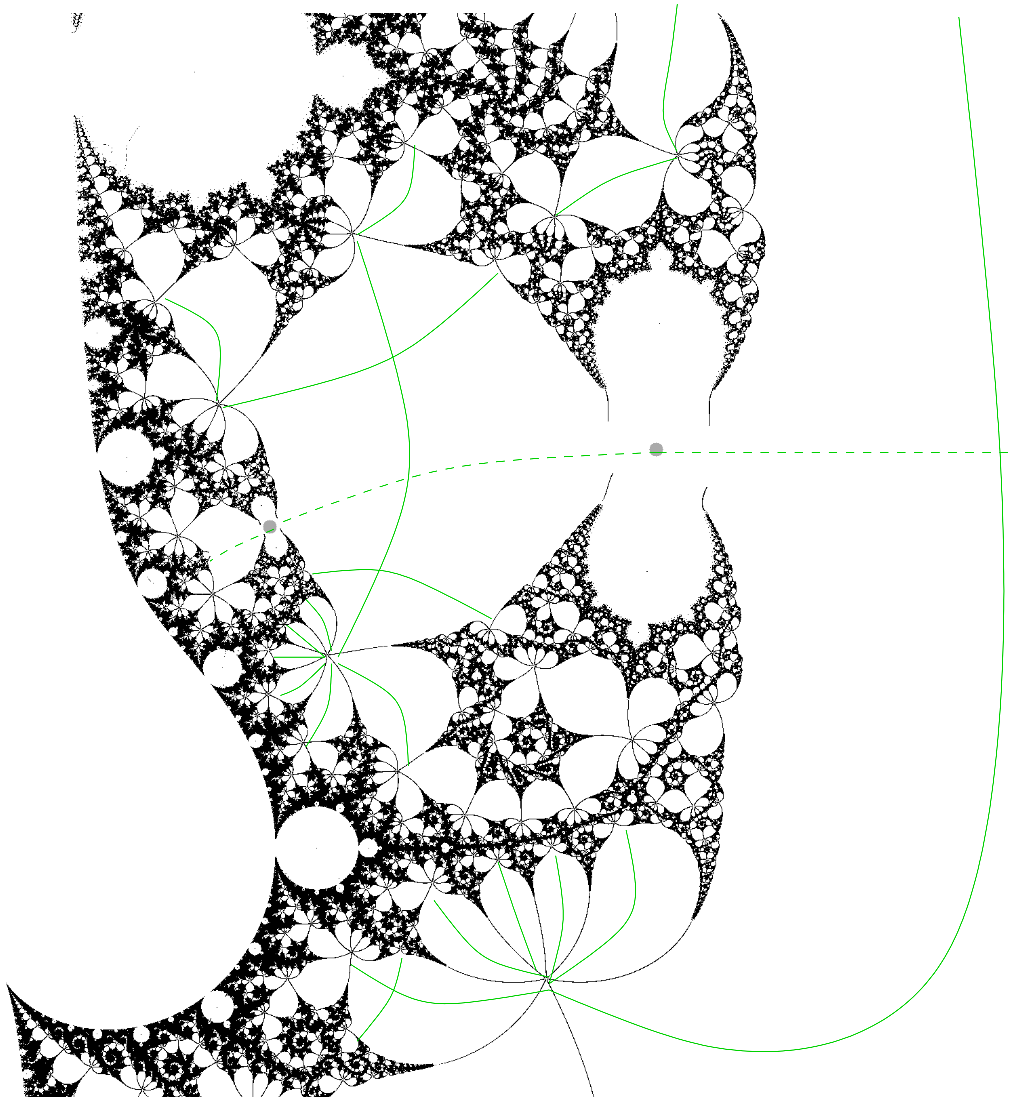
and simillarly Parameter Bubble rays.

Theorem 12 (Uhre, work in progress). *Any preperiodic ray in $\widehat{\mathcal{R}}^\omega$ lands at a relatively parabolic or Misieurewich parameter $\sigma \in M^\omega$.*

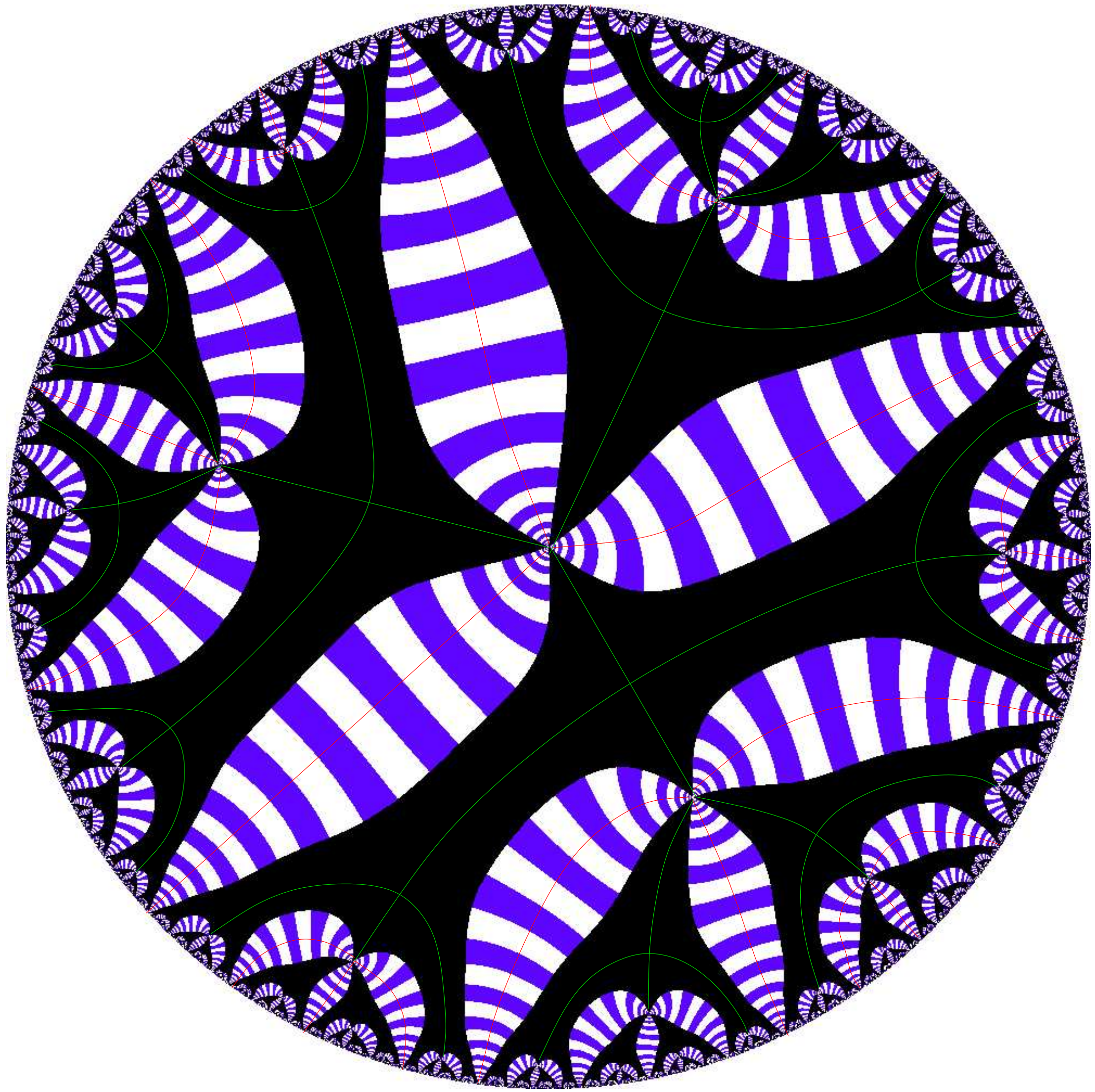
Actual $\widehat{\mathcal{R}}^{\omega_{1/3}}$ inverted



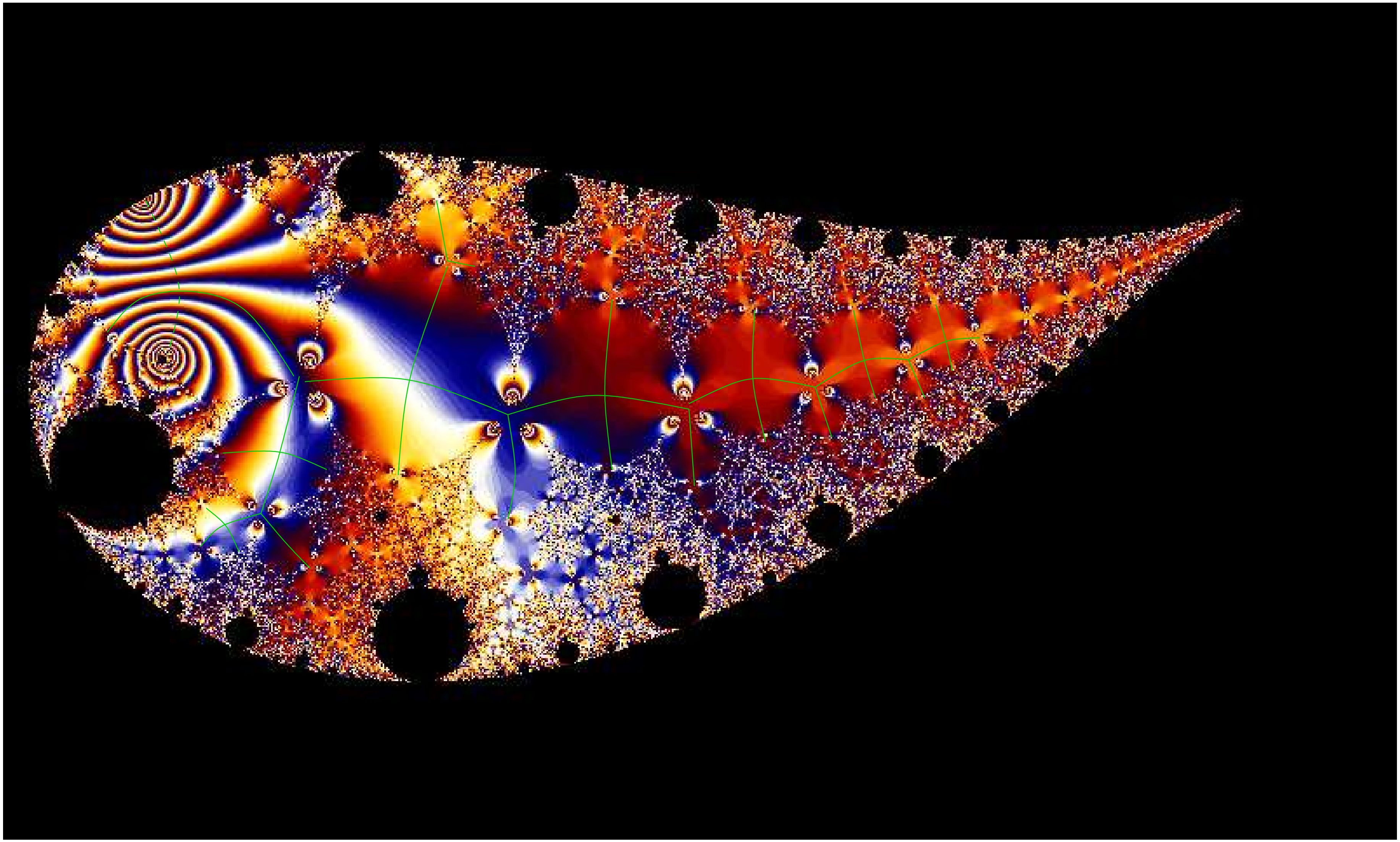
Actual $\widehat{\mathcal{R}}^{\omega_{1/7}}$



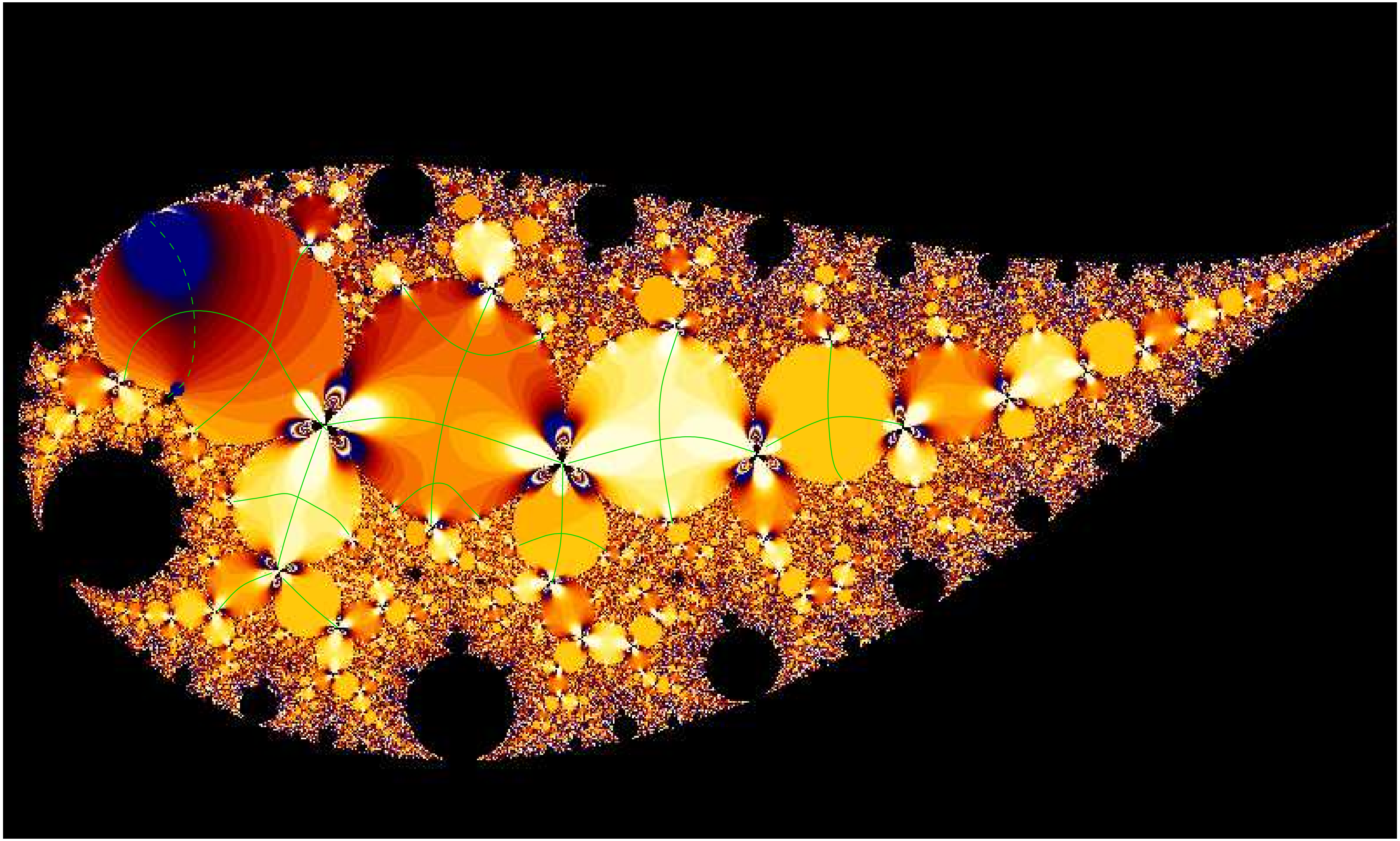
A model of $\widehat{\mathcal{R}}^{\lambda}$



Actual $\widehat{\mathcal{R}}^{\lambda}$ inverted



Actual $\widehat{\mathcal{R}}^{\omega_{1/3}}$ inverted



Actual $\widehat{\mathcal{R}}^\lambda$ inverted

Proof by movie.

The dynamical compactification

- Laura de Marco has defined a dynamical compactification of $\mathcal{M}_2$
- The algebraic compactification of $\mathcal{M}_2$ is $\mathbb{C}P^2$. In the de Marco compactification each ideal point

$$(\omega_{p/q}, \omega_{-p/q}, \infty) \in \mathbb{C}P^2$$

  is replaced by a copy of the Riemann's sphere, where each point on the sphere represents a measure on the sphere obtained as the weak limit of the measures of maximal entropi of degenerating maps.

Refined dynamic Compactification I

- In a joint but unfinished work joint with Adam Epstein we have studied the divergence of the limb $L^\lambda_{-p/q}$ as λ converges subtangentially to $\omega_{p/q}$. The principal idea is to prove that appropriately normalized representatives of the q-th iterate of maps in $L^\lambda_{-p/q}$ following the holomorphic motion by λ converges locally uniformly on $\mathbb{C}^*$ to a map in M^1 with the same combinatorics.
- Parallel studies have been and are undertaken by Anja Kabelka in her Thesis, and by Jan Kiwi.

Refined dynamic Compactification II

- The Conjectural refinement of the de Marco compactification is that each Riemann sphere in the dynamical compactification is naturally isomorphic to $Per_1(1)$. Moreover at the dyadic tips of M^1 in these spheres are attached additionally a countable number of Riemann spheres, which are naturally isomorphic to $Per_1(0)$, and which are enumerated by the different once renormalizeable primitive copies in the corresponding dyadic decoration of the $-p/q$ limb.

Refined dynamic compactification III

- Anja Kabelka has proved the real line version of the first part of the Conjectural refinement. Working on the real line she can apply kneeding sequences to control combinatorics.
- Jan Kiwi is to use non-Archimedean dynamics in his approach.
- Our approach is to use generalized Yoccoz puzzles.

Convergence of $Per_1(\lambda)$ to $Per_1(\omega_{p/q})$

- The limb $L^{\lambda}_{-p/q}$ diverges to ∞ under subtangential approach to $\omega_{p/q}$.
- But what about the other limbs $L^{\lambda}_{p'/q'}$, $p'/q' \neq -p/q$, do they remain bounded as λ converges subtangentially to $\omega_{p/q}$?
- **Theorem 13 (Uhre, work in progress).** *Yes, the other limbs remain bounded as λ converges subtangentially to $\omega_{p/q}$.*

THE END

THANK YOU TO THE ORGANIZERS