Parabolic implosion

from discontinuity to renormalization

Mitsuhiro Shishikura

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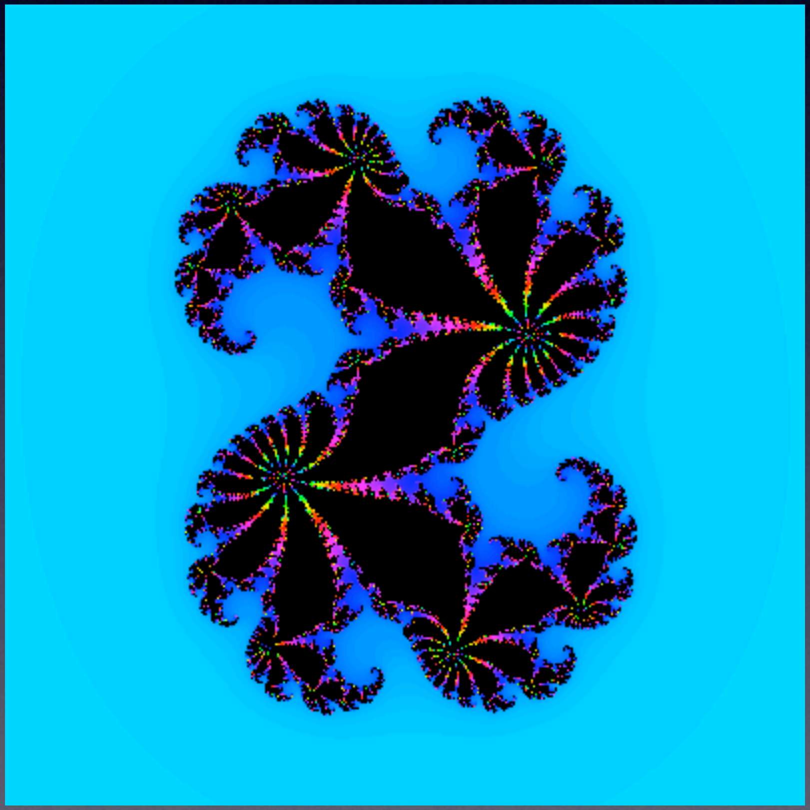
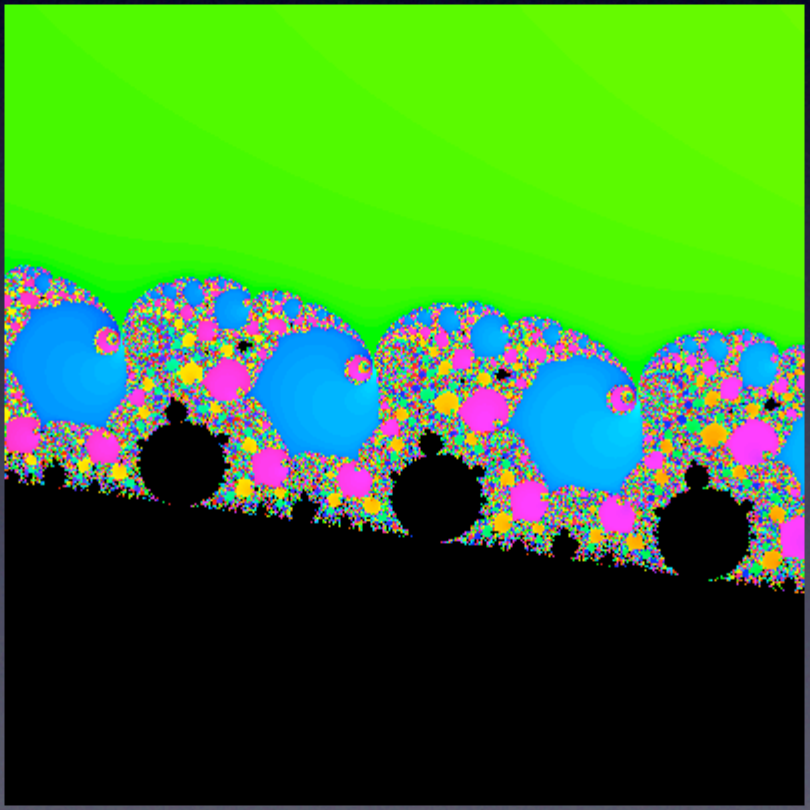


Discontinuity of Julia sets



“Periodicity’ in parameter space

Plan

Bifurcation of *parabolic* periodic points -- parabolic implosion

(saddle-node, intermittent chaos, periodic pts that bifurcate)

Local change of dynamics (“egg beater”) => Global effects

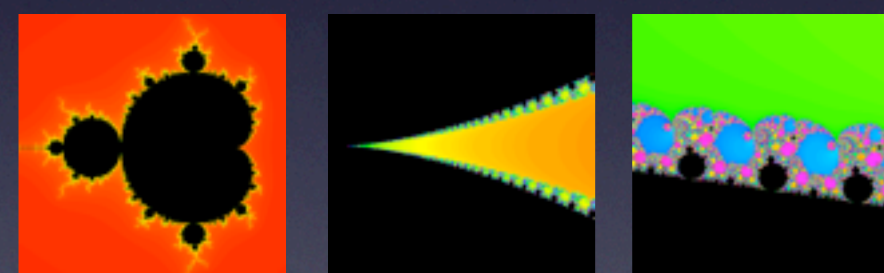
Main example: $f_0(z) = z + z^2 \; (\sim z^2 + \frac{1}{4})$ and its perturbation

Discontinuity of Julia sets

Richness of bifurcated Julia sets

when perturbed to “wild direction”

“Periodicity” in parameter space



Tools by Douady-Hubbard:

Fatou coordinates, Ecalle-Voronin cylinders, horn map, phase

Parabolic/Near-parabolic *renormalization*:

study of irrationally indifferent periodic points

Basic definitions

f polynomial (or just holomorphic mapping for local problems)

fixed point $f(z_0) = z_0$ multiplier $\lambda = f'(z_0)$

attracting $|\lambda| < 1$

indifferent $|\lambda| = 1$ **parabolic** $\alpha \in \mathbb{Q}$

$\lambda = e^{2\pi i\alpha}$ irrationally indifferent $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

repelling $|\lambda| > 1$

filled-in Julia set $K_f = \{z \in \mathbb{C} : \ \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}$

Julia set $J_f = \partial K_f = \text{closure of } \{\text{repelling periodic points}\}$

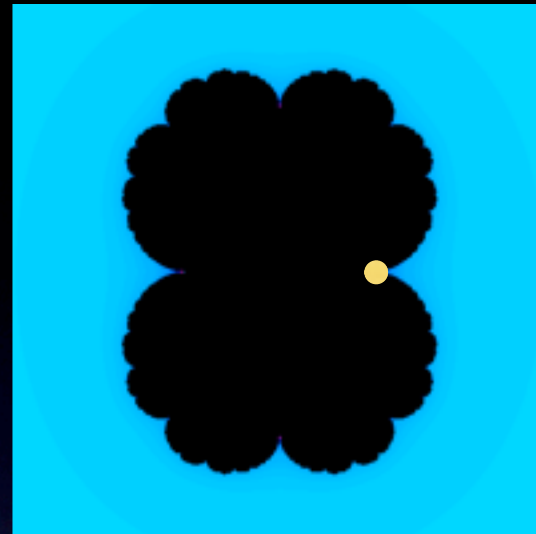
(chaotic part)

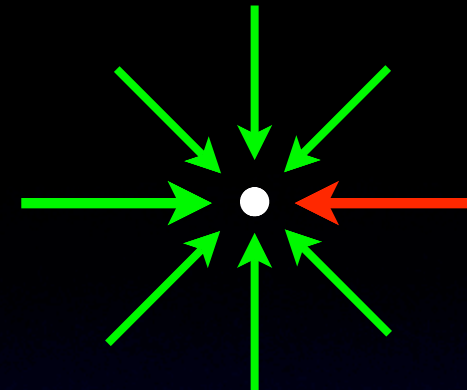
We consider the bifurcation of a parabolic fixed point and its global effect.

Tame and Wild directions

$f_\varepsilon(z) = z + z^2 + \ldots + \varepsilon$

0 is parabolic for f_0
with $\lambda_0 = 1$

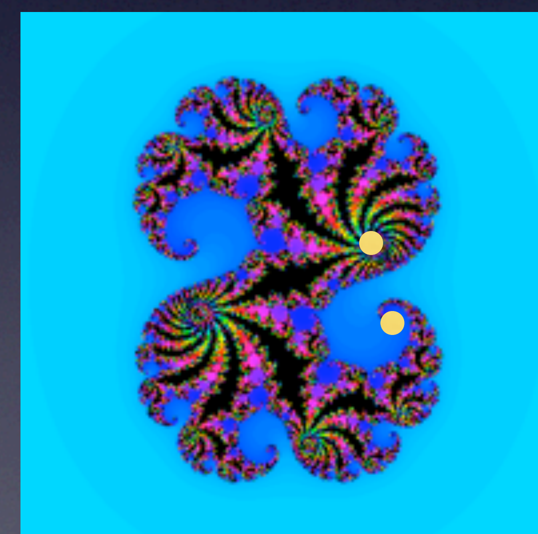




Tame direction
safe side

Wild direction
rich side

narrow chance to be rich

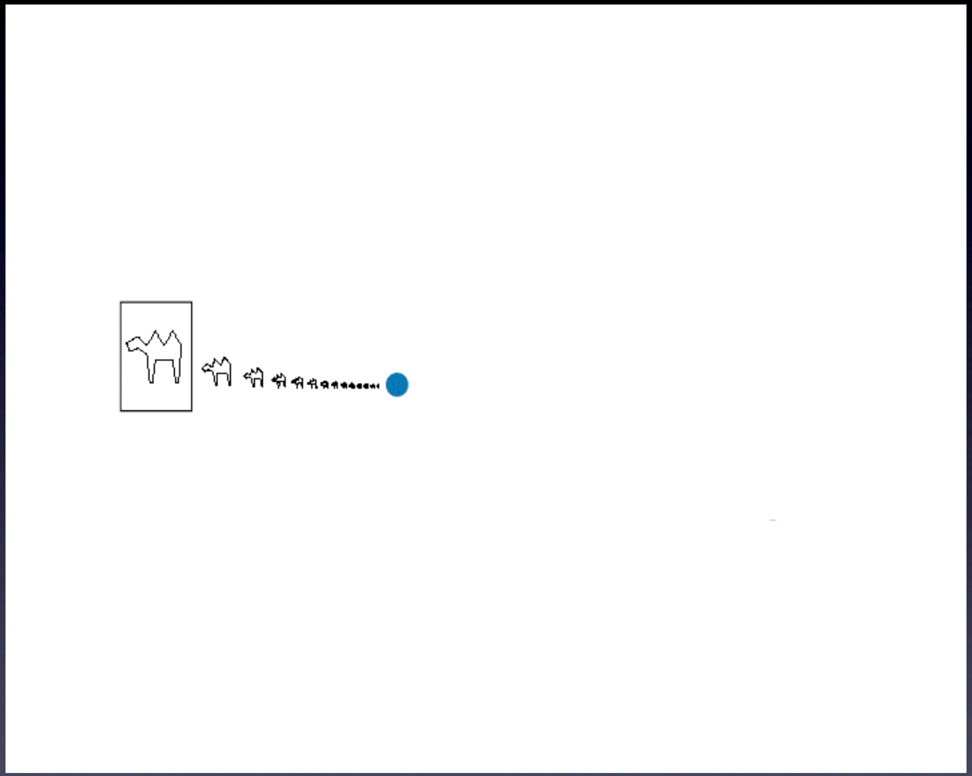


Continuous change
when restricted to tame direction

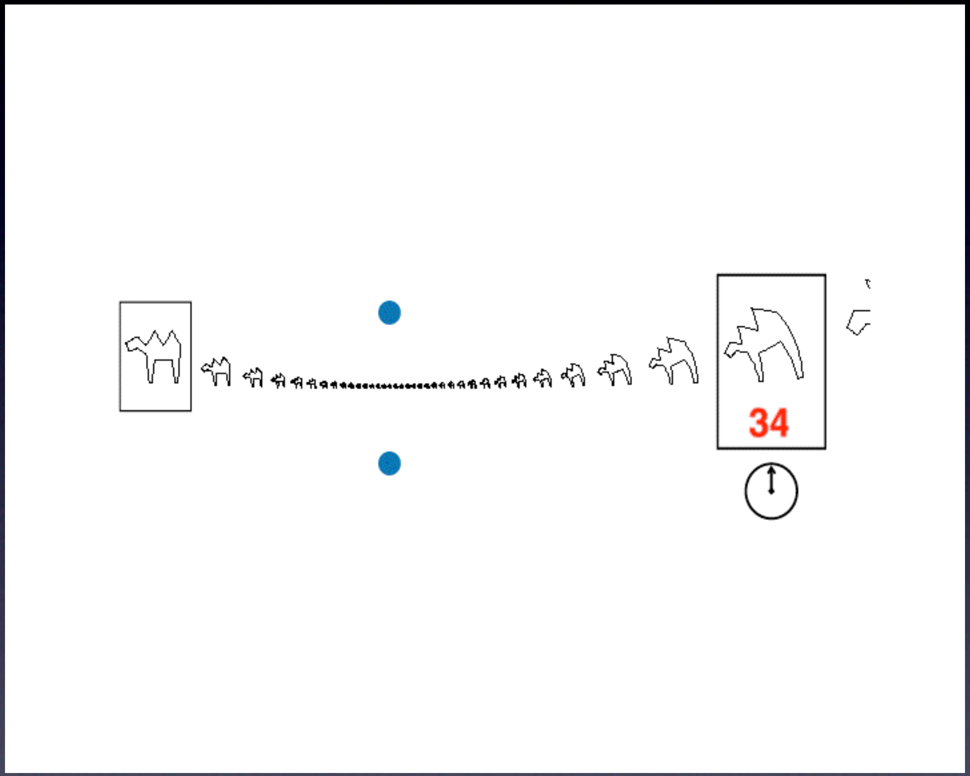
for example, $f(z) = e^{2\pi i\alpha} z + \ldots$
$\alpha \in \mathbb{R}$

Egg beater or Douady-Hubbard's get-rich-quick scheme®

before
f_0



after
perturbed into wild direction



Orbits from the left tend to the fixed point

Imagine that a point in the box on the right is:
escaping point
(inverse image of) repelling periodic point
inverse image of critical point

Theorem (discontinity of Julia sets)
Let $f_\varepsilon(z) = z + z^2 + \varepsilon$. Then

$$int\, K(f_0) = \{z :\ f^n \to 0 \text{ uniformly in a neighborhood of } z\}$$

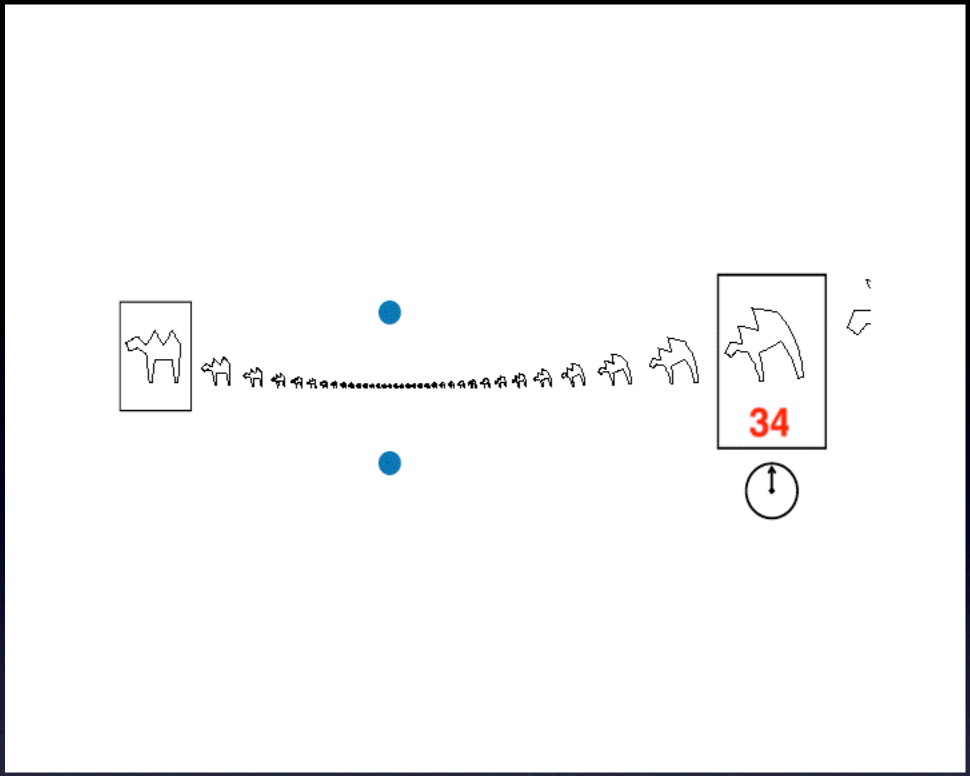
(parabolic basin).
Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \to 0$ and $\{\frac{\pi}{\sqrt{\varepsilon_n}}\}$ converges modulo $\mathbb{Z}$, i.e. for some integers $k_n \in \mathbb{Z}$,

$$\lim_{n\to\infty}\left(\frac{\pi}{\sqrt{\varepsilon_n}} - k_n\right) = -\beta$$

(for example, $\varepsilon_n = \frac{\pi^2}{(n-\beta)^2}$).
Then

$$J(f_0) \subsetneqq J(< f_0, g_\beta >) \subset \liminf_{n\to\infty} J(f_{\varepsilon_n})$$
$$\subset \limsup_{n\to\infty} K(f_{\varepsilon_n}) \subset K(< f_0, g_\beta >) \subsetneqq K(f_0)$$

where $g_\beta = \lim_{n\to\infty} f_{\varepsilon_n}^{k_n}$ in $int\, K(f_0)$, and
$< f_0, g_\beta >=$ "two generator dynamics" generated by f_0 and g_β.



Pick a point w in the box on the right which is escaping.

If a point z in the box on the left arrives at this point, i.e. $f_{\varepsilon_n}^{k_n}(z) = w$, then $z \in int\, K(f_0) \setminus K(f_{\varepsilon_n})$.

Hence $int\, K(f_0) \setminus \limsup_{n\to\infty} K(f_{\varepsilon_n}) \neq \emptyset$.

Pick a point w in the box on the right which is an inverse image of a repelling periodic point.

If a point z in the box on the left arrives at this point, i.e. $f_{\varepsilon_n}^{k_n}(z) = w$, then $z \in J(f_{\varepsilon_n}) \setminus J(f_0)$.

Hence $\liminf_{n\to\infty} J(f_{\varepsilon_n}) \setminus J(f_0) \neq \emptyset$.

The behavior of the critical point changes periodically. ("phase parameter")

$\Longrightarrow$ the "periodicity" in the parameter space ($\frac{\pi}{\sqrt{\varepsilon}}$ *modulo* $\mathbb{Z}$ matters).

Tools to analyze Egg beater dynamics

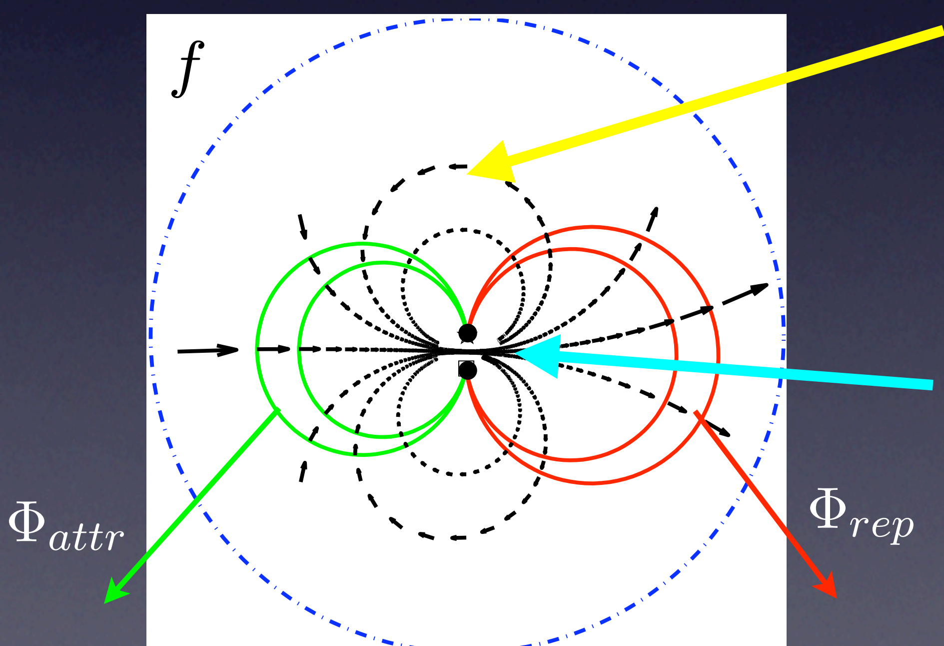
Difficulty: effects of perturbation on **unboundedly high** iterates of the map

Work in Fatou coordinates, which conjugate f to $T : z \mapsto z + 1$

Take a croissant-shaped fundamental strip, which is bounded by a curve ℓ and its image $f(\ell)$.

Glue ℓ and $f(\ell)$ by f to obtain a Riemann surface which is isomorphic to $\mathbb{C}/\mathbb{Z}$.

Lift of the map to $\mathbb{C}/\mathbb{Z}$ is a Fatou coordinate.



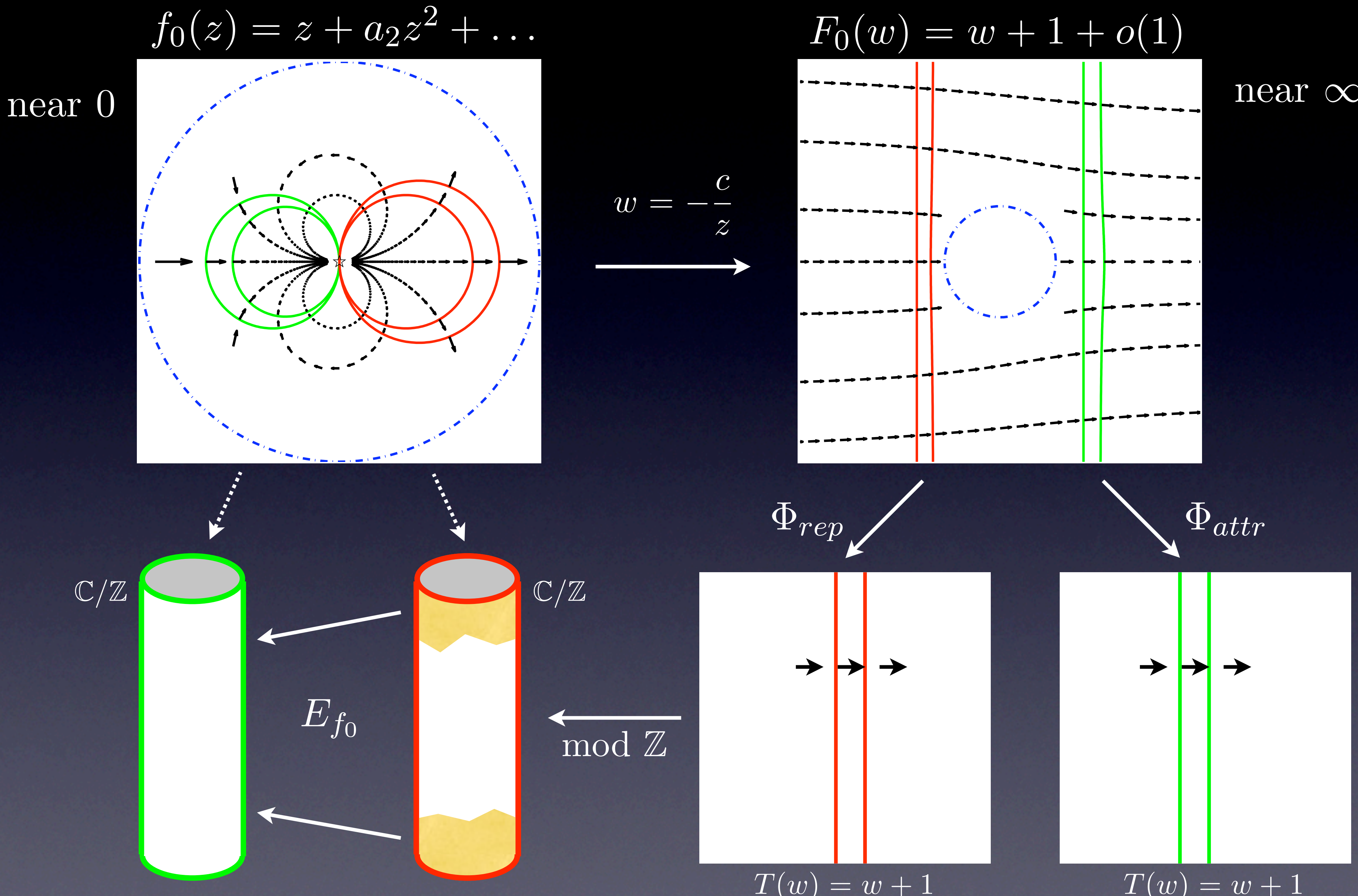
This part of dynamics is represented by a horn map E_f which is

- partially defined, non-linear
- continuous under perturbation.

This part of dynamics is represented by an identification between attracting and repelling Fatou coordinates which is

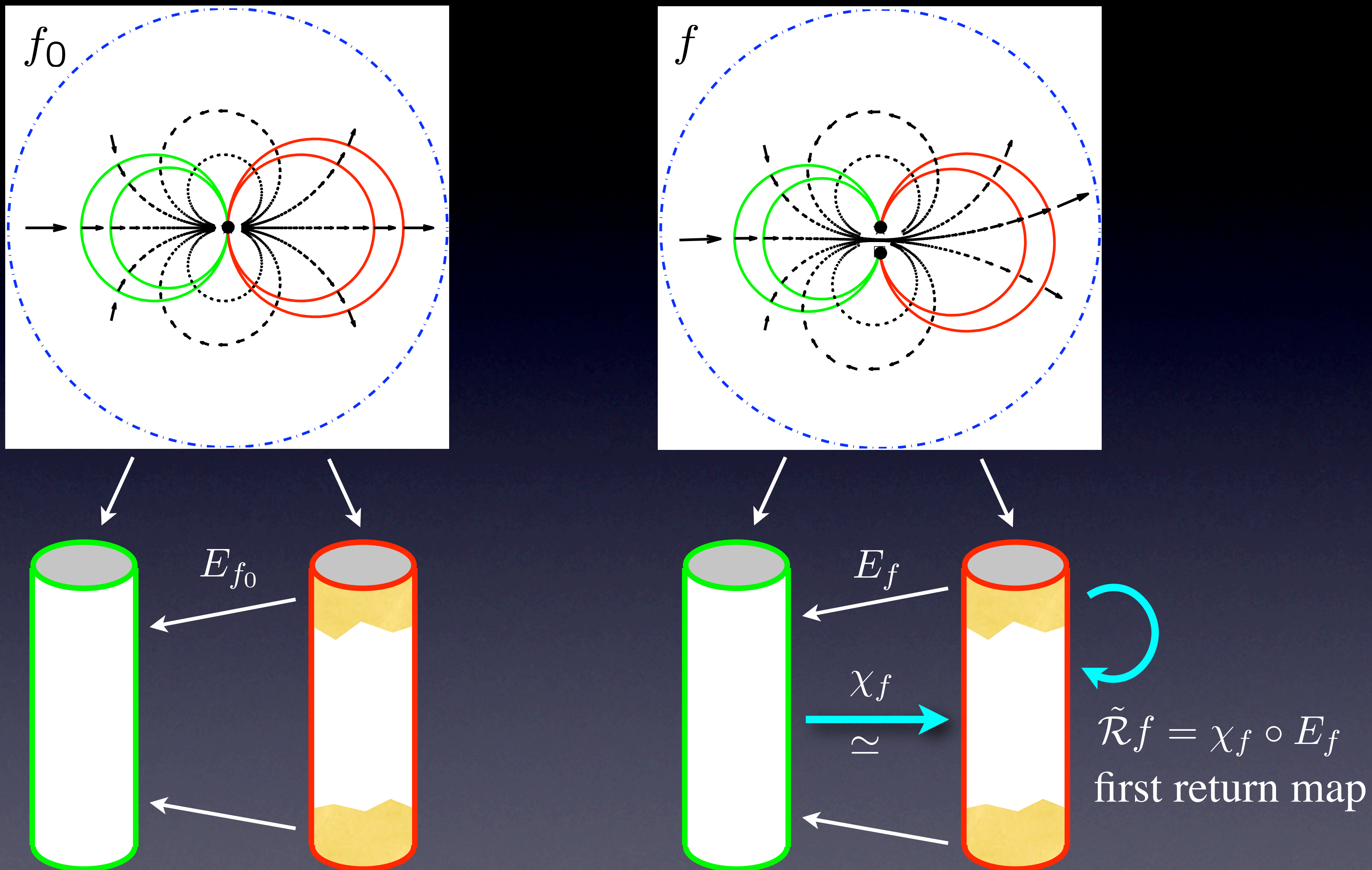
- an isomorphism
- sensitive wrt perturbation.

Fatou coordinates Φ_{attr}, Φ_{rep} and Horn map E_{f_0}



$$E_{f_0}(z) = \Phi_{attr} \circ \Phi_{rep}^{-1} \qquad \Phi_{\dots}(f_0(z)) = \Phi_{\dots}(z) + 1$$

Perturbation $f'(0) = e^{2\pi i \alpha}$, α small $|\arg \alpha| < \frac{\pi}{4}$ (wild direction)

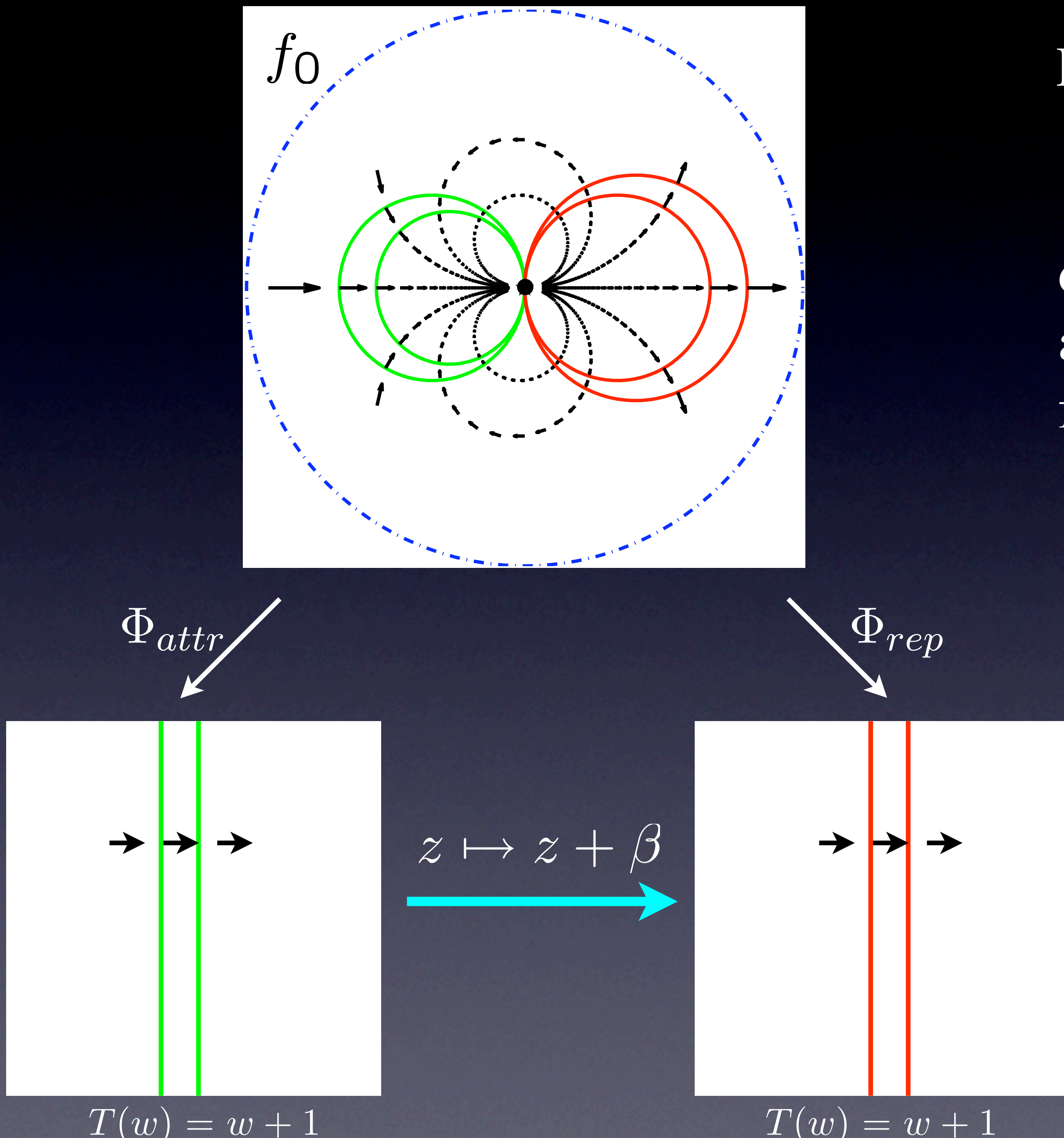


E_f depends continuously on f

(after a suitable normalization)

$$\chi_f(z) = z - \frac{1}{\alpha}$$

Lavaurs map



Lavaurs map

$$g_\beta(z) = \Phi_{rep}^{-1}(\Phi_{attr}(z) + \beta)$$

describes how the orbits entering attracting fundamental strip go out from repelling fundamental strip.

$$g_\beta = \lim_{n \to \infty} f_n^{k_n} \text{ in } int\, K(f_0)$$

if $f_n(z) = e^{2\pi i \alpha_n} z + \ldots,$

$$\lim_{n \to \infty} \left(\frac{1}{\alpha_n} - k_n \right) = -\beta,$$

or

if $f_n(z) = z + z^2 + \varepsilon_n,$

$$\lim_{n \to \infty} \left(\frac{\pi}{\sqrt{\varepsilon_n}} - k_n \right) = -\beta.$$

Limit dynamics

$$f_0 \circ g_\beta = g_\beta \circ f_0 = g_{\beta+1} \quad \text{in} \quad K_{f_0}$$

$$\langle f_0, g_\beta \rangle = \{ f_0^n \circ g_\beta^m : \ (m = 0 \text{ and } n \geq 0) \text{ or } (m > 0 \text{ and } n \in \mathbb{Z}) \}$$

(when $m > 0$, each element is defined in an open subset of $\mathbb{C}$)

$$K(\langle f_0, g_\beta \rangle) = \mathbb{C} \setminus \{ z : \exists h = f_0^n \circ g_\beta^m \in \langle f_0, g_\beta \rangle, h(z) \in \mathbb{C} \setminus K_{f_0} \}$$

$$J(\langle f_0, g_\beta \rangle) = \text{closure of } \{ \text{repelling fixed points of } h = f_0^n \circ g_\beta^m \in \langle f_0, g_\beta \rangle \}$$

$$J(f_0) \subsetneqq J(\langle f_0, g_\beta \rangle) \subset \liminf_{n \to \infty} J(f_n) \subset \limsup_{n \to \infty} K(f_n) \subset K(\langle f_0, g_\beta \rangle) \subsetneqq K(f_0)$$

$$K(\langle f_0, g_\beta \rangle) = J(\langle f_0, g_\beta \rangle) \implies \lim_{n \to \infty} K(f_n) = \lim_{n \to \infty} J(f_n) = J(\langle f_0, g_\beta \rangle)$$

$\exists h = f_0^n \circ g_\beta^m (\in \langle f_0, g_\beta \rangle)$ has an attracting fixed point

$$\implies \lim_{n \to \infty} K(f_n) = K(\langle f_0, g_\beta \rangle) \text{ and } \lim_{n \to \infty} J(f_n) = J(\langle f_0, g_\beta \rangle)$$

Further results

Discontinuity of straightening of polynomial-like mappings
(Douady-Hubbard, first published account on parabolic implosion)

Limit parameter space for $z^2 + c$ around $c_0 = 1/4$ (wrt $\frac{1}{\sqrt{c-1/4}}$)
versus parameter space of $\langle f_0, g_\beta \rangle$ (Lavaurs)

Non-local connectivity of connectedness locus for real/complex cubic polynomials (Lavaurs)

. . . . .

Parabolic/near-parabolic renormalization

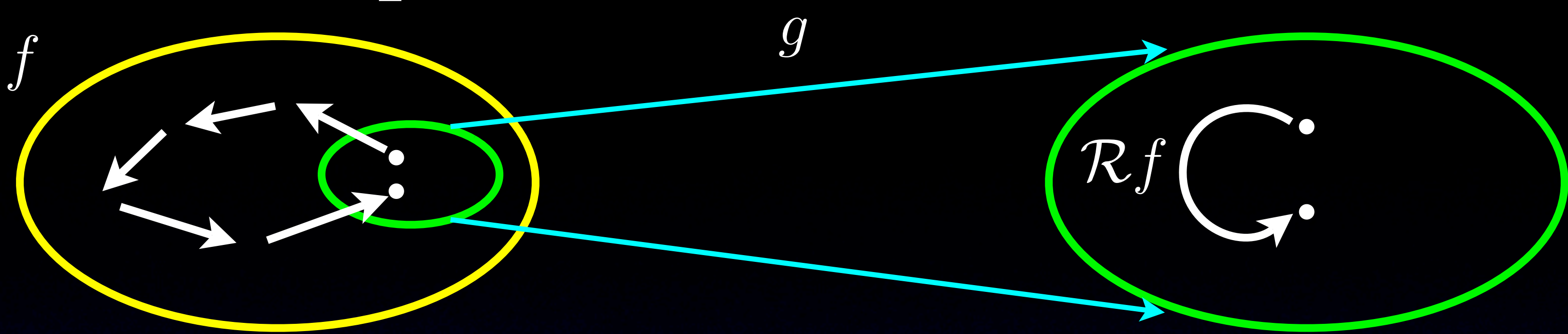
Renormalization for germs holomorphic functions with parabolic or near-parabolic fixed points (in wild direction)

S. with H. Inou: an invariant class of maps for the renormalization.

=> control on irrationally indifferent fixed points when
the continued fraction of the rotation number has large coefficients.

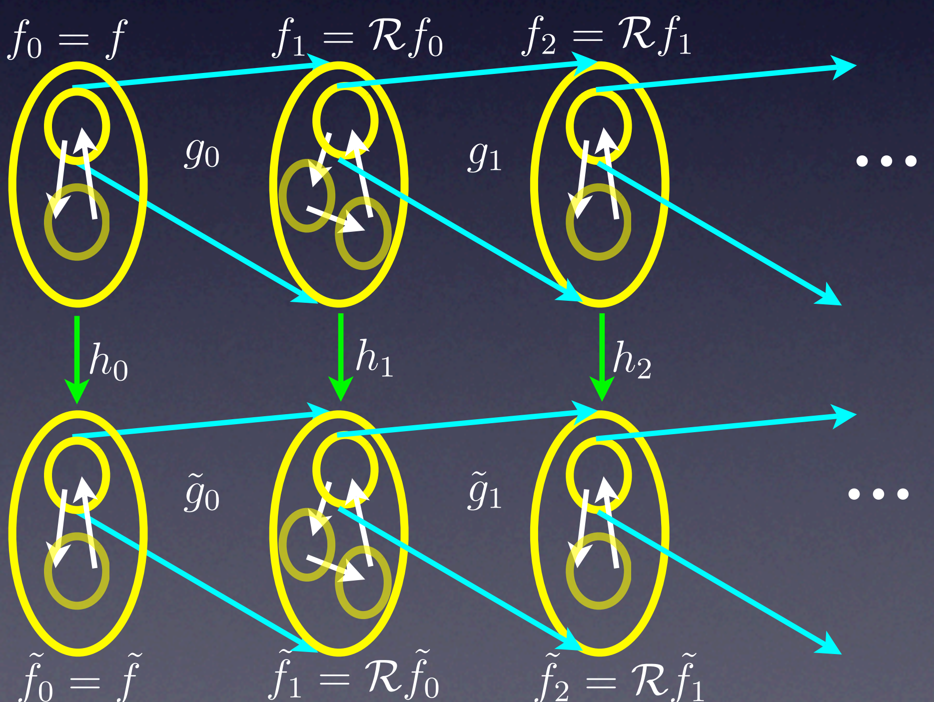
=> Buff-Cheritat's result on quadratic Julia sets with positive area.

Return map and Renormalization



Renormalization $f \rightsquigarrow \mathcal{R}f =$ first return map f (after rescaling)

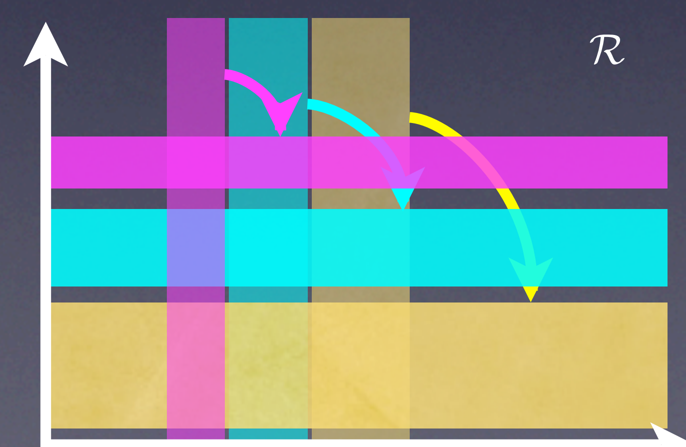
If f is infinitely renormalizable, ...



$f \rightsquigarrow \mathcal{R}f$

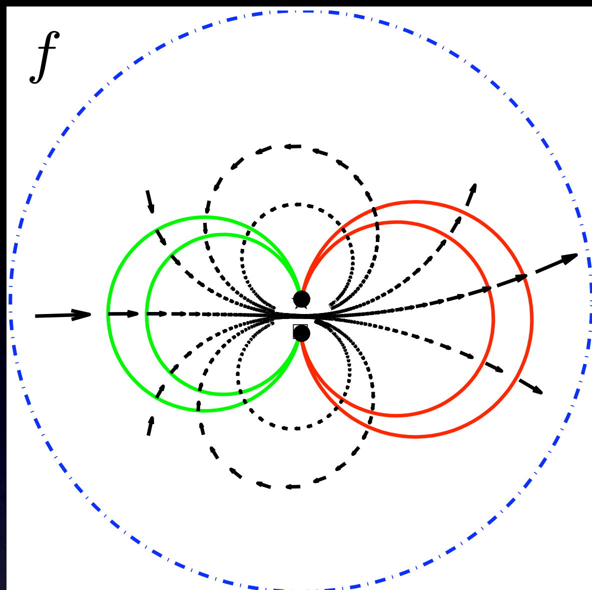
as a “meta dynamical system” on a space of dynamical systems

Often one expects a hyperbolic dynamics



rigidity: weak conj. upgraded to nicer one

Near-parabolic Renormalization (cylinder renorm.)



$\mathrm{Exp}^\sharp : \mathbb{C}/\mathbb{Z} \xrightarrow{\simeq} \mathbb{C}^*, \;\; z \mapsto e^{2\pi i z}$

$\mathcal{R}f$ is conjugate to the return map on red croissant via repelling Fatou coordinate and $\mathrm{Exp}^\sharp$

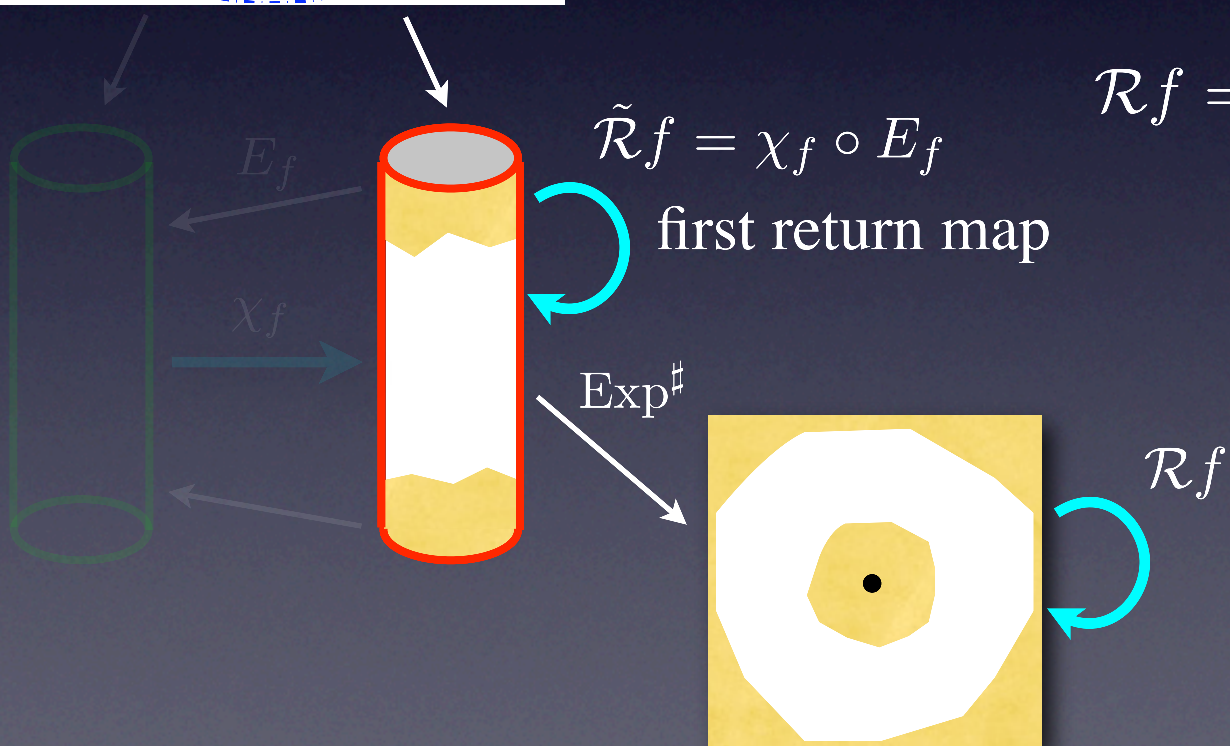
$$f = e^{2\pi i\alpha} h, \quad h = z + O(z^2)$$

$$f \;\; \leftrightarrow \;\; (\alpha, h)$$

$$\mathcal{R}f = e^{-2\pi i \frac{1}{\alpha}} h_1, \;\; h_1 = z + O(z^2)$$

$$\mathcal{R}f \;\; \leftrightarrow \;\; (-\tfrac{1}{\alpha}, h_1)$$

$$h_1 = \mathrm{Exp}^\sharp \circ E_{e^{2\pi i\alpha}h} \circ (\mathrm{Exp}^\sharp)^{-1}$$
$$= \mathcal{R}_\alpha h$$



cylinders and $\mathcal{R}f$ defined when $h''(0) \neq 0$ and α sufficiently small

Theorem. (H. Inou and S.) There exists a class $\mathcal{F}_1$ of maps with a parabolic fixed point (non-degenerate) and a unique critical point and a large number N such that the near-parabolic renormalization $\mathcal{R}$ is defined on

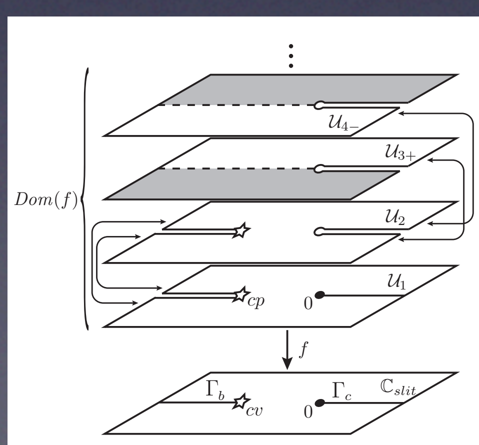
$$\{e^{2\pi i\alpha}h: \ \alpha\in\mathbb{R},\ 0<|\alpha|\le\tfrac{1}{N},\ h\in\mathcal{F}_1\}$$

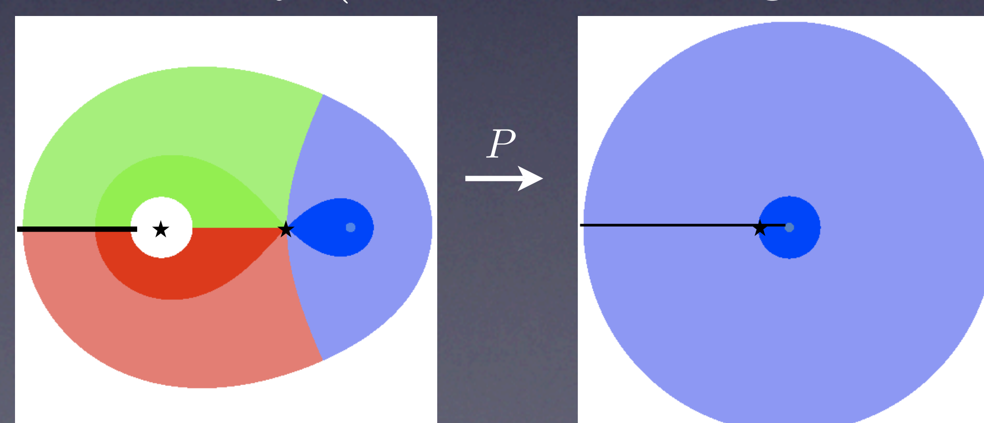
and hyperbolic. In fact, $\mathcal{R}$ can be considered as

$$\mathcal{R}:(\alpha,h)\mapsto(-\tfrac{1}{\alpha}\,mod\,\mathbb{Z},\mathcal{R}_\alpha h),$$

and $\alpha\mapsto-\frac{1}{\alpha}\,mod\,\mathbb{Z}$ is expanding. The “fiber direction” $\mathcal{F}_1$ is in one to one correspondence with the Teichmüller space of the punctured disk and the fiber map $\mathcal{R}_\alpha$ is holomorphic and uniform contracting with respect to the Teichmüller distance.

Invariant class $\mathcal{F}_1$ is characterized by (partial covering property.





Applications (work in progress)

Theorem. Let $f = e^{2\pi i\alpha}h$ and $\hat{f} = e^{2\pi i\alpha}\hat{h}$ where $h, \hat{h} \in \mathcal{F}_1$ (or $z + z^2$) and α is of high type (continued fraction coeffs $\geq N$). Then f and $\hat{f}$ are quasiconformally conjugate on the closure of the critical orbit. Moreover the conjugacy is $C^{1+\gamma}$-conformal on the critical orbit with some $\gamma > 0$.

Compare with McMullen's result on bounded type Siegel disks

Theorem. Let $f = e^{2\pi i\alpha}h$ where $h \in \mathcal{F}_1$ (or $z + z^2$) and α is of high type (continued fraction coefficientss $\geq N$). Then there exist open sets $U_n \ni 0$ and integers q_n $(n = 0, 1, \dots)$ such that f^{q_n} is defined on U_n and at most 3 to 1, $\bigcap_{n=0}^{\infty} U_n$ contains the critical orbit and consists of arcs ("hairs") which are disjoint from each other except at 0.

According to a discussion with Buff, Chéritat, Oversteegen, quadratic Julia sets with non-Bruno, high-type rotation number seem to be decomposable.

Merci!