Aeroplane Captures

Mary Rees

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Matings and Captures

- In the early 1980's, Adrien Douady and J.H. Hubbard defined the concept of *mating* of two polynomials of the same degree, giving a topological construction of certain rational maps.
- Adrien talked about mating in Minneapolis in 1983, illustrating this with "mating of the rabbits".
- A *capture* is another construction of a rational map from a polynomial, but this time from a single polynomial, together with a point in the backward orbit of a critical point of this polynomial.
- The term may first be used in writing in Wittner's thesis from Cornell in 1988, but was in use in the Hubbard school as least 5 years earlier.

Captures: Restrictions

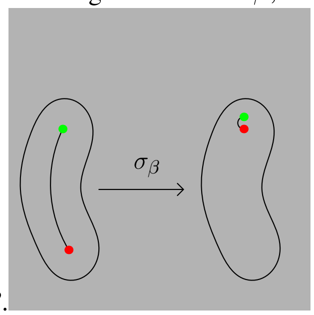
- In this talk I shall only consider critically finite hyperbolic rational maps.
- So it makes sense to describe captures up to Thurston equivalence.
- I shall also only consider rational maps of degree two.
- For a polynomial of degree two, I shall write v_1 for the finite critical value and v_2 for the infinite (fixed) critical point and value.

Captures and type III maps

- Captures are technically easier to study than matings, in a number of respects.
- One reason is that *type III maps* are easier to study than *type IV*.
- A critically finite branched covering f of degree two, with marked critical values v_1 and v_2 is *of type III* if v_1 is periodic and v_2 is nonperiodic but in the full orbit of v_1 under f.
- The map f is *type IV* if v_1 and v_2 are in distinct periodic orbits.
- Every capture and mating is of type III and type IV respectively, but the converse is not true.

Definition of σ_β

Given a directed arc β on the sphere, the homeomorphism σ_β is defined to be the identity outside a suitably small disc neighbourhood of β, and maps the first endpoint of β to the second endpoint of β.



Captures: Definition

- A (degree two preperiodic) *capture* is then described by a critically periodic quadratic polynomial f, and an arc β (up to suitable homotopy) in the dynamical plane of f, which we call the *capture path*.
- The arc β starts from v_2 and ends at a nonperiodic point $x \in \cup_{n\geq 2} f^{-n}(v_1)$ and *crosses the Julia set of f exactly once*, into the Fatou component containing x.
- The capture associated to f and β is then

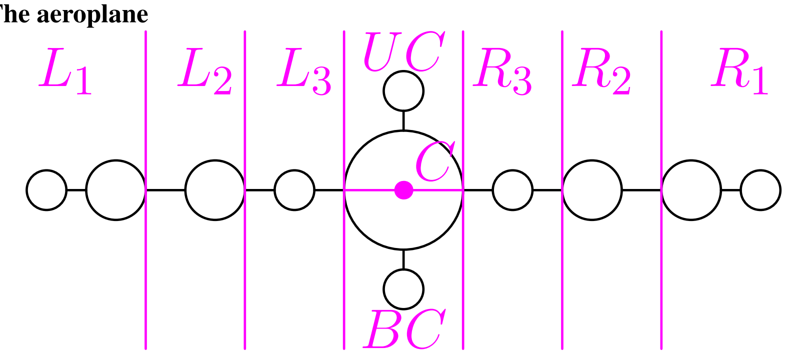
$$\sigma_\beta \circ f.$$

- Every type III map has such a form, but for a path β which may not be a capture path.

Special considerations for the rabbit

- If several Fatou components meet at the crossing point, then x should also be in one of the two Fatou components which is adjacent to β.
- We need to consider this for the rabbit polynomials, but not for the aeroplane.

The aeroplane



The capture and the second endpoint of the arc

- For a rabbit polynomial f, the second endpoint of a capture path β uniquely determines the capture $\sigma_\beta \circ f$ up to Thurston equivalence.
- For the aeroplane polynomial g, there can be two capture paths β_1 and β_2 with the same second endpoint such that $\sigma_{\beta_1} \circ g$ and $\sigma_{\beta_2} \circ g$ are *not* Thurston equivalent.
- But there are never more than two, and two happens relatively rarely with density tending to 0 as preperiod increases, for *some* second endpoints on the real line, where the crossing points have arguments in

$$\left(\frac{2}{7}, \frac{1}{3}\right) \cup \left(\frac{2}{3}, \frac{5}{7}\right)$$

Equivalence to rational maps

By the result of Tan Lei's thesis, a capture $\sigma_\beta \circ f$ is Thurston equivalent to a rational map if and only if the endpoint x of β is not in the limb of the Julia set containing the critical value v_1 of f. The number of possible endpoints of preperiod n is

- $\frac{3}{7}2^{n+1} + O(1)$ for each of the two rabbit polynomials,
- $\frac{1}{3}2^{n+1} + O(1)$ for the aeroplane polynomial.

Captures shared by the aeroplane with the rabbits

- But by the shared mating-capture construction of Wittner's thesis, $\frac{1}{7}2^{n+1} + O(1)$ of these endpoints for the aeroplane polynomial, those with arguments in

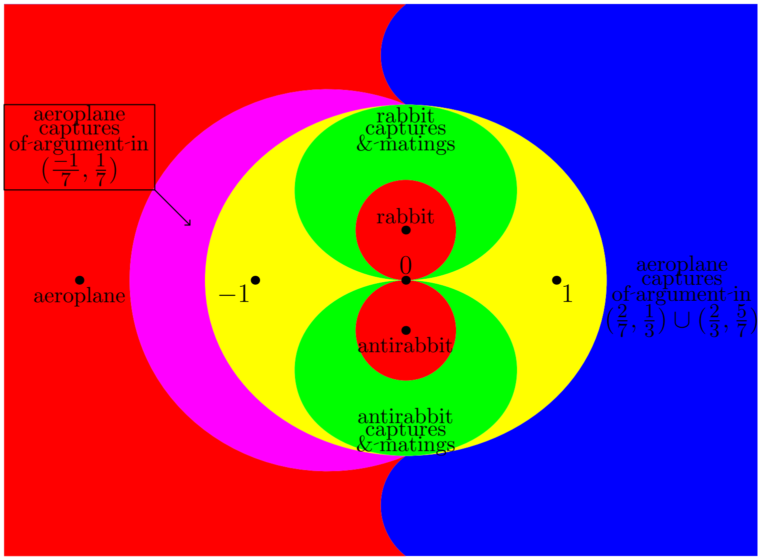
$$\left(\frac{1}{7}, \frac{2}{7}\right) \cup \left(\frac{5}{7}, \frac{6}{7}\right),$$

  are Thurston equivalent to rabbit captures.

- Therefore the number of Thurston equivalence classes of rabbit and aeroplane captures of preperiod n is at most

$$2^{n+1} + 2^{n+1}\frac{1}{21}(1 + o(1)).$$

The parameter space V_3 of quadratic rational maps with a marked critical value v_1 of period 3, modulo Möbius conjugacy, looks like this:



Rabbit captures

- It is relatively easy to show that rabbit/antirabbit captures for capture paths with distinct endpoints are not Thurston equivalent.
- This is small part of a larger picture. Each of the two rabbit Julia sets, minus the critical value limb, is a model for part of the parameter space V_3 (the space of quadratic rational maps with a marked critical point of period 3).
- In particular, all type III and IV hyperbolic components in the green regions are rabbit/antirabbit captures and matings, each one described in just one way.

Rabbit captures continued

- This has been known since the late 1980's and somewhat similar results for V_2 featured in Luo's Cornell thesis in 1995.
- Recently complete proofs for non-renormalisable parameter values in V_2 have been given by Aspenberger and Yampolsky, and in a slightly different context by Timorin.
- This is a topological picture only,
- in fact a bit less, because although the work by Timorin and Aspenberger/Yampolsky in a slightly different context and earlier work by Roesch in a somewhat different context implies that the common boundaries of the yellow and green regions are Jordan arcs, I think this is not known for the common boundaries of the yellow and purple, and similarly for the common boundary of the yellow and blue.

Aeroplane regions

- A more advanced theory shows that all type III components in the "easy" purple region are precisely the aeroplane captures with arguments in
$$\left(-\frac{1}{7},\frac{1}{7}\right)$$
each one represented as a capture in exactly one way.

- However this cannot be true in the "hard" blue region, because this region contains at most
$$\frac{1}{21}.2^{n+1}(1+o(1))$$
distinct captures of preperiod n, while the number of type III components in this region is
$$\frac{2}{21}.2^{n+1}+O(1).$$

The hard (blue) aeroplane region

- We write f for the aeroplane polynomial.
- By using a *fundamental domain* for $V_{3,n}$, the complement in V_3 of centres of type III hyperbolic components of preperiod $\leq n$, we can find a unique representation of each type III map in the blue region, and a possible topological model for this part of parameter space.
- This involves describing each type III map in the region as $\sigma_\beta \circ f$ (up to Thurston equivalence) for some path β *but the number of capture paths is only* $\geq \lambda^n$ *for some* $1, < \lambda < 2$.
- This raises a number of questions.

Questions

- We write g_x for the aeroplane capture in the hard (blue) region with endpoint at x. In the low density number of cases when there are two captures with endpoint x, we choose the capture with path crossing unit circle in the upper half-plane.
- We fix $n \geq 1$, and write $O_n(f)$ for the points in the full orbit of v_1 of preperiod n.
- The map $x \mapsto g_x$ is thus a map between two finite sets, the domain with $\frac{1}{21}2^{n+1}+O(1)$ elements, and the range with $\frac{2}{21}2^{n+1}+O(1)$ elements.
- We write
$$a_{n,r} = \#(\{x \in O_n(f) : \#(\{y : g_y = g_x\}) = r\}).$$
- What can one say about the asymptotics of the numbers $a_{n,r}$ as $n \to \infty$ for each $r \geq 1$?

Remarks

- In the "easy" regions (including both rabbit and antirabbit) the numbers corresponding to $a_{n,r}$ are 0 for all $r \geq 2$.
- The image size of the map $x \mapsto g_x$ is

$$\sum_r \frac{a_{n,r}}{r}.$$

Further questions

- Is the map $x \mapsto g_x$ typical?
- What sort of behaviour is typical?

Typical behaviour of elements of $\mathbf{n}^m$

- The set of maps from a set of m elements to a set of n elements identifies with $\mathbf{n}^m$ where $\mathbf{n} = \{1, \cdots n\}$.
- The mean image size is

$$n\left(1 - \left(1 - \frac{1}{n}\right)^m\right) \to (1 - e^{-a})n$$

as $m/n \to a$ and $n \to \infty$. This was shown to me by Jon Woolf, who gave a simple inductive proof.

- In fact, the image size of maps falls within $c_1\sqrt{n}$ of the mean, with probability tending to 1 as c_1 tends to 0, uniformly in n, for $m/n \to a$.
- This is one of many facts which can be proved using the powerful saddle-point method for estimating certain contour integrals, and the asymptotics of the *Stirling numbers of the second kind*

$$S(m, k),$$

the number of partitions of $\mathbf{m}$ into k unordered disjoint subsets.

Typical behaviour from the saddle point method

- These methods were used in the 1950's, for example by L.Moser and M. Wyman, who computed asymptotics of Stirling numbers of both first and second kind.
- We may hear something about Stirling numbers of the first kind in the next talk, possibly under a different name.
- Using the saddle point method we can also show that the maximal inverse image size of a map in $\mathbf{n}^m$ is $\geq \log n/2 \log\log n$ with probability tending to 1 as $n \to \infty$, if m/n is bounded from 0.

- We can also compute the mean number of points in $\mathbf{n}$ with exactly r preimages for maps in $\mathbf{n}^m$, for each r, and, again, for most maps the number with exactly r preimages falls within $c_1\sqrt{n}$ of the mean with probability $\to 1$ as $c_1 \to 0$, as $m/n \to a$ and $n \to \infty$.

Theorem 1. • *Let*

$$n_r = 8.5^r + 5.$$

- *For each $r \geq 0$, there are at least $r+2$ captures g_x with v_2 of preperiod n_r, with capture paths crossing the unit circle at arguments in $\left(\frac{2}{7}, \frac{9}{28}\right)$, which are all Thurston equivalent.*

It seems likely that this result is fairly sharp, in the sense that there is a constant c_2 such that at most $c_2 \log n$ captures with v_2 of preperiod n are Thurston equivalent.

Conjecture

For constants $c_r > 0$ and $1 < d_1 < d_2$,

$$a_{n,r} \sim c_r 2^n,$$

where

$$e^{-d_2^r} \leq c_r \leq e^{-d_1^r}.$$

Justification for conjecture

- For any capture path β in the hard aeroplane region, with endpoint x, there is a path γ in the fundamental domain such that $\sigma_\gamma \circ f$ is Thurston equivalent to $\sigma_\beta \circ f$.
- In fact there are exactly two such paths, because, for the boundary of this part of the fundamental domain, vertices connect in pairs, and this part of the boundary projects down to an interval minus finitely many points in $V_{3,n}$.
- $$x = \varphi_{1,x} \circ \varphi_{2,x} \circ \cdots \circ \varphi_{m(x),x}(y)$$

  where y is the endpoint of γ, and for a suitable action of the mapping class group on paths,

  $$\beta = \varphi_{1,x} \cdot \varphi_{2,x} \cdot \cdots \varphi_{m(x),x} \cdot \gamma$$

  and

  - $\varphi_{k,x}$ is a homeomorphism which is locally constant in x, on intervals $I_{k,x}$ decreasing exponentially in size with k.
  - For constants $c_3 > 0$ and $\lambda < 1$, for $k \leq c_3 \ell$, we have that $\varphi_{\ell+1,x} \circ \cdots \circ \varphi_{\ell+2k,x}$ is the identity except on a proportion $< \lambda^\ell$ of the points in $I_{2k-2,x}$.