Semiconjugacies and pinched Cantor bouquets

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Cantor bouquets in hedgehogs and transcendental iteration
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Outline

- Motivation & results
- Topological dynamics of $z \mapsto \pi \sinh z$
- Idea of proof

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire map.

- $\mathcal{J}(f)$: Julia set
- $\mathcal{F}(f)$: Fatou set

It is possible that

$$\mathcal{J}(f) = \mathbb{C}.$$

Question

Can we give a complete description of topological dynamics for a map with this property?

Yes, for a wide class of examples, including maps such as

$$z \mapsto \pi \sinh z.$$

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The escaping set of f is defined as

$$I(f) := \{z \in \mathbb{C} : f^n(z) \to \infty\}.$$

Question (Bergweiler)

Let $F_{a,b}(z) = a\,e^z + b\,e^{-z}$ such that both critical values are preperodic. Is $I(F_{a,b})$ connected?

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Strong subhyperbolicity

- $S(f)$: set of singular values
- $P(f) := \overline{\bigcup_{n\geq 0} f^n(S(f))}$: postsingular set

A transcendental entire map f is called subhyperbolic if

- $\mathcal{F}(f) \cap P(f)$ is compact,
- $\mathcal{J}(f) \cap P(f)$ is finite.

A subhyperbolic map f is called strongly subhyperbolic if

- $\mathcal{J}(f)$ contains no asymptotic values,
- the local degree of $f|_{\mathcal{J}(f)}$ is bounded by a finite constant.

Note that every hyperbolic map, i.e. a map f for which $P(f)$ is a compact subset of $\mathcal{F}(f)$, is strongly subhyperbolic.

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Theorem (Semiconjugacy)

Let f be strongly subhyperbolic, and let $\lambda \in \mathbb{C}$ be such that $g(z) := f(\lambda z)$ is hyperbolic with connected Fatou set (disjoint type). Then there is a continuous surjection $\phi : \mathcal{J}(g) \to \mathcal{J}(f)$ such that

$$f(\phi(z)) = \phi(g(z))$$

for all $z \in \mathcal{J}(g)$. Moreover, ϕ restricts to a homeomorphism between the escaping sets $I(g)$ and $I(f)$.

- The hypothesis will be automatically satisfied whenever λ is sufficiently small.
- Any two maps g and g' as in the theorem are qc-conjugate on their Julia sets, so it is sufficient to prove the theorem for any such map.

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★ This answers a question by Rempe on rigidity of Misiurewicz parameters in the cosine family.

★ The escaping set of a disjoint type map is disconnected.

Corollary 1

The escaping set of a strongly subhyperbolic transcendental entire function is disconnected.

★ Suppose that g is of disjoint type and has finite order, i.e. $\log\log|g(z)| = O(\log|z|)$ as $z \to \infty$.

Then it is known that $\mathcal{J}(g)$ is a Cantor bouquet; i.e. homeomorphic to a straight brush in the sense of Aarts & Oversteegen.

Corollary 2

Let f be a finite-order strongly subhyperbolic map. Then $\mathcal{J}(f)$ is a pinched Cantor bouquet; i.e. the quotient of a Cantor Bouquet by a closed equivalence relation defined on its endpoints.

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Previous results:

- When both critical values of a cosine map $z \mapsto a\,\mathrm{e}^z + b\,\mathrm{e}^{-z}$ are preperiodic, part of our result is due to Schleicher.
- For hyperbolic maps, our theorem is due to Rempe.

Restrictive assumptions:

- **Asymptotic values:** $E_1(z) := \frac{1}{e^2}e^z$ (disjoint type) and $E_2(z) := 2\pi i e^z$ (subhyperbolic) are not topologically conjugate on their escaping sets.
- **Unbounded degree:** No indication of what to expect for maps whose Julia sets contain no asymptotic values but sequences of points with unbounded local degree.

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$z \mapsto \pi \sinh z$

We want to define a model consisting of a

- topological space $\overline{X}$,
- $\mathcal{M} : \overline{X} \to \overline{X}$

such that if $g_\lambda : z \mapsto \lambda \sinh(z)$ is any disjoint type map then

- $\mathcal{J}(g_\lambda)$ is homeomorphic to $\overline{X}$,
- $\mathcal{M}|_{\overline{X}}$ is conjugate to $g|_{\mathcal{J}(g_\lambda)}$.

Our theorem tells us that $\mathcal{J}(f)$ is homeomorphic to $\overline{X}/\sim_p$, where $\sim_p$ is an equivalence relation defined on (the endpoints of) $\overline{X}$.

The combinatorial description of the dynamics of f on $\mathcal{J}(f)$ (Schleicher) tells us how $\sim_p$ is defined.

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Let $g_\lambda(z) := \lambda \sinh(z)$ with $\lambda > 0$.

- Critical values: $\{-\lambda i, \lambda i\}$, no asymptotic values.
- $g_\lambda : \mathbb{R} \to \mathbb{R}$ is a homeomorphism with $g_\lambda(0) = 0$.
- $\mathbb{R} \setminus \{0\} \subset I(g_\lambda)$, while $g_\lambda(i\mathbb{R}) \subset [-\lambda i, \lambda i]$.

For $f(z) = \pi \sinh z$ (i.e. $f = g_\pi$), 0 is a repelling fixed point with

$$f(\pi i) = 0 = f(-\pi i)$$

hence

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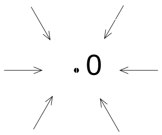

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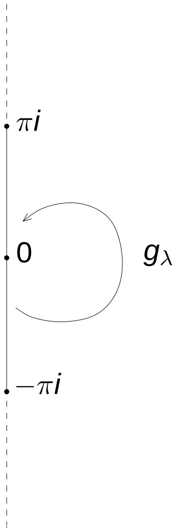

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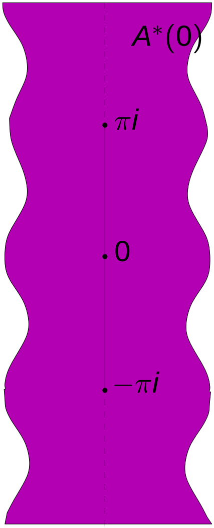

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$$\begin{aligned} S_{n_L} &:= \{z : \operatorname{Re} z < 0, \operatorname{Im} z \in ((n - 1/2)\pi, (n + 1/2)\pi)\}, \\ S_{n_R} &:= \{z : \operatorname{Re} z > 0, \operatorname{Im} z \in ((n - 1/2)\pi, (n + 1/2)\pi)\}. \end{aligned}$$

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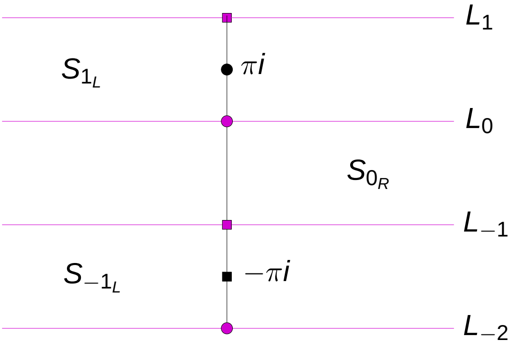

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The restrictions $g|_{S_{n_R}}$ and $g|_{S_{n_L}}$ are conformal isomorphisms onto the left or right half-plane, respectively.

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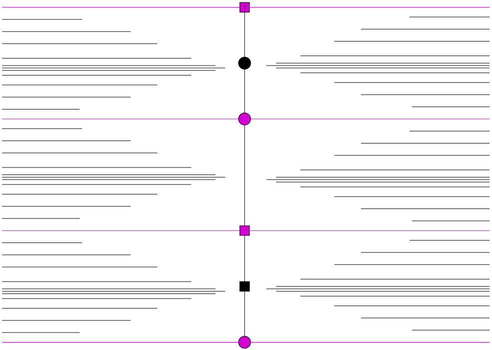

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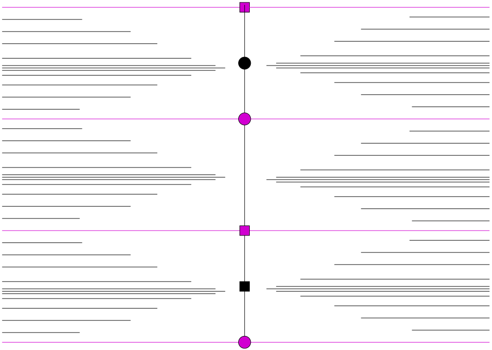

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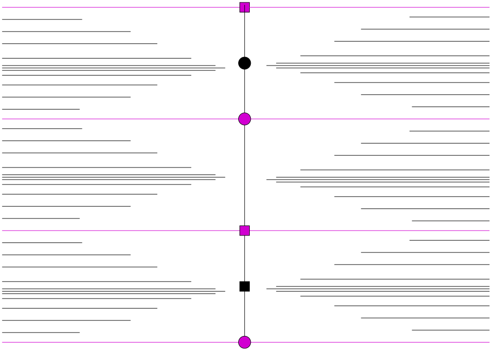

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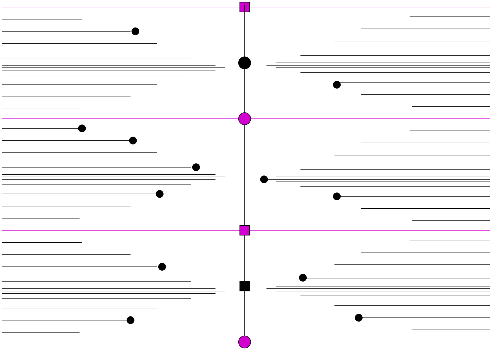

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Theorem (Schleicher)

If z is not on a dynamic ray then it is the landing point of one, two or four dynamic rays.

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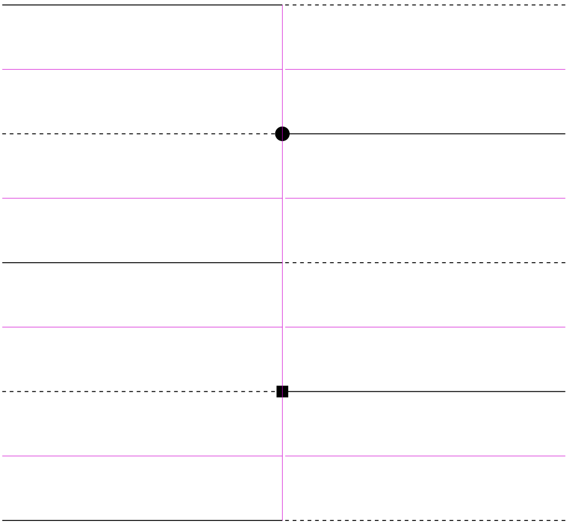

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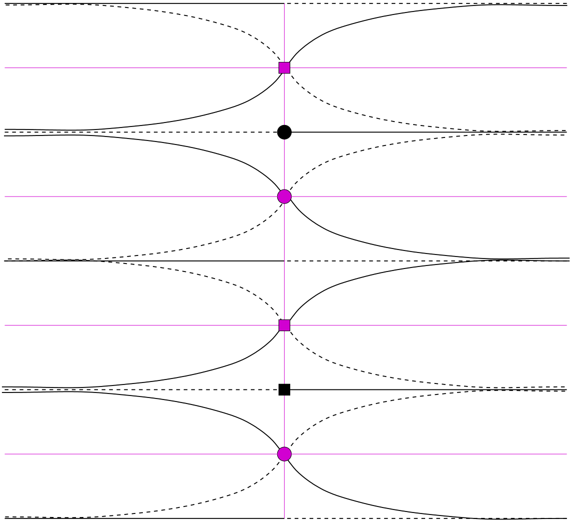

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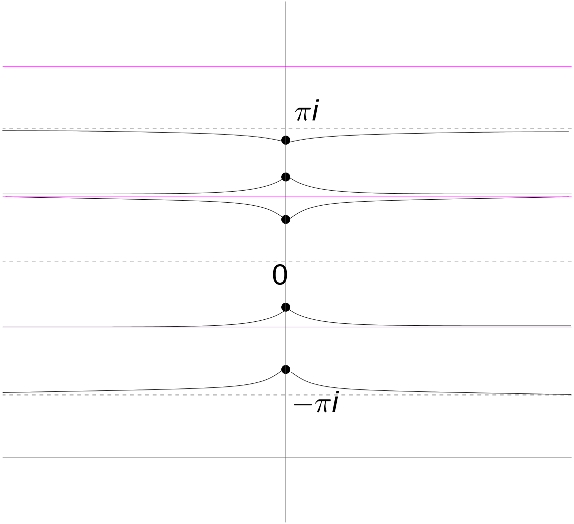

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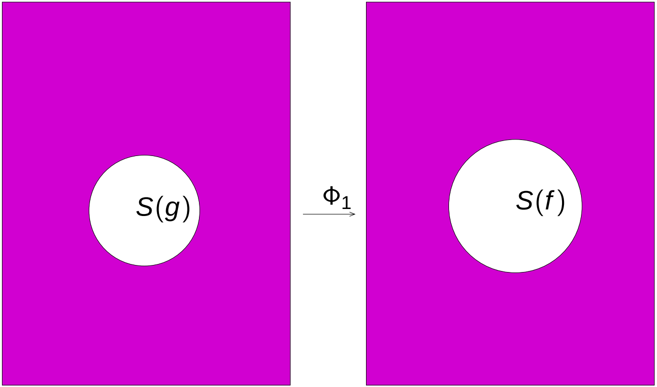

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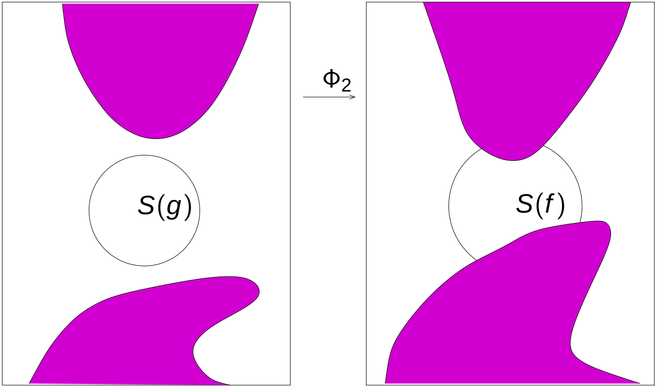

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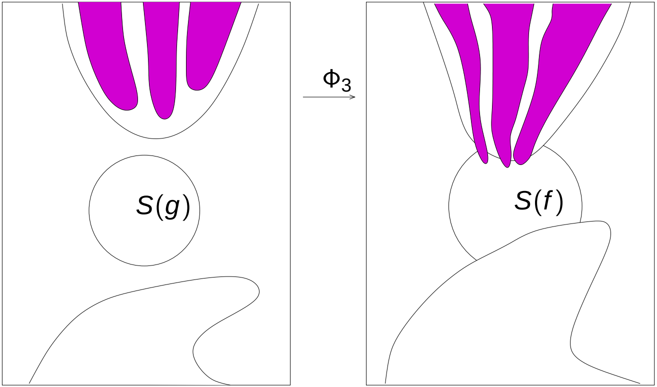

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- $V := f^{-1}(U) \subset U := \mathbb{C} \setminus K \Longrightarrow \rho_V(z) > \rho_U(z)$
- $f : V \to U$ is a local isometry $\Longrightarrow \rho_V(z) = \rho_U(f(z)) \cdot |f'(z)|$

f strongly subhyperbolic

As for hyperbolic f, we can find a domain $U \supset \mathcal{J}(f)$ such that

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- U is not necessarily hyperbolic ($U = \mathbb{C}$ is possible).
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Thank you for listening!