'Cantor bouquets' for non-entire meromorphic functions

Basic definition

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be *a transcendental meromorphic function*. Assume f is non-entire i.e.

$$f^{-1}(\infty) \neq \emptyset.$$

Let $\mathcal{P} = \{a \in \mathbb{C} : f(a) = \infty\}$ be the set of poles.

- the point $z \in f^{-n}(\infty)$ is called a *prepole* of order $n \in \mathbb{N}$. In particular the poles are prepoles of order 1.
- Thus $f^n : \mathbb{C} \setminus \bigcup_{k \leq n} P_k \to \mathbb{C}$ is holomorphic,
- f^n has poles at P_n,
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Conclusion

Let $f : \mathbb{C} \to \hat{\mathbb{C}}$ be a transcendental meromorphic function with poles. Then the family $\{f^n\}_{n\in\mathbb{N}}$ is well defined on

$$\mathbb{C} \setminus \bigcup_{k=0}^{\infty} P_k = \mathbb{C} \setminus O^-(\infty),$$

where $O^-(\infty) := \bigcup_{n\geq 0} f^{-n}(\infty)$ and $J(f) = \overline{O^-(\infty)}$

Fatou did not consider these function, since as he wrote *'there occurs a serious difficulty when one tries to generelize Julia's approach to the meromorphic case'*

see p. 358 at P. Fatou, Sur l'itération des fonctions transcendantes entèries, Acta Mat. 47 (1926), 337-360

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Basic properties

Fatou-Julia theory for meromorphic functions

The basic properties of Fatou and Julia sets of non-entire transcendental meromorphic functios are the same as for entire functions.

It was proved in a series of joint papers written by I.N. Baker(*), Y. Lü(*) and K.

The set of singularities of the inverse function f^{-1} of f

$$\mathrm{Sing}(f^{-1}) = \{a \in \mathbb{C} : \text{ a is a critical or an asymptotic value}\}$$

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Tangent function

- $f(z) = tan(z) = \frac{\sin z}{\cos z} = \frac{\sin 2x}{\cos 2x + \cosh 2y} + i \frac{\sinh 2y}{\cos 2x + \cosh 2y}$.
- f is a simply-periodic function $\forall_{z \in \mathbb{C}} f(z + \pi) = f(z)$
- The fundamental domains are the strips $L_k = \{z \in \mathbb{C} : \left(k - \frac{1}{2}\right)\pi < \mathrm{Re} z \leq \left(k + \frac{1}{2}\right)\pi\}, \quad k \in \mathbb{Z}$
- The inverse map $\arctan : \hat{\mathbb{C}} \setminus \{\pm i\} \to \mathbb{C}$ is defined as

$$\arctan w = \frac{1}{2i} \log \frac{1 + iw}{1 - iw}.$$

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- $f(z) \neq \pm i$, so i and $-i$ are the asymptotic values

  $H^+ = \{z \in \mathbb{C}; \mathrm{Im} z > y_0 > 0\}$ is an asymptotic tract corresponding to i.

  $H^- = \{z \in \mathbb{C}; \mathrm{Im} z < -y_0 < 0\}$ is an asymptotic tract corresponding to $-i$.

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- f has infinitely many poles i.e.

$$f(z) = \frac{\sin(z)}{\cos z} = \infty \Longleftrightarrow z = s_k := \left(k + \frac{1}{2}\right)\pi, \quad k \in \mathbb{Z}.$$

- ∞ is not an asymptotic value

Tangent family

Let $f_\lambda(z) = \lambda tan(z) = \lambda \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}, \quad \lambda \in \mathbb{C}^*.$

Singularities

$Sing(f^{-1}) = \{-\lambda i, +\lambda i\}$ - asymptotic values of f_λ.

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Structure of repelling periodic points in $J(f)$

Rational functions

Let f be a rational map. Then the set

$$Per_n(f) = \{z \in \mathbb{C} : \text{ z is a repelling periodic point of period n }\}$$

is finite. So it has no accumulation points.

Transcendental entire (Baker, Bergweiler)

An entire transcendental function f has infinitely many repelling periodic points of period n for all $n \geq 2$ and ∞ is the unique accumulation point of $Per_n(f)$

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Structure of repelling periodic points in $J(f)$

Transcendental non-entire functions (Baker-Lü-K.)

Suppose that f is a transcendental non-entire function. Then $J(f) = \overline{O^-(\infty)}$. Let

$$z_1, z_2, \ldots, z_5 \in O^-(\infty) \setminus \{\infty\}$$

are distinct. Define n_j by $f^{n_j}(z_j) = \infty$. Then there exists

$$j \in \{1, \ldots, 5\}$$

such that z_j is a limit point of repelling periodic points of minimal period $n_j + 1$. Their multipliers tends to ∞.

Structure of repelling periodic points in $J(f)$

Tangent family

If $v \neq \pm\lambda i$ is a prepole of oreder n, then there exists a sequence of points z_k, $k = 1, 2, \ldots$, such that

1. $f^{n+1}(z_k) = z_k$
2. $z_k \to v$ as $k \to \infty$
3. if m_k is a multiplier of the periodic cycle containing z_k then $|m_k| \to \infty$ as $k \to \infty$

Corollary

Unlike to entire functions the set $Per_n(f)$ has plenty of accumulation points. They are prepoles of order $n - 1$.

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Example 1

Let $\lambda = (\pi/2 + \epsilon)i$, $\epsilon > 0$ (small), then f_λ has two attracting cycles of period 2.

Example 2

Let $\lambda = (\pi/2 - \epsilon)i$, $\epsilon > 0$ small, then f_λ has one attracting cycles of period 4.

Example 3

Let $\lambda = (\pi/2)i$ then $J(f_\lambda) = \overline{\mathbb{C}}$ and attracting periodic cycles disappear.

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Hyperbolic components

$$\mathcal{H}_p = \{\lambda \in \mathbb{C}^* : f_\lambda \text{ has an attracting cycle of period p}\}.$$

The components of $\mathcal{H}_p$ we denote by Ω_p and call them hyperbolic components.

Virtual centers

Let $\mathcal{C}_0 = \{\infty\}$, $\mathcal{C}_p = \{\lambda : f_\lambda^p(\pm\lambda i) = \infty\}, p > 0$, $\mathcal{C} = \bigcup_0^\infty \mathcal{C}_p$.
Points in $\mathcal{C}_p$ are called *virtual centers* of order p.

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Theorem (Keen-K.)

- The virtual centers $\lambda_p \in \mathcal{C}_{p-1}$ are in one to one correspondence with pairs of hyperbolic components (Ω_p, Ω'_p).
- In Ω_p each function has a pair of periodic cycles of period p and each attracts the orbit of an asymptotic value whereas in Ω'_p each function has a single attracting cycle of period $2p$ which attracts both asymptotic values.
- The virtual center $\lambda_p \in \mathcal{C}_{p-1}$ is a common boundary point of (Ω_p, Ω'_p).

Remark

In our example $\lambda_1 = \frac{\pi i}{2}$ is a virtual center of two hyperbolic components Ω_2 and Ω'_2.

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The escaping set of f

- $I(f) = \{z : \lim_{n\to\infty} f^n(z) \to \infty\}$
- For $R > 0$ we also consider the set $I_R(f) = \{z \in \mathbb{C} : \liminf_{n\to\infty} |f^n(z)| \geq R\}$.
- Note that $I(f) = \bigcap_{R>0} I_R(f)$.

Remark

For tangent maps $I_R(f_\lambda)$ is a union of Cantor sets contained in $J(f_\lambda)$.

- Take $R > 0$ such that $\mathrm{Sing}(f^{-1}) = \{\pm\lambda i\} \subset D(0, R)$ and define $B(R) = \{z \in \overline{\mathbb{C}} : |z| > R\}$
- then all components U_k of $f^{-1}(B(R))$ are simply-connected, bounded and contain exactly one pole s_k of f_λ.

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- Take $R > 0$ such that $\mathrm{Sing}(f^{-1}) = \{\pm\lambda i\} \subset D(0, R)$ and define $B(R) = \{z \in \overline{\mathbb{C}} : |z| > R\}$
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Tangent family

The escaping set of f

- $I(f) = \{z : \lim_{n\to\infty} f^n(z) \to \infty\}$
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For

$$\lambda \in \mathcal{C}_{p-1} = \{\lambda : f_\lambda^{p-1}(\pm\lambda i) = \infty\}, p > 1,$$

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Remark

$I_p(f_\lambda)$ contains Cantor bouquets $\mathcal{C}_1 \neq \mathcal{C}_2$ such that:

- $\mathcal{C}_1$ is invariant for invariant for f^p,
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We define *the radial Julia set or conical set* $J_r(f)$ as the set of points z in $J(f)$ for which there exists a family of neighborhoods $B(z, r_j)$, $r_j \to 0$, which can be mapped by f with bounded distortion until the diameter of the image reaches a fixed size.

Estimates (K.-Urbański)

If $\lambda \in \Omega_p$ (resp. Ω'_p,$p > 1$) then

$$1 < \mathrm{HD}(J(f_\lambda)) = \mathrm{HD}(J_r(f_\lambda)) < 2$$

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If λ_0 is a virtual center of a pair (Ω_p, Ω'_p), $p > 1$, then $P(f_{\lambda_0}) := \{\pm\lambda_0 i, f_{\lambda_0}(\pm\lambda_0 i), \ldots, f_{\lambda_0}^{p-1}(\pm\lambda_0 i) = \infty\}$ and $J(f_{\lambda_0}) = \overline{\mathbb{C}}$. In this case

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Theorem (Bock)

If f is a transcendental meromorphic function then at least one of the following statement holds:

1. $\lim_{n\to\infty} dist_{\chi}(f^n(z), P(f)) = 0$ for almost all $z \in J(f)$;
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The class $\mathcal{R}$

Definition

Let $\mathcal{R}$ be the family of maps of the form

$$f(z) = R \circ exp(z),$$

where R is a non-constant rational function. Then f has two asymptotic values

$$\xi_1 := R(0), \quad \xi_2 := R(\infty).$$

We assume that

$$\xi_1 := R(0) \neq \infty, \quad \xi_2 := R(\infty) \neq \infty,$$

so f is non-entire.

The class $\mathcal{R}$

Subfamilies of $\mathcal{R}$

We define two subfamilies:

- $\mathcal{R}_1$ - only one asymptotic value e.g. ξ_1 is mapped onto a pole i.e. there is $q_1 \geq 1$ such that $f^{q_1}(\xi_1) = \infty$
- $\mathcal{R}_2$ - both asymptotic values ξ_1, ξ_2 are mapped onto poles i.e. there are $q_1 \geq 1, q_2 \geq 1$ such that $f^{q_1}(\xi_1) = \infty,\ f^{q_2}(\xi_2) = \infty,$

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1. $HD(I_{q_1}(f)) = 2$
2. If additionally $Sing(f^{-1}) \setminus \{\xi_1\} \subset F(f)$ then $meas(J(f)) = 0$.

Remark (Mc Mullen)

Compare with $f_\lambda(z) = \lambda e^z$ for $\lambda \in (0, 1/e)$

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Let $f \in \mathcal{R}_1$ (only one asymptotic value ξ_1 is mapped onto a pole $f^{q_1}(\xi_1) = \infty$). Let $I_{q_1}(f) := \{z \in J(f) : \lim_{k\to\infty} f^{kq_1}(z) = \infty\}$. Then

1. $HD(I_{q_1}(f)) = 2$
2. If additionally $Sing(f^{-1}) \setminus \{\xi_1\} \subset F(f)$ then $meas(J(f)) = 0$.

Remark (Mc Mullen)

Compare with $f_\lambda(z) = \lambda e^z$ for $\lambda \in (0, 1/e)$

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Theorem (Skorulski)

Let $f \in \mathcal{R}_2$ i.e. there are $q_1 \geq 1, q_2 \geq 1$ such that $f^{q_1}(\xi_1) = \infty$, $f^{q_2}(\xi_2) = \infty$. Then the set

$$P_{Asymp}(f) := \{\xi_1, \ldots, f^{q_1}(\xi_1) = \infty, \quad \xi_2, \ldots, f^{q_2}(\xi_1) = \infty\}$$

is a metric attractor i.e. the Lebesgue's measure of

$$\{z \in J(f) : \omega(z) \subset P_{Asymp}(f)\}$$

is positive.

The class $\mathcal{R}$

Theorem (Skorulski)

Let $f \in \mathcal{R}_2$ and $dist_\chi(P_2(f), J(f)) > 0$ where

$$P_2(f) := cl\{\Theta^+(Sing(f^{-1})) \setminus \Theta^+(\{\xi_1, \xi_2\})\}$$

then

- $meas(J(f))$ is positive, $J(f) \neq \mathbb{C}$
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Continuity of the Hausdorff dimension

If $\lambda \in \Omega_p$ (resp. Ω'_p) then $\overline{P(f_\lambda)} \subset F(f_\lambda)$ and the Julia $J(f_\lambda)$

1. has empty interior
2. the Lebesgue measure of the Julia set is *zero* since $dist(\overline{P(f_\lambda)}, J(f_\lambda)) > 0$ where $P(f_\lambda) = \bigcup_0^\infty f_\lambda^n(\pm\lambda i)$.

(2) follows e.g. from Stallard's results for meromorphic functions

Theorem (K.-Urbański)

If $\lambda \in \Omega_p$ (resp. Ω'_p), $p > 1$ then $\overline{P(f_\lambda)} \subset F(f_\lambda)$ and

$$1 < HD(J(f_\lambda)) = HD(J_r(f_\lambda)) < 2$$

The last estimate follows from our results concerning conformal measure for expanding meromorphic functions (Math. Annalen, 2002) and from (2).

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Theorem (S.)

- If λ_0 is a virtual center of a pair (Ω_p, Ω'_p) then for a.e. $z \in J(f_{\lambda_0}) = \overline{\mathbb{C}}$ we have $\omega(z) = P(f_{\lambda_0})\}$,
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- Let $h_\lambda := HD(J_r(f_\lambda))$
- $J(f_{\lambda_0}) = \overline{\mathbb{C}}$ then $HD(J(f_\lambda)) = 2$
- Question $h_{\lambda_0} = HD(J_r(f_{\lambda_0})) =?$

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Let $f \in \mathcal{R}$ such that at least one asymptotic value is mapped onto a pole. Then $1 < h_{\lambda_0} = HD(J_r(f_{\lambda_0})) < 2$.

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Question

Does $lim_{\lambda \to \lambda_0} h_\lambda = h_{\lambda_0} < 2$ for $\lambda \in \Omega_p$?

Theorem. (K.-Urbański)

For $p \geq 1$ let (Ω_p, Ω'_p) be a pair of hyperbolic components with virtual center at λ_0 then

$$lim_{\lambda \to \lambda_0} h_\lambda = h_{\lambda_0}$$

for $\lambda \in \Omega_p \cup \Omega'_p$.

Continuity of h_λ for $\lambda \in \Omega_p$ follows from the previous result proved for hyperbolic functions of the form

$$f(z) = H(exp(Q(z))).$$

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- Since f is periodic we can project f on a cylinder $\mathbb{C}/\sim$,
- We say $z_1 \sim z_2$ if there exists $k \in \mathbb{Z}$ such that $z_1 = z_2 + 2k\pi i$.
- Let Π be a projection of $\mathbb{C}$ onto $\mathcal{C}$ and $F : \Pi \circ f \circ \Pi^{-1}$.

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Let $f \in \mathcal{R}$ be as above. Then

- $1 < HD(J_r(F)) = h < 2$.
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Final conclusion

We consider a projection of f_λ and f_{λ_0} onto cylinder $\mathcal{C}$ i.e. $F_\lambda := \Pi \circ f_\lambda \circ \Pi^{-1}$ and $F_{\lambda_0} := \Pi \circ f_{\lambda_0} \circ \Pi^{-1}$. Then

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- It follows from Skorulski's results that $h_\lambda \geq s > 1$.
- Take $\{\lambda_n\}_0^\infty$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0$.
- Since $h_\lambda \in [s, 2]$, we may assume that the sequence $\{h_{\lambda_n}\}_0^\infty$ converges to some $t \in [s, 2]$.

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- the sequence of conformal measures m_{h_λ} for F_λ is tight (......), so it converges weakly to a Borel probability measure m.
- m appears to be t-conformal measure for F_{λ_0} and m is supported on the radial Julia set of F_{λ_0}.
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The Fatou functions

Definition

P.Fatou considered the function $f(z) = z + 1 + e^{-z}$. He proved that $\{z \in \mathbb{C} : \quad Rez > 0\}$ is an invariant domain. Today we call such a component an invariant Baker domain.

Baker domain

Let U be periodic component of $F(f)$. If there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case U is called Baker domain.

In this case $z_0 = \infty$.

Definition

For $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we consider the family of maps $f_\lambda(z) = z + e^{-z} + \lambda$

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Let U be periodic component of $F(f)$. If there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$, but $f^p(z_0)$ is not defined. In this case U is called Baker domain.

In this case $z_0 = \infty$.

Definition

For $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we consider the family of maps $f_\lambda(z) = z + e^{-z} + \lambda$

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To honor P.Fatou we will call all the functions f_λ Fatou functions.

We assume that $Re\lambda > 1$.

Proposition (K.-Urbański)

- The map f_λ has a Baker domain at ∞ and domain
$$\{z \in \mathbb{C} : Rez > \epsilon_\lambda > 0\}$$
is contained in Baker domain.
- The Julia set $J(f_\lambda)$ is the Cantor bouquet contained in
$$\bigcup_{n\in\mathbb{Z}} \{z \in \mathbb{C} : 2n\pi + \theta < Imz < 2(n+1)\pi - \theta\}$$
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The Fatou functions

The Fatou function f_λ has infinitely many critical values

$$c_k = 2\pi ki + 1 + \lambda, \quad k \in \mathbb{Z}$$

Theorem (Eremenko-Lyubich, Goldberg-Keen)

If transcendental entire function f has only finitely many critical and asymptotic values, then the Fatou set F(f) has no wandering domains.

$\Longrightarrow$ Fatou functions $f_\lambda(z) = z + e^{-z} + \lambda$ do not satisfy these assumptions. $f_\lambda \notin \mathcal{B}$.

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To prove this we applied the fact that e^{-z} is semiconjugacy between

$$f_\lambda(z) = z + e^{-z} + \lambda$$

and

$$g_\lambda(z) = e^{-\lambda} z e^{-z}$$

It is easy to show that for $Re\lambda > 0$ the function g_λ has

- exactly one attracting fixed point z=0
- its multiplier $g'_\lambda(0) = e^{-\lambda}$
- g_λ has only one singularity z=1 and it is a critical point
- g_λ is in class $\mathcal{S}$ satisfying assumptions of Sullivan's th.

The Fatou functions

Proposition (K.-Urbański)

- Let $I_\infty(f_\lambda) = \{z \in \mathbb{C} : \quad f^n(z) \to -\infty\}$. Then $I_\infty(f) \subset J(f_\lambda)$ and the Hausdorff dimension $HD(I_\infty(f_\lambda)) = 2$.
- Let $J_r(f_\lambda) := J(f_\lambda) \setminus I_\infty(f_\lambda)$. Then $1 < HD(J_r(f_\lambda)) < 2$.

Theorem (K.-Urbański)

If $Re\lambda > 1$ then the function

$$\lambda \to HD(J_r(f_\lambda))$$

is real analytic.

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Proof

To prove this theorem we apply thermodynamic formalism of potentials

$$-t log|F_\lambda'|,$$

where F_λ is the projection of f_λ to the infinite cylinder

$$Q = \mathbb{C}/\sim$$

and

$$w \sim z$$

iff $w - z = 2\pi i\mathbb{Z}$.

1 step

We fix parameter λ and omit it in the notations
For $t \geq 0$ and $z \in Q \setminus PC(F)$ we define the lower and upper topological pressure respectively by

$$\underline{P_z}(t) = liminf_{n\to\infty} \frac{1}{n} log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t}$$

$$\overline{P_z}(t) = limsup_{n\to\infty} \frac{1}{n} log \sum_{x \in F^{-n}(z)} |(F^n)'(x)|^{-t}.$$

Remark

$\underline{P_z}(t)$ and $\overline{P_z}(t)$ are independent of z. In fact for $t > 1$ we have $\underline{P}(t) = \overline{P}(t)$ Let $P(t)$ denote the common value of $\underline{P}(t)$ and $\overline{P}(t)$

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Proposition

The function $t \mapsto P(t)$, $t \geq 0$, has the following properties.
a) There exists $t \in (0,1)$ such that $0 \leq \mathrm{P}(t) < +\infty$.
b) $P(t) < +\infty$ for all $t > 1$.
c) The function $P(t)$ restricted to the interval $(1, +\infty)$ is convex, continuous and strictly decreasing.
d) $lim_{t \to +\infty} \mathrm{P}(t) = -\infty$.
e) There exists exactly one $t > 1$ such that $P(t) = 0$.

Let h denote the value of t for which $P(t) = 0$

Proposition (Bowen formula)

$h = HD(J_r(F))$.

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Proposition

For

$$(t, \lambda) \in (1, +\infty) \times \{\lambda \in \mathbb{C} : Re\lambda > 1\}$$

the function

$$(t, \lambda) \mapsto P_\lambda(t) = lim_{n\to\infty} \frac{1}{n} log \sum_{x \in F_\lambda^{-n}(z)} |(F_\lambda^n)'(x)|^{-t}$$

is continuous.

2 step

(*) Fix now $\lambda_0 \in \mathbb{C}$ with $Re\lambda_0 > 1$ and $t_0 \in (1, \infty)$.
(*) Let $C_b = C_b(J(F))$ be the space of all bounded continuous complex valued functions defined on $J(F)$.
(*) Fix $\alpha \in (0, 1]$. Given $g \in C_b$ let

$$v_\alpha = \inf\{L \geq 0 : |g(y) - g(x)| \leq L|y - x|^\alpha \ \text{ for all } \ x, y \in J(F)$$

with $|y - x| \leq \delta$ be the α-variation of the function g and let

$$||g||_\alpha = v_\alpha(g) + ||g||_\infty.$$

Clearly the space

$$H_\alpha = \mathrm{H}_\alpha(J(F)) = \{g \in C_b(J(F)) : ||g||_\alpha < \infty\}$$

endowed with the norm $|| \cdot ||_\alpha$ is a Banach space densely contained in C_b with respect to the $|| \cdot ||_\infty$ norm.

2 step

(*) For every $\lambda \in B(\lambda_0, r)$ and every $t > 1$ let

$$\mathcal{L}^0_{\lambda,t} : H_\alpha(J(F_{\lambda_0})) \to H_\alpha(J(F_{\lambda_0}))$$

be the generalized Perron-Frobenius operator defined as

$$\mathcal{L}^0_{\lambda,t} g(z) = \sum_{x \in F^{-1}_{\lambda_0}(z)} \left|F'_\lambda(h_\lambda(x))\right|^{-t} g(x),$$

where h_λ is q.c. conjugacy between F_{λ_0} and F_λ,
$|F'_\lambda \circ h_\lambda|^{-t}$ is called a potential.

2 step

(*) For $t \in (1, \infty)$ we consider the dual operator

$$\mathcal{L}^*_{\lambda,t} : H^*_\alpha(J(F_{\lambda_0})) \to H^*_\alpha(J(F_{\lambda_0}))$$

given by the formula

$$\mathcal{L}^*_{\lambda,t} m(g) = m(\mathcal{L}^0_{\lambda,t} g)$$

Then there exists a conformal measure $m_{\lambda_0,t}$ such that

$$\mathcal{L}^*_{\lambda,t}(m_{\lambda_0,t}) = e^{P_{\lambda_0}(t)} m_{\lambda_0,t}$$

In particular, if

$$t = h = HD(J_r(f_{\lambda_0}))$$

then

$$P_{\lambda_0}(h) = 0$$

so $e^{P_{\lambda_0}(h)} = 1$ is an engeinvalue of the Perron-Frobenius operator.

2 step

(*) Since the potential $|F'_\lambda \circ h_\lambda|^{-t}$ does not depend on

$$(\lambda, t) \in \mathbb{C}^2$$

in a holomorphic way, we have to embed λ into $\mathbb{C}^2$
and t into $\mathbb{C}$.
We embed the complex plane $\mathbb{C}$ into $\mathbb{C}^2$ by the formula

$$x + iy \mapsto (x, y) \in \mathbb{C}^2.$$

So, $\lambda \in \mathbb{C} = \mathbb{R}^2$ may be treated as an element of $\mathbb{C}^2$.

2 step

Proposition

Fix λ_0 with $Re\lambda_0 > 1$ and $t_0 > 1$. There then exist $R > 0$ and a holomorphic function

$$L : \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R) \to L(\mathrm{H}_\alpha(J(F(\lambda_0))))$$

such that for every $(\lambda, t) \in B(\lambda_0, R) \times B(t_0, R) \subset \mathbb{C} \times \mathbb{R}$

$$L(\lambda, t) = \mathcal{L}^0_{\lambda,t}.$$

3 step

(*) For $\lambda = \lambda_0$ applying theorem of Ionescu-Tulcea and Marinescu we can prove that for $t \in B(t_0, R)$

$$e^{P_{\lambda_0}(t)}$$

is a simple isolated eingevalue of the operator

$$L(\lambda_0, t) = \mathcal{L}^0_{\lambda_0,t}.$$

(*) Next applying perturbation theory for linear operators (Kato's results) we can show that there exists a holomorphic function

$$\gamma : \mathbb{D}_{\mathbb{C}^3}((\lambda_0, t_0), R) \mapsto \mathbb{C}$$

such that the number

$$\gamma(\lambda, t)$$

is a simple isolated eigenvalue of $L(\lambda, t)$, where

$$\gamma(\lambda_0, t) = e^{P_{\lambda_0}(t)}$$

3 step

(*) We can prove that $\gamma(\lambda, t)$ has the form

$$\gamma(\lambda, t) = e^{P_\lambda(t)}$$

$\Longrightarrow$ the function

$$(\lambda, t) \mapsto P_\lambda(t)$$

is real analytic for $(\lambda, t) \in B(\lambda_0, r') \times (t_0 - \rho, t_0 + \rho)$
(*) Since $P(h_\lambda) = 0$ where

$$h_\lambda = HD(J(F_\lambda)),$$

it follows from the Implicite Function Theorem that in order to conclude the proof it suffices to show that

$$\frac{\partial P_\lambda(t)}{\partial t} \neq 0$$

for all $(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$.

(*) Thus the map

$$(\lambda, t) \mapsto HD(J(F_\lambda)) = HD(J(f_\lambda)),$$

is real analytic for

$$(\lambda, t) \in B(\lambda_0, R_3) \times (t_0 - \rho, t_0 + \rho)$$

□