Simple topological models of Julia sets

L. Oversteegen

University of Alabama at Birmingham

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Denote by $\mathbb{C}$ the complex plane, by $\mathbb{C}^\infty$ the complex sphere $\mathbb{C} \cup \{\infty\}$, by $\mathbb{D}$ the unit disk and by $\mathbb{S} = \partial\mathbb{D}$.
Suppose J is the *connected* Julia set of a complex polynomial P and U^∞ is the unbounded component of $\mathbb{C}^\infty \setminus J$, then U^∞ is simply connected and there exists a conformal map $\varphi : \mathbb{D} \to U^\infty$ such that $\varphi(O) = \infty$ and $\varphi'(O) > 0$.

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Both $\Pi(\alpha)$ and $\mathrm{Imp}(\alpha)$ are subcontinua of ∂U and $\Pi(\alpha) \subset \mathrm{Imp}(\alpha)$.

Definition

A topological space X is *locally connected* at a point $x \in X$ if for each open set U containing x there exists an open and connected set V such that

$$x \in V \subset U.$$

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Lemma

A space X is locally connected if and only if every component of every open set is open.

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J is finitely Suslinian (For all $\varepsilon > 0$, any collection of pairwise disjoint subcontinua of diameter bigger than ε is finite).

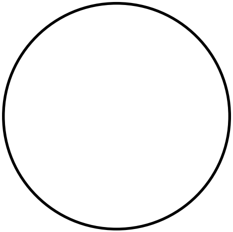
Let $\sigma_d : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be defined by $\sigma_d(z) = z^d$. Let $\varphi : \mathbb{D} \to U^\infty$ be the conformal map with $\varphi(O) = \infty$ and $\varphi'(O) > 0$. It is well known that if the degree of P is d then

$$P \circ \varphi = \varphi \circ \sigma_d.$$

If J is LC, this equality extends over $\mathbb{S}$. Hence, in the LC case, the dynamics of P on J is semi-conjugate to the dynamics of σ_d on $\mathbb{S}$.

We can visualize this as follows. Assume J is LC and φ is extended over $\mathbb{S}$.

For each $y \in J$, let L_y be the collection of all chords in the boundary of the convex hull of $\varphi^{-1}(y)$ in $\mathbb{D}$ and let $\mathcal{L} = \bigcup_{y \in J} L_y$. Then $\mathcal{L}$ is an invariant lamination in the unit disk.
Elements $\ell \in \mathcal{L}$ ar called *leaves* and components G of $\mathbb{D} \setminus \mathcal{L}$ *gaps.*



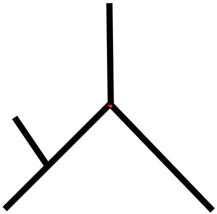
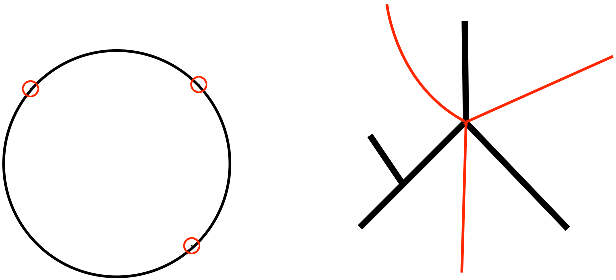


Figure: Lamination $\mathcal{L}$ on left, Julia set J on right.



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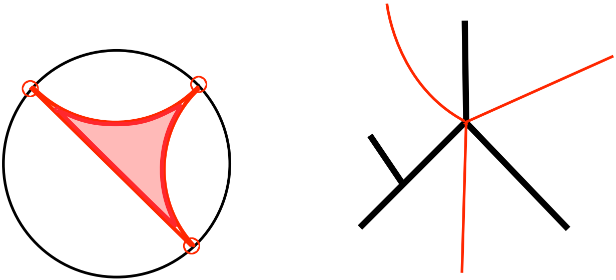

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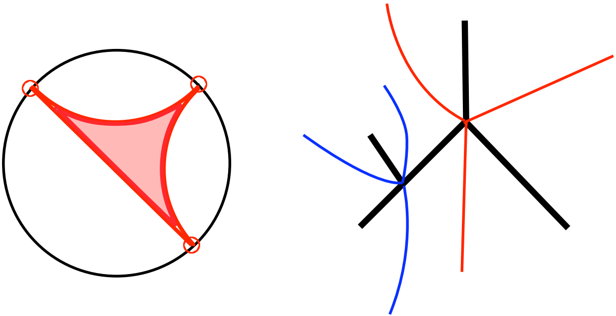

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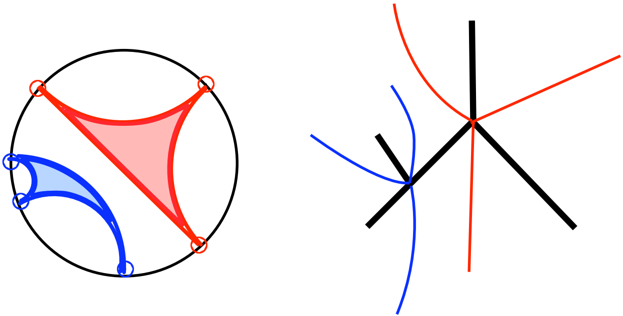

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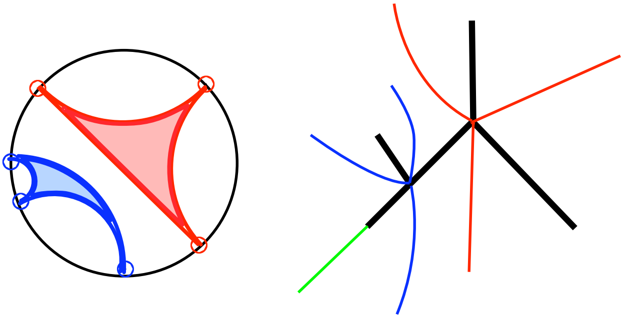

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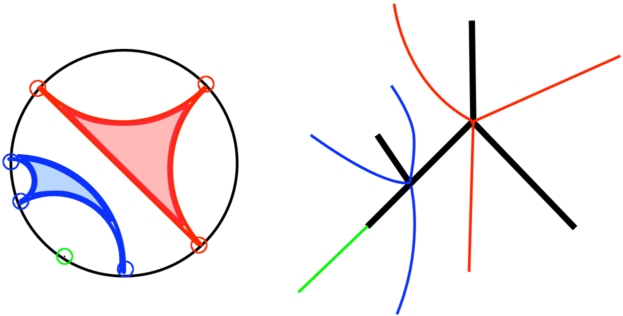

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Following Thurston we can define an invariant lamination abstractly as follows:

Definition

Suppose that $\mathcal{L}$ is a closed set of chords of the unit disk. Then $\mathcal{L}$ is called a d-invariant lamination if:

1. [non-crossing] for each $\ell_1 \neq \ell_2 \in \mathcal{L}$, $\ell_1 \cap \ell_2$ is at most a common endpoint.

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3. [onto] for each $\ell \in \mathcal{L}$ there exists $\ell' \in \mathcal{L}$ such that $\sigma(\ell') = \ell$,
4. [d-siblings] for each $\ell \in \mathcal{L}$ such that $\sigma(\ell)$ is a non-degenerate leaf, there exist d **disjoint** leaves $\ell_1, \ldots, \ell_d$ in $\mathcal{L}$ such that $\ell = \ell_1$ and $\sigma(\ell_i) = \sigma(\ell)$ for all i.

Leaf ℓ – element of $\mathcal{L}$.
Gap G– component of $\mathbb{D} \setminus \mathcal{L}$.

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Given a gap G we denote by $\sigma(G)$ the convex hull of the set $\sigma(\overline{G} \cap \mathbb{S})$ in $\mathbb{D}$.

Given an invariant lamination $\mathcal{L}$, we can extend σ linearly over all leaves in $\mathcal{L}$. We denote this extension by $\sigma^* : \mathcal{L} \cup \mathbb{S} \to \mathcal{L} \cup \mathbb{S}$.

Theorem (O.-Valkenburg)

Suppose that G is a gap of a d-invariant lamination $\mathcal{L}$. Then either

1. $\sigma(G)$ *is a point in* $\mathbb{S}$ *or a leaf of* $\mathcal{L}$*,*
2. $\sigma(G) = H$ *is also a gap of* $\mathcal{L}$ *and the map* $\sigma^* : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ *is the positively oriented composition of a monotone map* $m : \mathrm{Bd}(G) \to S$*, where* S *is a simple closed curve, and a covering map* $g : S \to \mathrm{Bd}(H)$*.*

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Equivalence classes are maybe proper or the entire circle, $J_{\text{top}} = \mathbb{S}/\approx$ is called a *topological Julia set* and the map $g : J_{\text{top}} \to J_{\text{top}}$ induced by σ_d a *topological polynomial.*

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Finite gaps correspond to branch points of J_{top} and uncountable gaps to “Fatou domains.”

In general it is difficult to decide if a lamination $\mathcal{L}$ containing a full set of critical leaves corresponds to a non-degenerate equivalence relation.
(see *Non-degenerate quadratic laminations* by A. Blokh, D. Childers, J. Mayer and O. for the quadratic case.)

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Thurston has shown that the space of all 2-invariant laminations is itself a lamination whose quotient space is a locally connected model for the boundary of mandelbrot set $\mathcal{M}$.

A map $m : X \to Y$ is *monotone* if $m^{-1}(y)$ is connected for each $y \in Y$.

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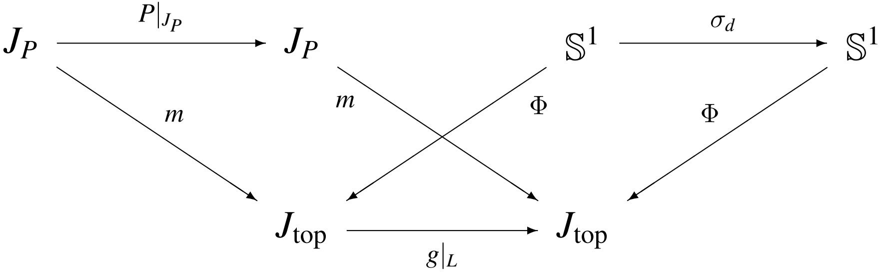
Theorem (Blokh-Curry-O.)

All Julia sets J have a locally connected model J_{top}. (I.e., there exists a finest monotone surjection $m : J \twoheadrightarrow J_{\mathrm{top}}$ such that J_{top} is locally connected and for every monotone surjection $m' : J \twoheadrightarrow X$, where X is locally connected, there exists a monotone map $m'' : J_{\mathrm{top}} \to X$ such that $m' = m'' \circ m$.)

It follows from Kiwi's work that a non-degenerate locally connected model always exists when P has no irrational neutral points.

Since J_{top} is locally connected, it induces a lamination $\mathcal{L}$ in $\mathbb{D}$ whose quotient space is J_{top}.



In certain cases the locally connected model L for a Julia set J is a point. For example, this is the case when: $\deg(P) = 2$ and P has a fixed Cremer point. We will call such polynomials *basic Cremer polynomials*.

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Theorem (Blokh-O.)

If P is a basic Cremer polynomial and $m : J \twoheadrightarrow L$ is a monotone surjection, where L is LC, then L is a point.

Theorem (Blokh-Curry-O.)

Let P be any polynomial with connected Julia set. Then the finest LC model of J is not degenerate if and only if at least one of the following properties is satisfied.

1. *The filled-in Julia set K_P contains a parattracting Fatou domain.*
2. *The set of all repelling bi-accessible periodic points is infinite.*
3. *The polynomial P admits a Siegel configuration.*

Let N be the number of cycles of bounded Fatou domains of P plus the number of Cremer cycles of P

Theorem (Fatou, Doaudy-Hubbard, Shishikura)

$$N \leq d - 1$$

All bounded Fatou domains and all Cremer cycles “attract”attract a critical point.
It is known that wandering branch points also attract critical points, allowing an improvement of the above inequality.

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3. *for each arc $A \subset J_\alpha$ there exists n such that $P^n(b_\alpha) \in A$.*
4. *the set of all cubic critical WT-portraits is a dense, uncountable, first category subset of the set of all cubic critical portraits.*

A *wandering* – $P^n(A) \cap P^m(A) = \emptyset$ for any $n \neq m$.
A *precritical* – an eventual forward image of A contains a critical point;
non-precritical – otherwise.

$\mathrm{Val}_X(Y) = \mathrm{Val}(Y) = |\mathrm{Comp}(X \setminus Y)|$ – *valence of* Y *(in* X*)*.
If $\mathrm{Val}(Y) > 1$ we call Y a *cut-continuum* (of X).

If J_P is locally connected, the valence $\mathrm{Val}(x)$ of a point $x \in J$ equals the number of (external) rays landing at x.

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Theorem (Blokh-Childers-Levin-O.-Schleicher)

Suppose that J_P *is connected. If* $\Gamma \neq \emptyset$ *is a wandering collection of non-precritical continua of valences* $M_1 > 2, \ldots, M_k > 2$ *then* $\sum_\Gamma (M_i - 2) + N \leq |C_{wr}| - 1 \leq d - 2.$

With thanks to Shishikura

N is the number of non-repelling periodic orbits plus the number of Cremer cycles;
N^∞ is the number of repelling orbits without periodic dynamic rays landing on them;
Given a set Q, denote χ_Q to be 1 if Q is non-empty and 0 otherwise.
Also, set $\sum_{i=1}^m (M_i - 2) = 0$ if $m = 0$.

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Theorem (Blokh-Childers-Levin-O.-Schleicher)

Let P be any polynomial. Consider a wandering collection Γ of non-precritical continua/points of P with $\mathrm{Val}(W) > 2$ for $W \in \Gamma$. Then

$$\sum_{W \in \Gamma} (\mathrm{Val}(W) - 2) + N + N_\infty \leq d - 1 - \chi(\Gamma).$$

Definition

A polynomial P is said to be a *basic uniCremer polynomial* if it has a Cremer periodic point and no repelling/parabolic periodic point of P is biaccessible (by results of Kiwi and Goldberg-Milnor then the Cremer point must be fixed).

Basic uniCremer polynomials have degenerate locally connected models.

Definition

A topological space X is *connected im kleinen* (CIK) at $x \in X$ if for each open set U, containing x there exists a connected set C such that:

$$x \in \mathrm{Int}(C) \subset C \subset U.$$

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Lemma

If X is CIK (at every point), then X is LC.

Theorem (Blokh-O.)

Let P be a basic Cremer polynomial. Then its Julia set J must be one of the following two types.

Solar Julia set J has the following equivalent properties:

1. there is an impression not containing the Cremer point;
2. there is a degenerate impression;
3. the set Y of all K-separate angles with degenerate impressions contains all angles with dense orbits and a dense in S^1 set of periodic angles, and the Julia set J is CIK at the landing points of these rays;
4. there is a point at which the Julia set is CIK.

Red dwarf Julia set Every impression contains the Cremer point p. Then J has the following properties:

1. the (non-empty) intersection of all impressions contains all forward images of all critical points,
2. J is nowhere connected im kleinen.

Moreover, in this case no point of J is biaccessible and p is not accessible from $\mathbb{C} \setminus J$.

Building on results by Inou and Shishikura, Buff and Chéritat have shown that there exist basic Cremer polynomials P (i.e., of $\deg(P) = 2$ and with a fixed Cremer point) whose Julia sets J have positive Lebesgue area.

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Theorem (Blokh, Buff, Chéritat and O.)

There exist basic Cremer polynomials with solar Julia sets of positive area.

Theorem (Kiwi, Grispolakis-Mayer-O.)

Suppose P is a basic Cremer polynomial with solar Julia set J, critical point c, Cremer fixed point p and $P'(p) = e^{2\pi i\alpha}$. Then there exists a building block $B \subset J$ and a Cantor set $A \subset [\frac{\theta}{2}, \frac{\theta+1}{2}] \subset \mathbb{S}$ such that:

1. *B is a nowhere dense subcontinuum of J,*
2. *$P(B) = B$,*
3. $p \cup P^{-1}(p) \cup \mathcal{O}(c) \subset B$
4. *$\sigma(A) = A$, minimally, with rotation number α,*
5. *$B = \bigcup_{\gamma \in A} \mathrm{Imp}(\gamma)$.*

Note $\{c, p, -p\} \subset \mathrm{Imp}(\theta/2) \cap \mathrm{Imp}(\theta/2 + 1/2)$.

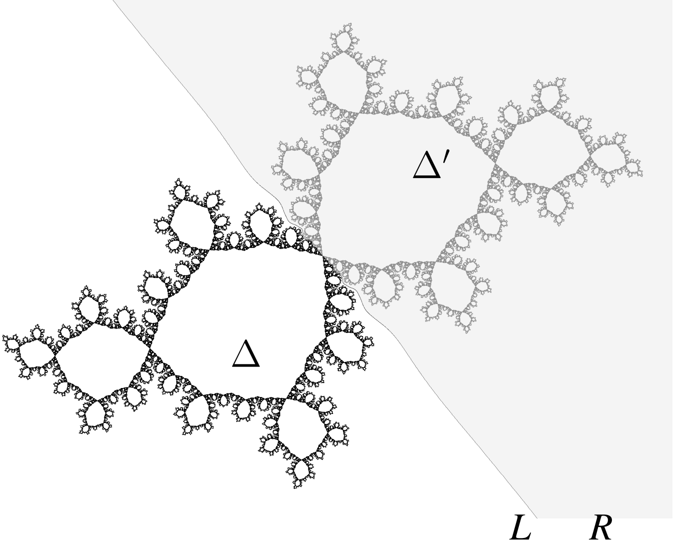

Figure: Example of a locally connected basic Siegel polynomial Julia set.

Building blocks contain *hedgehogs* constructed by Peréz Marco:
For each open set Ucontaining the Cremer fixed-point p such that $\overline{U}$ does not contain the critical point c, there exists an invariant continuum H with $p \in H$ and $H \cap \partial U \neq \emptyset$.
Let $\Delta = \{H \mid H \text{ is a hedgehog}\}$ and let

$$M = \overline{\bigcup_{H \in \Delta} H}.$$

Then H is called the *mother hedgehog*.

Theorem (Childers)

M is connected, contains the critical point c and $\omega(c) = M \subset B$.

Recently Shishikura has shown that there exists a maximal hedgehog MH such that $p, c \in MH$ and $P|_{MH} : MH \to MH$ is a homeomorphism.

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$$H \subset M \subset MH \subset B$$

where H is any hedgehog
$M = \omega(c)$ is he mother hedgehog
MH is the maximal hedgehog and
$B = \bigcup_{\theta \in A} \mathrm{Imp}(\theta)$ is the building block.

Clearly $H \neq M$ and $MH \neq B$.

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Is the mother hedgehog equal to the maximal hedgehog,

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Shishikura has shown that MH is a Cantor bouquet:

A *dendroid* is an arcwise connected continuum such that the intersection of any two subcontinua is connected. Equivalently, a dendroid is an arcwise connected tree-like continuum. An endpoint e, of a dendroid X, is a point such that for each arc $A \subset X$ which contains e, e is an endpoint of A. The cone over the Cantor set is a dendroid with exactly one vertex, O, and a (closed) Cantor set of endpoints.

Peréz Marco has shown that the cone over a Cantor set cannot be a hedgehog.

All hedge hogs must admit arbitrary small irrational rotations.

There exists a *Lelek function*:
$\ell : [0,1] \to [0,1]$ is USC such that:

1. for a dense set $D_0 \subset [0,1]$, for each $d \in D_0$, $\ell(d) = 0$ and $\ell(0) = \ell(1) = 0$,
2. for a dense set $D_{>0} \subset [0,1]$, for each $d \in D_{>0}$, $\ell(d) > 0$,
3. for each $x \in (0,1)$ there exists $y_n \uparrow x$, $y'_n \downarrow x$ and $\lim \ell(y_n) = \lim \ell(y'_n) = \ell(x)$.

Definition (Aarts-O.)

Given ℓ as above, the set

$$\mathbb{H} = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq \ell(x)\}$$

is called the *basic hairy arc with base* $B = [0,1] \times \{0\}$.

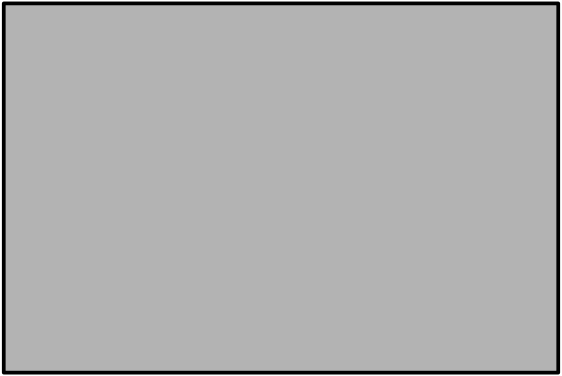
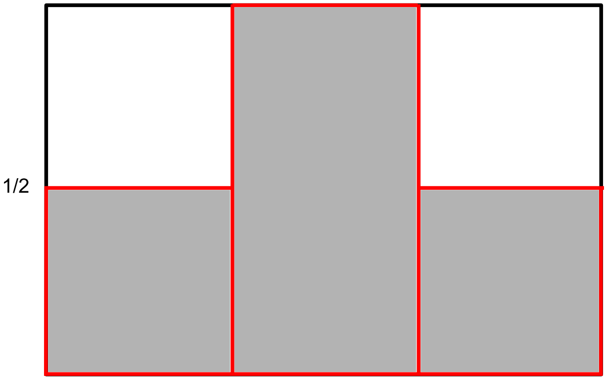


Figure: The Hariry arc.



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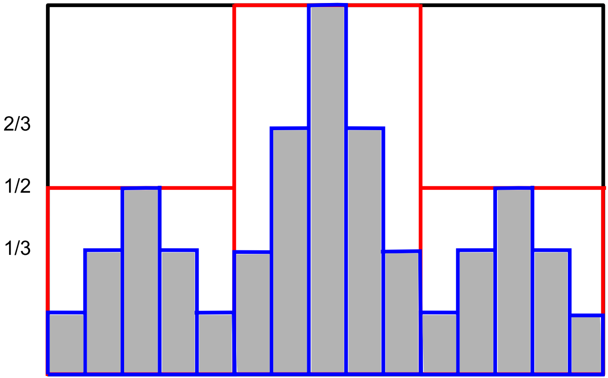

Figure: The Hariry arc.

Any space homeomorphic to $\mathbb{H}/B$ is called a *Cantor bouquet.* Any space $X \subset \mathbb{C}$ homeomorphic to $\mathbb{H}$, with all hairs on the same side of the base, is called a *hairy arc* and any space $X \subset \mathbb{C}$ homeomorphic to $\mathbb{H}/\{(0,0),(1,0)\}$, with all hairs in the unbounded component of the image of the base, a *hairy circle.*

Cantor bouquets were first constructed by Lelek. It follows from work by Devaney that a Cantor bouquet is homeomorphic to the Julia set of the exponential map λe^z, for λ small, in the sphere.

It is known that all Cantor bouquets are homeomorphic (Charatonik and Bula-O) (even under homeomorphisms of the entire plane (Aarts-O.)) if all hairs are limits from both sides)

The set of endpoints E of a Cantor bouquet is a one-dimensional and totally disconnected. Moreover, E is homeomorphic to the set of points in ℓ_2 all of whose coordinates are irrational (Kawamura-O.-Tymchatyn).

By (unpublished) results of Shishikura, Buff and Chéritat there exist basic Cremer polynomials whose Julia sets contain a Cantor bouquet whose vertex is the fixed Cremer point.

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Lemma (Shishikura)

There exist basic Cremer Julia setssuch that the maximal hedgehog MH is a Cantor bouquet. Hence, there exists an arc joining the Cremer point and its pre-image.

It follows that there exists a second category subset D of J, which includes all repelling periodic points and their preimages, such that for each $d \in D$ there exists an arc A from d to the fixed Cremer point p.

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P is hereditarily equivalent and homogeneous.

The construction of a Cantor bouquet can be changed so that ever arc is replace by a pseudoarc. We will call this continuum a *pseudo Cantor bouquet*.

Do there exist Cremer Julia sets which contain pseudo Cantor bouquets?

Theorem (Childers– Mayer–Rogers)

The connected Julia set J of a polynomial is indecomposable
iff
The impression of every external angle is the entire Julia set J iff
The impression of one external ray has interior in J.

Question: does there exist an indecomposable Julia set??

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Theorem (Curry, Mayer, Rogers)

The Makienko conjecture is true if there are no indecomposable Julia sets.

The residual Julia set of a rational function is defined as its Julia set minus the boundaries of its Fatou components. It is a well-known fact that, when a component of the Fatou set is fully invariant under some power of the map, the residual Julia set is empty. Based on Sullivan's dictionary, Peter M. Makienko conjectured that the converse is true: when the residual Julia set of a rational map is empty, there is a Fatou component which is fully invariant under a power of the map.