Local invariant sets of irrationally indifferent fixed points of high type

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Workshop on
Cantor bouquets in hedgehogs and transcendental iteration
Université Paul Sabatier
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Plan

Want to understand the dynamics of a quadratic polynomial f when it has an irrational indifferent fixed point *of high type*:

$$f(z) = e^{2\pi i\alpha} z + z^2, \qquad \alpha = \pm\cfrac{1}{a_1 \pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}} \quad (a_i \in \mathbb{N},\ a_i \geq N \ \text{ large})$$

(also applies to $e^{2\pi i\alpha} z(z+1)^n$, $e^{2\pi i\alpha} z e^z$)

Goal:

Topological description of invariant sets around the fixed point

Hedgehog, the boundary of Siegel disk

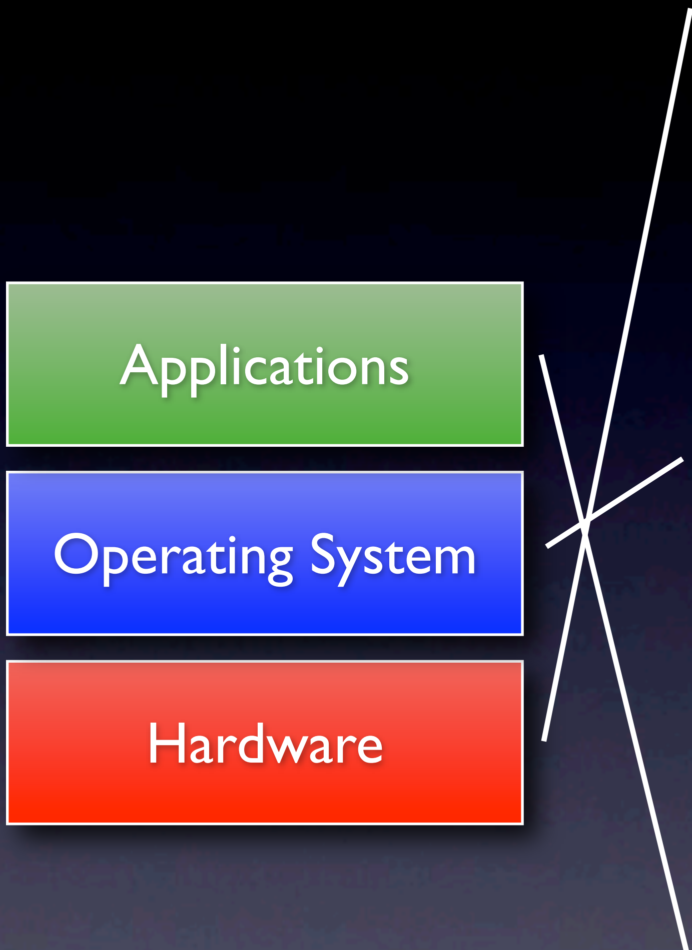
Tools:

Near-parabolic renormalization $f \mapsto \mathcal{R}f$

Inou-S. "uniform lower bound on the nonlinearity of $\mathcal{R}^n f$"

Reconstructing f from $\mathcal{R}f, \mathcal{R}^2 f, \ldots$

Plan of 3 talks



Talk 1: Inou-Shishikura Theorem

Class $\mathcal{F}_1$ and its invariance under the near-parabolic renormlaization $\mathcal{R}$

Truncated checkerboard pattern Ω_f and its relation to $\mathcal{F}_1$

Talk 2: Reconstructing (part of f) from $\mathcal{R}^n f$

$\Omega_{f,k}$'s within Ω_f, their gluing and the dynamics

the combinatorics of rotation $r_{\alpha,n} : A_n \to A_n$, with $A_n \subset \mathbb{Z}^n$

$\Omega_{f,k_1,\ldots,k_n}$ for $(k_1, \ldots, k_n) \in A_n$

Talk 3: Applications

Cantor bouquets, hairs, hedgehogs and the boundary of Siegel disks

Compare dynamics

Easy: Contractions

Nice: Expanding maps (inverse: multivalued contraction)

Lifting argument by inverse branches via appropriate homotopy

→ structural stability (homotopical stability)

Hölder continuity of conjugacy

symbolic dynamics, topological model

Hyperbolic rational maps $\hat{\mathbb{C}} = F_f \cup J_f$

F_f =basin of attracting periodic points; f is expanding on J_f.

J_f connected $\Longrightarrow$ locally connected

↕ opposite

Nasty(?): maps with irrationally indifferent fixed points

not expanding at the fixed point

Julia set contains a critical point, which is recurrent (Mañé)

Easy: Contractions

Nice: Hyperbolic rational maps

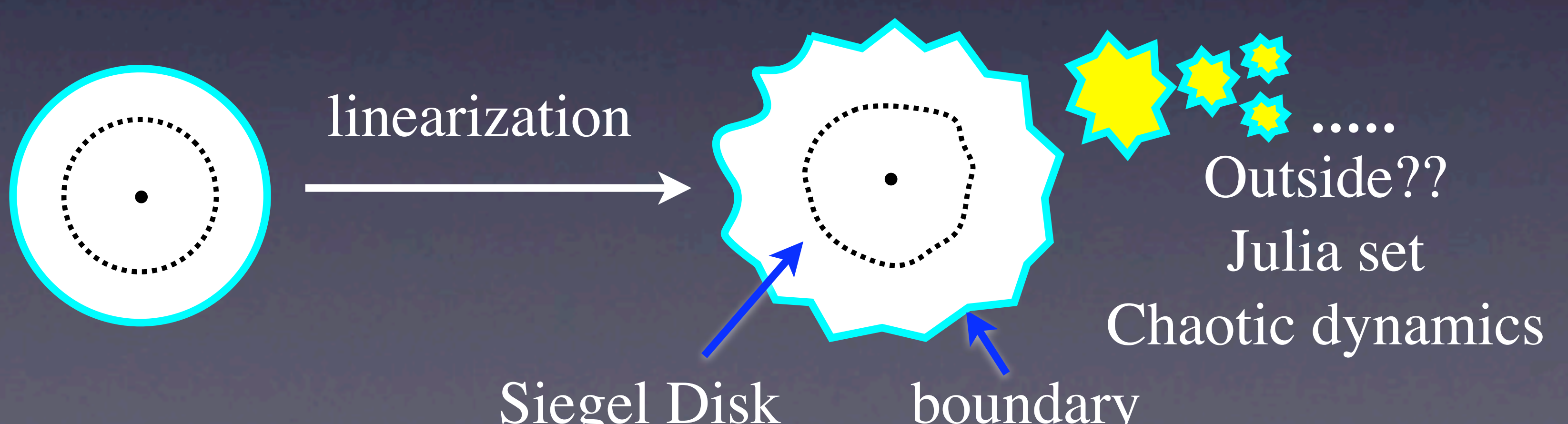
Nasty(?): maps with irrationally indifferent fixed points
- not expanding at the fixed point
- Julia set contains a critical point, which is recurrent (Mañé)

rotation numbers $\{\text{bounded type}\} \subset \{\text{Diophantine}\} \subset \{\text{Brjuno}\}$

Brjuno rotation # $\longrightarrow$ linearizable (Siegel-Brjuno-Yoccoz)

Siegel disk = domain of linearization

bounded type $\longrightarrow$ boundary of Siegel disk is Jordan curve
Julia set is locally connected
(Herman, Petersen, Petersen-Zackeri)



Physicists expect a "universal phenomenon" at the boundary of SD

Easy: Contractions

Nice: Hyperbolic rational maps

Nasty(?): maps with irrationally indifferent fixed points

bounded type Brjuno rotation #

Nastier: rotation number with large continued fraction coefficients

Liouville rotation #, non-Brjuno or high type

non-Brjuno → non-linearizable fixed pt (Cremer pt)

for some rot #, bdry of SD is Jordan curve, but no crit pt (Herman)

In these cases, Julia sets is NOT locally connected.

Questions: bdry of SD = J?

J = indecomposable continuum?

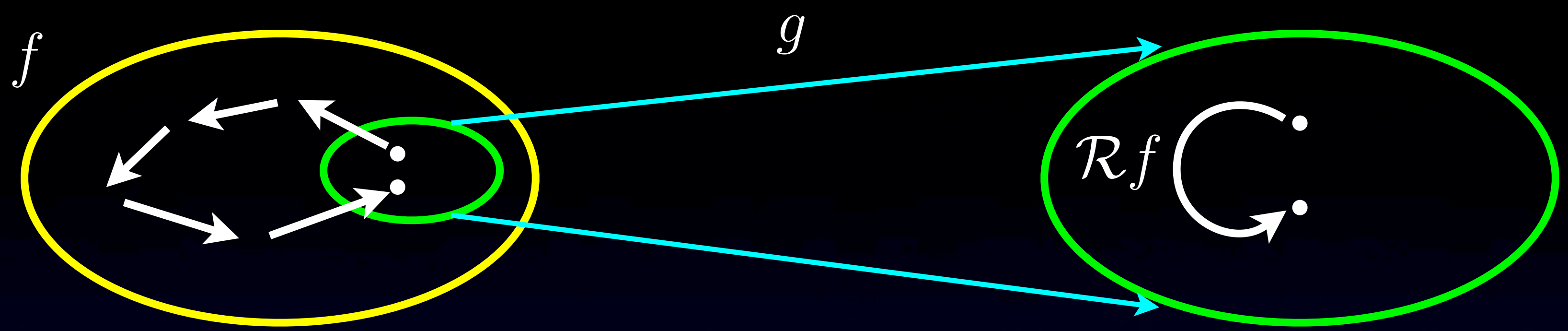
impression of 0-ray = J?

How can we describe the topology of J?

Are they *Monsters*?

We are going to deal with this case (high type).

Irrationally indifferent fixed points or rotation-like dynamics
study via renormalization (constructed as a return map)



Successive construction of $\mathcal{R}f$, $\mathcal{R}^2 f$, ..., helps to understand the dynamics of f (orbits, invariant sets, rigidity, bifurcation, ...)

For bounded type (or Dioph., Brjuno), the number of iteration needed in the construction of $\mathcal{R}f$ is not too big.

+ upper bounds on the non-linearity of the renormalizations

⟶ solution of linearization problem, etc...

For high type, the number of iteration will be very big and the return map (renormalization) $\mathcal{R}f$ is close to identity.

identity: the most difficult map to study (if you want to study perturbation)

Non-linearity helps! Need *lower bound* on non-linearity.

More on renormalization for irrationally indifferent fixed points

$f(z) = e^{2\pi i\alpha} z + O(z^2)$ → [to be defined later] → $\mathcal{R}f(z) = e^{2\pi i\alpha_1} z + O(z^2)$

$$\alpha = \pm \cfrac{1}{a_1 \boxed{\pm \cfrac{1}{a_2 \pm \cfrac{1}{\ddots}}}} \qquad \boxed{\cdots} = \alpha_1$$

high type $\Rightarrow \alpha, \alpha_1, \alpha_2, \ldots$ small

Want: non-linear term of $\mathcal{R}^n f$ not too small

Inou-S.: If $f(z) = e^{2\pi i\alpha} z + z^2$ and α is of sufficiently high type, then $\mathcal{R}^n f$ are defined and $|(\mathcal{R}^n f)''(0)| \geq \exists c > 0 \quad (n = 0, 1, 2, \ldots)$.

Applications

Theorem 1 (structure): Let $f(z) = e^{2\pi i \alpha} h(z)$, where $h(z) = z + z^2$ or $h \in \mathcal{F}_1$ with α sufficiently high type.

Then there exist domains $\Omega^{(0)} \supset \Omega^{(1)} \supset \Omega^{(2)} \supset \ldots$, such that $\Omega^{(n)} \setminus \{0\} = \bigcup_{(k_1,\ldots,k_n)\in A_n} \Omega^{(n)}_{k_1,\ldots,k_n}$, where $\Omega^{(n)}_{k_1,\ldots,k_n}$'s are "almost cyclically permuted" by f and the intersection $\Lambda_f = \bigcap_{n=0}^{\infty} \Omega^{(n)}$ is a closed, forward invariant set containing 0 and the forward critical orbit. Every point in Λ_f is recurrent and f is injective on this set.

more description on $\Omega^{(n)}_{k_1,\ldots,k_n}$ and the action of f will be explained in Talk 2.

Theorem 2 (hairs): Let f and $\Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ be as in Theorem 1. For an "allowable" sequence $k_1, k_2, \ldots$, the intersection $\cap_{n=1}^{\infty} \Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ is either empty or an arc tending to 0 (closed arc when 0 is added). The set of these arcs are cyclically permuted by f. In particular, there is an arc in Λ_f from the critical point to 0.

Applications (continued)

Theorem 3: Let f be a quadratic polynomial as in Theorem 1. Then the Julia set J_f is decomposable and locally connected at every periodic point except 0.

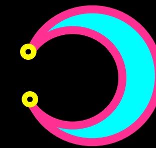
Theorem 4: Let f be as in Theorem 1. Then Λ_f contains all "hedgehogs" in Perez Marco's sense.

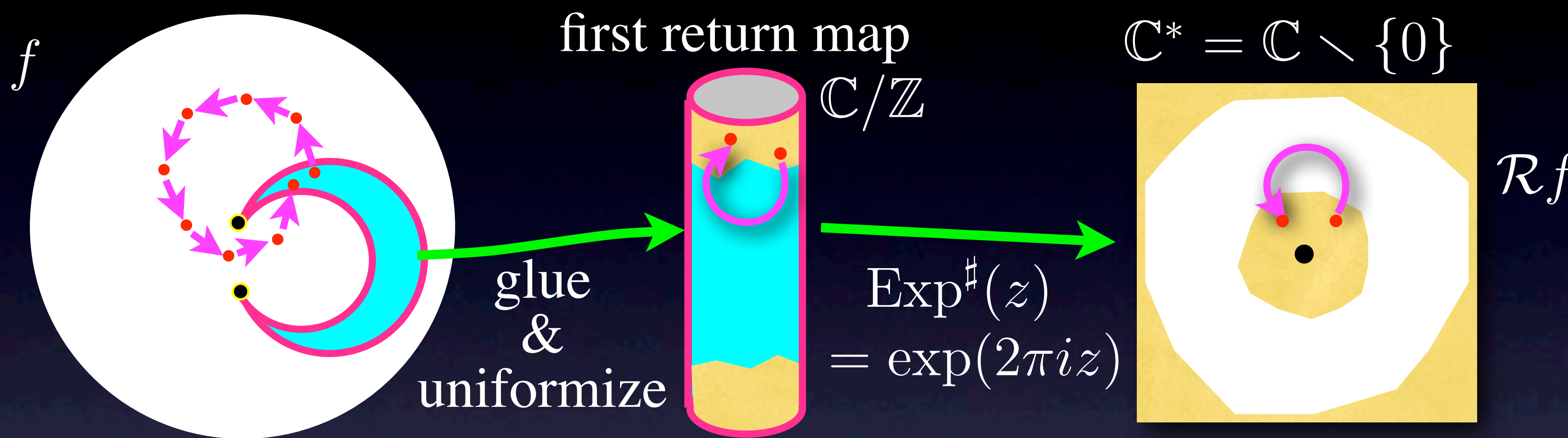
Theorem 5 (boundary of Siegel disk): Let f be as in Theorem 1, and assume that α is a Brjuno number. By Siegel-Brjuno, f is linearizable and has a Siegel disk Δ_f.
Then the boundary $\partial\Delta_f$ is a Jordan curve.
Furthermore, one can give a bound on the modulus of continuity in terms of continued fraction expansion of α.

(Earlier results by Herman, Petersen, Petersen-Zackeri, via surgery.)

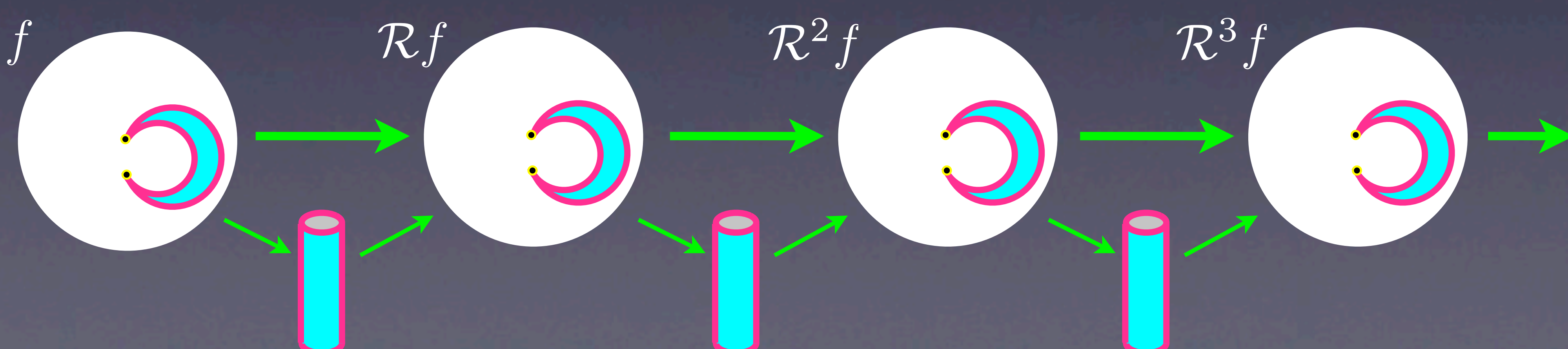
Theorem 6: In Theorem 5, $\partial\Delta_f$ contains the critical point if and only if $\alpha \in \mathcal{H}$.

Definition of Renormalization $\mathcal{R}f$

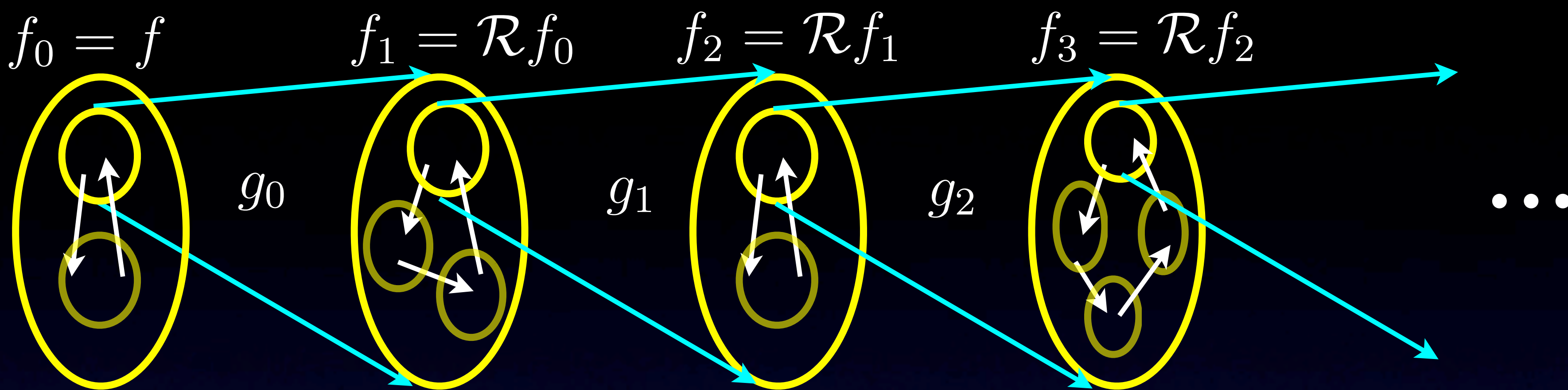
If one can define a “fundamental region”  so that its quotient is isomorphic to $\mathbb{C}/\mathbb{Z}$, then the renormalization $\mathcal{R}f$ can be defined.



Inou-S.: For f as in the theorem, we have the sequence:



Key idea in renormalization



f may be very recurrent, non-expanding, non-linear, has critical pt

The sequence of "renormalizers" (coordinate changes between consecutive renormalizations) is like iteration of expanding maps.

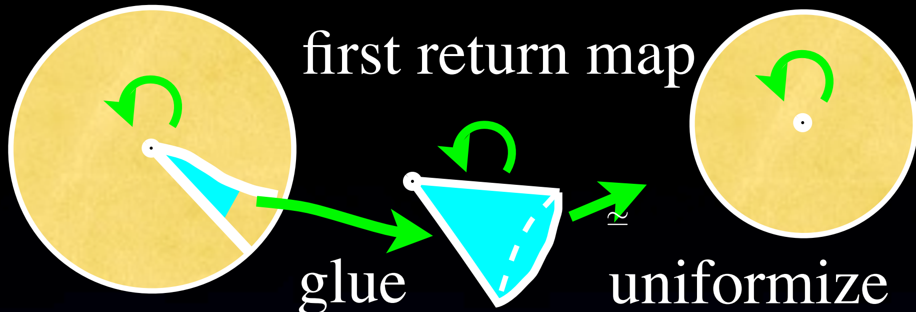
Nice "dynamics"!

In the limit $N \to \infty$, g_i's are "like" exponential maps (parabolic renormalization).

quadratic polynomials are transcendental!

(if you consider renormalizations)

Yoccoz sectorial renormalization



works for any germ, any rot. #

may lose a lot by cut-off, when rot. # is small

no critical points

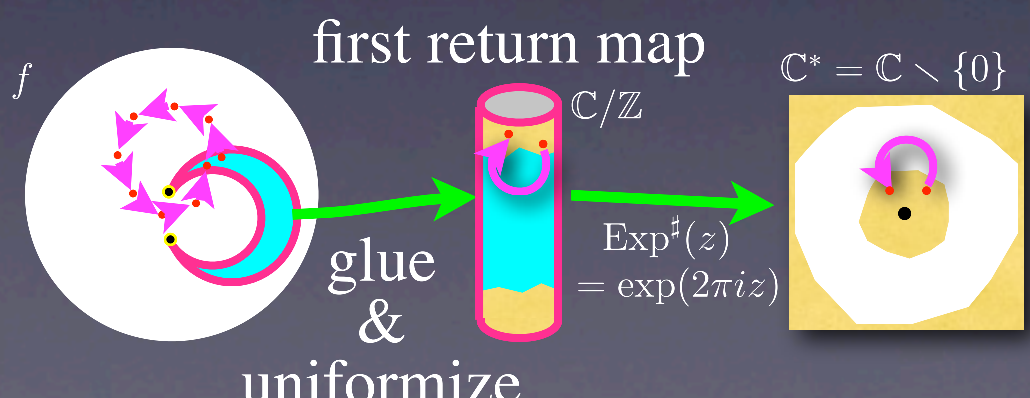
Perez Marco renormalization for quadratic type germs

works for quadratic type

need to show the existence

no critical points

Near-parabolic renormalization



works only for $f = e^{2\pi i\alpha}h$

$h \in \mathcal{F}_1$ or $h = z + z^2$

α of high type

invariant class for renormalization implies QTC

the map has a critical point

Theorem (IS): Let $P(z) = z(1+z)^2$. There exists a Jordan domain V (with $V \ni 0, -\frac{1}{3}, \not\ni -1$) and large N such that the following holds for the class

$$\mathcal{F}_1 = \left\{ h = P \circ \varphi^{-1} : \varphi(V) \to \mathbb{C} \,\middle|\, \begin{array}{c} \varphi : V \to \mathbb{C} \text{ is univalent} \\ \varphi(0) = 0,\ \varphi'(0) = 1 \end{array} \right\}.$$

(0) If $h \in \mathcal{F}_1$, then $h(z) = z + O(z^2)$, $|h''(0)| \geq c > 0$, h has a unique critical point $(= \varphi(-\frac{1}{3}))$;

(1) If $f = e^{2\pi i\alpha} h$ with $h(z) = z + z^2$ or $h \in \mathcal{F}_1$ and α is of high type ($a_i \geq N$), then $\mathcal{R}f$ is defined and can be written as $\mathcal{R}f = e^{2\pi i \alpha_1} h_1$ with $h_1 \in \mathcal{F}_1$ and $\alpha_1 = \pm\{\frac{1}{\alpha}\}$.

Outline of Proof:

For f as above, one can find a "truncated checkerboard pattern" Ω_f (in pre-Fatou coordinate). *justified by numerical estimates*

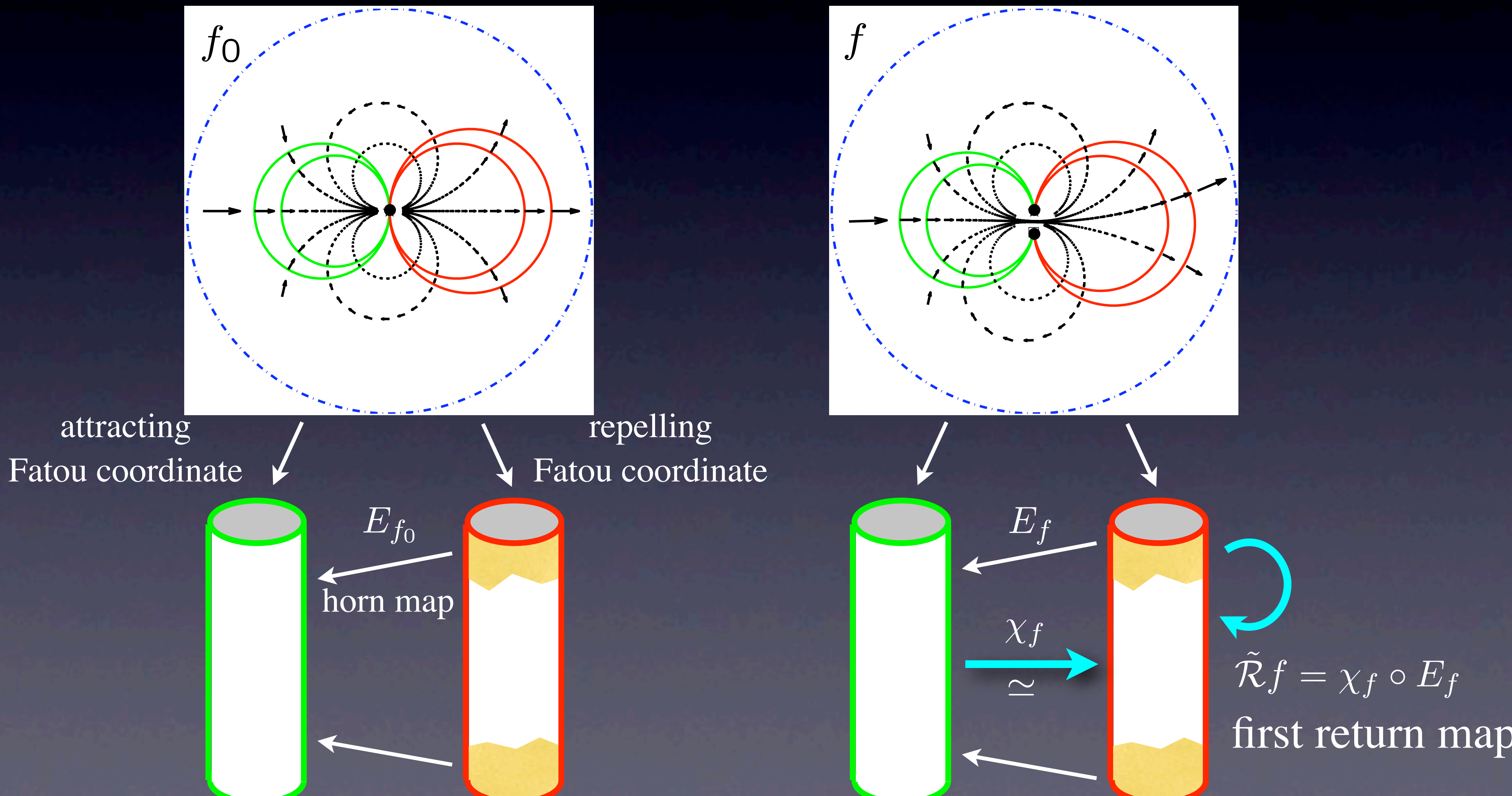
If there is a truncated checkerboard pattern, then $\mathcal{R}f$ can be written by $h_1 \in \mathcal{F}_1$. *proof by picture*

Why Non-linearity (or non-zero second derivative) helps?

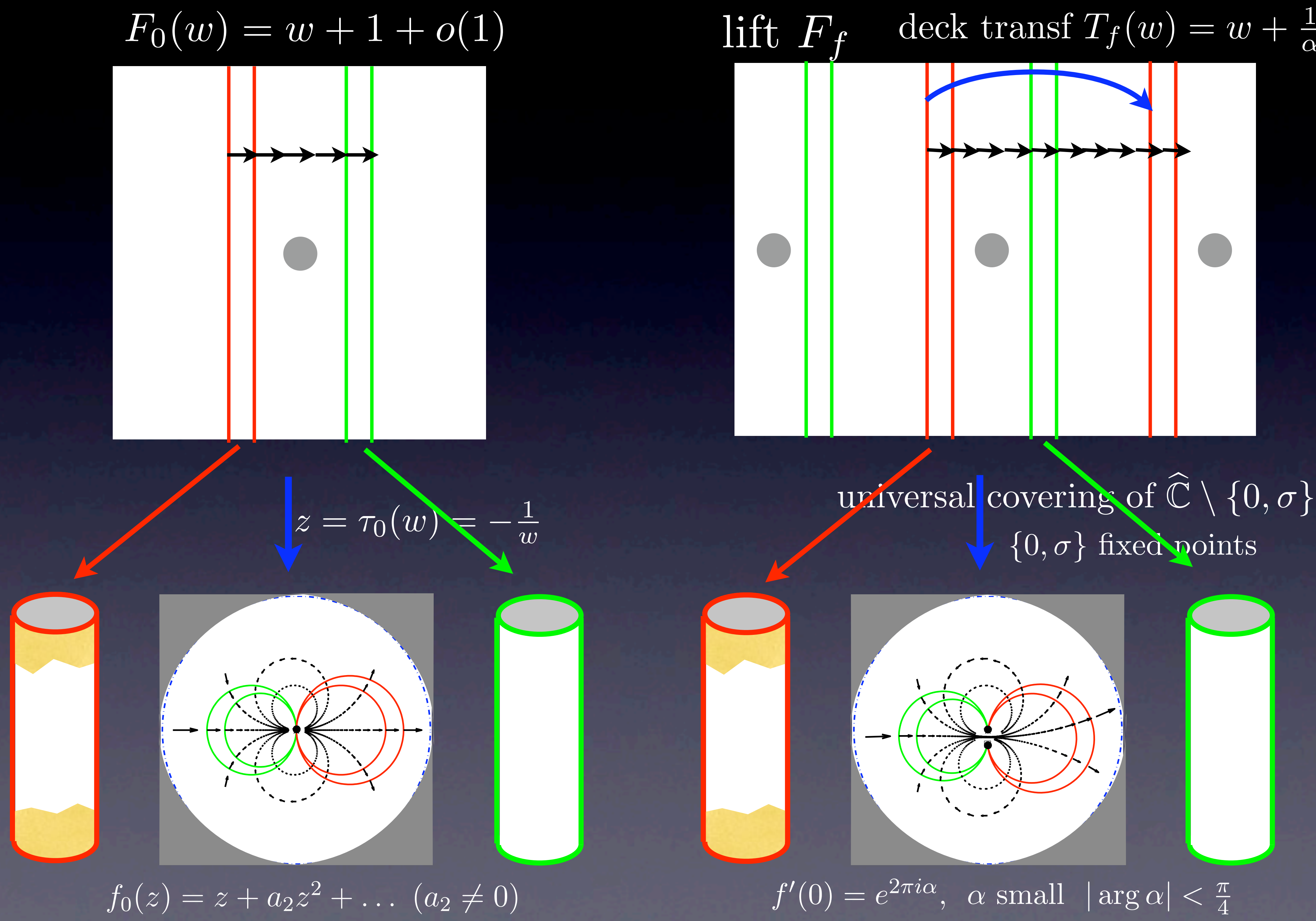
If $f''(0)$ not small and $f'(0) = e^{2\pi i\alpha}$, with α high type, then

Can use Douady-Hubbard-Lavaurs theory of parabolic implosion.

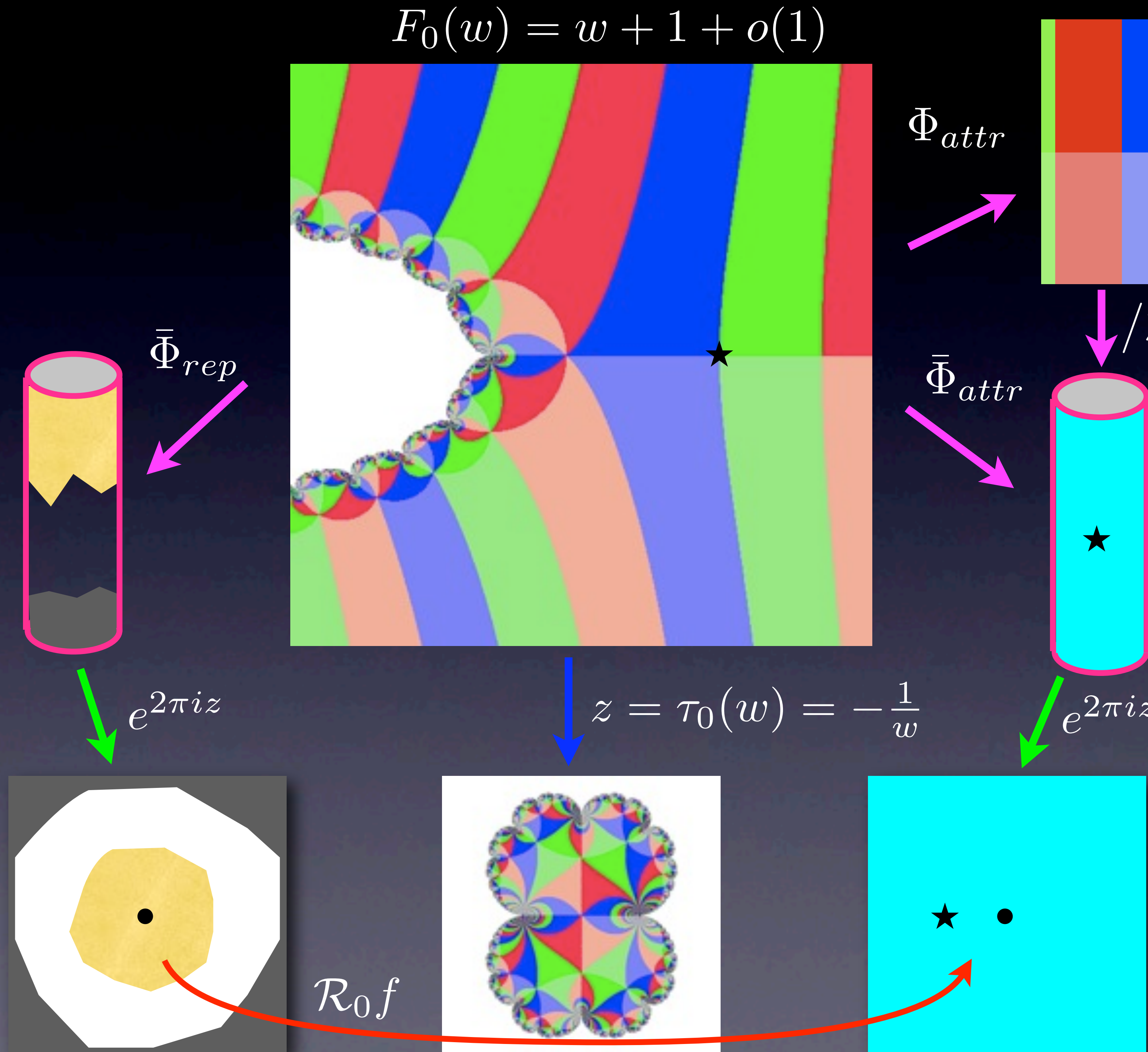
$f_0(z) = z + a_2 z^2 + \ldots \quad (a_2 \neq 0)$ $\qquad$ $f'(0) = e^{2\pi i\alpha}$, α small $|\arg\alpha| < \frac{\pi}{4}$



pre-Fatou coordinate and the lift of f



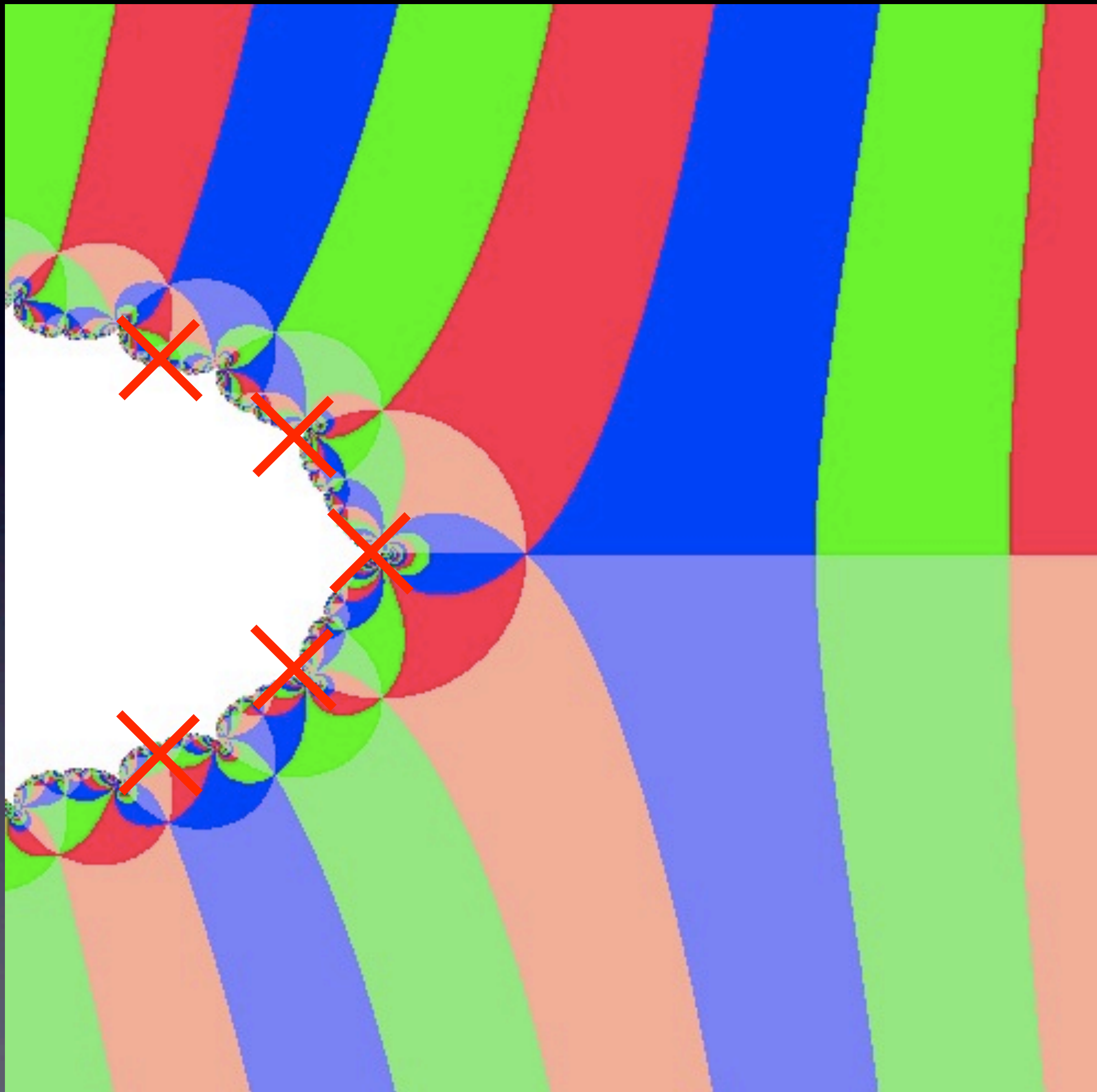
Basic checkerboard pattern for parabolic map



If a parabolic basin contains only one simple critical point, then the checkerboard pattern (and the dynamics) in the basin is the same

$g = \mathcal{R}_0 f$ is again in the class $\mathcal{F}_0$, i.e.
$g : Dom(g) \setminus \{0\} \to \mathbb{C} \setminus \{0\}$
is a branched covering with only one critical value (with all crit. pts simple)

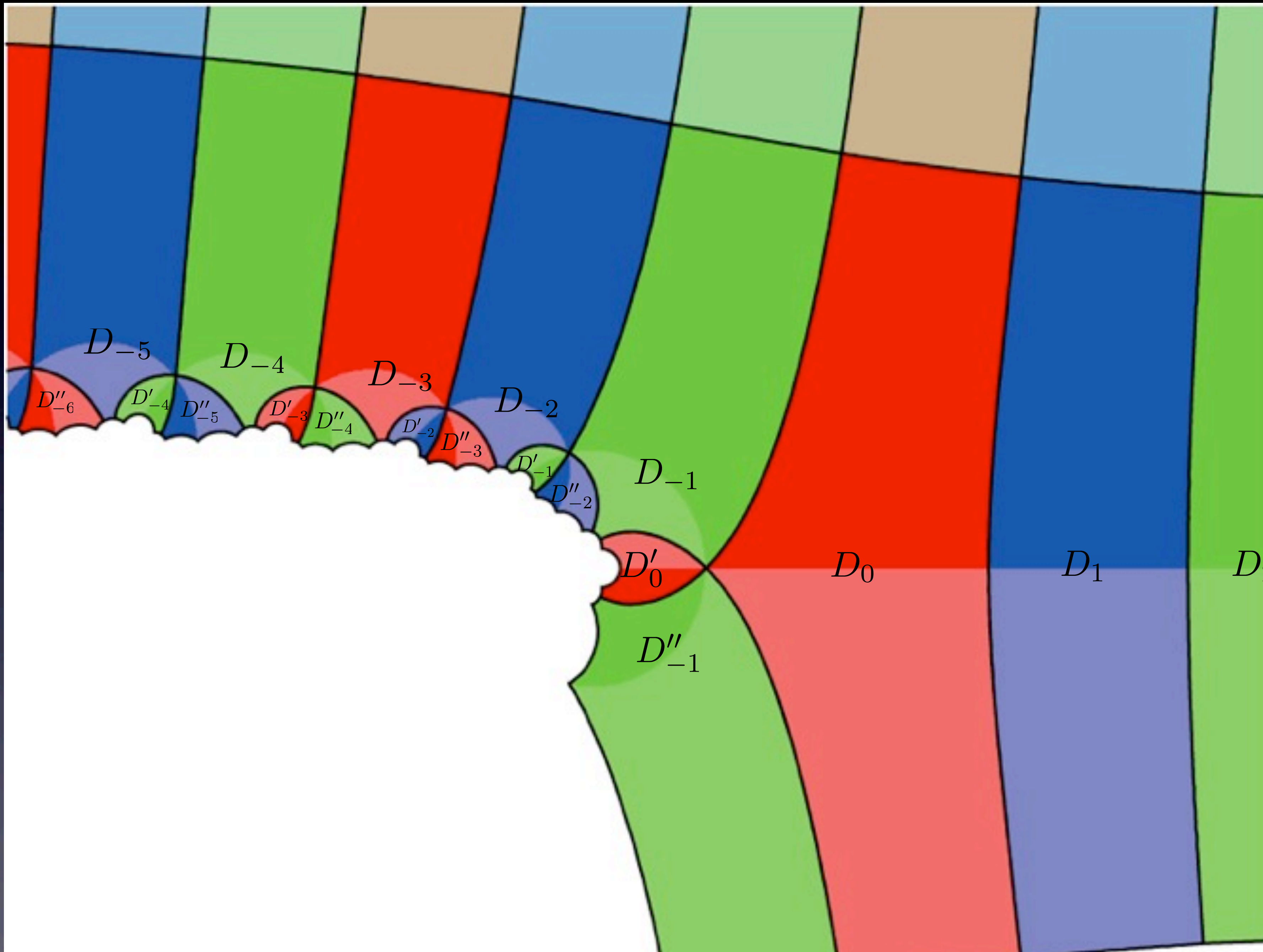
Basic checkerboard pattern for parabolic map 2



When the map is only partial defined or perturbed to non-parabolic,
not every detail of the pattern is preserved.

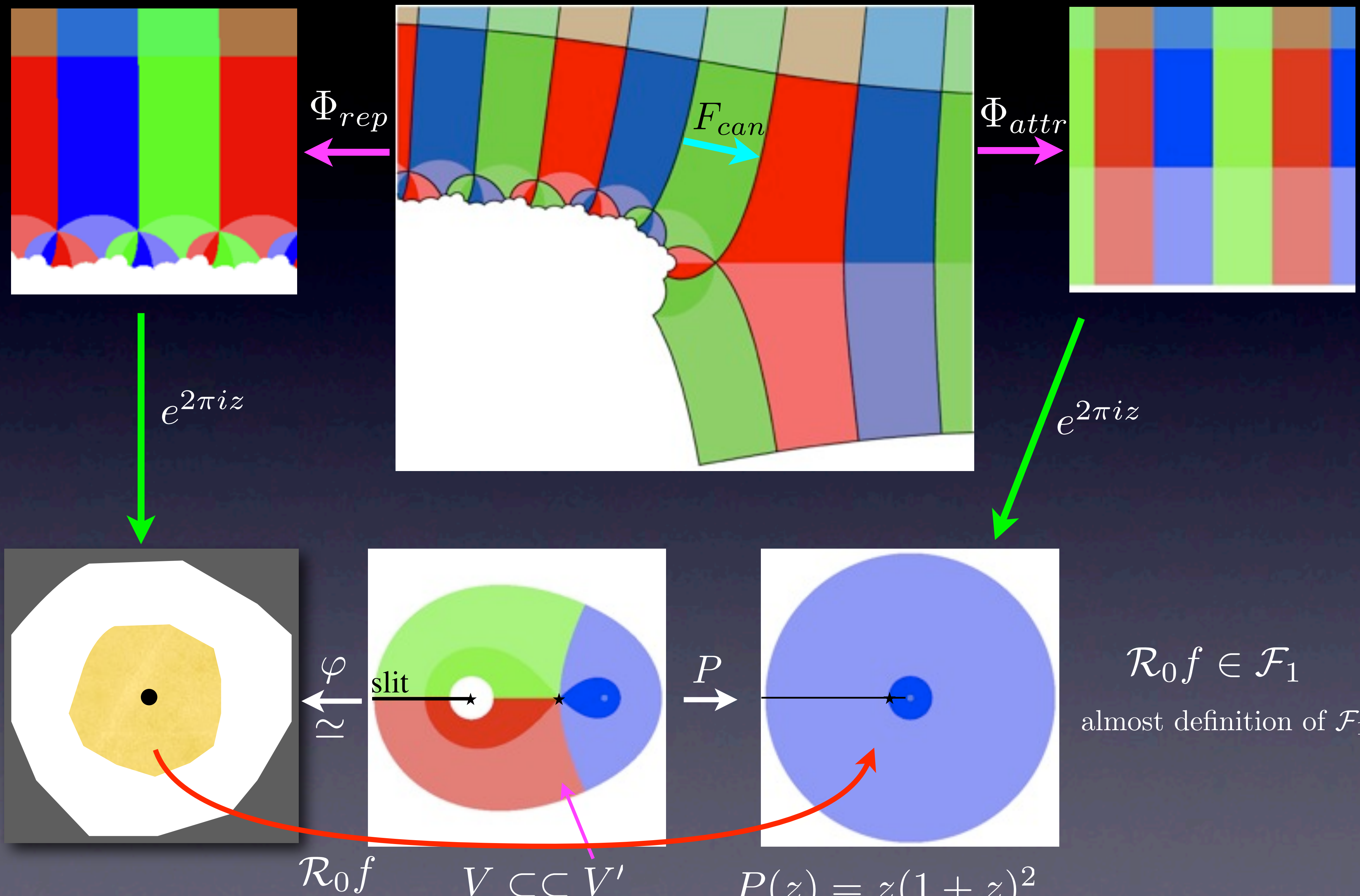
The pattern persists to some extent.

Truncated checkerboard pattern



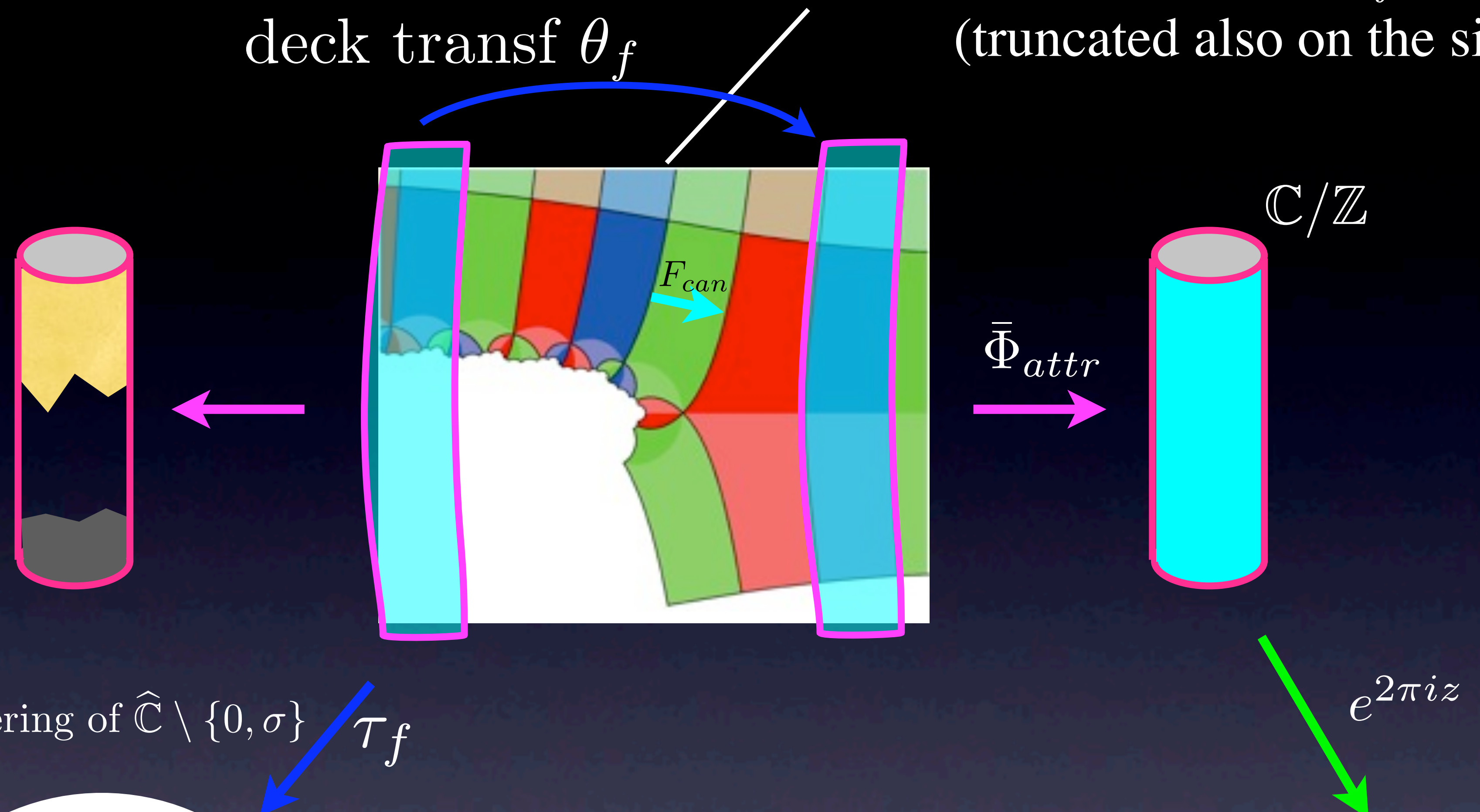
Truncated portions $D_{-n}, D'_{-n}, D''_{-n}$ are likely to remain after a perturbation

Truncated pattern induces a cubic-like covering



Near-parabolic case:

Truncated checkerboard pattern $\Omega_f = \Omega_f^{(0)}$ (truncated also on the side)



f

This shows that $\mathcal{R}f \in e^{2\pi i \alpha_1}\mathcal{F}_1$.

Instead of f itself, one should consider the canonical map F_{can} on $\Omega_f / \underset{\theta_f}{\sim}$, where θ_f is the gluing which depends on f.

$0, \sigma$ fixed pts

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

One more thing ...

Inou-S.: the invariant class $\mathcal{F}_1$ under near-parabolic renormalization

$$f = e^{2\pi i \alpha} h \longmapsto \mathcal{R} f = e^{2\pi i \alpha_1} h_1$$

$$\mathcal{F}_1 \ni h \overset{\mathcal{R}_\alpha}{\longmapsto} h_1 \in \mathcal{F}_1$$

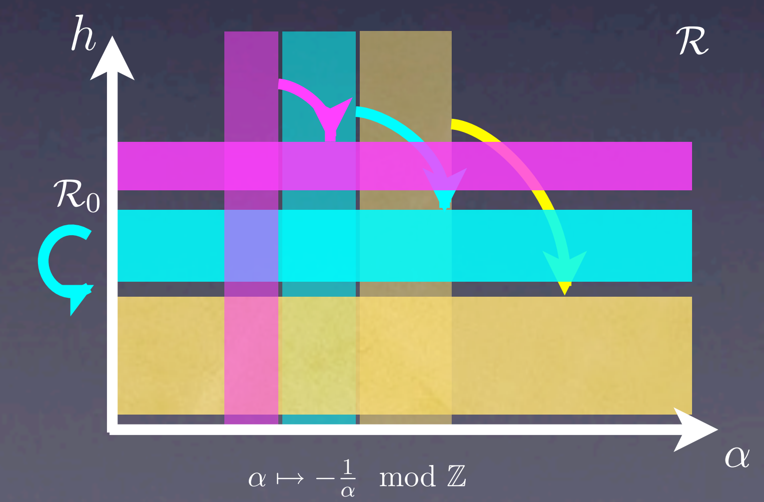
$\Longleftrightarrow$ a priori bound

$\mathcal{F}_1$ is in one to one correspondence with a Teichmüller space (of a punctured disk).

by Royden-Gardiner theorem = Schwarz lemma for Teichmüller space

⟶ $\mathcal{R}_\alpha$ is a contraction
$\mathcal{R}$ is hyperbolic
for α high type

Nice dynamics!



Prove one, get another one free!*

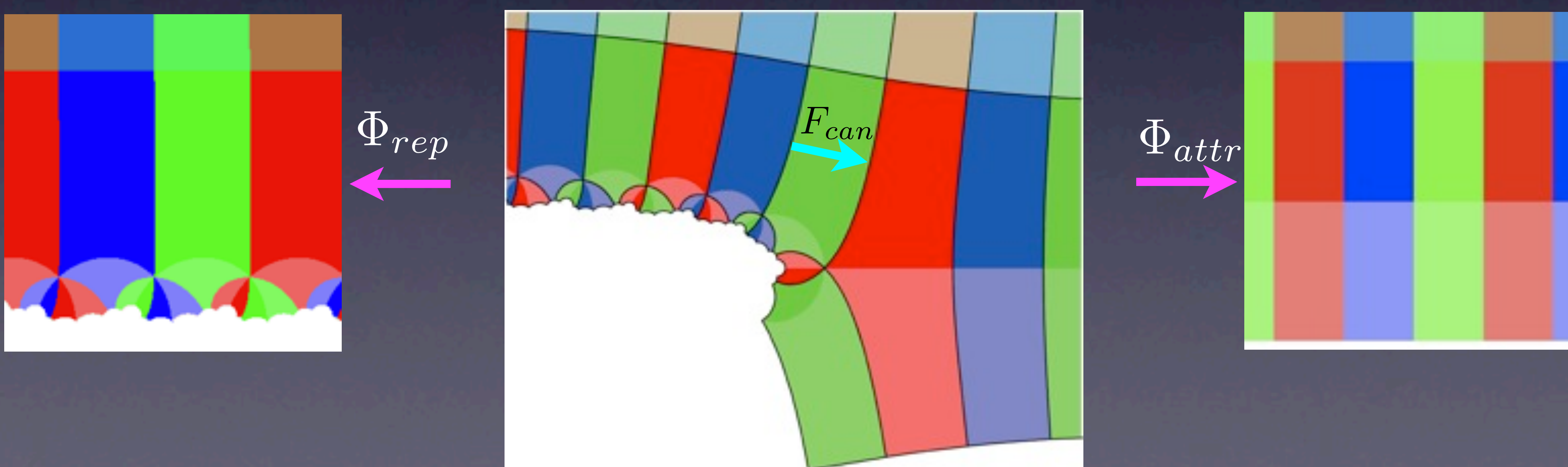
*— Requires slight improvement of domain of h_1, estimate in the cotangent space of Teichmüller space and an isoperimetric inequality for quadratic differentials.

À suivre...

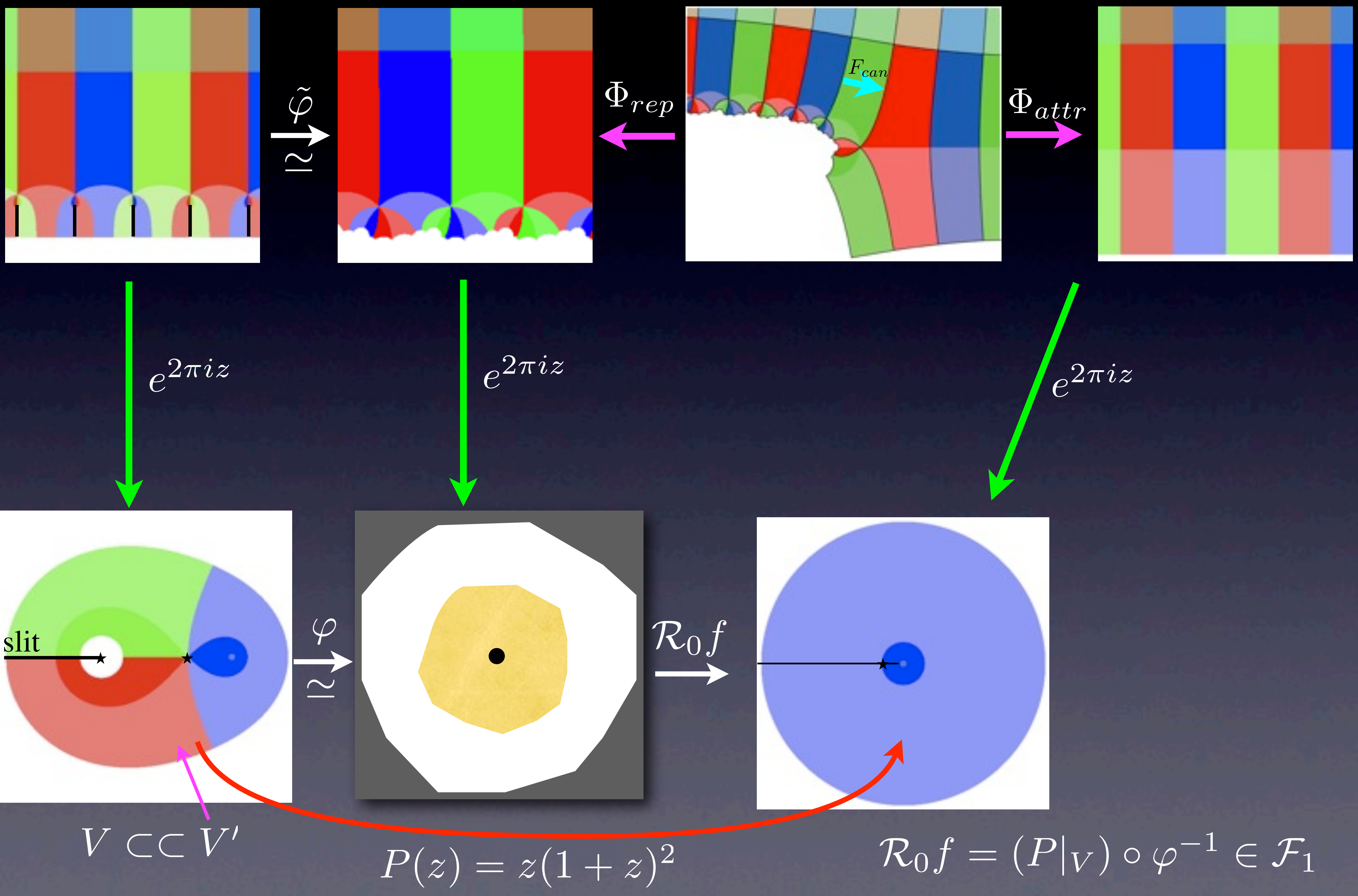
Assumption: $f = e^{2\pi i \alpha} h$ with $h(z) = z + z^2$ or $h \in \mathcal{F}_1$ and α is of high type ($a_i \geq N$).

Then $\mathcal{R}f, \mathcal{R}^2 f, \ldots$ are defined and can be written as $\mathcal{R}^n f = e^{2\pi i \alpha_n} h_n$ with $h_n \in \mathcal{F}_1$.

For a parabolic h (whose second derivative is not too small), one can find a "truncated checkerboard pattern" Ω_f. With a help of numerical estimates, one can give estiamtes on attracting Fatou coordinate Φ_{attr} and define associaeted rectangles etc. and their finite number of inverse images via (the region with critical point) until they arrive in the region where repelling Fatou coordinate Φ_{rep} is defined.



If you see Truncated checherboard pattern Ω_f, it induces a "cubic-like map" $\mathcal{R}_0 f$ from $\mathbb{C}^*$ (on repelling side) to $\mathbb{C}^*$ (on attracting side)

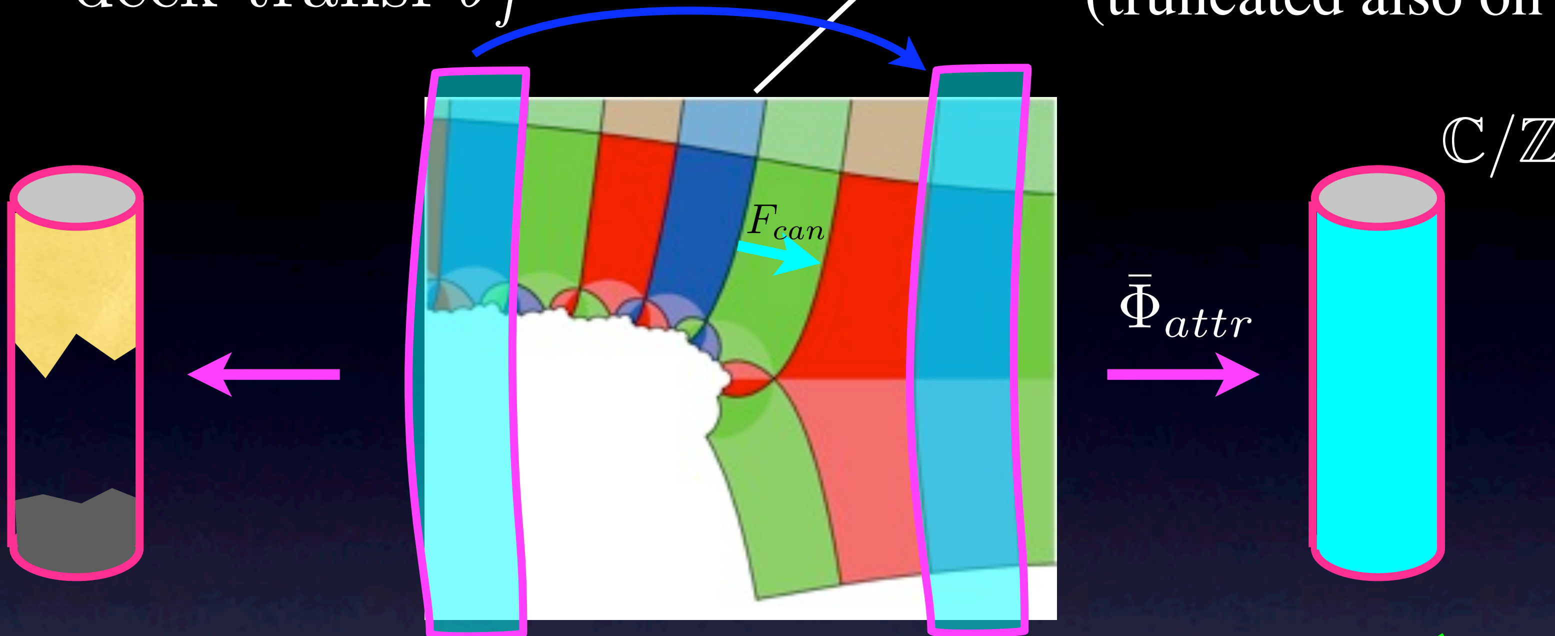


Near-parabolic case: work in pre-Fatou coordinate (deck transf added)

We still see truncated checkerboard pattern $\Omega_f = \Omega_f^{(0)}$

(truncated also on the side)

deck transf θ_f



universal covering of $\widehat{\mathbb{C}} \setminus \{0, \sigma\}$ τ_f

$e^{2\pi i z}$

$0, \sigma$ fixed pts

This shows that $\mathcal{R}f \in e^{2\pi i \alpha_1} \mathcal{F}_1$.

Instead of f itself, one should consider the canonical map F_{can} on $\Omega_f / \underset{\theta_f}{\sim}$, where θ_f is the gluing which depends on f.

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

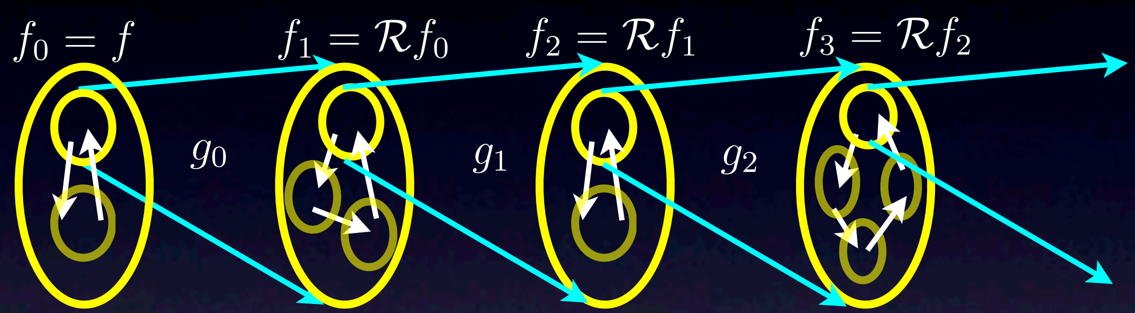
Talk 2: Reconstructing (part of f) from $\mathcal{R}^n f$

$\Omega_{f,k}$'s within Ω_f, their gluing and the dynamics
the combinatorics of rotation $r_{\alpha,n} : A_n \to A_n$, with $A_n \subset \mathbb{Z}^n$
$\Omega_{f,k_1,\ldots,k_n}$ for $(k_1, \ldots, k_n) \in A_n$

How can one conclude something about f by knowing that the renormalizations $\mathcal{R}f, \mathcal{R}^2 f, \ldots$ are defined and not too bad?

How can we understand f (or part of it) from $\mathcal{R}f$, or from $\mathcal{R}^2 f, \ldots$?

Why non-trivial?

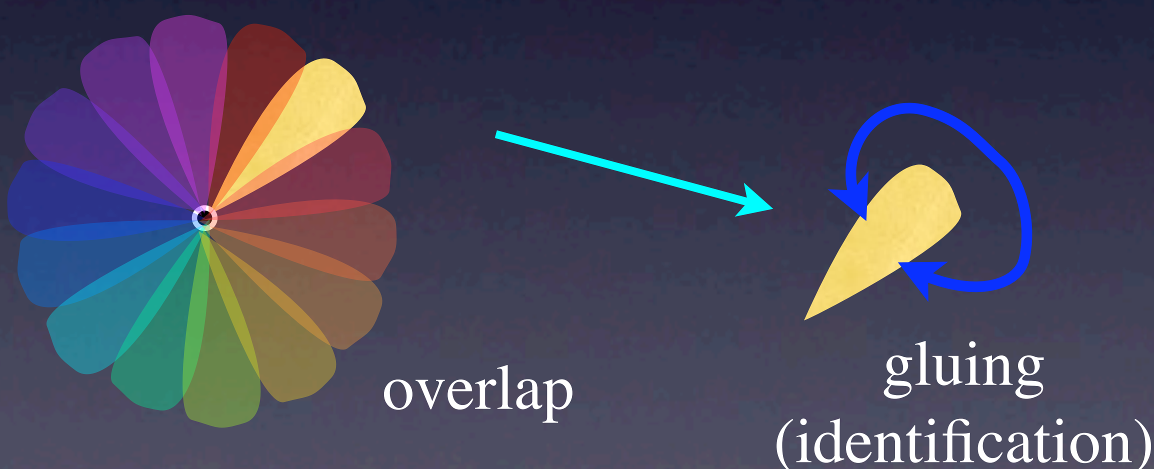


FCT renromalization

adding machine

$(\mathbb{Z}/a_1\mathbb{Z}) \times (\mathbb{Z}/a_2\mathbb{Z}) \times (\mathbb{Z}/a_3\mathbb{Z}) \times \ldots$

approximate period ~~=~~ $a_1 a_2 \ldots a_n$

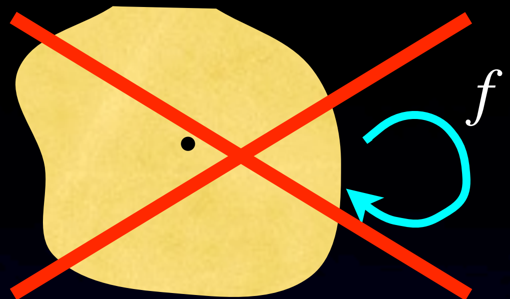


not trivial to go back to f from $\mathcal{R}f$

Zen question:

What was you SELF when your parents were not yet born?

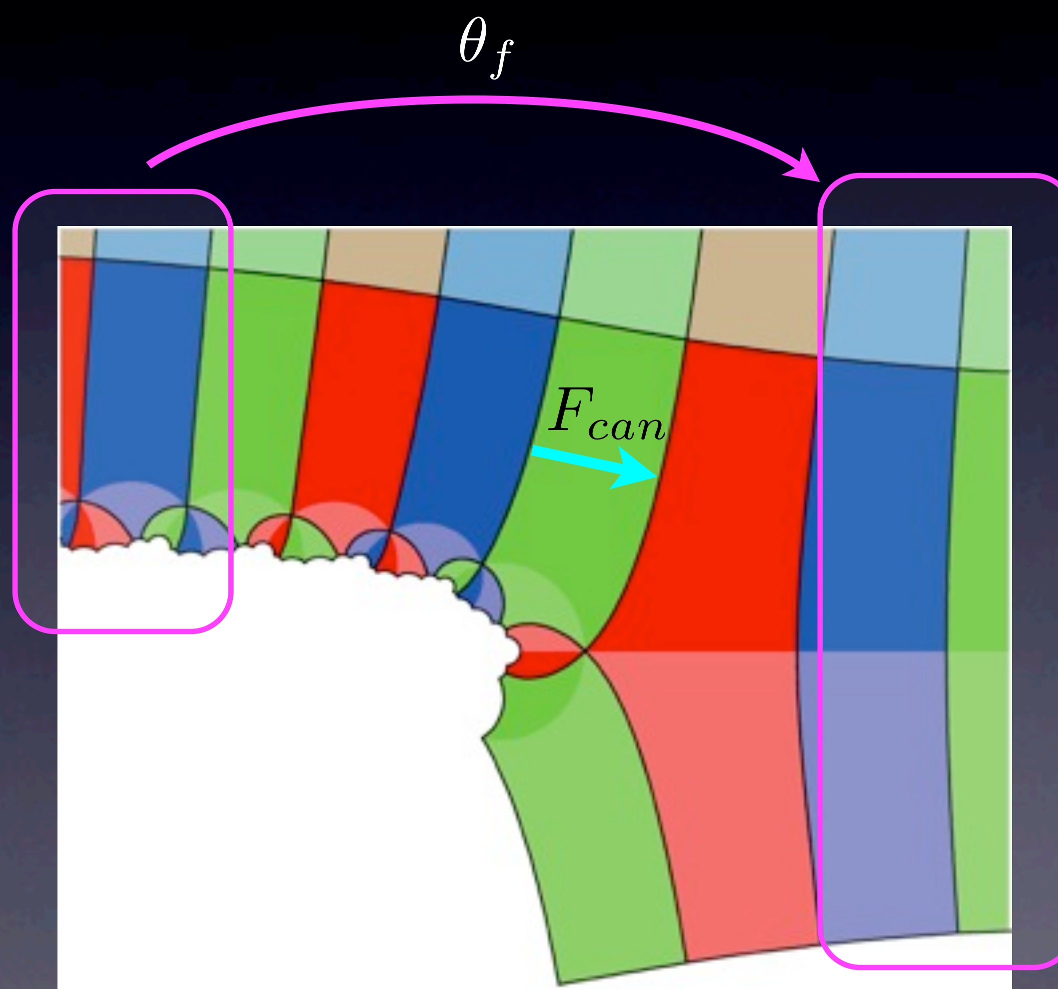
Need to *understand* what the dynamics f really *is*



F_{can} on Ω_{can} + θ_f

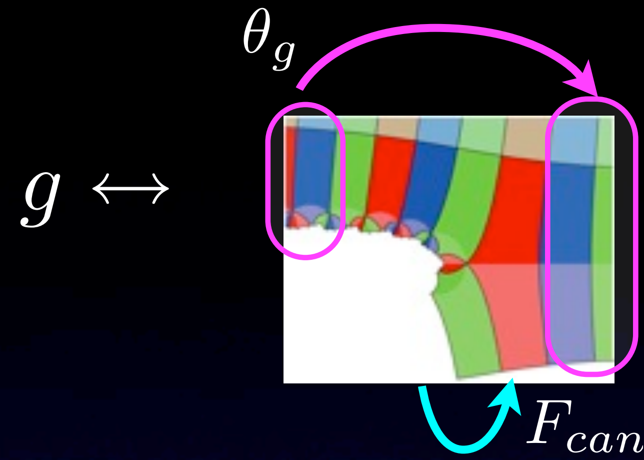
canonical map | trunc. pattern | gluing which commutes with F_{can}

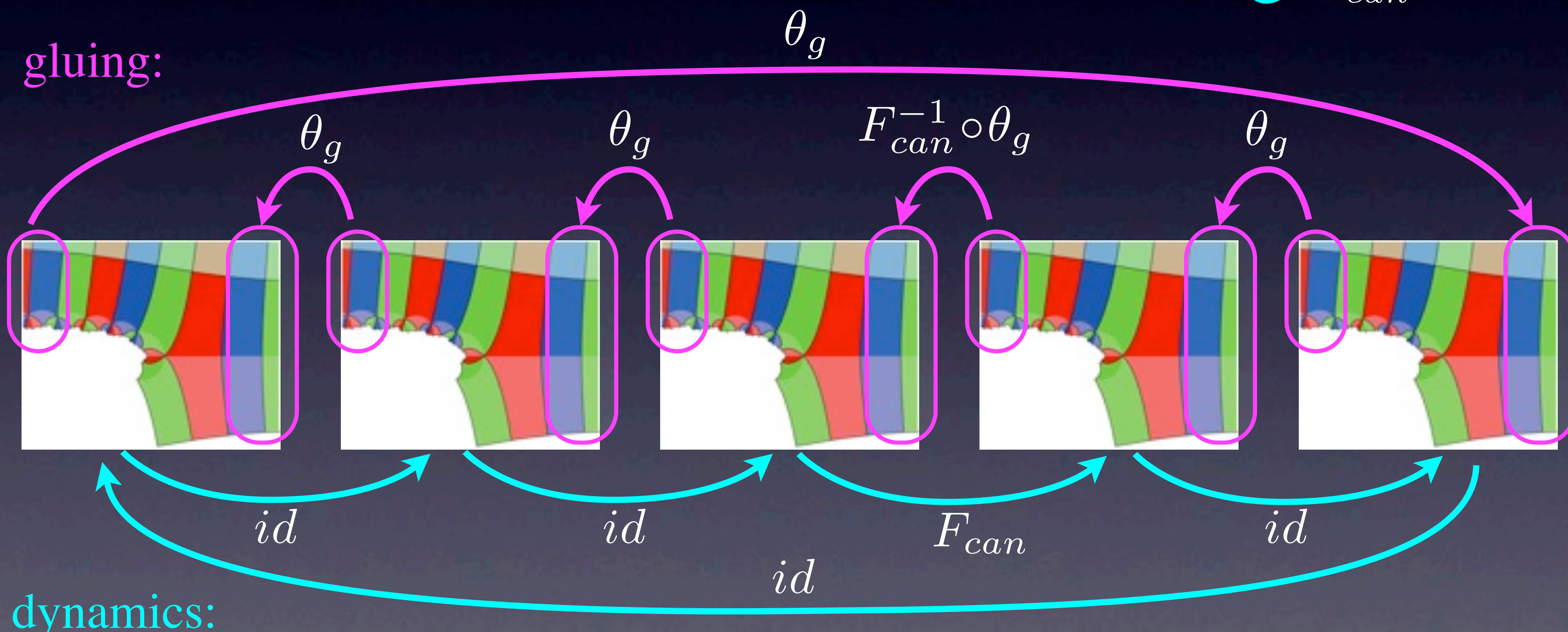
=



How did the dynamics of $g = \mathcal{R}f$ appear within the dynamics of f ?

We build a heuristic model, an abstract model for which $g \leftrightarrow F_{can}$ on $\Omega_{can}/\underset{\theta_f}{\sim}$ appears as the return map.





1. well-defined after gluing   2. return map is F_{can} modulo θ_g 3. this picture embeds into f

K. Kodaira's Essay
on his theory of elliptic surfaces

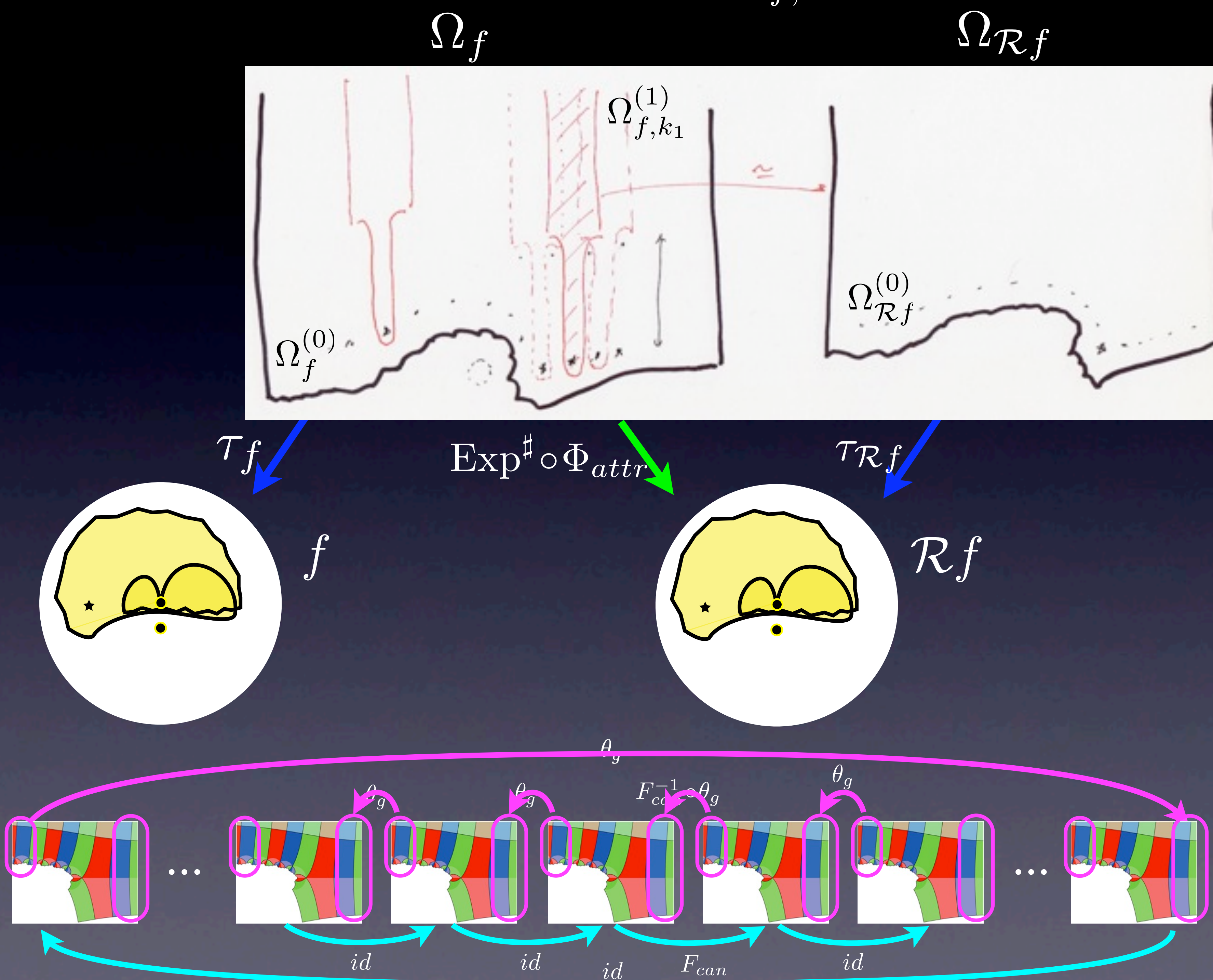
Michelangelo (1475-1564)

For Michelangelo, the job of the sculptor was to free the forms that were already inside the stone. He believed that every stone had a sculpture within it, and that the work of sculpting was simply a matter of chipping away all that was not a part of the statue.

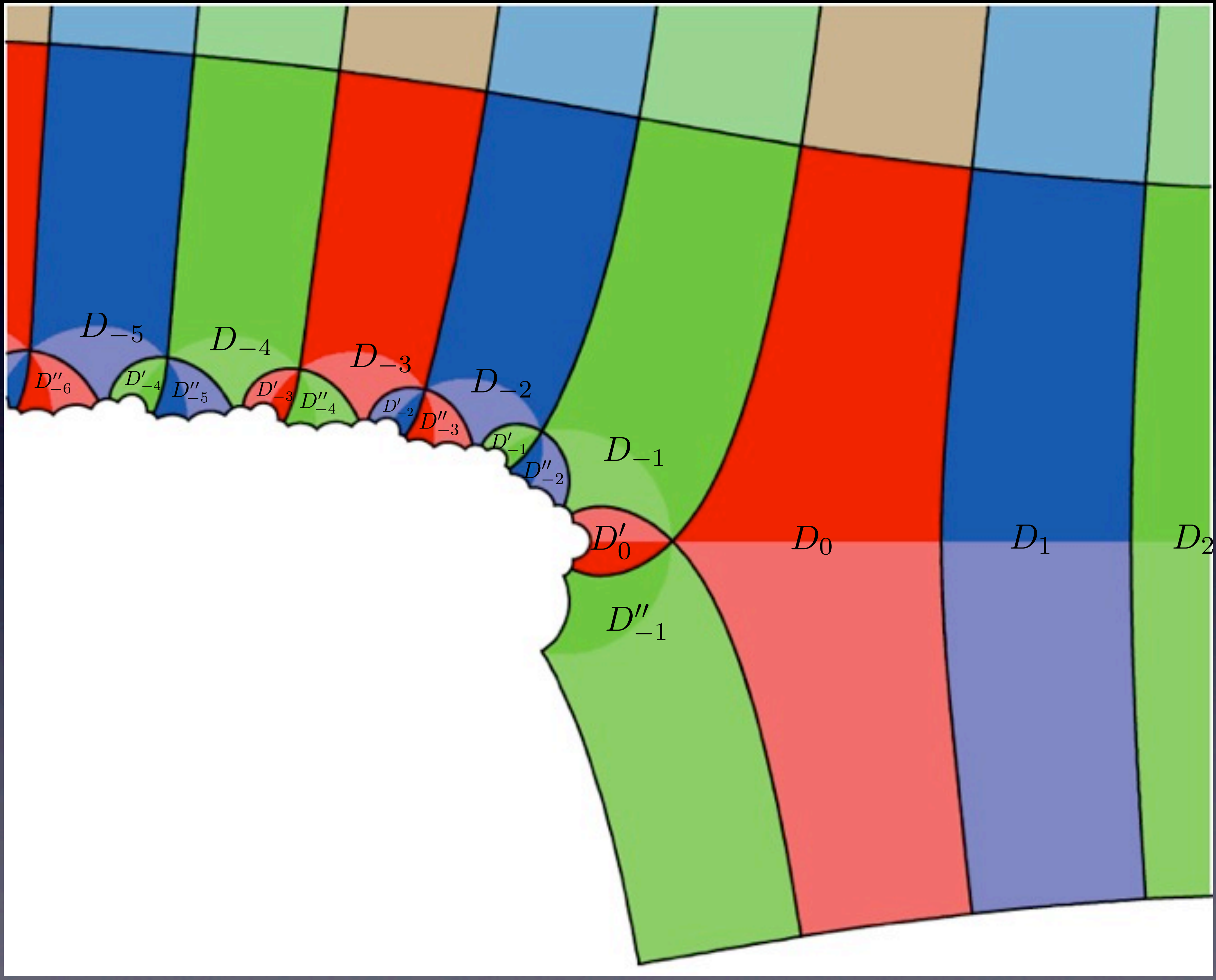
Unkei (? -1224)

(according to Soseki Natsume's novel)

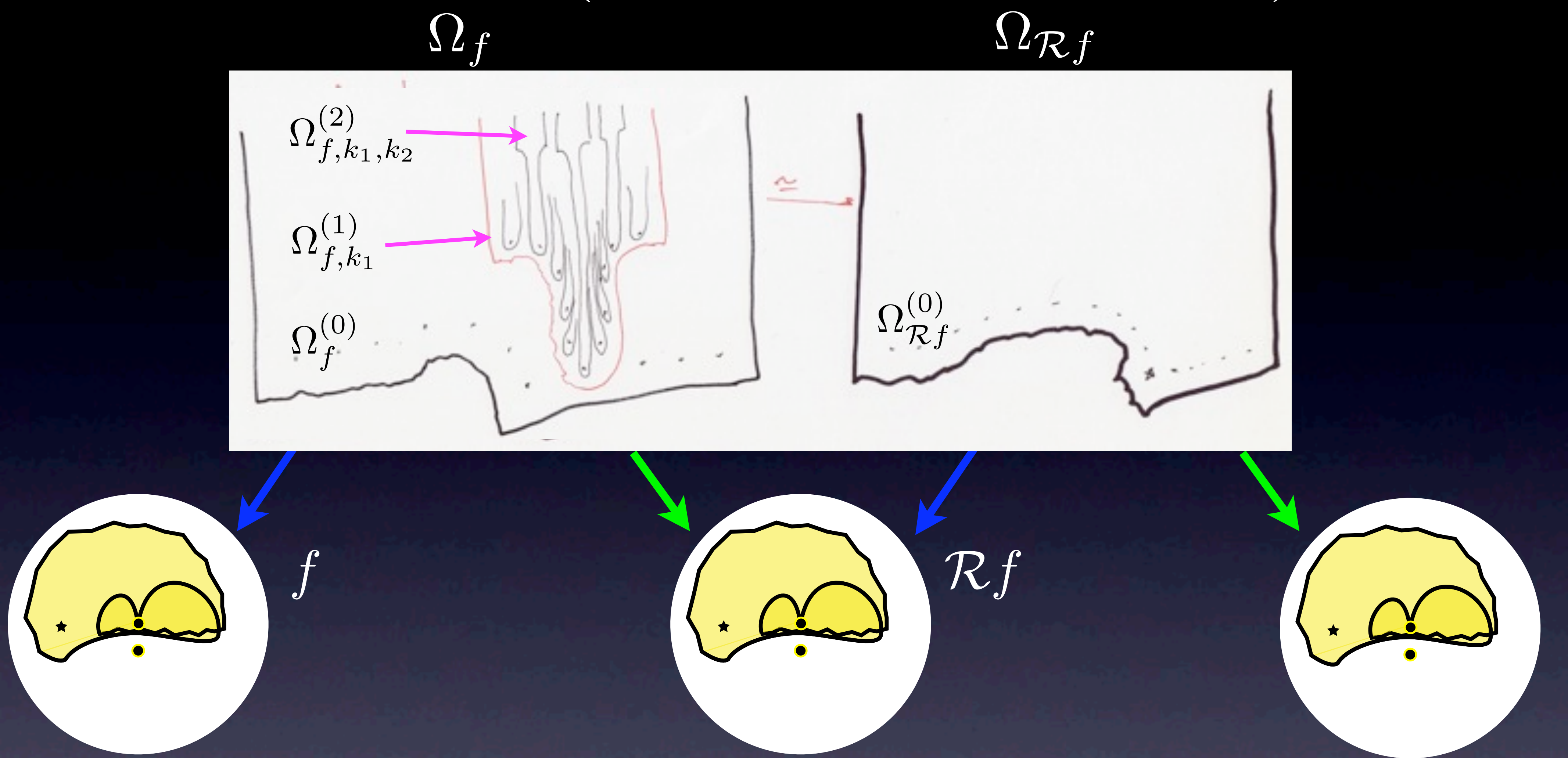
Construction of $\Omega^{(1)}_{f,k}$ within Ω_f



Truncated checkerboard pattern



Construction (Theorem 1: Structure Theorem)



$$\Omega^{(0)} \supset \Omega^{(1)}_{k_1} \supset \cdots \supset \Omega^{(n)}_{k_1,k_2,\ldots,k_n} \supset \Omega^{(n+1)}_{k_1,k_2,\ldots,k_n,k_{n+1}} \supset \ldots$$

each $\Omega^{(n)}_{k_1,k_2,\ldots,k_n}$ is isomorphic to truncated checkerboard pattern $\Omega_{\mathcal{R}^n f}$ they are glued via $\theta_{\mathcal{R}^n f}$

$$\Lambda_f = \bigcap_{n=0}^{\infty} \bigcup_{(k_1,\ldots,k_n)\in A_n} \Omega^{(n)}_{f,\,k_1,k_2,\ldots,k_n}$$ is an invariant set containing the critical orbit "maximal hedgehog"

Continue with blackboard

Merci!