An entire transcendental family with two singular values and a persistent Siegel disk

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Setup

We introduce the family of entire transcendental functions

$$f_a(z) = \lambda a(e^{z/a}(z - (a - 1)) + (a - 1)),$$

where $z \in \mathbb{C}$, $a \in \mathbb{C}^*$ and $\lambda = e^{i\theta}$, $\theta \in (\mathbb{R} \setminus \mathbb{Q}) \cap \mathcal{B}$ is FIXED.

- $f_a(0) = 0$ and $f_a'(0) = \lambda \ \Rightarrow \ f_a$ has a Siegel disk Δ_a around $z = 0$.
- f_a has two singular values

  simple crit. value $f_a(c)$ where $c = -1$ is a critical point.

  asymp. value $v_a = \lambda a(a-1)$. It has one finite preimage at $p_a = a - 1$.
- One of the two singular orbits must accumulate on $\partial\Delta_a$, but they may alternate.

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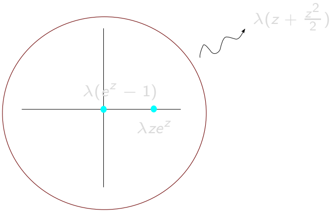
Motivation 1

This family "contains" three very important examples.

the semistandard map $f_1(z) = \lambda z e^z$;

the exponential family $f_a(z) \underset{a \to 0}{\longrightarrow} \lambda(e^z - 1)$;

the quadratic polynomial $f_a(z) \underset{a \to \infty}{\longrightarrow} \lambda(z + \frac{z^2}{2})$



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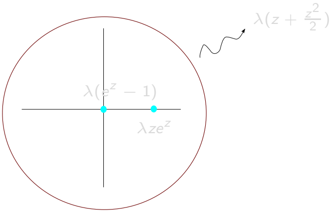

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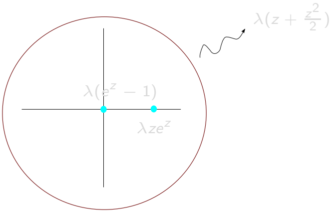

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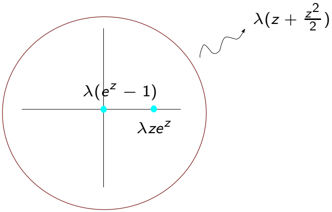

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In fact, if we conjugate by $u = z/a$, we obtain

$$g_a(u) = \lambda(e^u(au - (a-1)) + (a-1))$$

Then, if we write $a = a_0 + \varepsilon$, the perturbation is of the form

$$g_a(z) = g_{a_0}(z) + \varepsilon u^2 h(u),$$

with $h(0) \neq 0$.

This type of perturbations were used to relate the semistandard map to the quadratic family and, in particular, to prove the necessity of the Brjuno condition for the semistandard map (see [Geyer01]).

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Motivation 2

$$f_a(z) = \lambda a(e^{z/a}(z-(a-1))+(a-1)),$$

This family contains all ETF functions (up to conformal conjugacy) with the following properties

- finite order,
- one asymptotic value v_a, with exactly one finite preimage p_a of v_a,
- a fixed point (at 0) of multiplier $\lambda \in \mathbb{C}$
- a simple critical point (at $z = -1$) and no other critical points.

It follows that $v_a = \lambda a(a-1)$ and $p_a = a-1$.

One parameter family, but no singular orbit has a predetermined behaviour.

Previous work: S. Zakeri, Dynamics of cubic Siegel polynomials, *Comm. Math. Phys.*, 1999.

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Long term goal: to find a path linking the quadratic polynomial with the semistandard map (or other functions), to study properties of $\partial\Delta_a$.

More inmediate goals:

- To study the possible scenarios for the dynamical plane of f_a;
- To investigate the parameter space: regions of $J-$stability and their boundaries, capture components, semi-hyperbolic components,....
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Possible scenarios

- At least one of the singular orbits (SO) must accumulate on $\partial\Delta_a$. We see different dynamical planes depending on which SO is accumulating.
- The other SO is free.
  - If the free SO is attracted to an attracting periodic orbit, we say that a is a *semihyperbolic parameter* and $a \in H = H^c \cup H^v$.
  - If the free SO intersects the Siegel disc Δ_a we say that a is a *capture parameter*, and $a \in C = C^c \cup C^v$.
  - If the free SO escapes to infinity, we say that a is an escaping parameter and $a \in E^c \cup E^v$.
  - The six sets H^c, H^v, C^c, C^v, E^c and E^v are pairwise disjoint.

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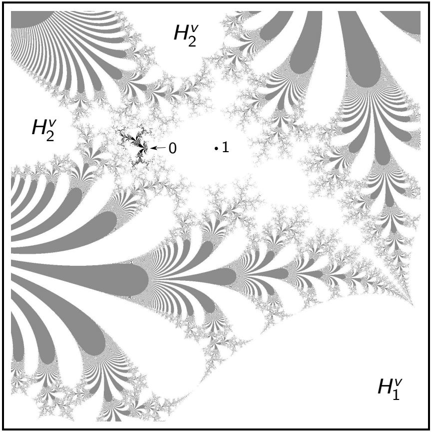
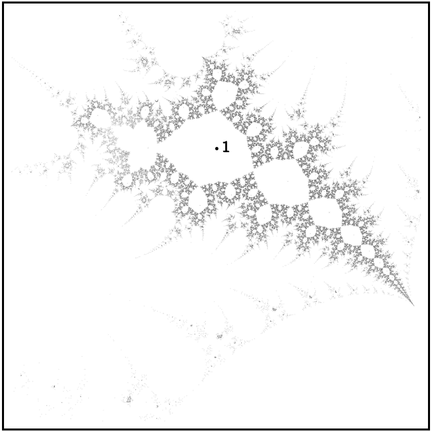
Parameter plane



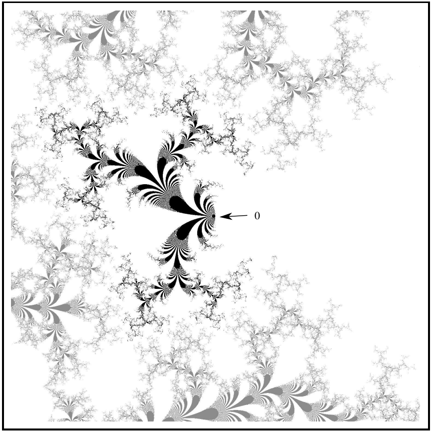
Escape algorithm.

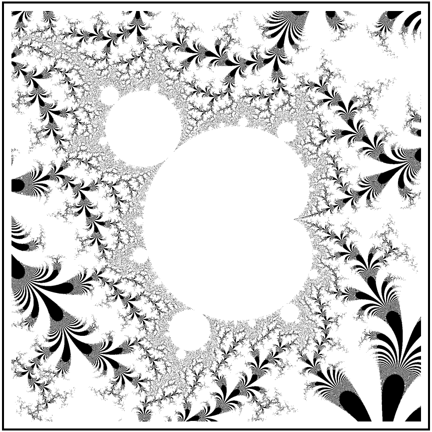


Main capture component C_0^v

$$\lambda = e^{2\pi i\theta},\ \theta = \frac{1+\sqrt{5}}{2}.$$

Parameter plane





E^c (black) and E^v (grey)

Components of H^c

Parameter plane

Theorem

a) *All components of $H \cup C$ are open and simply connected.*

b) *Every component of H^v is unbounded while every component of H^c is bounded.*

c) *If $a \in H \cup C$, then f_a is J-stable. Hence, in any component of $H \cup C$, the boundary $\partial\Delta_a$ moves holomorphically with the parameter.*

This allows us to spread "properties" to whole components of $\mathcal{J}-$stability, as long as they are satisfied for one parameter value.

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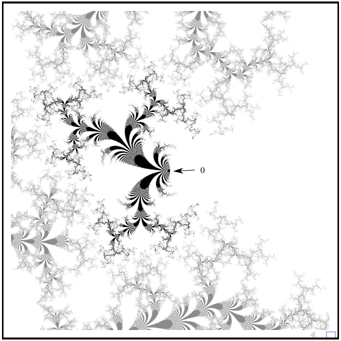
proof

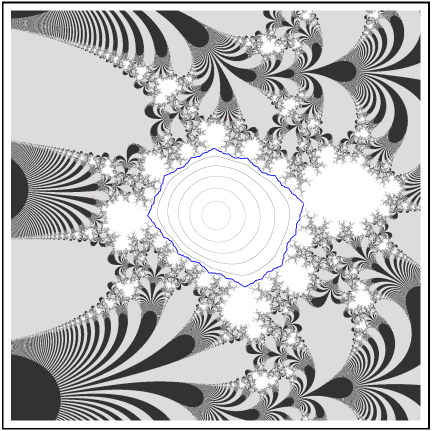
Most arguments for this proof are standard but it needs the following fact.

Proposition

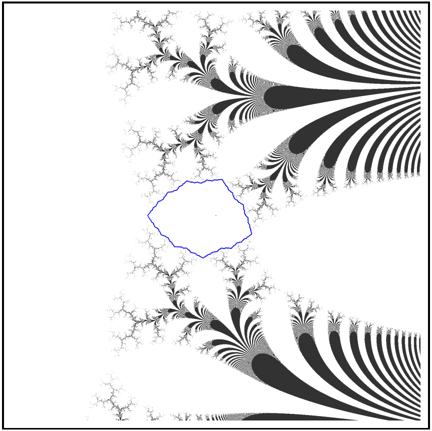
The set E^c (escaping parameters for the critical orbit) contains curves $a(t) \to 0$ as $t \to \infty$. As a consequence, no component of $H \cup C$ can suround $a = 0$.



Dynamical planes

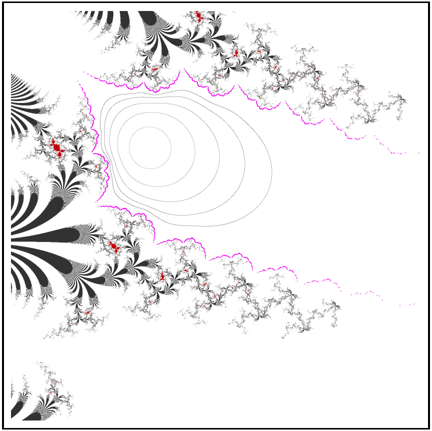


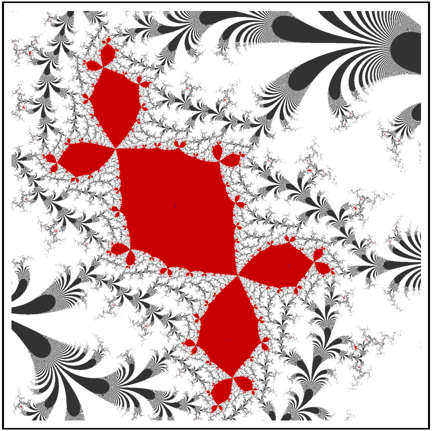
$a \in H^v$



$a = 1 \in C_0^v$ (SS Map)

Dynamical plane: $a \in H^c$





Unbounded Siegel disk and attracting basin for $a \in H^c$

Large values of $|a|$ (for any λ)

Theorem

There exists $M > 0$, such that $f_a(z)$ is polynomial-like of degree two for $|a| > M$. Moreover the small filled Julia set (and in particular Δ_a) is contained in $D(0, R)$ with R independent of a.

Corollary

- *The main capture component $C_0^v = \{a \in \mathbb{C} \mid v_a \in \Delta_a\}$ is bounded*
- *The set $H^c \cup C^c \cup E^c$ is bounded.*
- *For $|a| > M$ and $\theta \in CT$, the boundary of Δ_a is a quasicircle containing the critical point. By $J-$stability, this is true for all $a \in H^v$, for example.*
- *In fact, for $|a| > M$, the map $f_{a,\theta}$ is linearizable iff Q_θ is linearizable and, moreover, the two Siegel disks are "quasiconformally related".*

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Unbounded Siegel disks

This situation is in contrast with what happens for $|a| < M$, where we find unbounded Siegel disks, and hence with non-locally connected boundary [Baker+Dominguez]. Recall $CT \subset \mathcal{D} \subset \mathcal{H} \subset \mathcal{B}$.

Proposition

Let $\theta \in \mathcal{H}$.

(a) *If $a \in E^c$. Then Δ_a is unbounded and $v_a \in \partial\Delta_a$.*

(b) *If $a \in H^c \cup C^c$, then Δ_a is unbounded or $\partial\Delta_a$ is an indecomposable continuum.*

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Proof ingredients

Theorem (Herman 85)

Suppose $\theta \in \mathcal{H}$ and Δ bounded. If $f|_{\partial\Delta}$ is a homeomorphism, then $\partial\Delta$ contains a critical point.

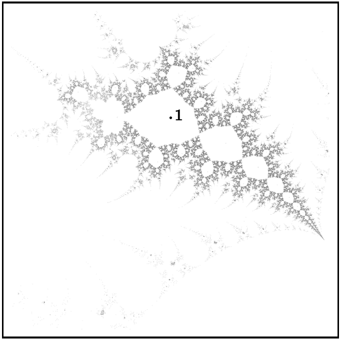
Theorem (Rogers 92, generalized)

Let $f \in \mathcal{B}$ and Δ be a bounded Siegel disk of f. If $\partial\Delta$ is a decomposable continum, then $\partial\Delta$ separates $\mathbb{C}$ into exactly two complementary domains.

Theorem (Rottenfusser 08?)

If $f \in \mathcal{B}$, then $I(f) \cup \{\infty\}$ is arc-connected.

Question 1: Is the boundary of C_0^v a Jordan curve? (curve where both singular orbits are on the boundary of Δ_a) – Yes for cubics [Zakeri'99].



Question 2: What is the nature of $\partial\Delta_a$ when $a \in \partial C_0^v$?

Two questions

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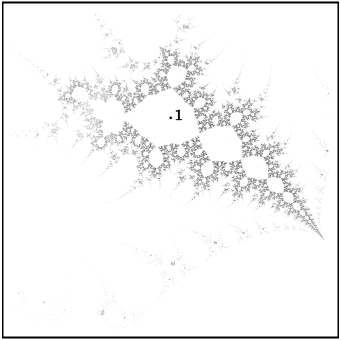
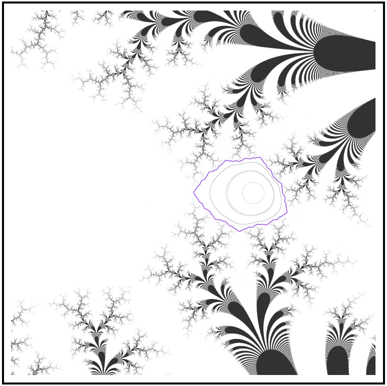


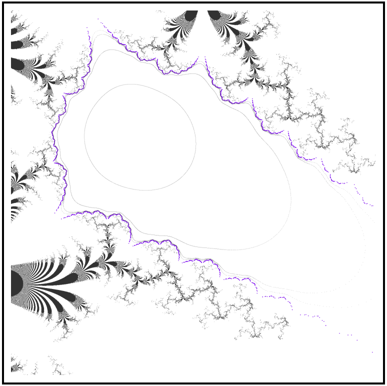
Question 2: What is the nature of $\partial\Delta_a$ when $a \in \partial C_0^v$?

Two examples

$a_1, a_2 \in \partial C_0^v$. Both singular values are conjectured to be on the boundary.



f_{a_1}



f_{a_2}

Thank you for your attention!!