Siegel Disks in a Family of Entire Maps

Saeed Zakeri

June 19, 2009

Introduction

- f a non-linear entire or rational map of degree ≥ 2
- $f(0) = 0$ and $f'(0) = \lambda = e^{2\pi i\theta}$, where θ is an irrational number
- f linearizable near 0, with the **Siegel disk** Δ

The unique conformal isomorphism $\zeta : \mathbb{D} \xrightarrow{\cong} \Delta$ which satisfies $\zeta(0) = 0$ and $\zeta'(0) > 0$ linearizes f in Δ:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{R_\theta} & \mathbb{D} \\ \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \zeta} \\ \Delta & \xrightarrow{f} & \Delta \end{array}$$

Introduction

- f a non-linear entire or rational map of degree ≥ 2
- $f(0) = 0$ and $f'(0) = \lambda = e^{2\pi i\theta}$, where θ is an irrational number
- f linearizable near 0, with the **Siegel disk** Δ

The unique conformal isomorphism $\zeta : \mathbb{D} \xrightarrow{\cong} \Delta$ which satisfies $\zeta(0) = 0$ and $\zeta'(0) > 0$ linearizes f in Δ:

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{R_\theta} & \mathbb{D} \\ \downarrow \zeta & & \downarrow \zeta \\ \Delta & \xrightarrow{f} & \Delta \end{array}$$

Question 1. Is $\partial\Delta$ always a Jordan curve?

Question 2. What can be said about the metric properties of $\partial\Delta$ when it is a Jordan curve?

Question 3. What are the possible obstructions to enlarging the linearization domain Δ?

Question 1. Is $\partial\Delta$ always a Jordan curve?

Question 2. What can be said about the metric properties of $\partial\Delta$ when it is a Jordan curve?

Question 3. What are the possible obstructions to enlarging the linearization domain Δ?

Question 1. Is $\partial\Delta$ always a Jordan curve?

Question 2. What can be said about the metric properties of $\partial\Delta$ when it is a Jordan curve?

Question 3. What are the possible obstructions to enlarging the linearization domain Δ?

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda\, z\, e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + a z^2)\, e^z$

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda\, z\, e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + az^2)\, e^z$

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda\, z\, e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + az^2)\, e^z$

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda\, z\, e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + az^2)\, e^z$

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda\ z\ e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + az^2)\ e^z$

Let $\theta = [a_1, a_2, a_3, \ldots]$ be an irrational of **bounded type** in the sense that $\sup a_n < +\infty$.

The boundary $\partial\Delta$ is known to be a quasicircle containing a critical point when

- (Douady-Herman-Swiatek, 1986) f is a quadratic polynomial
- (Z., 1998) f is a cubic polynomial
- (Shishikura, unpublished) f is a polynomial of arbitrary degree ≥ 2
- (Geyer, 2001) $f(z) = \lambda \, z \, e^z$
- (Keen-Zhang, 2007) $f(z) = (\lambda z + az^2) \, e^z$

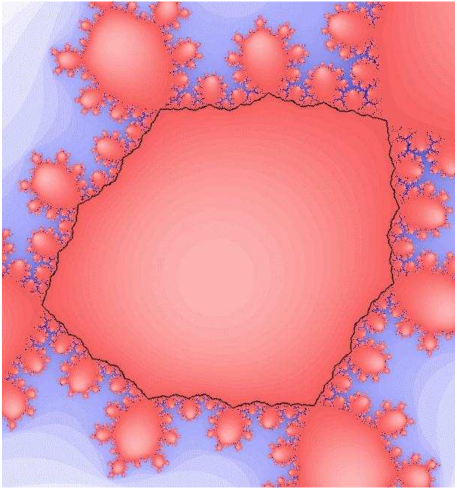


Figure: $z \mapsto \lambda z + z^2$

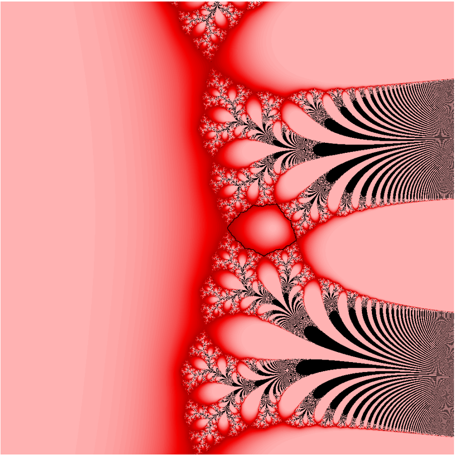


Figure: $z \mapsto \lambda\, z\, e^{z}$

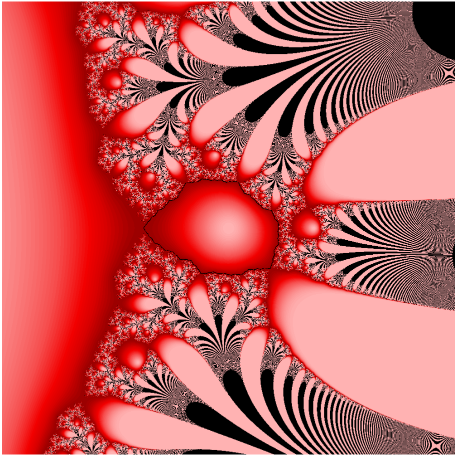


Figure: $z \mapsto \lambda\, z\, (1 - 2z/3)\, e^z$

Let $\mathcal{E}^{p,q}(\theta)$ be the family of all non-linear entire maps of the form

$$f(z) = P(z)\, e^{Q(z)}$$

where P, Q are polynomials with

- $\deg P = p \geq 1$, $\deg Q = q \geq 0$
- $P(0) = Q(0) = 0$
- $P'(0) = f'(0) = e^{2\pi i\theta}$.

These maps have finitely many zeros and critical points. In the transcendental case $q > 0$, they have a single asymptotic value at 0.

Let $\mathcal{E}^{p,q}(\theta)$ be the family of all non-linear entire maps of the form

$$f(z) = P(z)\, e^{Q(z)}$$

where P, Q are polynomials with

- $\deg P = p \geq 1$, $\deg Q = q \geq 0$
- $P(0) = Q(0) = 0$
- $P'(0) = f'(0) = e^{2\pi i\theta}$.

These maps have finitely many zeros and critical points. In the transcendental case $q > 0$, they have a single asymptotic value at 0.

Main Theorem. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ where θ is of bounded type. Then $\partial\Delta$ is a quasicircle in $\mathbb{C}$ and contains at least one critical point of f.*

Strategy of the proof (following Shishikura): show that the invariant curves

$$\gamma_r := \zeta(\{z : |z| = r\}) \qquad 0 < r < 1$$

are K-quasicircles for a $K > 1$ independent of r.

Main Theorem. *Suppose $f \in \mathcal{E}^{p,q}(\theta)$ where θ is of bounded type. Then $\partial\Delta$ is a quasicircle in $\mathbb{C}$ and contains at least one critical point of f.*

Strategy of the proof (following Shishikura): show that the invariant curves

$$\gamma_r := \zeta(\{z : |z| = r\}) \qquad 0 < r < 1$$

are K-quasicircles for a $K > 1$ independent of r.

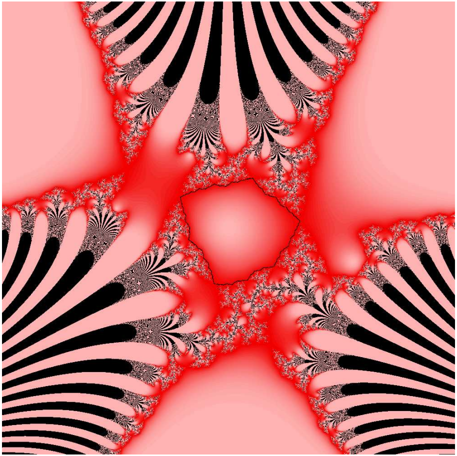


Figure: $z \mapsto \lambda\, z\, (1 - (11 + 3i)z/13)\, e^{iz^3}$

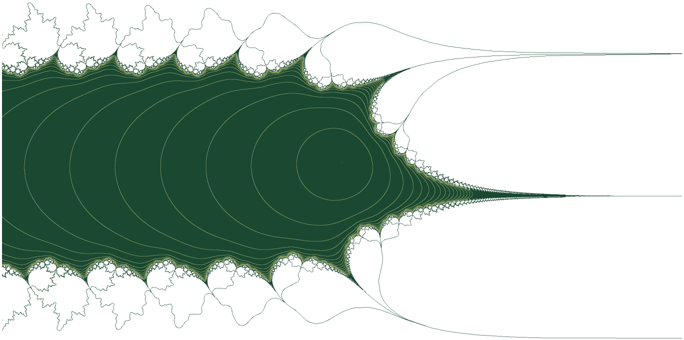


Figure: (Courtesy of A. Cheritat) $z \mapsto \lambda\, e^{z-\lambda}$

Symmetrizing f

Let $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal reflection such that

- I fixes γ_r pointwise.
- $I(0) = \infty$.
- I is anti-holomorphic off a small neighborhood of γ_r.

Define

$$F := \begin{cases} f & \text{in } \operatorname{ext}(\gamma_r) \\ I \circ f \circ I & \text{in } \operatorname{int}(\gamma_r) \smallsetminus \{0\} \end{cases}$$

The quasiregular map $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ commutes with I. It has a "quasiconformal Herman ring" with γ_r as an invariant curve.

Symmetrizing f

Let $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiconformal reflection such that

- I fixes γ_r pointwise.
- $I(0) = \infty$.
- I is anti-holomorphic off a small neighborhood of γ_r.

Define

$$F := \begin{cases} f & \text{in } \operatorname{ext}(\gamma_r) \\ I \circ f \circ I & \text{in } \operatorname{int}(\gamma_r) \smallsetminus \{0\} \end{cases}$$

The quasiregular map $F : \mathbb{C}^* \to \widehat{\mathbb{C}}$ commutes with I. It has a "quasiconformal Herman ring" with γ_r as an invariant curve.

Straightening F

There is a conformal structure μ of bounded dilatation on $\widehat{\mathbb{C}}$ which is invariant under both F and I.

Let $\xi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\xi^* \mu_0 = \mu$, normalized so that $\xi(0) = 0, \xi(\zeta(r)) = 1$.

The map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ defined by

$$G := \xi \circ F \circ \xi^{-1}$$

is holomorphic and commutes with $z \mapsto 1/\overline{z}$. It has a Herman ring of rotation number θ with the unit circle $\mathbb{T}$ as an invariant curve.

Straightening F

There is a conformal structure μ of bounded dilatation on $\widehat{\mathbb{C}}$ which is invariant under both F and I.

Let $\xi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\xi^* \mu_0 = \mu$, normalized so that $\xi(0) = 0, \xi(\zeta(r)) = 1$.

The map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ defined by

$$G := \xi \circ F \circ \xi^{-1}$$

is holomorphic and commutes with $z \mapsto 1/\overline{z}$. It has a Herman ring of rotation number θ with the unit circle $\mathbb{T}$ as an invariant curve.

Straightening F

There is a conformal structure μ of bounded dilatation on $\widehat{\mathbb{C}}$ which is invariant under both F and I.

Let $\xi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\xi^* \mu_0 = \mu$, normalized so that $\xi(0) = 0, \xi(\zeta(r)) = 1$.

The map $G : \mathbb{C}^* \to \widehat{\mathbb{C}}$ defined by

$$G := \xi \circ F \circ \xi^{-1}$$

is holomorphic and commutes with $z \mapsto 1/\overline{z}$. It has a Herman ring of rotation number θ with the unit circle $\mathbb{T}$ as an invariant curve.

Explicit form of G:

$$G(z) = \tau \; z^p \; B(z) \; e^{\alpha(z) - \overline{\alpha(1/\overline{z})}},$$

where

- $|\tau| = 1$.
- B is a degree $p-1$ Blaschke product with the same zeros as G.
- α is a polynomial of degree q with $\alpha(0) = 0$.

Theorem. *There are constants* $\delta = \delta(p, q) > 1$ *and* $M = M(p, q) > 0$ *such that*

$$\left| \frac{zG'(z)}{G(z)} \right| \leq M$$

in the annulus $\delta^{-1} < |z| < \delta$.

Corollary. $G : \mathbb{T} \to \mathbb{T}$ *is conjugate to* R_θ *by a k-quasisymmetric map, where* $k = k(p, q, \theta)$.

Theorem. *There are constants* $\delta = \delta(p, q) > 1$ *and* $M = M(p, q) > 0$ *such that*

$$\left| \frac{zG'(z)}{G(z)} \right| \leq M$$

in the annulus $\delta^{-1} < |z| < \delta$.

Corollary. $G : \mathbb{T} \to \mathbb{T}$ *is conjugate to* R_θ *by a* k*-quasisymmetric map, where* $k = k(p, q, \theta)$.

Modifying G

Let $h : \mathbb{T} \to \mathbb{T}$ be the normalized linearizing map of $G : \mathbb{T} \to \mathbb{T}$.

Let $H : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal extension of h fixing 0 and 1. We can take $K = K(p, q, \theta)$.

Define $\hat{G} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{G} := \begin{cases} G & \text{outside } \mathbb{D} \\ H^{-1} \circ R_\theta \circ H & \text{inside } \mathbb{D} \end{cases}$$

Thus $\hat{G}$ has a "quasiconformal Siegel disk" on $\mathbb{D}$.

Modifying G

Let $h : \mathbb{T} \to \mathbb{T}$ be the normalized linearizing map of $G : \mathbb{T} \to \mathbb{T}$.

Let $H : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal extension of h fixing 0 and 1. We can take $K = K(p, q, \theta)$.

Define $\hat{G} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{G} := \begin{cases} G & \text{outside } \mathbb{D} \\ H^{-1} \circ R_\theta \circ H & \text{inside } \mathbb{D} \end{cases}$$

Thus $\hat{G}$ has a "quasiconformal Siegel disk" on $\mathbb{D}$.

Modifying G

Let $h : \mathbb{T} \to \mathbb{T}$ be the normalized linearizing map of $G : \mathbb{T} \to \mathbb{T}$.

Let $H : \mathbb{D} \to \mathbb{D}$ be a K-quasiconformal extension of h fixing 0 and 1. We can take $K = K(p, q, \theta)$.

Define $\hat{G} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{G} := \begin{cases} G & \text{outside } \mathbb{D} \\ H^{-1} \circ R_\theta \circ H & \text{inside } \mathbb{D} \end{cases}$$

Thus $\hat{G}$ has a "quasiconformal Siegel disk" on $\mathbb{D}$.

Straightening $\hat{G}$

There is a $\hat{G}$-invariant conformal structure ν of bounded dilatation on $\mathbb{C}$. Let $\psi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\psi^* \mu_0 = \nu$, normalized so that $\psi(0) = 0, (\psi \circ H^{-1})'(0) > 0$.

The conjugate map

$$g := \psi \circ \hat{G} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}.$$

belongs to $\mathcal{E}^{p,q}(\theta)$.

Straightening $\hat{G}$

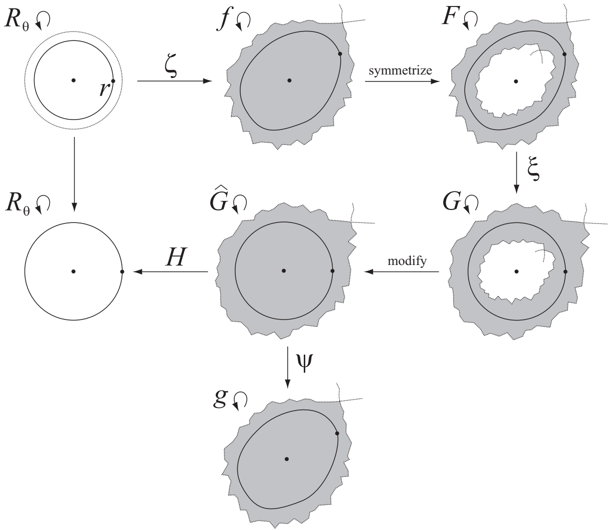
There is a $\hat{G}$-invariant conformal structure ν of bounded dilatation on $\mathbb{C}$. Let $\psi : \mathbb{C} \to \mathbb{C}$ be the unique quasiconformal solution of $\psi^* \mu_0 = \nu$, normalized so that $\psi(0) = 0, (\psi \circ H^{-1})'(0) > 0$.

The conjugate map

$$g := \psi \circ \hat{G} \circ \psi^{-1} : \mathbb{C} \to \mathbb{C}.$$

belongs to $\mathcal{E}^{p,q}(\theta)$.

This defines a **surgery map** $\mathcal{S}_r : f \mapsto g$.



The invariant curve $\gamma_{g,r} \subset \Delta_g$ is a K-quasicircle for a constant $K = K(p, q, \theta)$. The Main Theorem would follow if we knew g is the map f that we started with.

Question. Does the surgery map $\mathcal{S}_r$ act as the identity?

Answer. No!

The problem arises when f has critical points which are **captured** by its Siegel disk.

The invariant curve $\gamma_{g,r} \subset \Delta_g$ is a K-quasicircle for a constant $K = K(p, q, \theta)$. The Main Theorem would follow if we knew g is the map f that we started with.

Question. Does the surgery map $\mathcal{S}_r$ act as the identity?

Answer. No!

The problem arises when f has critical points which are **captured** by its Siegel disk.

The invariant curve $\gamma_{g,r} \subset \Delta_g$ is a K-quasicircle for a constant $K = K(p, q, \theta)$. The Main Theorem would follow if we knew g is the map f that we started with.

Question. Does the surgery map $\mathcal{S}_r$ act as the identity?

Answer. No!

The problem arises when f has critical points which are **captured** by its Siegel disk.

Maps with no free capture spots

Rigidity Theorem. *If every captured critical point of f eventually lands at 0, then $\mathcal{S}_r(f) = f$.*

Thus, the Main Theorem holds for such f.

Maps with no free capture spots

Rigidity Theorem. *If every captured critical point of f eventually lands at 0, then $\mathcal{S}_r(f) = f$.*

Thus, the Main Theorem holds for such f.

Maps with one free capture spot

Now suppose there is an $\omega \in \Delta \setminus \{0\}$ such that every captured critical point of f eventually lands at 0 or ω.

- Embed f in a holomorphic family $\{f_t : t \in \mathbb{D}^*\}$ of quasiconformally conjugate maps in $\mathcal{E}^{p,q}(\theta)$ by changing the conformal position of ω.
- Show that there is a $K = K(p, q, \theta)$ such that $\gamma_{t,r}$ is a K-quasicircle when $|t| < 1/2$ or $|t| > r$.
- Apply the maximum principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ to conclude that $\gamma_{t,r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$.

Maps with one free capture spot

Now suppose there is an $\omega \in \Delta \smallsetminus \{0\}$ such that every captured critical point of f eventually lands at 0 or ω.

- Embed f in a holomorphic family $\{f_t : t \in \mathbb{D}^*\}$ of quasiconformally conjugate maps in $\mathcal{E}^{p,q}(\theta)$ by changing the conformal position of ω.
- Show that there is a $K = K(p, q, \theta)$ such that $\gamma_{t,r}$ is a K-quasicircle when $|t| < 1/2$ or $|t| > r$.
- Apply the maximum principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ to conclude that $\gamma_{t,r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$.

Maps with one free capture spot

Now suppose there is an $\omega \in \Delta \smallsetminus \{0\}$ such that every captured critical point of f eventually lands at 0 or ω.

- Embed f in a holomorphic family $\{f_t : t \in \mathbb{D}^*\}$ of quasiconformally conjugate maps in $\mathcal{E}^{p,q}(\theta)$ by changing the conformal position of ω.
- Show that there is a $K = K(p, q, \theta)$ such that $\gamma_{t,r}$ is a K-quasicircle when $|t| < 1/2$ or $|t| > r$.
- Apply the maximum principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ to conclude that $\gamma_{t,r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$.

Maps with one free capture spot

Now suppose there is an $\omega \in \Delta \smallsetminus \{0\}$ such that every captured critical point of f eventually lands at 0 or ω.

- Embed f in a holomorphic family $\{f_t : t \in \mathbb{D}^*\}$ of quasiconformally conjugate maps in $\mathcal{E}^{p,q}(\theta)$ by changing the conformal position of ω.
- Show that there is a $K = K(p, q, \theta)$ such that $\gamma_{t,r}$ is a K-quasicircle when $|t| < 1/2$ or $|t| > r$.
- Apply the maximum principle to a suitable cross-ratio function $\mathbb{D}^* \to \mathbb{C}$ to conclude that $\gamma_{t,r}$ is a K-quasicircle for all $t \in \mathbb{D}^*$.

The general case

Reduce to one of the two previous cases as follows:

- For every component U of $f^{-n}(\Delta)$ which contains postcritical points, choose a **center** c_U such that
  - $f^{\circ n}(c_U)$ is 0 if U is bounded and is some $\omega \in \Delta \smallsetminus \{0\}$ if U is unbounded.
  - $f(c_U) = c_{f(U)}$.
- Modify dynamics on U so that the new map $U \to f(U)$ is branched at c_U and ramified over $c_{f(U)}$ only.
- Straighten the resulting quasiregular dynamics to obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot.

The general case

Reduce to one of the two previous cases as follows:

- For every component U of $f^{-n}(\Delta)$ which contains postcritical points, choose a **center** c_U such that
  - $f^{\circ n}(c_U)$ is 0 if U is bounded and is some $\omega \in \Delta \smallsetminus \{0\}$ if U is unbounded.
  - $f(c_U) = c_{f(U)}$.
- Modify dynamics on U so that the new map $U \to f(U)$ is branched at c_U and ramified over $c_{f(U)}$ only.
- Straighten the resulting quasiregular dynamics to obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot.

The general case

Reduce to one of the two previous cases as follows:

- For every component U of $f^{-n}(\Delta)$ which contains postcritical points, choose a **center** c_U such that
  - $f^{\circ n}(c_U)$ is 0 if U is bounded and is some $\omega \in \Delta \smallsetminus \{0\}$ if U is unbounded.
  - $f(c_U) = c_{f(U)}$.
- Modify dynamics on U so that the new map $U \to f(U)$ is branched at c_U and ramified over $c_{f(U)}$ only.
- Straighten the resulting quasiregular dynamics to obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot.

The general case

Reduce to one of the two previous cases as follows:

- For every component U of $f^{-n}(\Delta)$ which contains postcritical points, choose a **center** c_U such that
  - $f^{\circ n}(c_U)$ is 0 if U is bounded and is some $\omega \in \Delta \smallsetminus \{0\}$ if U is unbounded.
  - $f(c_U) = c_{f(U)}$.
- Modify dynamics on U so that the new map $U \to f(U)$ is branched at c_U and ramified over $c_{f(U)}$ only.
- Straighten the resulting quasiregular dynamics to obtain a map $g \in \mathcal{E}^{p,q}(\theta)$ with at most one free capture spot.