Arithmetical Hedgehogs

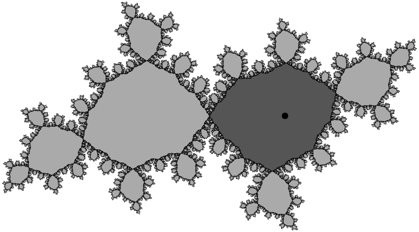
Xavier Buff

Université de Toulouse

Toulouse, 18/06/2009

The quadratic family

- $R_\alpha(z) = e^{i2\pi\alpha} z$
- $P_\alpha(z) = e^{i2\pi\alpha} z(1+z)$.
- P_α fixes 0 and the linear part is the rotation R_α.
- P_α is linearizable if it is conjugate to R_α at the origin.
- When P_α is linearizable, the Siegel disk Δ_α is the largest rotation domain containing 0.



Motivation

Question

For which values of α is there a Siegel disk Δ_α whose boundary contains the critical point of P_α ?

Question

When there is a non-empty Siegel disk Δ_α, is its boundary a Jordan curve ?

Question

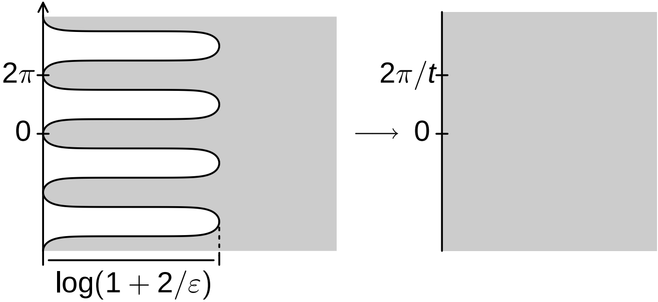
Is it possible to describe the closure of the postcritical set $\mathcal{P}_\alpha$ up to a homeomorphism ?

A toy model

Given $\varepsilon > 0$ and $t > 0$, consider the transformation

$$\Phi_{\varepsilon,t} : Z \mapsto \frac{1}{t} \log \frac{1+\varepsilon \exp(Z)}{1+\varepsilon}.$$

which lifts $z = \exp(-Z) \mapsto \left(\frac{z(1+\varepsilon)}{z+\varepsilon}\right)^{1/t}$.



A toy model

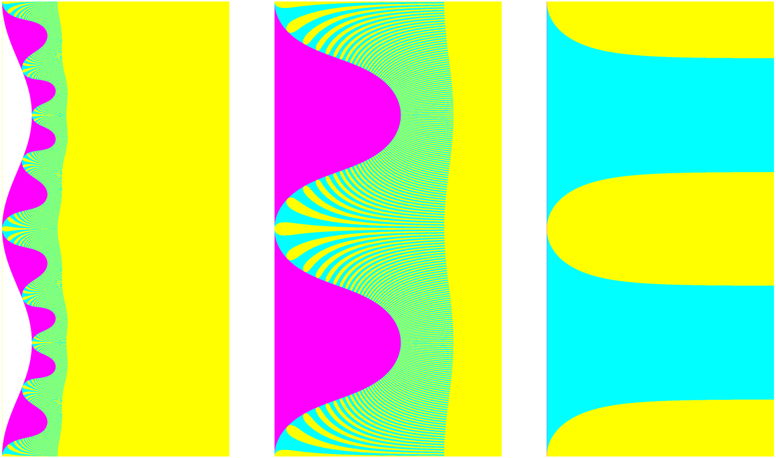
Given sequences (ε_n) and (t_n) set :

$$K_n = \{Z \in \mathbb{H}\ ;\ \Phi_{\varepsilon_n,t_n} \circ \cdots \circ \Phi_{\varepsilon_0,t_0}(Z) \in \mathbb{H}\}$$

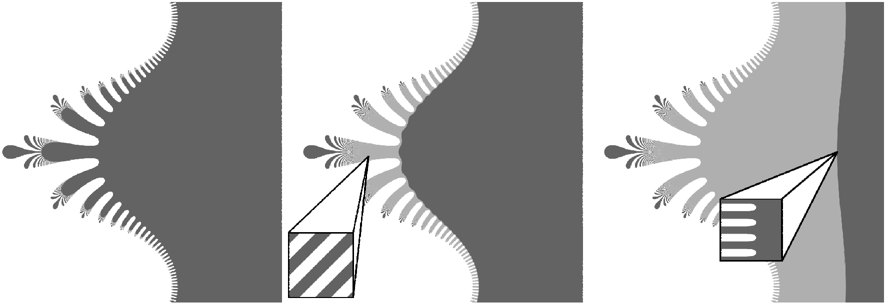
and

$$K = \bigcap_{n\geq 0} K_n.$$

The construction



Examples of sets K_3

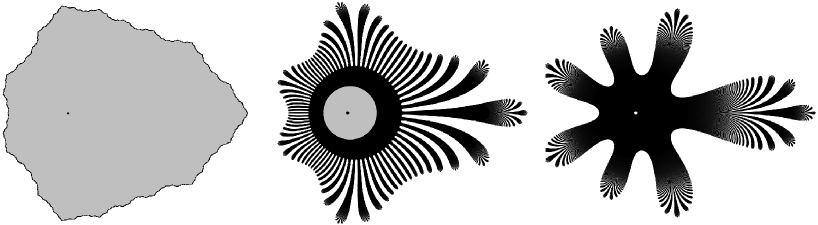


In all frames, $\varepsilon_n = 2t_n$, $t_0 \approx 50$, $t_1 \approx 10^6$.
Left : $t_2 \approx e^{30} \approx 10^{13}$. Middle : $t_2 \approx e^{10^6}$. Right : $t_2 \approx e^{10^8}$.

Given an irrational number α with approximants p_n/q_n, we will be particularly interested by the following choice of sequences :

$$t_n = \frac{q_{n+1}}{q_n} \quad \text{and} \quad \varepsilon_n = 2\pi t_n.$$

We denote by K_α the corresponding set.
K_α is invariant by translation by $2i\pi$ and projects to $\mathbb{C}^*$.



Arithmetical conditions

- α is a Brjuno number if

$$B(\alpha) := \sum \frac{\log q_{n+1}}{q_n} < +\infty.$$

- α is Herman if for all $n_0 \geq 0$ there is a $n > n_0$ such that

$$\Phi_{t_{n-1}} \circ \cdots \circ \Phi_{t_{n_0}}(1) > B(\alpha_n).$$

Cantor bouquets and hairy disks

- A *radial brush* is a set of the form

$$E = \left\{re^{2i\pi\theta} \in \mathbb{C}\ ;\ 0 < r < R(\theta)\right\}$$

  where $R : \mathbb{R}/\mathbb{Z} \to [0, +\infty)$ is a function having the following properties
  - both $R^{-1}(0)$ and its complement are dense subsets of $\mathbb{R}/\mathbb{Z}$,
  - $\limsup_{\theta \searrow \theta_0} R(\theta) = \limsup_{\theta \nearrow \theta_0} R(\theta) = R(\theta_0)$ for every $\theta_0 \in \mathbb{R}$.
- A *brush* is a set homeomorphic to a radial brush.
- A *Cantor bouquet* is a set homeomorphic $E \cup \{0\}$.
- A *hairy disk* is a set homeomorphic to $F(E) \cup \overline{\mathbb{D}}$ with $F(re^{i\theta}) = (1 + r)e^{i\theta}$.

Topology of K_α

Proposition

Let $\alpha \in \mathbb{R} - \mathbb{Q}$ *:*

1. 0 *is in the interior of* K_α *if and only if* α *is a Brjuno number. In that case, the interior* U_α *of* K_α *is connected and simply connected.*
2. *If* α *satisfies the Herman condition, then* $K_\alpha = \overline{U}_\alpha$ *and is a closed topological disk.*
3. *If* α *does not satisfy the Herman condition, then* $K_\alpha - \overline{U}_\alpha$ *is a brush, hence non empty. Moreover,* K_α *is a hairy disk if* α *is a Brjuno number or a Cantor bouquet otherwise.*