Geometric Limits in Holomorphic Dynamics

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Overview

- Various interesting discussions in holomorphic dynamics involve notions which do not depend continuously on parameters :
  - Julia sets
  - Quasiconformal deformations
- This *parabolic implosion* phenomenon has important applications in connection with the geography of parameter space :
  - Hausdorff dimension 2, for the boundary of the Mandelbrot set
  - Failure of local connectivity, for higher degree Connectedness loci
- Such considerations are often said to concern *geometric limits*, especially in connection with analogous issues for Kleinian groups. This terminology represents **informal** usage. Our objective is to formalize the notion.

Overview

Such formalization entails a certain investment in abstraction, with foundations in :

- General topology - *hyperspaces* of subsets of given spaces
- Complex analysis - dynamics and deformations of *finite type maps*, and of *finite type towers*

The reward for this investment is a precise language, which may be used to classify the underlying "limit dynamics" which is ultimately responsible. This furnishes a precise description of the existing applications, and serves as a guide to generating new ones.

Overview

Analogous considerations in the theory of Kleinian groups (and their quotient hyperbolic 3-manifolds) are explained, and exploited, via disambiguation of “convergence” notions :

- *Algebraic convergence* : convergence of generators, for representations of a fixed (finitely generated) abstract group
- *Geometric convergence* : convergence of dynamical systems and their quotients, as subgroups of $\mathbf{PSL}_2\mathbb{C}$

There is a body of theory concerning the relation between algebraic and geometric convergence, and there is a corresponding body of applications. For safety and comfort, these convergence notions may be presented in terms of associated topologies.

Overview

- For rational maps (of fixed degree D) the "algebraic" topology is the usual one given by coefficients : $\mathbf{Rat}_D$ is a Zariski open subset of $\mathbb{P}^{2D+1}$.
- The corresponding "geometric" topology is more problematic. One needs to specify, and suitably topologize, an appropriate underlying set (which will be strictly larger than the original set of rational maps). We introduce a space **CDS** whose underlying set consists of all (closed) conformal dynamical systems on $\mathbb{P}^1$.
- The iterates of a rational map f constitute a conformal dynamical system $\langle f \rangle$. This yields a map of sets

$$\mathbf{Rat} \hookrightarrow \mathbf{CDS}$$

which is injective on $\bigcup_{D=2}^{\infty} \mathbf{Rat}_D$.

Overview

We endow the space **CDS** with a topology which is compatible with the heuristic definition of geometric convergence.

- *A priori* there can be at most one such topology, and its existence has concrete consequences, *e.g.* iterated limits.
- The space **CDS** is compact and metrizable.
- For $D \geq 1$, the canonical injection $\mathbf{Rat}_D \hookrightarrow$ **CDS** is not continuous, and its image is not closed.

Thus, any algebraically convergent sequence $f_n \to f$ in $\mathbf{Rat}_D$ admits a geometrically convergent subsequence $\langle f_{n_k} \rangle \to \mathcal{F}$ in **CDS**.

- What is the relation between $\langle f \rangle$ and $\mathcal{F}$?
- Does $\mathcal{F}$ admit a “synthetic” description in terms of constructions on f itself ?
- What is the behavior of Julia sets, quasiconformal deformations *etc.* under geometric convergence ?

Overview

Even to sketch the answer will require the careful formulation of further definitions and the development of various intermediate results. The subsequent lectures will survey this background material (finite type maps and towers) and outline its application to the formulation and proof of a Structure Theorem for these geometric limit systems.

Grosso modo the argument is as follows :

- The geometric limit has an initial segment which is a tower obtained by countably iterated parabolic enrichment of the algebraic limit.
- The tower so obtained is (up to minor augmentation on rotation domains) the entire geometric limit.

The remainder of this lecture is devoted to the precise formulation of the motivating question. This task entails a discussion of what is meant by geometric convergence of conformal dynamical systems.

Conformal dynamical systems

Let X be a complex 1-manifold. A *conformal dynamical system* $\mathcal{F}$ on X consists of the assignment to each nonempty open $U \subseteq X$ a collection $\mathcal{F}[U]$ of open analytic maps $U \to X$ such that :

- If $f \in \mathcal{F}[U]$ and $V \subseteq U$ then $f_{|V} \in \mathcal{F}[V]$.
- If $f \in \mathcal{F}[U]$ and $g \in \mathcal{F}[f(U)]$ then $g \circ f \in \mathcal{F}[U]$.
- $\mathrm{I}_U \in \mathcal{F}[U]$.

Examples :

- Any subgroup $\Gamma \subseteq \mathbf{PSL_2}\mathbb{C}$ determines a system $\langle\Gamma\rangle$ on $\mathbb{P}^1$:

$$\langle\Gamma\rangle[U] = \{\gamma_{|U} : \gamma \in \Gamma\}.$$

- Any analytic map $f : W \to \mathbb{P}^1$, in particular any rational map of $\mathbb{P}^1$, determines a system $\langle f\rangle$ on $\mathbb{P}^1$:

$$\langle f\rangle[U] = \{f^n_{|U} : n \in \mathbb{N} \text{ and } U \subseteq \mathrm{dom}\ f^n\}.$$

Geometric convergence

The definition originates in the classical construction of Hausdorff. Let $(\mathfrak{X}, d)$ be a compact metric space.

- For $F, G \subseteq \mathfrak{X}$, the quantity

$$\delta(F, G) = \max\left(\sup_{f\in F}\inf_{g\in G} d(f,g),\ \sup_{g\in G}\inf_{f\in F} d(f,g)\right)$$

  is a nonnegative real number, and zero precisely when $\overline{F} = \overline{G}$.
- δ is a pseudometric on

$$P(\mathfrak{X}) = \{F : F \subseteq \mathfrak{X}\}$$

  and a metric on

$$\mathbf{P}(\mathfrak{X}) = \{F \in P(\mathfrak{X}) : F \text{ is closed in } \mathfrak{X}\}.$$

- $(\mathbf{P}(\mathfrak{X}), \delta)$ is a compact metric space.
- There is a canonical isometric embedding $(\mathfrak{X}, d) \hookrightarrow (\mathbf{P}(\mathfrak{X}), \delta)$ given by $x \mapsto \{x\}$.

Geometric convergence

More generally, let $(\mathfrak{X}, d)$ be a locally compact second countable metric space.

- Application of the above construction to suitable exhaustions yields a countable collection of pseudometrics on $\mathbf{P}(\mathfrak{X})$.
- One obtains a metric which depends on the various choices made, and an underlying compact topology which does not.
- For this topology,

$$F_n \to F \quad \Longleftrightarrow \quad \liminf F_n = F = \limsup F_n$$

where

$$\begin{aligned} \liminf F_n &= \{f : f_n \to f \text{ for some } f_n \in F_n\}, \\ \limsup F_n &= \bigcup_{\text{subsequences } F_{n_k}} \liminf F_{n_k}. \end{aligned}$$

Geometric convergence

Chabauty exploited this idea to topologize the collection of closed subgroups of a locally compact group G :

- The set of all subgroups is closed in the space of all subsets of G.
- The space of closed subgroups of G is compact and metrizable.

This discussion applies in particular to Lie groups, *e.g.* $\mathbf{PSL}_2\mathbb{C}$.

Consider representations $\rho : \Gamma \to G$, for a finitely generated group Γ. Convergence on generators gives a notion of *algebraic convergence* of representations. One may compare this with the notion of *geometric convergence*, meaning Hausdorff-Chabauty convergence of the image subgroups.

Proposition

If ρ_n converges algebraically then $\rho_{n_k}(\Gamma)$ converges geometrically, for some subsequence ρ_{n_k}. In this situation, the geometric limit contains (the image of) the algebraic limit.

Geometric convergence

We define geometric convergence of conformal dynamical systems by means of this $\liminf = \limsup$ prescription. There are a number of technical issues which require care :

- *A priori* it is not sufficient to specify the convergent *sequences*, so the definition is given in terms of *nets* :

$$\mathcal{F}_\eta \to \mathcal{F} \quad \Longleftrightarrow \quad \liminf \mathcal{F}_\eta = \mathcal{F} = \limsup \mathcal{F}_\eta$$

  where

$$\begin{aligned} \liminf \mathcal{F}_\eta &= \{f : f_\eta \to f \text{ for some } f_\eta \in \mathcal{F}_\eta\}, \\ \limsup \mathcal{F}_\eta &= \bigcup_{\text{subnets } \mathcal{F}_{\eta_\nu}} \liminf \mathcal{F}_{\eta_\nu}. \end{aligned}$$

- Such a definition makes reference to convergence of maps in

$$\mathcal{H}_X = \{f : U \to X : f \text{ is analytic and } U \neq \emptyset \text{ is open in } X\}.$$

  What topology is implied here ?

Spaces of analytic maps

In our setting, $f_\eta \to f$ should mean uniform convergence on compact sets. We have a choice of conventions for convergence of domains :

1. Any compact subset of $\mathrm{dom}\, f$ eventually belongs to $\mathrm{dom}\, f_\eta$.
2. The above, and also conversely.

Convention 1 yields a space $|\mathcal{H}_X|$ which is locally compact but not Hausdorff. Convention 2 yields a space $||\mathcal{H}_X||$ which is metrizable but not locally compact. Fortunately, this dilemma has a resolution :

- Classical *hyperspace* theory recommends that we give up the Hausdorff property in favor of local compactness.
- In any event, we are not concerned with all (compositionally closed) sets of maps, but only with those that are saturated with respect to restriction. Evidently, the closed subsets of $|\mathcal{H}_X|$ are precisely the saturated closed subsets of $||\mathcal{H}_X||$.

Spaces of analytic maps

We endow $|\mathcal{H}_X|$ with the weakest topology such that

$$O_\Upsilon = \{f : U \text{ is compactly contained in } \mathrm{dom}\, f, \text{ and } f_{|U} \in \Upsilon\}$$

is open, for every nonempty open subset U of X, and every open subset Υ of the space of (open) analytic maps $U \to X$.

Theorem

For any complex 1-manifold X :

- *The space $|\mathcal{H}_X|$ is locally compact, but neither Hausdorff nor regular.*
- *If X has countably many components then $|\mathcal{H}_X|$ is second countable.*

Spaces of analytic maps

- The poor separation properties of $|\mathcal{H}_X|$ (neither Hausdorff nor regular) are due to the fact that

$$\overline{\{f\}} = \{f_{|U} : U \subseteq \mathrm{dom}\, f\} \neq \{f\}.$$

- The proper definition of local compactness is crucial :
  - We say that a space is *locally compact* if every open neighborhood of a point contains a compact subneighborhood.
  - For spaces which are Hausdorff or regular, an *a priori* weaker condition is equivalent to the *a priori* stronger requirement that every open neighborhood of a point contains a closed compact subneighborhood.
  - Local compactness of $|\mathcal{H}_X|$ is a consequence of Montel's Theorem.
- The second countability assertion for $|\mathcal{H}_X|$ follows from the second countability of Riemann surfaces (Radó's Theorem).

Hyperspaces

Let $\mathfrak{X}$ be a topological space, and consider the *hyperspaces*

- $P(\mathfrak{X}) = \{F : F \subseteq \mathfrak{X}\}$,
- $\mathbf{P}(\mathfrak{X}) = \{F \in P(\mathfrak{X}) : F \text{ is closed}\}$.

Note that there is an inclusion $\mathbf{P}(\mathfrak{X}) \hookrightarrow P(\mathfrak{X})$ and there is a retraction $P(\mathfrak{X}) \to \mathbf{P}(\mathfrak{X})$ given by $F \mapsto \overline{F}$. We topologize $\mathbf{P}(\mathfrak{X})$, and then give $P(\mathfrak{X})$ the weakest topology which makes that retraction continuous.

We desire $\mathbf{P}(\mathfrak{X})$ to be a compact Hausdorff space, with convergence given by the $\liminf = \limsup$ prescription. Evidently, this can be done in at most one way.

- As discussed, if $\mathfrak{X}$ is a locally compact second countable metric space then the Hausdorff-Chabauty recipe may be applied.
- Vietoris gave a purely topological description of the construction applicable to compact Hausdorff spaces.
- Fell appropriately generalized this to (not necessarily Hausdorff) locally compact spaces.

Hyperspaces

Fell equips $\mathbf{P}(\mathfrak{X})$ with the weakest topology such that the subset

$$\mathfrak{E}_S = \{F \in \mathbf{P}(\mathfrak{X}) : F \cap S \neq \emptyset\}$$

is open in $\mathbf{P}(\mathfrak{X})$ for every nonempty open $S \subseteq \mathfrak{X}$, and closed in $\mathbf{P}(\mathfrak{X})$ for every compact $S \subseteq \mathfrak{X}$.

Theorem

For any topological space $\mathfrak{X}$, the topological space $\mathbf{P}(\mathfrak{X})$ is compact. Moreover, if $\mathfrak{X}$ is locally compact then :

1. *The space $\mathbf{P}(\mathfrak{X})$ is Hausdorff.*
2. *$F_\eta \to F$ if and only if $\liminf F_\eta = F = \limsup F_\eta$.*
3. *If $\mathfrak{X}$ is second countable then $\mathbf{P}(\mathfrak{X})$ is second countable, hence metrizable.*

Geometric limits of conformal dynamical systems

We apply Fell's construction to the locally compact space $|\mathcal{H}_X|$, and deduce the following :

Theorem

Let X be a complex 1-manifold.

1. *There is a unique topology on* $\mathbf{CDS}(X)$ *such that* $\mathcal{F}_\eta \to \mathcal{F}$ *if and only if* $\liminf \mathcal{F}_\eta = \mathcal{F} = \limsup \mathcal{F}_\eta$.
2. *The space* $\mathbf{CDS}(X)$ *is compact and Hausdorff.*
3. *If X has countably many components then* $\mathbf{CDS}(X)$ *is second countable, hence metrizable.*

Overview

In this lecture we will :

- Survey the theory of *finite type* complex analytic maps.
- Discuss fundamental examples arising via parabolic renormalization.

In the final lecture we will :

- Study the *towers* of finite type complex analytic maps obtained through (iterated) parabolic enrichment.
- Discuss the relation between these towers and the geometric limits considered in the previous lecture.

Definitions

An analytic map of complex 1-manifolds

$$f : W \to X$$

is of *finite type* if :

- X is compact,
- f is open,
- f has no isolated removable singularities,
- $S(f)$ is finite.

Here $S(f)$ is the set of *singular values* :

- By definition, $x \in X$ belongs to the complement of $S(f)$ if and only if there exists an *evenly covered* open neighborhood of x.
- In general, $S(f) = \overline{C(f) \cup A(f)}$, where $C(f)$ is the set of *critical values* and $A(f)$ is the set of *asymptotic values*, hence $S(f) = C(f) \cup A(f)$ for f of finite type.

Examples

- Any nonconstant analytic map between compact Riemann surfaces : in particular, any rational map $\mathbb{P}^1 \to \mathbb{P}^1$.
- Finite type maps $\mathbb{C} \to \widehat{\mathbb{C}}$:
  - entire : exp, sin, cos, $\int^z P(\zeta) \exp(Q(\zeta))\, d\zeta$ for polynomials $P, Q, \ldots$
  - meromorphic : tan, $\wp, \ldots$
  - Finite type maps $\mathbb{C}^* \to \widehat{\mathbb{C}^*}$: $z \mapsto az \exp b(z + \frac{1}{z}), \ldots$
- The elliptic modular functions $j, \lambda : \mathbb{H} \to \mathbb{P}^1$.
- *Skinning maps* on boundaries of hyperbolic 3-manifolds.

New maps from old

- Let $f : W \to X$ and $g : Y \to Z$ be finite type maps with $Y \subseteq X$. Then
$$g \circ f : f^{-1}(Y) \to Z$$
is of finite type. In particular, if $W \subseteq X$ then the iterates f^n are of finite type.
- If $f : W \to X$ is a finite type map, and if Z is a connected component of X which intersects the image of f, then the first visit to Z gives a finite type map $f^{\{Z\}} : W^{\{Z\}} \to Z$.
- Let $f : W \to X$ be a finite type map. If f is *postsingularly finite* then the linearizers for repelling periodic points of f are finite type maps $\Lambda \to X$ for appropriate $\Lambda \subseteq \mathbb{C}$. In particular, for a postsingularly finite entire map $\mathbb{C} \to \widehat{\mathbb{C}}$, the linearizers are finite type entire maps $\mathbb{C} \to \widehat{\mathbb{C}}$.
- Let $f : W \to X$ be a finite type map. The parabolic renormalization construction yields a finite type analytic map $\mathcal{W}_f \to \mathcal{X}_f$.

Islands and tracts

Let $f : W \to X$ be a finite type analytic map, and let $B \subseteq X$ be a Jordan domain whose boundary is disjoint from $S(f)$.

- If B is disjoint from $S(f)$ then the components of $f^{-1}(B)$ are Jordan domains : *(simple) islands* over B.
- If $B \cap S(f)$ consists of a single point then every component D of $f^{-1}(B)$ is simply connected. If D is compactly contained in W then D is a Jordan domain (*island*). Otherwise, D is a *tract*.
- The preimage of the boundary of B is dense in the boundary of any tract over B.

Islands property

Theorem

Let $f : W \to X$ be a finite type analytic map. Assume that X is connected, and let $B \subseteq X$ be a Jordan domain whose closure is disjoint from $S(f)$. Suppose further that $W \subseteq Y$, and let U be a connected open subset of Y which intersects ∂W. Then any connected component of $U \cap W$ contains infinitely many islands over B.

Compare this to the Ahlfors Five Islands Theorem for (not necessarily finite type) meromorphic $\mathbb{C} \to \widehat{\mathbb{C}}$.

Fatou and Julia sets

Let $f : W \to X$ be an analytic map. Assume that $W \subset X$, and consider the sequence of iterates $f^n : W_n \to X$: note that

$$X = W_0 \supseteq W_1 \supseteq W_2 \supseteq \dots$$

- The *Fatou set* $\Omega(f)$ consists of all points $x \in X$ possessing an open neighborhood U such that :
  - Either there exists $N \geq 0$ such that $U \subseteq W_n$ for $n \leq N$ and $U \subseteq X \setminus W_n$ for $n > N$,
  - Or $U \subseteq W_n$ for every n, and moreover the family $\{f^n_{|U} : n \in \mathbb{N}\}$ is normal.
- The *Julia set* $J(f)$ is the complement $X \setminus \Omega(f)$.
- $\Omega(f)$ is open and $J(f)$ is closed.
- $\Omega(f)$ and $J(f)$ are invariant : if $x \in W$ then x belongs to $\Omega(f)$ or $J(f)$ if and only if $f(x)$ does.

Typical and exceptional maps

Let $f : W \to X$ be an analytic map, where X is connected. We say that f is *typical* if $\bigcap_{n=1}^{\infty} W_n$ contains a nonhyperbolic Riemann surface, and *exceptional* otherwise.

Proposition

- *If f is typical then $J(f) = \overline{\bigcup_{n=1}^{\infty} \partial W_n}$.*
- *If f is exceptional then, up to analytic conjugacy, f is :*
    - *(algebraic) a rational endomorphism of $\mathbb{P}^1$, or an affine toral endomorphism, or an automorphism of a higher genus surface ;*
    - *(transcendental) a map $\mathbb{C} \to \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$, or a map $\mathbb{C}^* \to \mathbb{C}^* \hookrightarrow \widehat{\mathbb{C}^*}$, or a map $\mathbb{C} \to \widehat{\mathbb{C}}$ whose second iterate is a map $\mathbb{C}^* \to \mathbb{C}^* \hookrightarrow \widehat{\mathbb{C}^*}$.*

Stratification of $J(f)$

Let $f : W \to X$ be a finite type analytic map.
The islands property has the following consequences :

Proposition

Let $f : W \to X$ be a finite type analytic map.

1. *$J(f)$ is the disjoint union of the sets $f^{-(n-1)}(\partial W)$, for $n \geq 1$, and the set $J_+(f) = J(f) \cap \bigcap_{n=1}^{\infty} W_n$.*
2. *For each component Z of X, either $Z \subseteq J(f)$ or else $J(f)$ is nowhere dense in Z.*
3. *$J(f)$ is an uncountable perfect set, if X is connected and f is not an automorphism.*

Density of repelling periodic points

Theorem

Let $f : W \to X$ be a finite type analytic map. Assume that X is connected. Then ∂W is the accumulation of the set of repelling fixed points of f.

Corollary

Under these assumptions, $J(f)$ is the closure of the set of repelling periodic points of f.

Geometric finiteness conditions

Let $f : W \to X$ be a finite type analytic map. By definition :

- f is *weakly geometrically finite* if every infinite postsingular orbit lies in the basin of some periodic cycle.
- f is *strongly geometrically finite* if f is weakly geometrically finite and every singular value in $J(f)$ is preperiodic.

Note that a transcendental exceptional map may be weakly geometrically finite, but never strongly geometrically finite.

Theorem

Let $f : W \to X$ be a finite type analytic map.

- *If f is weakly geometrically finite then $J_+(f)$ carries no invariant linefield.*
- *If f is strongly geometrically finite, and if $J(f)$ is nowhere dense in X, then $J_+(f)$ has measure 0.*

Classification of Fatou components

The domain W may have nonempty exterior $X \setminus \overline{W}$, and Fatou components which eventually map to this exterior are said to *escape*. The remaining Fatou components are classified precisely as in the rational case.

Theorem

Let f be a finite type analytic map.

- *Every component of $\Omega(f)$ which does not escape is eventually periodic.*
- *Every periodic component of $\Omega(f)$ is a superattracting, attracting, or parabolic basin, or a rotation domain.*

In short : there are no wandering domains and no Baker domains.

Fatou-Shishikura inequality

Let $f : W \to X$ be a finite type analytic map. By definition :

- $\gamma(f)$ is the number of cycles of periodic points, with multiplicities

$$\gamma_{\langle x\rangle}(f) = \begin{cases} 0 & \text{if } \langle x\rangle \text{ is repelling or superattracting} \\ 1 & \text{if } \langle x\rangle \text{ is attracting or irrationally indifferent} \\ \nu & \text{if } \langle x\rangle \text{ is parabolic-repelling} \\ \nu + 1 & \text{if } \langle x\rangle \text{ is parabolic-attracting} \\ & \quad \text{or parabolic-indifferent .} \end{cases}$$

- $\hbar(f)$ is the number of Herman ring cycles.
- $\delta(f)$ is the number of infinite tails of postsingular orbits.

Theorem

Let $f : W \to X$ be a finite type analytic map. Then $\gamma(f) + 2\hbar(f) \leq \delta(f)$, provided that no return to any component of X is an automorphism.

Parabolic renormalization

Overview

In this lecture we will :

- Describe the *parabolic enrichment* process which adds a new generator to a dynamical system with parabolic cycles.
- Disuss how this procedure may be iterated, yielding countably generated *towers* of finite type complex analytic maps.
- State Structure and Realization theorems which recognize the geometric limits considered in the previous lecture as (essentially) these towers.
- Outline the conceptual steps in our program for proving these theorems.

Parabolic Enrichment

Let $f : W \to X$ be an analytic map, and assume that f has at least one parabolic cycle. Recall :

- The quotient of the attracting petals by f is a disjoint union of Riemann surfaces which are conformally equivalent to $\mathbb{C}/\mathbb{Z}$. We denote by $\mathcal{X}_f$ the union of these (compactified) cylinders.
- The quotient of the repelling petals by f is a disjoint union of Riemann surfaces which are conformally equivalent to $\mathbb{C}/\mathbb{Z}$. We denote by $\mathcal{Y}_f$ the union of these (compactified) cylinders.
- The local parabolic dynamics of f induces an analytic map $\mathcal{E}_f : \mathcal{W}_f \to \mathcal{X}_f$ whose domain contains the ends of $\mathcal{Y}_f$.
- The specification of a transit isomorphism $\Phi : \mathcal{X}_f \to \mathcal{Y}_f$ creates new dynamics on $\mathcal{X}_f \cup \mathcal{Y}_f$.

Towers