About Inou and Shishikura's near parabolic renormalization

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Lemma: A rational map with a simply connected periodic Fatou component is conjugated there to a Blaschke product on $\mathbb{D}$ with $J = \partial\mathbb{D}$.

Consequence: The dynamics on simply connected immediate parabolic basins containing only one critical value is *unique*.

... because ...

by Riemann-Hurwitz there must be only one critical point and, up to Möbius conjugacy, there is only one unicritical Blaschke product of degree d with $J = \partial\mathbb{D}$ and having a parabolic point:

$$B(z) = \frac{z^d + a_d}{1 + a_d z^d},\ a_d = \frac{d-1}{d+1}$$

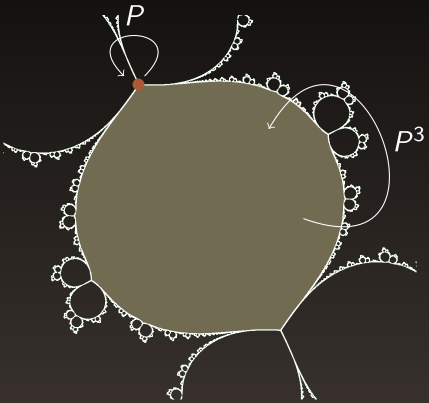
In particular, for a fixed d, they all have the same Fatou coordinates: if U is the immediate basin of a petal, p its period, $\phi : U \to \mathbb{D}$ an isomorphism conjugating f^p on U to B on $\mathbb{D}$ then

$$\mathsf{Fatou}_{\mathsf{attr}}[f] = \mathsf{Fatou}_{\mathsf{attr}}[B] \circ \phi$$

where $\mathsf{Fatou} = \mathsf{Fatou}_{\mathsf{attr}}[f] : U \to \mathbb{C}$ denotes the extended attracting Fatou coordinates associated to f^p on U: $\mathsf{Fatou} \circ f = T_1 \circ \mathsf{Fatou}$ where $T_1(z) = z + 1$.

The parabolic che[ss/cker]board

A nice way to visualize the extended Fatou coordinates is to make use of the parabolic graph and chessboard.



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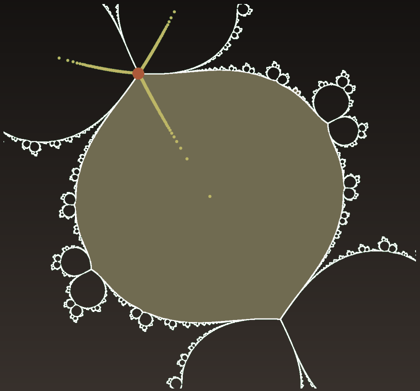

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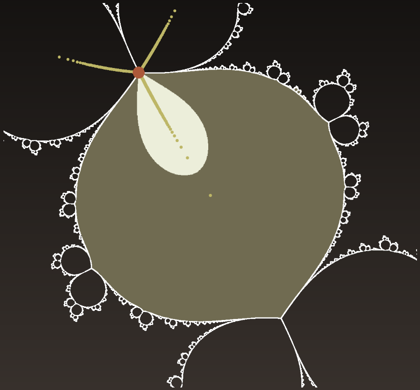

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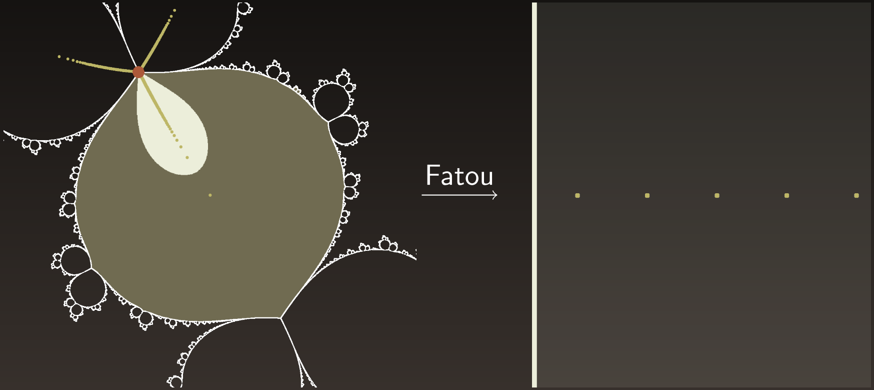

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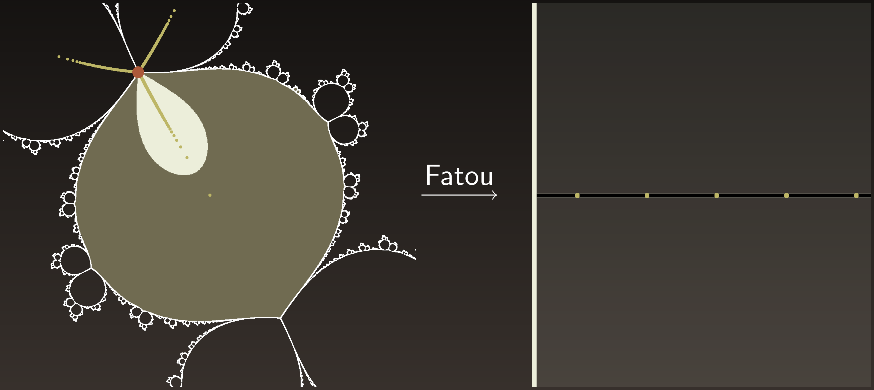

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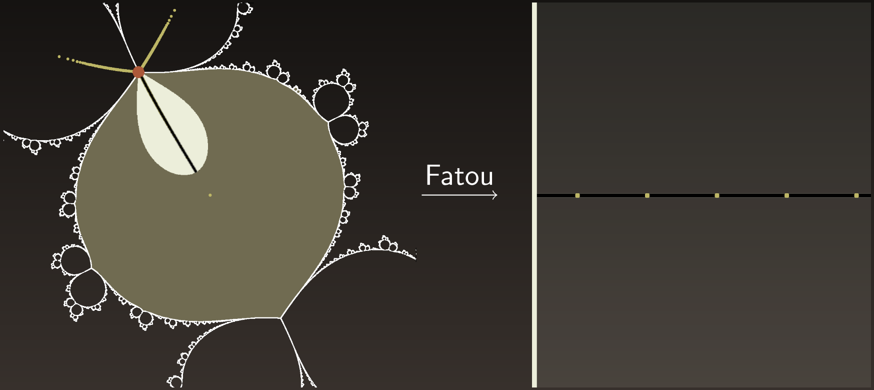

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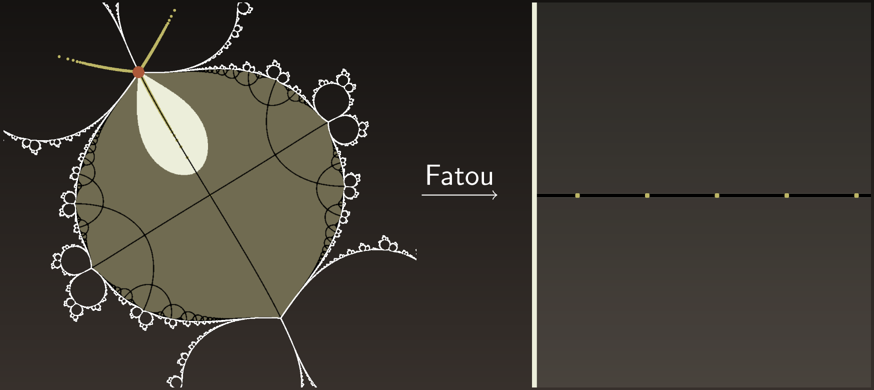

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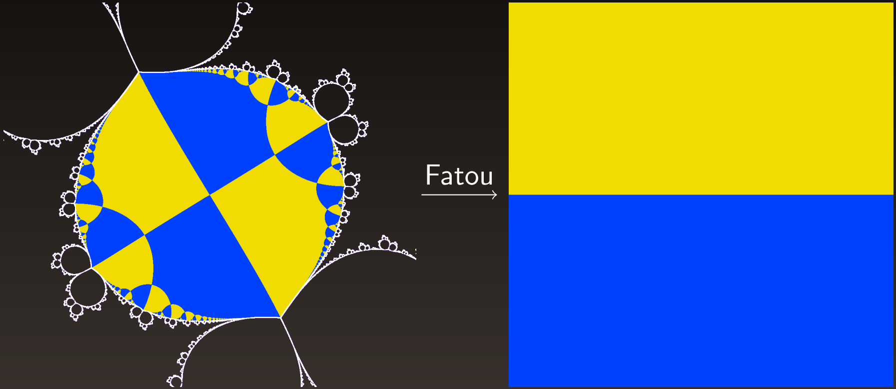

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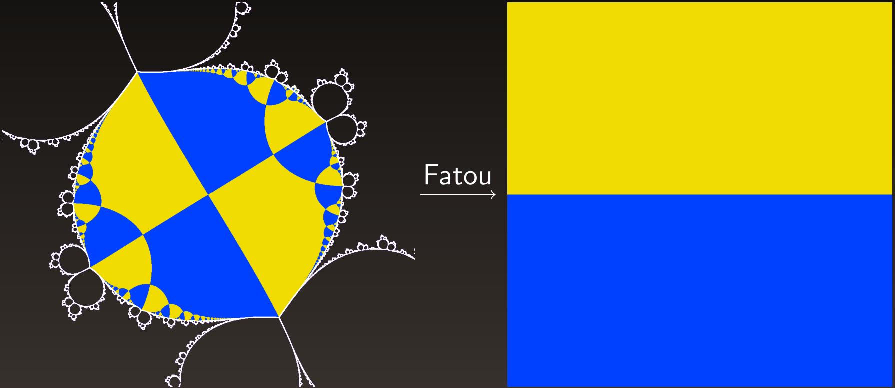
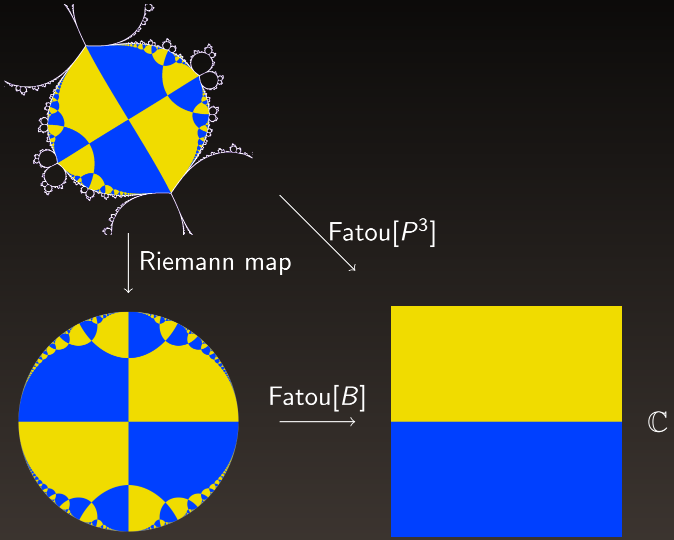


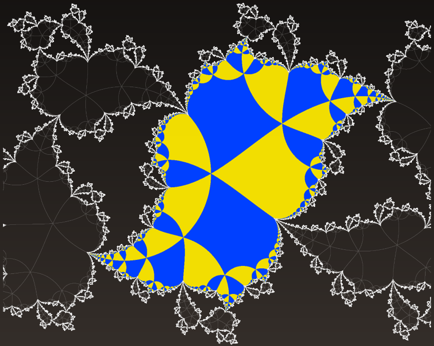
Extended attracting Fatou coordinates have nice covering properties.

Universality, in pictures



Universality, in pictures

$d = 3$



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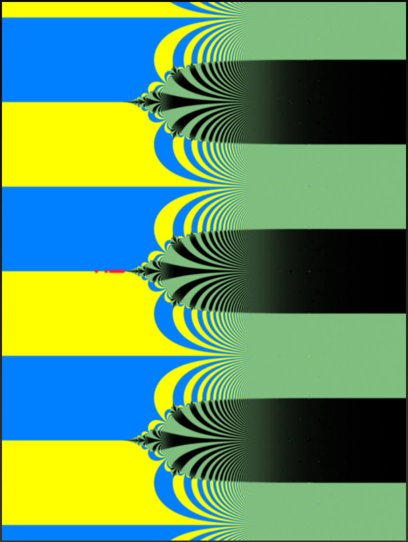
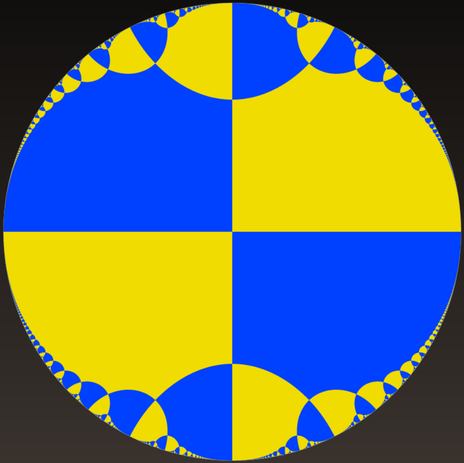
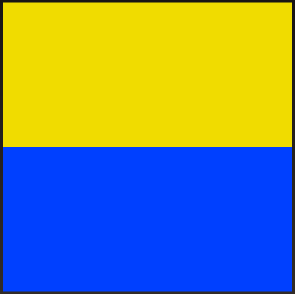


one asymptotic value

The Fatou coords of the Blaschke prod

More eye candy





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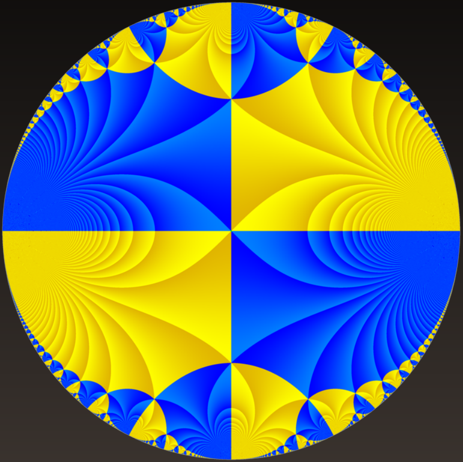

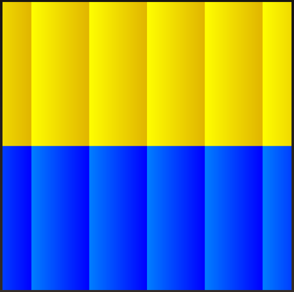

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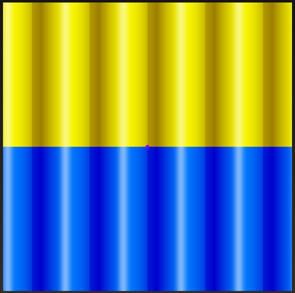
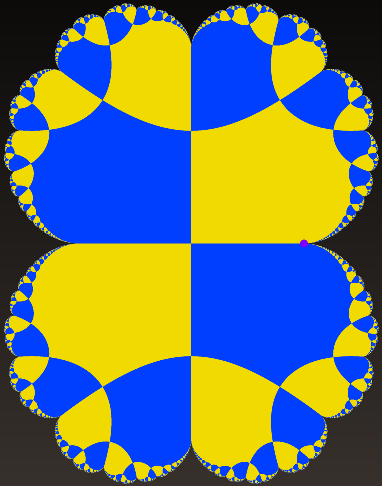


The horn map

I just hope extended horn maps have already been defined at this point of the workshop. . .

The horn map



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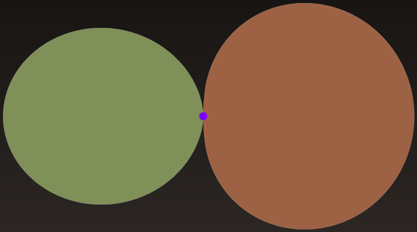

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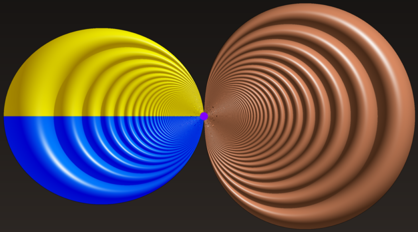

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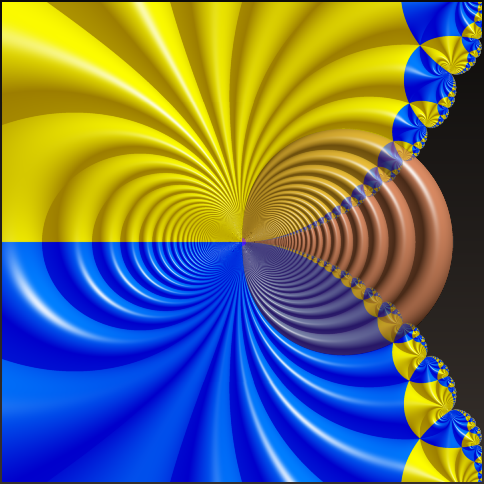

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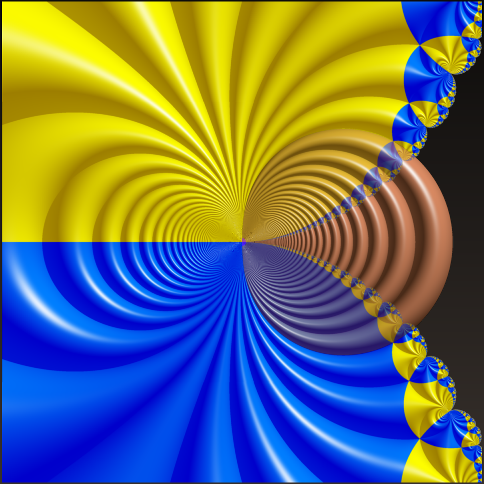
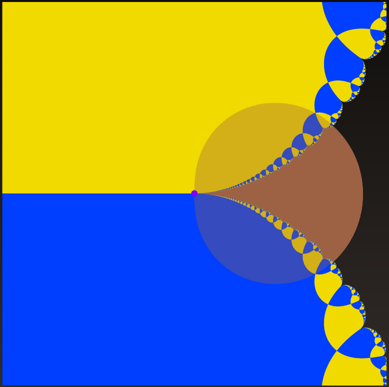


Maybe I'm going too far with candy

The horn map



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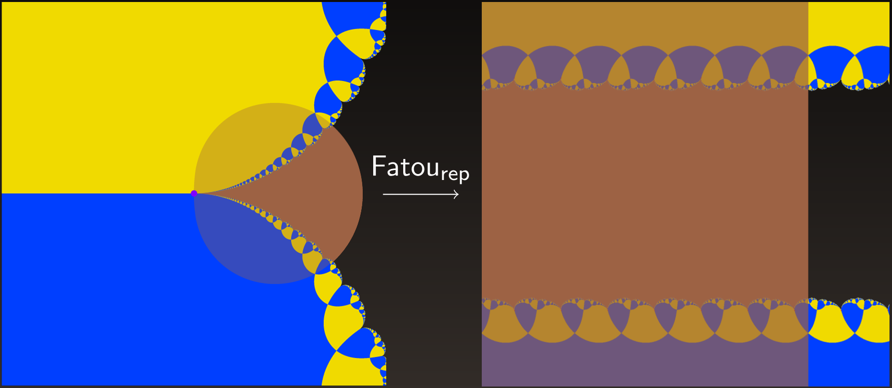
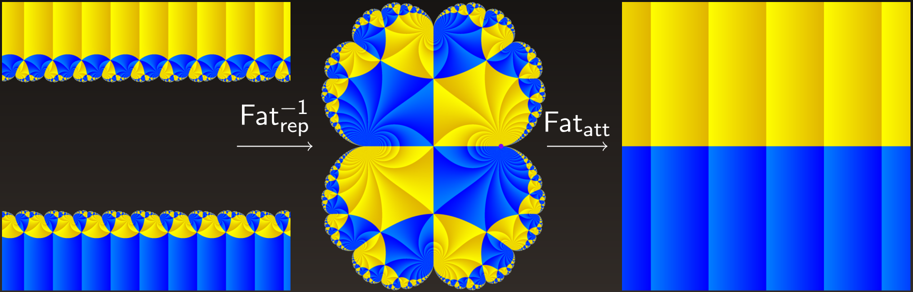


The horn map

The extended horn map



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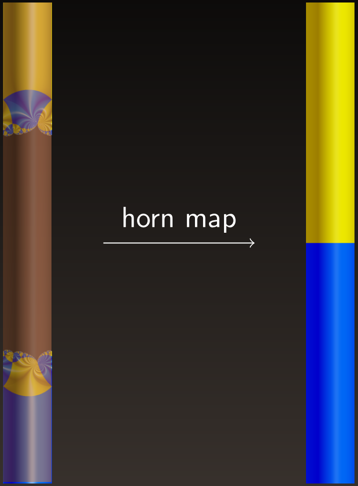
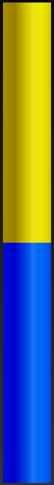


The horn map

horn map →



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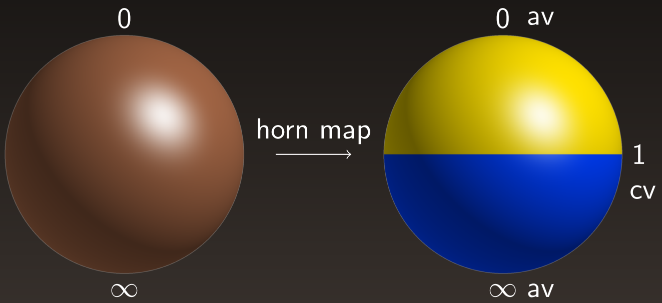
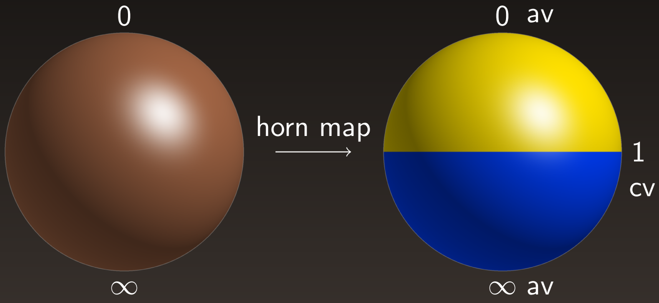


Movie!

The horn map

The horn map is a finite type map in the sense of Epstein over $\widehat{\mathbb{C}}$: it has only 3 singular values. As a cover, it can be understood with the parabolic chessboard.



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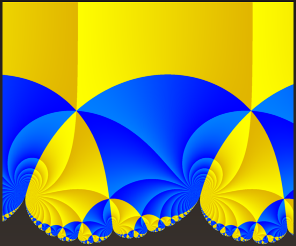
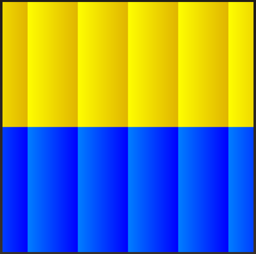
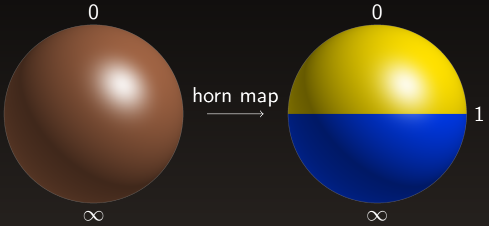


horn map $\longrightarrow$



Parabolic renormalization

The horn maps on $\mathbb{C}^*$ have an analytic extension at 0 and ∞, fixing each.



They are defined up to pre and post composition by complex multiplications. We can *normalize* by taking the unique representative whose unique critical value is $z = 1$ and which has derivative one at $z = 0$.

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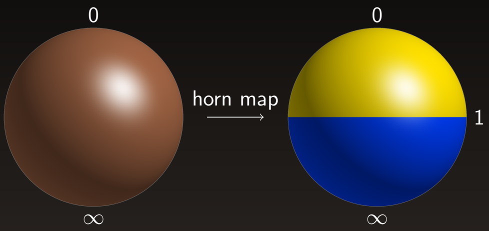

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The restriction to the component containing 0 of this normalized horn map is called the *parabolic renormalization* of the map we started with.

Universality for unisingular simply connected parabolic basins: conjugacy

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$$\Longrightarrow$$

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a universality for horn maps: cover equivalence.

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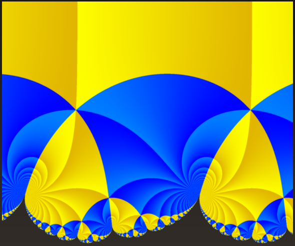
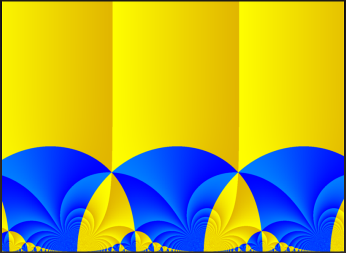


$\xrightarrow{\text{conformal}}$



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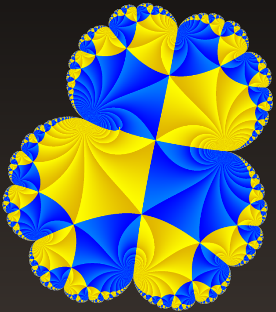

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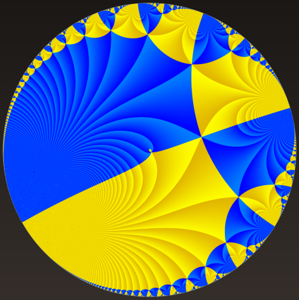

Renormalizing the renormalized

It may at first look odd to want to iterate horn maps, but the theory of parabolic implosion tells us that they really occur as geometric limits. Horn maps naturally come in simple families $h_\sigma(z) = e^{2i\pi\sigma} h_0(z)$. This family also undergoes parabolic implosions, so if a horn map has a parabolic periodic point, the horn maps of these points will also tell us something about the geometric limits of the initial family.

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The renormalized map (we really should find something easier to pronounce), call it $\mathcal{R}(f)$, is parabolic at the origin. As a consequence of its covering properties, Shishikura proved that it has only one petal, that its basins of attraction U contains the critical value and is simply connected and h is proper of degree 2 from U to U.

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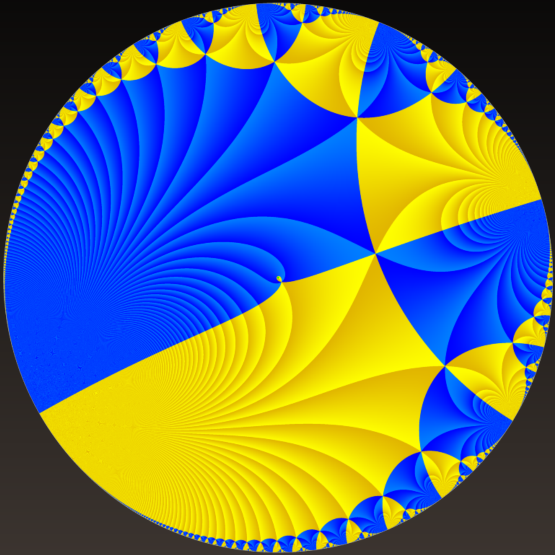
Therefore... (drumroll)... It is also conjugated to the Blaschke product!

Shishikura's invariant class



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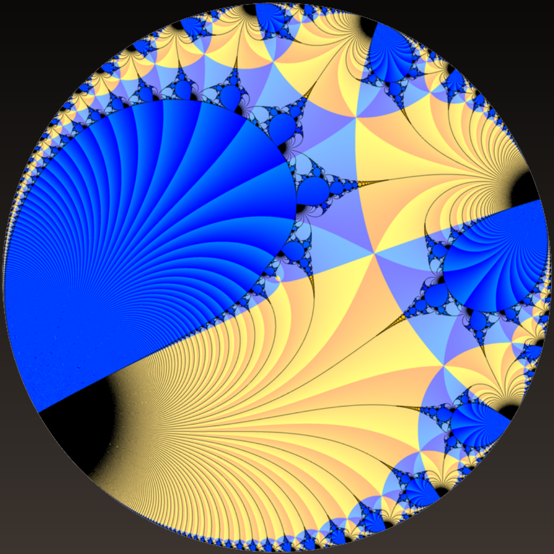

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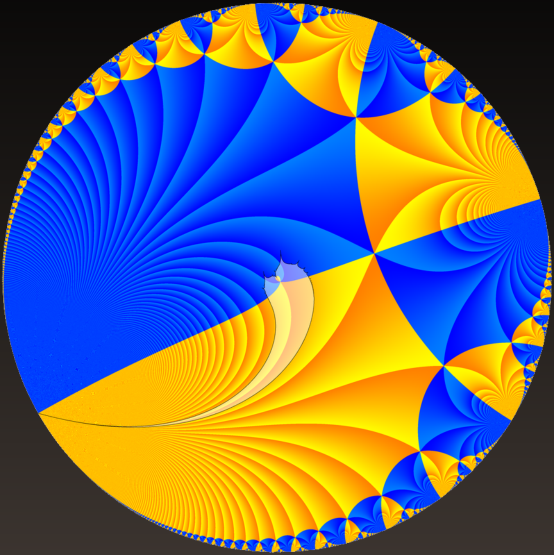

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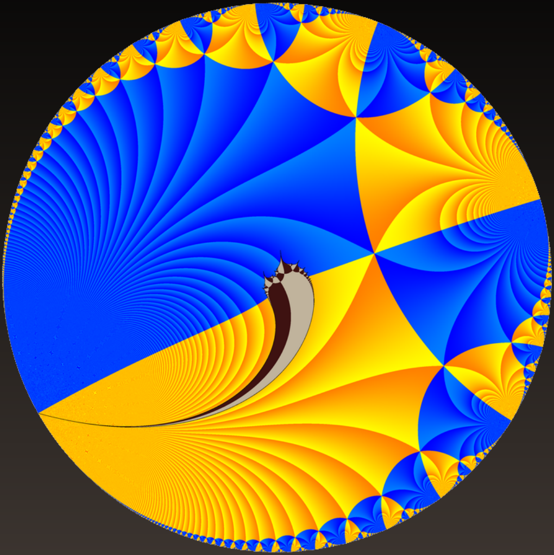

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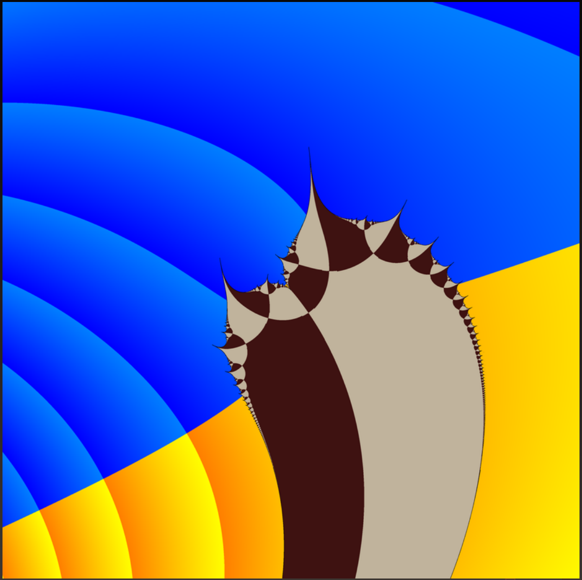

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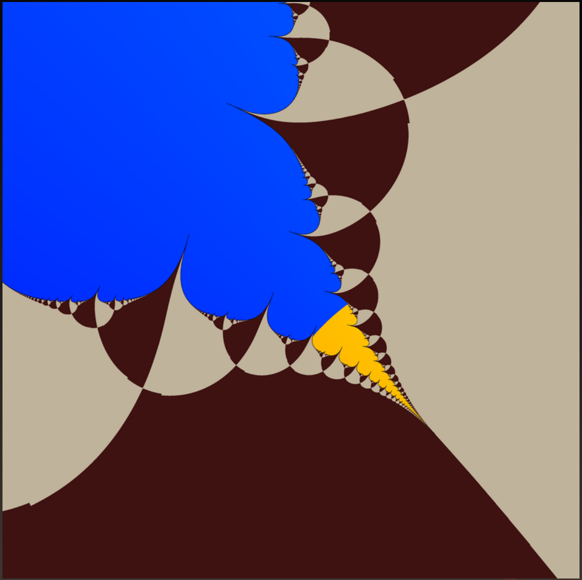

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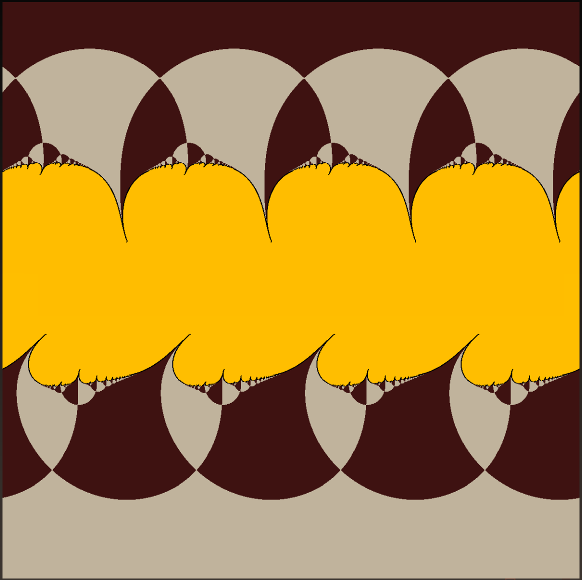

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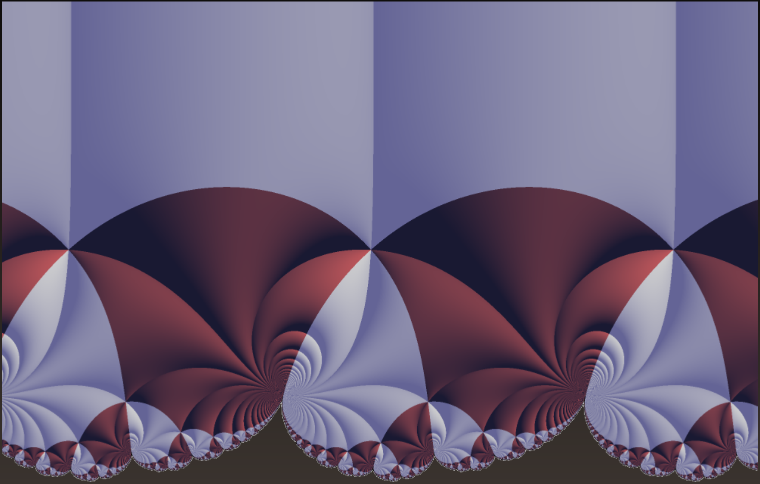
Shishikura's invariant class has the following form: $\mathcal{S} = \left\{ h[B] \circ \phi^{-1} \right\}$ with $\phi \in$ Schlicht. Then $\mathcal{R} : \ \mathcal{S} \to \mathcal{S}$. There is a high chance that $\mathcal{R}$ is *contracting* on $\mathcal{S}$, but it is not so easy to prove because $\mathcal{S}$ is not a complex (Banach) variety.

Inou and Shishikura's flexible invariant class



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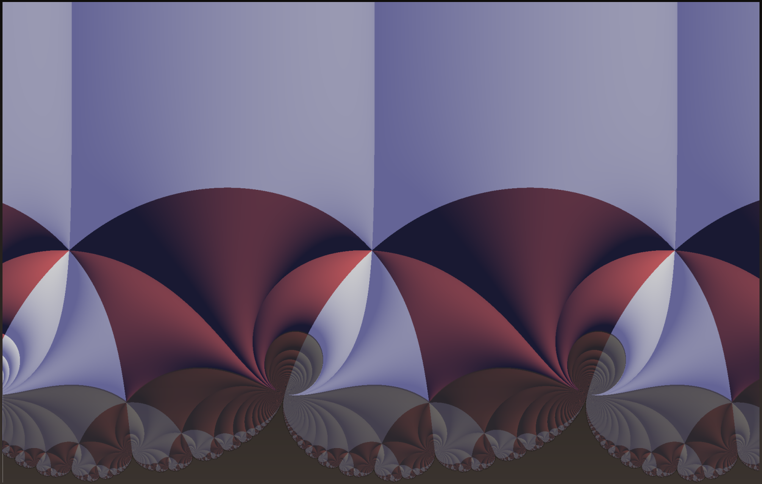

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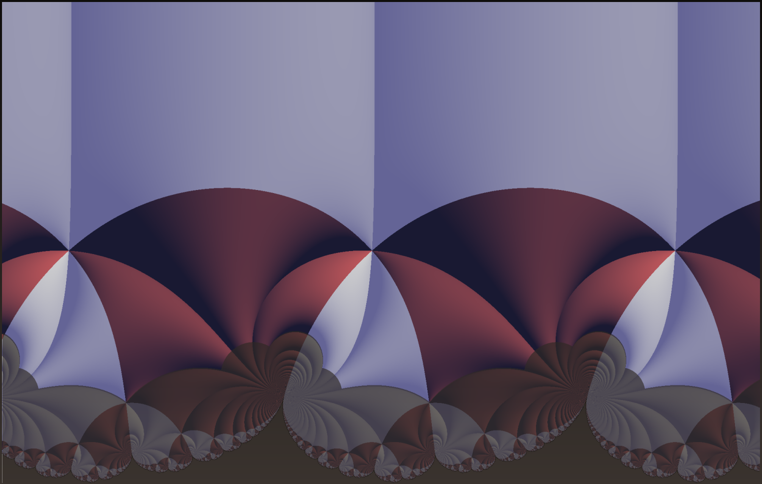

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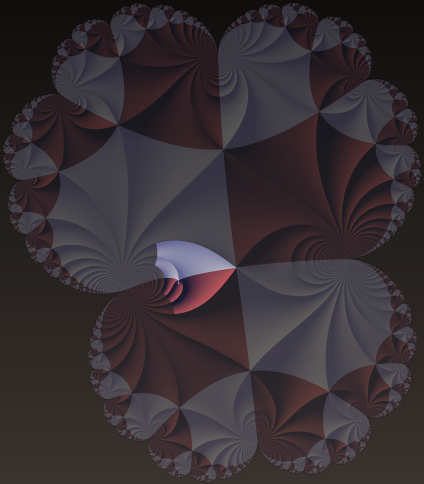

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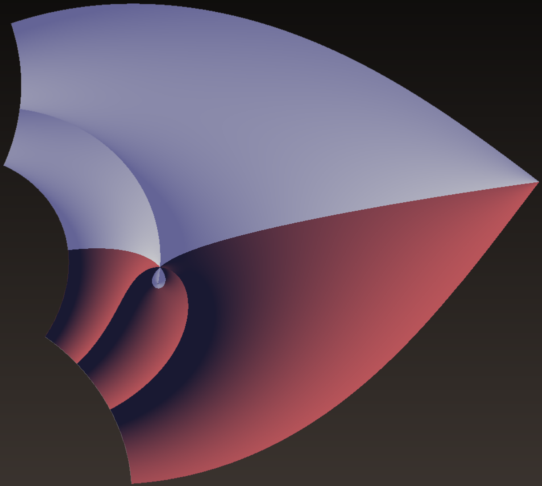

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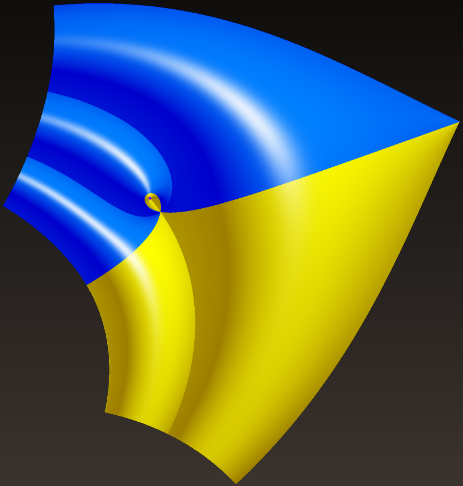

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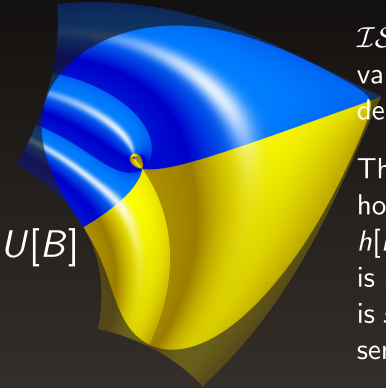


$\mathcal{IS} = \{f = h[B] \circ \phi^{-1}\}$ where ϕ is now univalent on a cutout $U[B]$ (which is independent of f) and still satisfies $\phi(z) = z +$h.o.t..

They proved that there $\forall f \in \mathcal{IS}$, it has a horn map which has a restriction of the form $h[B] \circ \psi^{-1}$ where $\psi(z) = z + h.o.t.$ but which is defined and univalent on a set $V[B]$ that is *strictly bigger* than the cutout $U[B]$ in the sense that $U[B] \Subset V[B]$.

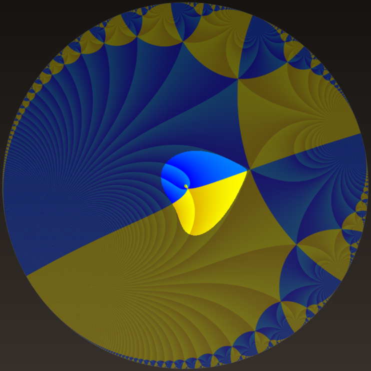
Use

Gain:

- contraction
- perturbability: there is still some renormalization even when the multiplier at 0 is not exactly 1.
- powerful understanding of the post critical set of some quadratic maps
- positive measure Julia sets

Trade-off: since it is a perturbation argument one has to restrict to small rotation numbers that stay small when renormalized, i.e. all entries in the continued fraction need to be $>$ some universal N.

An attempt at flexibilizing for higher degree



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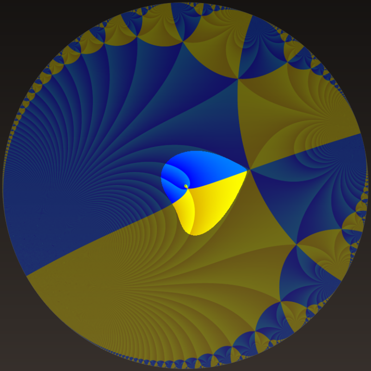

Do you see all this structure that was left away?

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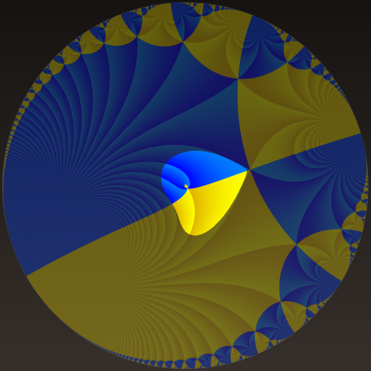

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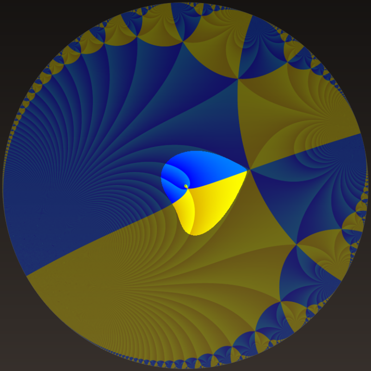

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Strategy

Two steps:

1. For $f \in \mathcal{S}$ (i.e. with the full structure), denote by $B[f](d)$ the hyperbolic disk of center 0 and radius d in the set of definition of f. Prove that there exists a function $D(d)$ with $D \ll d$ as $d \longrightarrow +\infty$ all map $f \in \mathcal{S}$, then the image by the extended repelling Fatou coordinate of $B[\mathcal{R}(f)](d)$ is contained in $B[f](D)$.

2. Use a specific perturbation to deduce from this that for some d (big enough), maps with structure corresponding to the hyperbolic disk of radius d are stable by renormalization. The perturbation consists in starting from a map with this reduced structure, and then continuously deform it into a map having the full structure, keeping the set of definition of f unchanged but "feeding in" back the structure missing. The hope is, since $D << d$, the map f will almost not change at points of $B[f](D)$.