Limits of sub semigroups of $\mathbb{C}^*$ and Siegel enrichments

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$\mathcal{SG}(\mathbb{C}^*) = \{\Gamma \cup \{0, \infty\} \mid \Gamma \text{closed sub semigroup of } \mathbb{C}^*\} \subset Comp(\mathbb{P}^1)$

$\mathcal{SG}(\mathbb{C}^*)$ has naturally the Hausdorff topology on compact subsets of $\mathbb{P}^1$.

Limits of closed semigroups are closed semigroups.

For $z \in \mathbb{C}^*$, $\Gamma_z = \{z, z^2, ..., z^k, ...\}$.

$$\mathcal{SG}_1(\mathbb{C}^*) = \overline{\{\Gamma_z \cup \{0, \infty\} \mid z \in \mathbb{C}^*\}}.$$

Topological model for $\overline{\{\Gamma_z \mid z \in \mathbb{C}^*\}} \subset \mathcal{SG}(\mathbb{C}^*)$

.

$\frac{r}{s} \in \mathbb{Q}/\mathbb{Z}$ with $\gcd(r,s) = 1$ let :
$\mathcal{D}_{\frac{r}{s}} \subset \mathbb{C} \setminus \mathbb{D}$: the open disc of radius $\frac{1}{s^2}$ and tangent to $\mathbb{S}^1$ at $e^{2i\pi\frac{r}{s}}$.
$\partial\mathcal{D}_{\frac{r}{s}}$: $p \mapsto z_{\frac{r}{s}}(p)$ the point intersection of $\partial\mathcal{D}_{\frac{r}{s}}$ and the half line through $e^{2i\pi\frac{r}{s}}$ making slope $p \in [-\infty, +\infty]$ with the line $\theta = \frac{r}{s}$.

$$X_1 = \left(\mathbb{C} \setminus \bigcup_{\frac{r}{s}} \mathcal{D}_{\frac{r}{s}}\right) / \sim_1 \ , \ \ X_2 = \left(\mathbb{C} \setminus \bigcup_{\frac{r}{s}} S \cdot \mathcal{D}_{\frac{r}{s}}\right) / \sim_2$$

- the non trivial $\sim_1$-classes consist of $\overline{\mathbb{D}}$ and for all $p \in [-\infty, +\infty]$, all $s \in \mathbb{N}^*$ the set $\{z_{\frac{r}{s}}(p) \mid r \in \{1, .., q-1\} \text{ s.t. } \gcd(r,s) = 1\}$.
- $\sim_2$ is defined by $z \sim_2 z'$ if and only if $S(z) \sim_1 S(z')$, where $S : z \mapsto \frac{1}{z}$.

The model for $\overline{\{\Gamma_z \mid z \in \mathbb{C} \setminus \overline{\mathbb{D}}\}}$

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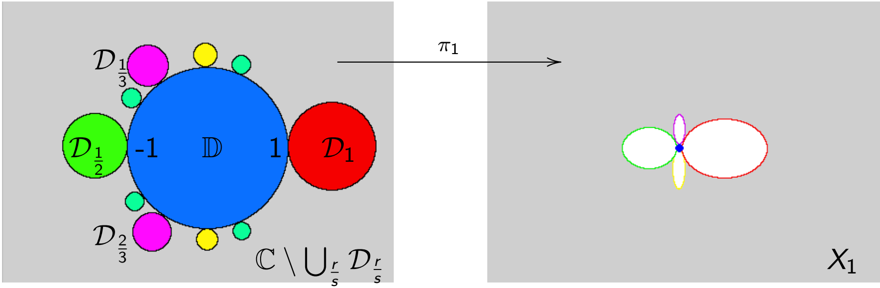

Topological model for $\overline{\{\Gamma_z \mid z \in \mathbb{C}^*\}} \subset \mathcal{SG}(\mathbb{C}^*)$

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Let X be the disjoint union of X_1, X_2 and $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ endowed with the discrete topology on $\mathbb{N}$ making ∞ its unique accumulation point. Then

Theorem

The topological space X is compact and homeomorphic to $\mathcal{SG}_1(\mathbb{C}^*) = \overline{\{\Gamma_z \mid z \in \mathbb{C}^*\}} \subset \mathcal{SG}(\mathbb{C}^*)$.

Let $\pi_1 : \mathbb{C} \to X_1$, $\pi_2 : \mathbb{C} \to X_2$ be the canonical projections and denote $\overline{S} : X \to X$ the involution induced by S.

Differents notions of convergence to $\mathbb{S}^1$

.

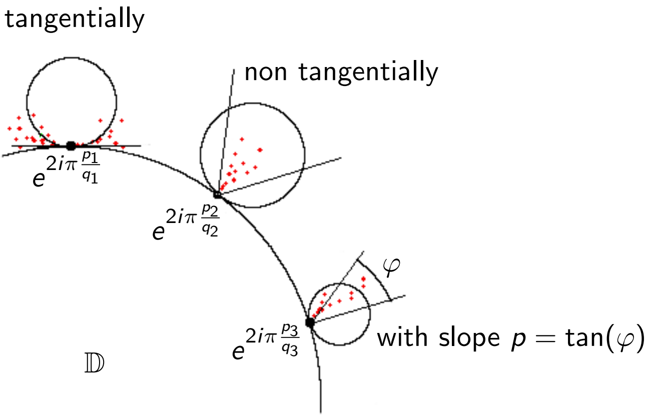
Definition

Let $(z_j = \rho_j e^{2i\pi\theta_j}) \subset \mathbb{C} \setminus \mathbb{S}^1$ *and* $e^{2i\pi\theta} \in \mathbb{S}^1$, *we say*

1. $|z_j| \to 1$ *with infinite slope w.r.t the rationals if* $\forall r \in \mathbb{Q}/\mathbb{Z}$ $\left(\frac{\theta_j - r}{\ln(\rho_j)}\right)$ *is unbounded.*
2. $z_j \to e^{2i\pi\theta}$ *tangentially if* $\left(\frac{\theta_j - \theta}{\ln(\rho_j)}\right)$ *is unbounded.*
3. $z_j \to e^{2i\pi\theta}$ *non tangentially if* $\left(\frac{\theta_j - \theta}{\ln(\rho_j)}\right)$ *is bounded.*
4. $z_j \to e^{2i\pi\theta}$ *with slope* $p \in \mathbb{R}$ *if* $\frac{\theta_j - \theta}{\ln(\rho_j)} \to p$.

Observation

Given any $|z_j| \to 1$ *up to a subsequence* (z_j) *falls in one of the 3 above cases.*


tangentially
non tangentially
$e^{2i\pi \frac{p_1}{q_1}}$
$e^{2i\pi \frac{p_2}{q_2}}$
$e^{2i\pi \frac{p_3}{q_3}}$
φ
with slope $p = \tan(\varphi)$
$\mathbb{D}$

Accumulation points of (Γ_{z_j})

. $K_{>1} = \{j \mid \rho_j > 1\}$, $K_{<1} = \{j \mid \rho_j < 1\}$, $K_{=1} = \{j \mid \rho_j = 1\}$.
$\mathcal{S}_p : \rho = \left(\frac{1}{p}\right)^{\theta}$ (logarithmic spiral based at 1).

Proposition (Possible accumulation points of (Γ_{z_j}))

1. *Either* $Acc(z_j) \cap \mathbb{S}^1 = \emptyset$ *then* $Acc(\Gamma_{z_j}) \subset \{\Gamma_z \mid z \in \mathbb{C} \setminus \mathbb{S}^1\}$.
2. *Either* $\exists e^{2i\pi\theta} \in Acc(z_j) \cap \mathbb{S}^1$ *and*
   - 2.1 *either* $|K_{<1} \cup K_{=1}| < \infty$ *and then* $\Gamma_{z_j} \to \mathbb{C} \setminus \mathbb{D}$ *iff* $|z_j| \to 1$ *with infinite slope w.r.t the rationals or* $\Gamma_{z_j} \to \left(\bigcup_{k=0}^{s-1} e^{2i\pi\frac{k}{s}}\mathcal{S}_p\right) \cap \mathbb{C} \setminus \mathbb{D}$ *iff* $Acc(z_j) \subset < e^{\frac{2i\pi}{s}} >$ *and all the limits accumulates with slope* $p \in \mathbb{R}$,

Accumulation points of (Γ_{z_j})

.

2.2 either $|K_{>1} \cup K_{=1}| < \infty$ and then
$\Gamma_{z_j} \to \overline{\mathbb{D}}$ or $\Gamma_{z_j} \to \left(\bigcup_{k=0}^{q-1} e^{2i\pi\frac{k}{q}}\mathcal{S}_p\right) \cap \overline{\mathbb{D}}$ (with symmetric cond.),

2.3 either $|K_{>1} \cup K_{<1}| < \infty$ and then $\Gamma_{z_j} \to \mathbb{S}^1$ iff $|z_j| \to 1$ with infinite slope w.r.t the rationals or $\Gamma_{z_j} \to < e^{\frac{2i\pi}{q}} >$ iff $|\{j \mid z_j \in < e^{2i\pi\frac{1}{q}} >\}| = \infty$.

In all other cases the sequence Γ_{z_j} does not converge ! But all the possible accumulation points are precisely those describe above.

Key Lemma

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Lemma

If $(z_j) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$ *and* $\exists \theta \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ *s.t* $e^{2i\pi\theta} \in \liminf \Gamma_{z_j}$*, then* $\Gamma_{z_j} \to \mathbb{C} \setminus \mathbb{D}$*.*

Démonstration.

1. $\limsup \Gamma_{z_j} \subset \mathbb{C} \setminus \mathbb{D}$,
2. $e^{2i\pi\theta} \in \liminf \Gamma_{z_j}$, thus $\mathbb{S}^1 \subset \liminf \Gamma_{z_j}$
3. Need to prove $[1, +\infty[\subset \liminf \Gamma_{z_j}$.
   Suppose $z_{j_k}^{n_k} = \left(\rho_{j_k} e^{2i\pi\theta_{j_k}}\right)^{n_k} \to e^{2i\pi\theta}$ and take $\rho \geqslant 1$.
   Then $\left(\rho_{j_k}^{n_k}\right)^{\left[\frac{\ln(\rho)}{\ln\left(\rho_{j_k}^{n_k}\right)}\right]} \to \rho$.

Definition of $\Phi : \mathcal{SG}_1(\mathbb{C}^*) \to X$

.

1. $\mathbb{C} \setminus \left(\bigcup_{\frac{r}{s}} \partial\mathcal{D}_{\frac{r}{s}}\right) \simeq^{\pi_1} \pi_1\left(\mathbb{C} \setminus \left(\bigcup_{\frac{r}{s}} \partial\mathcal{D}_{\frac{r}{s}}\right)\right) =: \mathrm{int}(X_1)$.
   $\mathbb{C} \setminus \left(\bigcup_{\frac{r}{s}} \partial\mathcal{D}_{\frac{r}{s}}\right) \simeq^{\varphi_1} \mathbb{C} \setminus \overline{\mathbb{D}}$ because $\overline{\mathbb{D}} \cup \left(\bigcup \overline{\mathcal{D}_{\frac{r}{s}}}\right)$ is comp., conn, loc conn and full (choose φ_1 tangent to id at ∞).
   $\mathbb{C} \setminus \overline{\mathbb{D}} \simeq^{\iota_1} \{\Gamma_z \mid z \in \mathbb{C} \setminus \overline{\mathbb{D}}\}$: $\iota_1 : z \mapsto \Gamma_z$ is cont. and $\iota_1^{-1}(z) = z_\Gamma$ where $|z_\Gamma| = \inf |\Gamma|$ is also cont.
   $$\Phi(\Gamma) := \pi_1 \circ \varphi_1^{-1} \circ \iota_1^{-1}(\Gamma).$$
2. $\Phi\left(\left(\bigcup_{k=0}^{q-1} e^{2i\pi\frac{k}{q}}\mathcal{S}_p\right) \cap \mathbb{C} \setminus \mathbb{D}\right) := \pi_1(z_{\frac{r}{s}}(p))$ for $p \in \mathbb{R}$.
3. $\Phi(\mathbb{C} \setminus \mathbb{D}) := p_1$ the point corresponding to the $\sim_1$-class of $\overline{\mathbb{D}}$.
4. $\Phi \circ \overline{S} = \overline{S} \circ \Phi$,
5. $\Phi(q) := < e^{\frac{2i\pi}{q}} >$ and $\Phi(\infty) = \mathbb{S}^1$.

Φ is a homeomorphism

.

Theorem

$\Phi : \mathcal{SG}_1(\mathbb{C}^*) \to X$ is a homeomorphism.

Démonstration.

1. $\Phi_| : \{\Gamma_z \mid z \in \mathbb{C} \setminus \overline{\mathbb{D}}\} \to \mathrm{int}(X_1)$ homeo ok
2. at $\left(\bigcup_{k=0}^{q-1} e^{2i\pi\frac{k}{q}}\mathcal{S}_p\right) \cap \mathbb{C} \setminus \mathbb{D}$ or $\left(\bigcup_{k=0}^{q-1} e^{2i\pi\frac{k}{q}}\mathcal{S}_p\right) \cap \overline{\mathbb{D}}$:
   2.1 on ∂X : slope moves continuously $=>$ spirals moves continuously in Hausdorff topology ok
   2.2 Γ_{z_j} cv to the spiral $<=>$ $z_j \to e^{2i\pi\frac{r}{s}}$ with slope $p \in \mathbb{R}$ $=>$ ok
3. $\Gamma_{z_j} \to p_i \in X_i$ $i = 1, 2$ $<=>$ $|z_j| \to 1$ with infinite slope w.r.t rationals $=>$ $\varphi_1^{-1}(z_j)$ enters all the neighbourhoods of $\mathbb{S}^1$ $=>$ $\Phi(\Gamma_{z_j}) \to p_i$.

Conformal dynamic in the sens of Douady-Epstein

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Definition

A conformal dynamic on $\mathbb{C}$ is a set $\mathcal{G} = \{(g, U) \mid U \subset \mathbb{C} \text{ open and } g : U \to \mathbb{C} \text{ holomorphic}\}$ which is closed under restrictions and compostions.

Let $\mathcal{P}oly_d$ be the space of monic centered polynomials of degre $d > 1$.

Conformal dynamic generated by a polynomial $f \in \mathcal{P}oly_d$

$$[f] = \{(f^n, U) \mid U \subset \mathbb{C} \ , \ n > 0\}.$$

Enrichments

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Definition

Let $g : U \to \mathbb{C}$ be holomorphic with $U \subset \mathbb{C}$ open. We say that (g, U) is an enrichment of the dynamic $[f]$ if for every connected component W of U there exists a sequence $((f_i^{n_i}, W_i)) \subset \prod [f_i]$, s.t.

1. *$f_i \to f$ uniformly on compacts sets,*
2. *$(f_i^{n_i}, W_i) \to (g, W)$ in the sens of Carathéodory.*

Observation

If $int(K(f)) = \emptyset$ then there are no enrichment of $[f]$.

Enrichments

.

Proposition

1. *For any enrichment* (g, U) *of* $[f]$ *there exists a unique enrichment of [f] defined on an* f*-stable open subset of* $int(K(f))$ *extending* (g, U).
2. *Any enrichment of* $[f]$ *defined on an* f*-stable open subset of* $int(K(f))$ *commutes with* f.

Démonstration.

$f_i \circ f_i^{n_i} = f_i^{n_i} \circ f_i$ cv unif on some compact sets $=> f \circ g = g \circ f$. □

Siegel Enrichments

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Let $f \in \mathcal{P}oly_d$ with a irrationnally indifferent periodic point a of period m and multiplier $e^{2i\pi\theta}$, where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a Brujno number. Let $\triangle$ be the Siegel disc with center a and $< \triangle >= \triangle \cup f(\triangle) \cup ... \cup f^{m-1}(\triangle)$ the cycle of Siegel discs. Let $\phi :< \triangle > \longrightarrow \mathbb{D}$ be the linearising coordinate and denote $\mathcal{U} = \bigcup_{n>0} f^{-n}(< \triangle >)$.

Δ-LLC maps

.

Definition

Let $U \subset \mathcal{U}$ open and $g : U \to <\triangle>$ be a holomorphic map. We say (g, U) is LLC (with respect to f and $<\triangle>$) if for any c.c. W of U and for (any !) $n \in \mathbb{N}$ s.t. $f^n(W) \subset <\triangle>$, the map $\phi \circ g \circ (\phi \circ f^n_{|W})^{-1}$ is linear.

Observation

(g, U) is LLC iff it is the restriction of some map defined in an f-stable domain of int$(K(f))$ that commutes with f (power series argument).

Siegel enrichments

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Theorem

The enrichments of $[f]$ with domain of definition in $\mathcal{U}$ are exactly the Δ-LLC maps.

PROOF :

First enrichment implies (eventually) commutes with f implies LLC.

Conversly Suppose $U \subset \mathcal{U}$ and (g, U) is LLC and let us work with a c.c W of U. Define $n_W := \min\{n > 0 \mid f^n(W) \subset < \triangle >\}$ and $\lambda_{g,W} := \widetilde{g}'(0)$ the derivative at 0 of the linear map induces by $g \circ (f^{n_W}_{|W})^{-1}$.

Siegel enrichments

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Lemma (cf accumulation points of (Γ_{z_j}))

$\exists(\lambda_i) \subset \mathbb{C}$ *and* $\exists(n_i) \subset \mathbb{N}$ *s.t*

1. $\lambda_i \longrightarrow e^{2i\pi\theta}$,
2. $\lambda_i^{n_i} \longrightarrow \lambda_{g,W}$.

Furtermore according to whether $|\lambda_g| \leqslant 1$ or $\geqslant 1$, we can choose $(\lambda_i) \subset \mathbb{D}$ or $\mathbb{C} - \mathbb{D}$. Let us suppose $(\lambda_i) \subset \mathbb{D}$.

By a standard Implicit Function Theorem argument, we can follow the Siegel cycle of f holomorphically.

i.e : $\exists \mathcal{W}_f \in \mathcal{N}(f) \subset \mathcal{P}oly_d$ and $\xi_f : W_f \longrightarrow \mathbb{C}$ s.t.

1. $h \mapsto (h^m)'(\xi_f(h))$ is holomorphic, non-constant (thus open !)
2. $\forall h \in \mathcal{W}_f \;\; \xi_f(h)$ is a periodic point of period m for h,
3. $\xi_f(f) = a$.

So we can choose $f_i \in \mathcal{W}_f$ in order to have $f_i \to f$ and $\xi_f(f_i) = \lambda_i$. Let $(\phi_i, \mathcal{D}om(\phi_i))$ be the linearizing coordinates of the attracting cycle $< \xi_f(f_i) >$ and its domain of definition.

Proposition

The linear coordinates $\phi_i : \mathcal{D}om(\phi_i) \to \mathbb{D}$ converge in the sens of Carathéodory to the linearizing map $\phi :< \triangle >\to \mathbb{D}$.

This implies that one can reduces the bifurcation $f_i \to f$ to $\Gamma_{\lambda_i} \to \mathbb{C} \setminus \mathbb{D}$.
$\lambda_i^{n_i} \to \lambda_{g,W} => \exists W_i \subset W$ open s.t $(f_i^{n_i} \circ f^{n_w}, W_i) \to (g, W)$ in the sens of Carathéodory. QED.

Let $f = e^{2i\pi\theta}z + z^2$ with $\theta \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ a Brunjo number, Δ the Siegel disc of 0 and denote

1. $[f]_{\mathbb{C}\setminus\mathbb{D}} := [f] \cup \{(g, U)\ \Delta\text{-LLC maps with } \lambda_{g,W} \in \mathbb{C} \setminus \mathbb{D}\}$,
2. $[f]_{\overline{\mathbb{D}}} := [f] \cup \{(g, U)\ \Delta\text{-LLC maps with } \lambda_{g,W} \in \overline{\mathbb{D}}\ \}$,
3. $[f]_{\mathbb{S}^1} := [f] \cup \{(g, U)\ \Delta\text{-LLC maps with } \lambda_{g,W} \in \mathbb{S}^1\ \}$.

Corollary

Let $(\lambda_n) \subset \mathbb{C}$ s.t. $\lambda_n \to e^{2i\pi\theta}$. Then for the geometric convergence topology on degree 2 polynomial dynamics the possible accumulation points of $[\lambda_n z + z^2]$ are :

$$[f]_{\mathbb{C}\setminus\mathbb{D}}, [f]_{\overline{\mathbb{D}}}\ or\ [f]_{\mathbb{S}^1}.$$

THANKS !