An application of Thurston's theorem on branched coverings

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Toulouse, November 22, 2010

Plan

Thuston's theorem on branched covering:
Characterization of rational maps in terms of growth condition on weighted pull-backs of simple closed curves.

(a branched covering not equivalent to a rational map if and only if it has a *Thurston obstruction*, which is a collection of s.c.c. with growth.)

Need to check for infinitely many collections of simple closed curves!

Levy cycle: a special type of obstruction which often can be detected by a finite combinatorial procedures.

Successful examples: Polynomials (Hubbard-Schleicher, Poirier), Matings of degree 2 (Tan Lei, Rees), Newton's method of degree 3 (Head, Tan Lei).

In this talk, we try to present an example which can be shown to have no Thurston obstruction without Levy cycle theorem.

Preparation for Thurston's theorem(1): Thurston equivalence

Definition. Suppose $f : S^2 \to S^2$ is a branched covering. We always assume that branched coverings in this paper are orientation preserving and of degree grater than one. Let

$$\Omega_f = \{\text{critical points of } f\} \text{ and } P_f = \bigcup_{n \geq 1} f^n(\Omega_f).$$

A branched covering f is called *postcritically finite*, if P_f is finite.

Two postcritically finite branched coverings f and g are *equivalent*, $f \sim g$, if there exist two orientation preserving homeomorphisms $\theta_1, \theta_2 : S^2 \to S^2$ such that

$\theta_i(P_f) = P_g \ (i = 1, 2)$, $\theta_1 = \theta_2$ on P_f, θ_1 and θ_2 are isotopic relative to P_f,

and the following diagram commutes:

$$\begin{array}{ccc} S^2 & \xrightarrow{\theta_1} & S^2 \\ f \big\downarrow & & \big\downarrow g \\ S^2 & \xrightarrow{\theta_2} & S^2. \end{array}$$

conjugate up to isotopy

Preparation for Thurston's theorem (2): Thurston matrix

Definition. Let $f : S^2 \to S^2$ be a postcritically finite branched covering. A simple closed curve in $S^2 - P_f$ is called *peripheral* if it bounds a disc containing at most one point of P_f. A *multicurve* Γ is a collection of disjoint simple closed curves in $S^2 - P_f$, such that none of them is peripheral and no two curves are homotopic to each other in $S^2 - P_f$. A multicurve Γ is f*-invariant*, if

$$f^{-1}(\Gamma) = \{\text{connected components of } f^{-1}(\gamma) | \gamma \in \Gamma\}$$

consists of peripheral curves and curves which are homotopic to curves in Γ.

For an f-invariant multicurve Γ, the *Thurston's linear transformation* f_Γ is a linear map from $\mathbb{R}^\Gamma = \{\sum_{\gamma \in \Gamma} c_\gamma \gamma | \ c_\gamma \in \mathbb{R}\}$ to itself defined by

$$f_\Gamma(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{deg(f : \gamma' \to \gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma,$$

weighted pull-back of s.c.c

where the sum is over all non-peripheral components γ' of $f^{-1}(\gamma)$ and $[\gamma']_\Gamma$ denotes the curve in Γ homotopic to γ', if there is one, otherwise $[\gamma']_\Gamma = 0$. We denote by λ_Γ the leading eigenvalue of f_Γ.

Thurston's theorem

Theorem (Thurston)**.** *Suppose $f : S^2 \to S^2$ is a postcritically finite branched covering with a hyperbolic orbifold. Then f is equivalent to a rational map, if and only if there is no f-invariant multicurve Γ with $\lambda_\Gamma \geq 1$.*

Remark. The definition of *hyperbolic orbifold* is omitted. If the orbifold is not hyperbolic, then $f^{-1}(P_f) \subset \Omega_f \cup P_f$ and $\#P_f \leq 4$. Therefore brached coverings with non-hyperbolic orbifolds are considered to be exceptional.

Definition. An f-invariant curve Γ with $\lambda_\Gamma \geq 1$ is called a *Thurston obstruction.*

a collection of s.c.c. which grows under weighted pull-backs

The proof of Thurston's theorem is given by looking at the action of f on the Teichmüller space of $S^2 \smallsetminus P_f$:

$$\begin{aligned} Teich(S^2 \smallsetminus P_f) &= \{ \text{ conformal structures on } S^2 \smallsetminus P_f \text{ with marking } \}/\sim \\ &= \{\varphi : S^2 \to \widehat{\mathbb{C}}\}/\sim_{\text{Möbius+isotopy rel } P_f} \end{aligned}$$

The pull-back f^* acts on $Teich(S^2 \smallsetminus P_f)$. f is Thurston equivalent to a rational map if and only if f^* has a fixed point in $Teich(S^2 \smallsetminus P_f)$.

Applications of Thurston's Theorem

From a given dynamical information, branched coverings are easier to construct than rational maps.

On the other hand, in order to use Thurston's theorem to obtain a rational map, one has to check the condition for eigenvalues for *infinitely many* multicurves.

So it will be nice to reduce the criterion to a finitely checkable conditions.

Definition. A multicurve $\gamma_1, ..., \gamma_n$ is called a *Levy cycle*, if each $f^{-1}(\gamma_{i+1})$ contains a component γ_i' homotopic to γ_i and $f : \gamma_i' \to \gamma_{i+1}$ is of degree one ($i = 0, ..., n-1$), where $\gamma_0 = \gamma_n$. Any Levy cycle is contained in a Thurston obstruction.

Theorem (Levy, Rees?). *For a topological polynomial f (i.e. $f^{-1}(\infty) = \{\infty\}$) or a branched covering f of degree 2, f has a Thurston obstruction if and only if it has a Levy cycle.*

Levy cycles are much easier to detect combinatorially.

Successful examples: Polynomials (Hubbard-Schleicher, Poirier), Matings of degree 2 (Tan Lei, Rees), Newton's method of degree 3 (Head, Tan Lei).

More general cases?

However this Levy cycle theorem does not hold for branched coverings in general.

Theorem (S.-Tan). *There exists a mating of cubic polynomials such that it has a Thurston obstruction, but has no Levy cycle.*

In this talk, we try to present an example which can be shown to have no Thurston obstruction (hence equivalent to a rational map) without using Levy cycle theorem.

The example will be constructed by a plumbing construction from a tree and a piecewise linear map on it. So it has a stable multicurve which is not a Thurston obstruction.

This non-obstruction actually helps up to conclude that there is no Thurston obstruction.

Key tools are geometric intersection number of curves and unweighted and effective Thurston matrices (or operators).

Geometric intersection number

Definition. Let α and β be non-peripheral simple closed curves in $S^2 \smallsetminus P_f$. Define the geometric intersection number to be

$$\alpha \cdot \beta = \min\{\#(\alpha' \cap \beta') | \alpha' \sim \alpha, \beta' \sim \beta\},$$

where the minimum is always attained (for example by hyperbolic geodesics in the homotopy classes). Obviously this number can also be defined for the homotopy classes of simple closed curves, and naturally extends bilinearly to $\mathbb{R}^{\underline{\alpha}} \times \mathbb{R}^{\underline{\beta}}$ for multicurves. $\underline{\alpha}, \underline{\beta}$ Instead of simple closed curves, one can take one of α and β to be simple arcs in $S^2 \smallsetminus P_f$ joining points of P_f.

Lemma. *Let α and β be non-peripheral simple closed curves in $S^2 \smallsetminus P_f$. Let α' be a connected component of $f^{-1}(\alpha)$ such that $f : \alpha' \to \alpha$ is a covering of degree k. Then we have*

$$\alpha' \cdot f^{-1}(\beta) \leq k\alpha \cdot \beta.$$

Definition (Unweighted Thurston matrix and μ_Γ)**.** Let us define the unweighted Thurston operator $f_\Gamma^\#$ by

$$f_\Gamma^\#(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma.$$

Denote the leading eigenvalue of $f_\Gamma^\#$ by μ_Γ.

Remark. It is obvious from the definition that $\lambda_\Gamma \leq \mu_\Gamma$.

Definition (Reduced multicurve)**.** An invariant multicurve Γ is called *reduced* if all the coefficients of the eigenvector of Thurston operator are positive. From any invariant multicurve, one can extract a reduced with the same eigenvalue.

Theorem. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\lambda_{\underline{\alpha}}\, \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

Theorem. *Let $\underline{\beta}$ be reduced invariant multicurve and $\underline{\alpha}$ a Levy cycle (or a simple cycle or arcs joining points in P_f) for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\mu_{\underline{\beta}} \leq 1.$$

In particular, either $\underline{\beta}$ is not a Thurston obstruction, or it contains a Levy cycle. (Head, S.-Tan, Pilgrim-Tan)

Proof. Let $u_{\underline{\alpha}}$, $v_{\underline{\beta}}$ be positive eigenvectors for $f_{\underline{\alpha}}$ and $f^{\#}_{\underline{\beta}}$, hence $f_{\underline{\alpha}}(u_{\underline{\alpha}}) = \lambda_{\underline{\alpha}} u_{\underline{\alpha}}$ and $f^{\#}_{\underline{\beta}}(v_{\underline{\beta}}) = \mu_{\underline{\beta}} v_{\underline{\beta}}$ Lemma 5 applied to f^n implies that (note that $P_{f^n} = P_f$) for each component $\alpha' \subset f^{-n}(\alpha)$, we have

$$\alpha' \cdot f^{-n}(\beta) \leq \deg(f^n : \alpha' \to \alpha)\alpha \cdot \beta.$$

Hence

$$\mu^n_{\underline{\beta}} \frac{1}{\deg(f^n : \alpha' \to \alpha)} \alpha' \cdot v_{\underline{\beta}} \leq \alpha \cdot v_{\underline{\beta}}.$$

Now denote N_n be the maximum number of non-peripheral components of $f^{-n}(\alpha)$ for $\alpha \in \underline{\alpha}$. By multiplying the coefficients of $u_{\underline{\alpha}}$ and adding (**??**) for all components $\alpha' \subset f^{-n}(\alpha)$ and $\alpha \in \underline{\alpha}$, we obtain

$$\lambda^n_{\underline{\alpha}} \mu^n_{\underline{\beta}} u_{\underline{\alpha}} \cdot v_{\underline{\beta}} \leq N_n u_{\underline{\alpha}} \cdot v_{\underline{\beta}}.$$

By Perron-Frobenius Theorem, we have $N_n \leq C\mu^n_{\underline{\alpha}}$ for some $C > 0$. Hence $\lambda^n_{\underline{\alpha}} \mu^n_{\underline{\beta}} \leq N_n \leq C\mu^n_{\underline{\alpha}}$. Taking n-th root and the limit, we conclude that

$$\lambda_{\underline{\alpha}} \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

$\square$

Decomposition/Construction of branched coverings to/from tree maps

Theorem. *For a reduced invariant multicurve* Γ*, there exist a finite* $\mathbb{R}$*-tree* $T = T_\Gamma$ *and a piecewise linear map* $F = F_\Gamma : T \to T$ *such that*

- *each edge of* T *corresponds to a curve in* Γ*;*
- *each vertex if* T *corresponds to a connected component of* $S^2 \setminus \Gamma$*;*
- *each edge decomposes to sub-edges corresponding to non-peripheral component* γ' *of* $f^{-1}(\Gamma)$*,* F *maps this sub-edge to the edge corresponding to* $f(\gamma')$ *with linear factor* $\lambda_\Gamma \deg F$*, where* $\deg F$ *is integer=values function whose value on this sub-edge is the degree of* f *on* γ'*.*

Theorem. *To the vertices (and sub-vertices)* x *of* T*, one can associate a copy* S^2_x *of 2-sphere, and a branched covering* $g_x : S^2_x \to S^2_{F(x)}$ *such that*

- *each* S^2_x *has marked points corresponding to edges emanating from* x*;*
- *the local degree of* g_x *at a marked point is equal to* $\deg F$ *on the corresponding edge;*
- *The collection* $\{g_x\}$ *is postcritically finite.*

Definition. A reduced Thurston obstruction is called *intersecting* if it intersects with another Thurston obstruction. Otherwise it is called *non-intersecting.*

With a little more work in the proof of Thurston's theorem, one can show that whenever there is a Thurston obstruction, there exists a non-intersecting Thurston obstruction.

Theorem. *A branched covering decomposes into a tree map with* $\lambda \geq 1$ *corresponding to all non-intersecting Thurston obstructions and local models* $g_x : S^2_x \to S^2_{F(x)}$ *such that periodic part of local models are equivalent to rational maps, homeomorphisms, or branched coverings with non-hyperbolic orbifolds. The intersecting obstructions should come from pseudo-Anosov homeomorphisims or Lattès maps. (cf. Pilgrim's canonical decomposition)*

Conversely, from a tree map and a collection of local models one can construct a branched covering. There are some ambiguities on Dehn twists along associated curves and some of postcritical orbits. In fact, the first non-Levy cycle obstruction in S.-Tan was constructed this way.

Definition. Let α be a non-peripheral simple closed curve in $S^2 \smallsetminus P_f$. Let α' be a connected component of $f^{-1}(\alpha)$. The *effective degree* eff-deg$(f : \alpha' \to \alpha)$ is the smallest $k \geq 1$ such that for any non-peripheral simple closed curve β in $S^2 \smallsetminus P_f$, the following holds:

$$\alpha' \cdot f^{-1}(\beta) \leq k\alpha \cdot \beta.$$

Example. Suppose α and $\alpha'(\subset f^{-1}(\alpha))$ bound disks D_1 and D_0 such that $f(D_0) = D_1$ and f has only one critical point ω in D_0. If $P_f \cap D_1 = \{f(\omega), y\}$ and $\#(P_f \cap f^{-1}(y)) = k$, then

$$\text{eff-deg}(f : \alpha' \to \alpha) \leq k.$$

Definition (Effective Thurston matrix and μ_Γ)**.** Let us define the effective Thurston operator $f_\Gamma^{\$}$ by

$$f_\Gamma^{\$}(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\text{eff-deg}(f : \gamma' \to \gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma.$$

Denote the leading eigenvalue of $f_\Gamma^{\$}$ by ν_Γ.

Remark. It is obvious from the definition that

$$1 \leq \text{eff-deg}(f : \gamma' \to \gamma) \leq \deg(f : \gamma' \to \gamma) \text{ and } \lambda_\Gamma \leq \nu_\Gamma \leq \mu_\Gamma.$$

Thurston effective unweighted

As before one can prove:

Theorem. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\nu_{\underline{\alpha}}\, \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

Corollary. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\nu_{\underline{\alpha}}\, \nu_{\underline{\beta}} \leq 1.$$

Since $\lambda_{\underline{\beta}} \leq \nu_{\underline{\beta}}$, we have

Theorem. *Let $\underline{\alpha}$ be a reduced invariant multicurve for f such that $(\lambda_{\underline{\alpha}} <) 1 < \nu_{\underline{\alpha}}$. Then f has no Thurston obstruction intersecting $\underline{\alpha}$.*

If a branched covering is constructed from a tree map $F : T \to T$ with $\lambda < 1$ but with the effective eigenvalue $\nu_F > 1$ and the local models are rational maps, then it has no Thurston obstruction, hence is equivalent to a rational map.

Merci!