Germs of analytic families of diffeomorphisms unfolding a parabolic point (I)

Christiane Rousseau

Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier

Structure of the mini-course

- Statement of the problem (first lecture)
- The preparation of the family (first lecture)
- Construction of a modulus of analytic classification in the codimension 1 case (second lecture)
- The realization problem in the codimension 1 case (third lecture)

We consider germs of generic k-parameter families f_ϵ of diffeomorphisms unfolding a parabolic point of codimension k

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

When are two such germs conjugate?

Conjugacy of two germs of families

Two germs of families of diffeomorphisms f_ϵ and $\tilde{f}_{\tilde{\epsilon}}$ are conjugate it there exists $r, \rho > 0$ and analytic functions

$$h : \mathbb{D}_\rho \to \mathbb{C}, \qquad H : \mathbb{D}_r \times \mathbb{D}_\rho \to \mathbb{C}$$

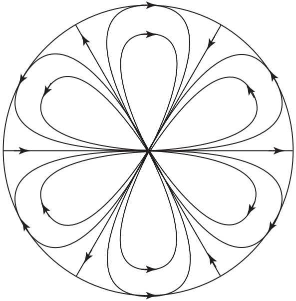
such that

- h is a diffeomorphism and for each fixed ϵ, $H_\epsilon = H(\cdot, \epsilon)$ is a diffeomorphism;
- for all $\epsilon \in \mathbb{D}_\rho$ and for all $z \in \mathbb{D}_r$, then

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ (H_\epsilon)^{-1}$$

The choice of $\mathbb{D}_r$

$\mathbb{D}_r$ is chosen so that the behaviour of f_0 on the boundary is as in



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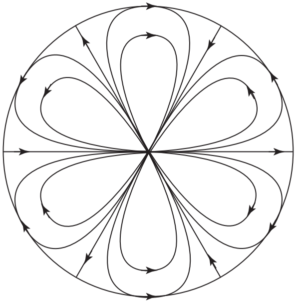

$\mathbb{D}_\rho$ is chosen sufficiently small so that f_ϵ has the same behaviour near the boundary. In particular, all fixed points of f_ϵ remain inside the disk.

A natural strategy: the use of normal forms

A germ of generic k-parameter family f_ϵ unfolding a parabolic point of codimension k is formally conjugate to the time-1 map of a vector field

$$v_\epsilon = \frac{P_\epsilon(z)}{1 + a(\epsilon)z^k}\frac{\partial}{\partial z}$$

where

$$P_\epsilon(z) = z^{k+1} + \epsilon_{k-1}z^{k-1} + \cdots + \epsilon_1 z + \epsilon_0$$

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Problem: the change to normal form diverges.
What does it mean?

Can we exploit the formal normal form despite its divergence?

Let us look at the case $k = 1$:

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Two singular points $\pm\sqrt{\epsilon}$ with eigenvalues

$$\mu_\pm = \frac{\pm 2\sqrt{\epsilon}}{1 \pm a(\epsilon)\sqrt{\epsilon}}$$

The parameter is an analytic invariant of the vector field!

Indeed, we have

$$\frac{1}{\mu_+} + \frac{1}{\mu_-} = a(\epsilon)$$
$$\frac{1}{\mu_+} - \frac{1}{\mu_-} = \frac{1}{\sqrt{\epsilon}}$$

Hence, can we hope to bring the system to a "prenormal" form in which the parameter is invariant?

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Advantage: a conjugacy between prepared families must preserve the *canonical* parameters.

Theorem

We consider a diffeomorphism with a parabolic point of codimension k:

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

For any generic k-parameter unfolding f_η, there exists an analytic change of coordinate and parameter $(z,\eta) \mapsto (Z,\epsilon)$ in a neighborhood of the origin transforming the family into the prepared form

$$F_\epsilon(Z) = Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)K(Z,\epsilon))$$

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$$F_\epsilon(Z) = Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)K(Z,\epsilon))$$

such that, if $Z_1, \dots Z_{k+1}$ *are the fixed points, then*

$$F'_\epsilon(Z_j) = \exp\left(\frac{P'_\epsilon(Z_j)}{1 + a(\epsilon)Z_j^k}\right)$$

This determines almost uniquely the parameters!

The only freedom will be inherited from a rotation of order k in Z

$$Z \mapsto \tau Z; \qquad \tau^k = 1$$

which yields the corresponding change on ϵ:

$$(\epsilon_{k-1}, \epsilon_{k-2} \dots, \epsilon_0) \mapsto (\tau^{2-k}\epsilon_{k-1}, \tau^{1-k}\epsilon_{k-2}, \dots, \tau\epsilon_0)$$

Proof of the theorem

We consider a diffeomorphism with a parabolic point of codimension k:

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

A k-parameter unfolding can be written in the form

$$f_\eta(z) = z + p_\eta(z) g_\eta(z),$$

with $g_\eta(z) = 1 + O(\eta, z)$.

Using the Weierstrass division theorem on the rest allows to write f_η in the form

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$$q_\eta(z) = c_0(\eta) + c_1(\eta)z + \cdots + c_k(\eta)z^k.$$

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Genericity condition: the Jacobian

$$\frac{\partial \nu}{\partial \eta}$$

is invertible.

Since

$$f_\eta(z) = z + p_\eta(z)(1 + q_\eta(z) + p_\eta(z)h_\eta(z))$$

the fixed points z_j of f_η are the zeroes of p_η.

The strategy

The formal normal form is the time one map of a vector field

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Hence the the fixed points of f_η must be sent to the singular points Z_j of V_ϵ.
Moreover we need have

$$f_\eta'(z_j) = \exp(V_\epsilon'(Z_j))$$

How do we find the formal invariant $a(\epsilon)$?

Let

$$\lambda_j = f_\eta'(z_j)$$

We have that

$$\sum 1/\ln(\lambda_j) = a(\epsilon).$$

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There exists a polynomial $r_\eta(z)$ of degree $\leq k$ such that at the points z_j we have

$$\ln\left(f_\eta'(z_j)\right) = p_\eta'(z_j)(1 + r_\eta(z_j)).$$

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(Such a polynomial is found by the Lagrange interpolation formula for distinct z_j. The limit exists when two fixed points coallesce (codimension 1 case). We can fill in for the other values of η by Hartogs's Theorem.)

The reparameterization

By Kostov theorem, there exists a change of coordinate and parameter transforming the vector field:

$$p_\eta(z)(1+r_\eta(z))\frac{\partial}{\partial z} = v_\eta(z)$$

into:

$$P_\epsilon(Z)/(1+a(\epsilon)Z^k)\frac{\partial}{\partial Z} = V_\epsilon(Z),$$

where

$$P_\epsilon(Z) = Z^{k+1} + \epsilon_{k_1}Z^{k-1} + \epsilon_1 Z + \epsilon_0.$$

We apply this change of coordinate and parameter to f_η.

Claim: this brings f_η to a prepared form F_ϵ

- It sends the zeros z_j of $p_\eta(z)$ to the zeroes of $P_\epsilon(Z)$. Since the z_j are the fixed points of f_η, their images are the fixed points Z_j of F_ϵ.

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- Hence

$$\begin{aligned} F_\epsilon(Z) &= Z + P_\epsilon(Z)K_\epsilon(Z) \\ &= Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)H_\epsilon(Z)) \end{aligned}$$

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- Let be a fixed point. Then

$$F'_\epsilon(Z_j) = \lambda_j = f'_\eta(z_j) = \exp(v'_\eta(z_j)) = \exp(V'_\epsilon(Z_j))$$

which is what we need for a prepared family.

The parameters are (almost) canonical

We have

$$F_\epsilon(Z) = Z + P_\epsilon(Z)K_\epsilon(Z)$$
$$= Z + P_\epsilon(Z)(1 + Q_\epsilon(Z) + P_\epsilon(Z)H_\epsilon(Z))$$

Claim: P_ϵ, Q_ϵ and ϵ are unique up to the change

$$Z \mapsto \tau Z; \qquad \tau^k = 1$$

and the corresponding change on ϵ:

$$(Z, \epsilon_{k-1}, \epsilon_{k-2} \dots, \epsilon_0) \mapsto (\tau Z, \tau^{2-k}\epsilon_{k-1}, \tau^{1-k}\epsilon_{k-2}, \dots, \tau\epsilon_0)$$

The proof

Let us suppose that two prepared families $f_\epsilon(z)$ and $\tilde{f}_{\tilde{\epsilon}}(\tilde{z})$ are conjugate under a map $(\tilde{\epsilon},\tilde{z}) = (h(\epsilon), H_\epsilon(z))$:

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ H_\epsilon^{-1}$$

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f_ϵ

Fixed points z_j are those of $v_\epsilon(z) = P_\epsilon(z)/(1+az^k)\frac{\partial}{\partial z}$

$\tilde{f}_{\tilde{\epsilon}}$

Fixed points $\tilde{z}_j$ are those of $\tilde{v}_{\tilde{\epsilon}}(\tilde{z}) = \tilde{P}_{\tilde{\epsilon}}(\tilde{z})/(1+a\tilde{z}^k)\frac{\partial}{\partial \tilde{z}}$

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Note that the formal invariants are the same.

Then H_ϵ sends the fixed points z_j to the fixed points $\tilde{z}_j$. Hence

$$H_\epsilon^*(\tilde{v}_{h(\epsilon)})(z) = P_\epsilon(z) U_\epsilon(z) \frac{\partial}{\partial z} = w_\epsilon(z)$$

where $U \neq 0$.

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where $U \neq 0$.

v_ϵ and w_ϵ have the same singular points with same eigenvalues! Hence

$$w_\epsilon = P_\epsilon(z)\left(\frac{1}{1+az^k} + P_\epsilon(z)M_\epsilon(z)\right)\frac{\partial}{\partial z}$$
$$= v_\epsilon(1+P_\epsilon(z)N_\epsilon(z))\frac{\partial}{\partial z}.$$

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Then $(K_\epsilon^{-1} \circ H_\epsilon)^*(\tilde{v}_{h(\epsilon)}) = v_\epsilon$. The result follows from the following theorem proved with L. Teyssier.

Theorem (RT)

We consider a germ of an analytic change of coordinates $\Psi : (z,\epsilon) = (z,\epsilon_0,\ldots,\epsilon_{k-1}) \mapsto (\varphi_\epsilon(z), h_0(\epsilon),\ldots,h_{k-1}(\epsilon)) = (z,h)$ *at* $(0,0,\cdots,0) \in \mathbb{C}^{1+k}$*. The following assertions are equivalent :*

1. *the families* $\left(\frac{P_\epsilon(z)}{1+a(\epsilon)z^k}\frac{\partial}{\partial z}\right)_\epsilon$ *and* $\left(\frac{P_h(z)}{1+\tilde{a}(h)z^k}\frac{\partial}{\partial z}\right)_h$ *are conjugate under* Ψ*,*
2. *there exist* τ *with* $\tau^k = 1$ *and* $T(\epsilon)$ *an analytic germ such that, if* $R_\tau(z) = \tau z$
   - $\varphi_\epsilon(z) = \Phi_{v_\epsilon}^{T(\epsilon)} \circ R_\tau(z)$
   - $\epsilon_j = \tau^{j-1} h_j(\epsilon)$,
   - $a(\epsilon) = \tilde{a}(h(\epsilon))$.

Reduction to the case $\tau = 1$

If $\varphi_0'(0) = \tau$ we need have $\tau^k = 1$ in order to preserve the form of v_0.

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So we can compose $\Psi(z, \epsilon)$ with R_τ and the corresponding change of parameters $\epsilon_j = \tau^{j-1} h_j(\epsilon)$ and only discuss the composed family.

Hence we can suppose that $\Psi(z, \epsilon)$ is such that $\varphi_0'(0) = 1$.

The case $\epsilon = 0$

It is easy to check that the only changes of coordinates tangent to the identity which preserve v_0 are the maps $\Phi_{v_0}^t$.

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Indeed, such changes of coordinates have the form $z(1+m_t(z^k))$ with $m_t(z) = tz^k + o(z^k)$. The function $m_t(z)$ is completely determined by $m_t'(0) = t$. This is exactly the form of the family $\Phi_{v_0}^t$.

Reduction to the case $\frac{\partial^{k+1}\varphi_\epsilon}{\partial z^{k+1}}(0)=0$

We correct φ to

$$G(z,t,\epsilon):=\Phi^t_{v_\epsilon}\circ\varphi_\epsilon(z)$$

with $t(\epsilon)$ well chosen.

Reduction to the case $\frac{\partial^{k+1}\varphi_\epsilon}{\partial z^{k+1}}(0) = 0$

We correct φ to

$$G(z,t,\epsilon) := \Phi^t_{v_\epsilon} \circ \varphi_\epsilon(z)$$

with $t(\epsilon)$ well chosen.

Let

$$H(z,t,\epsilon) := \frac{\partial^{k+1}G}{\partial z^{k+1}}(z,t,\epsilon)$$

$$K(t,\epsilon) := H(0,t,\epsilon)$$

K is analytic and

$$\frac{\partial K}{\partial t}(0,0) = (k+1)! \neq 0.$$

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K is analytic and

$$\frac{\partial K}{\partial t}(0,0) = (k+1)! \neq 0.$$

Let t_0 be such that $K(t_0,0) = 0$. By the implicit function theorem, there exists $t(\epsilon)$ unique such that $t(0) = t_0$ and $K(t(\epsilon),\epsilon) \equiv 0$.

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$$K(t,\epsilon) := H(0,t,\epsilon)$$

K is analytic and

$$\frac{\partial K}{\partial t}(0,0) = (k+1)! \neq 0.$$

Let t_0 be such that $K(t_0,0) = 0$. By the implicit function theorem, there exists $t(\epsilon)$ unique such that $t(0) = t_0$ and $K(t(\epsilon),\epsilon) \equiv 0$.

Composing φ_ϵ with $\Phi^{t(\epsilon)}_{X_\epsilon}$ we can suppose that the original family Ψ is such that $\frac{\partial^{k+1}\varphi_\epsilon}{\partial z^{k+1}}(0) = 0$.

The rest of the argument is an infinite descent

We introduce the ideal

$$I = \langle \epsilon_0, \ldots, \epsilon_{k-1} \rangle.$$

We have

$$\varphi_\epsilon(z) := z + \sum_{n \geq 0} f_n(\epsilon) z^n$$

where $f_n \in I$ and $f_{k+1} \equiv 0$.
We must solve

$$\left(1 + a(\epsilon) z^k\right)\left(\varphi_\epsilon^{k+1}(z) + h_{k-1}\varphi_\epsilon^{k-1}(z) + \cdots + h_0\right) \\ - \left(1 + \tilde{a}(h)\varphi_\epsilon^k(z)\right)\left(z^{k+1} + \epsilon_{k-1} z^{k-1} + \cdots + \epsilon_0\right)\varphi_\epsilon'(z) = 0.$$

It is then clear that $h_j(\epsilon) \in I$ and $f_j(\epsilon) \in I$.

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We have

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where $f_n \in I$ and $f_{k+1} \equiv 0$.
We must solve

$$\left(1 + a(\epsilon) z^k\right)\left(\varphi_\epsilon^{k+1}(z) + h_{k-1}\varphi_\epsilon^{k-1}(z) + \cdots + h_0\right) \\ - \left(1 + \tilde{a}(h)\varphi_\epsilon^k(z)\right)\left(z^{k+1} + \epsilon_{k-1}z^{k-1} + \cdots + \epsilon_0\right)\varphi'_\epsilon(z) = 0.$$

It is then clear that $h_j(\epsilon) \in I$ and $f_j(\epsilon) \in I$.

Let $g_j z^j$ be the term of degree j. We will play with the infinite set of equations $g_j = 0$, $j \geq 0$.

The equations $g_j = 0$ with $0 \leq j \leq k-1$ yield

$$h_j - \epsilon_j \in I^2,$$

since all other terms in the expression of g_j belong to I^2.

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The equation $g_{k+j} = 0$ with $0 \leq j \leq k$ yields $f_j \in I^2$.

Looking at the linear terms in the equations $g_\ell = 0$ with $\ell > 2k+1$ yields $f_{\ell-k} \in I^2$.
So we have that $f_j \in I^2$ for all j.

The general step by induction

We suppose that $h_j - \epsilon_j \in I^n$ when $0 \leq j \leq k-1$ and $f_j \in I^n$ whenever $j \geq 0$.

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To show that $h_j - \epsilon_j \in I^{n+1}$ for $0 \le j \le k-1$ we consider again the corresponding equations $g_j = 0$, where the only linear terms are $h_j - \epsilon_j$. Hence all other terms of the equation belong to I^{n+1} yielding $h_j - \epsilon_j \in I^{n+1}$.

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We suppose that $h_j - \epsilon_j \in I^n$ when $0 \leq j \leq k-1$ and $f_j \in I^n$ whenever $j \geq 0$.

To show that $h_j - \epsilon_j \in I^{n+1}$ for $0 \leq j \leq k-1$ we consider again the corresponding equations $g_j = 0$, where the only linear terms are $h_j - \epsilon_j$. Hence all other terms of the equation belong to I^{n+1} yielding $h_j - \epsilon_j \in I^{n+1}$.

For the same reason the equation $g_{k+j} = 0$ with $0 \leq j \leq k$ yields $f_j \in I^{n+1}$ and the equations $g_\ell = 0$ with $\ell > 2k+1$ yields $f_{\ell-k} \in I^{n+1}$.