Germs of analytic families of diffeomorphisms unfolding a parabolic point (II)

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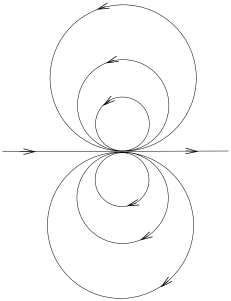
Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier

Structure of the mini-course

- Statement of the problem (first lecture)
- The preparation of the family (first lecture)
- Construction of a modulus of analytic classification in the codimension 1 case (second lecture)
- The realization problem in the codimension 1 case (third lecture)

The parabolic germ



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Comparing is constructing a change of coordinates to normal form.

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So we cover $\mathbb{D}_r$ with two sectors $U_\pm$. Over each sector the comparison is almost unique (up to a symmetry of the model, which is a time t map of v_ϵ.)

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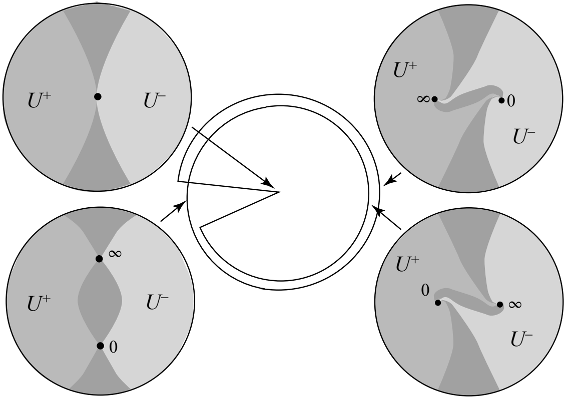
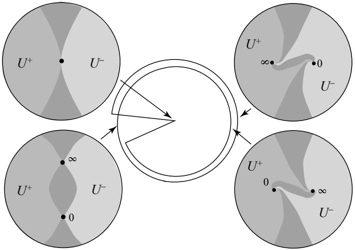
The modulus is given by the comparison of the two normalizations over $U_\cap = U_+ \cap U_-$. It is a symmetry of the model. (Because $U_\cap$ is small, there are many more symmetries .)

The choice of the sectors $U_\pm$



The underlying idea



The dynamics is transversal to the inner part of the boundary of the sectors (except at the fixed point).

- It goes from U_+ to U_- on the part not joining the fixed points.
- It goes from U_- to U_+ on the part joining the fixed points.

The choice of the sector in ϵ

We work with a ramified covering of a neighborhood $\mathbb{D}_\rho$ of the origin in ϵ-space.

$$V_\delta = \{\hat{\epsilon} : |\hat{\epsilon}| < \rho, \arg \hat{\epsilon} \in (-\pi + \delta, 3\pi - \delta)\}$$

for $\delta \in (0, \pi)$.

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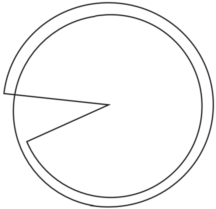

δ can be taken arbitrarily small. The smaller δ, the smaller ρ. In particular, the opening is always smaller than 4π.

The lift to the time coordinate

In practice we prefer to work with the time coordinate Z of the vector field

$$w_\epsilon = (z^2 - \epsilon)\frac{\partial}{\partial z},$$

namely we make the multivalued change of coordinate

$$Z = p_\epsilon^{-1}(z) = \begin{cases} \frac{1}{2\sqrt{\epsilon}} \ln \frac{z-\sqrt{\epsilon}}{z+\sqrt{\epsilon}} & \epsilon \neq 0 \\ -\frac{1}{z} & \epsilon = 0 \end{cases}$$

with period $\alpha = \frac{\pi i}{\sqrt{\hat{\epsilon}}}$ when $\epsilon \neq 0$.

The underlying idea

If we allow complex time, then all points $z \in \mathbb{D}_r$ are in the trajectory of a unique point z_0. Hence, $z = \phi_{w_0}^{Z}(z_0)$ for some Z.

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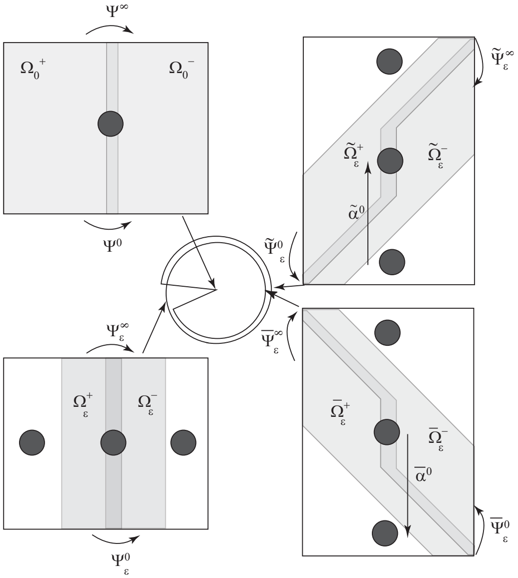
The boundaries of the spiral sectors come from lines in Z-space.

The sectors in Z-space



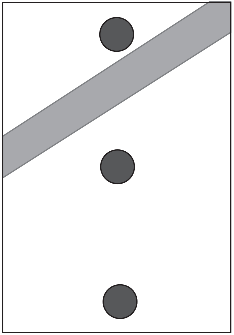
The Fatou coordinates

Let $\Omega^{\pm}_{\hat{\epsilon}}$ be the sectors in Z-space. We construct Fatou coordinates $\Phi^{\pm}_{\hat{\epsilon}} : \Omega^{\pm}_{\hat{\epsilon}} \to \mathbb{C}$ which conjugate F_ϵ (the lifting of f_ϵ in Z-coordinate) to T_1 the translation by 1:

$$\Phi^{\pm}_{\hat{\epsilon}} \circ F_\epsilon = T_1 \circ \Phi_{\hat{\epsilon}}$$

The construction of Fatou coordinates

We take a slanted line ℓ in $\Omega^{\pm}_{\hat{\epsilon}}$ so that $\ell \cap F_{\epsilon}(\ell) = \emptyset$, and so that the strip S between ℓ and $F(\ell)$ is included in $\Omega^{\pm}_{\hat{\epsilon}}$.



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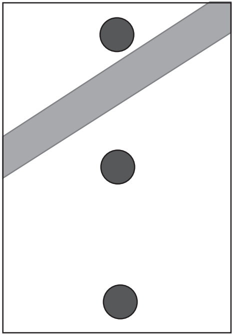
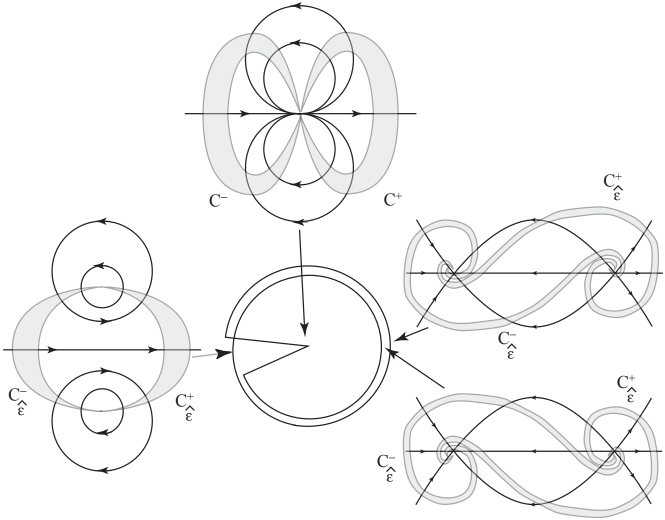


We construct $\phi : S \to \mathbb{C}$ conjugating F_{ϵ} with T_1 by linear interpolation. We extend ϕ. The map ϕ is quasi-conformal. We correct ϕ to a conformal map by Ahlfors-Bers theorem.

The strips are fundamental domains in z-space



Dependence of the Fatou coordinates on $\hat{\epsilon}$

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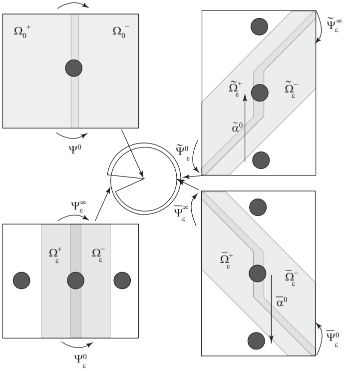
In particular, they are uniquely defined by a base point such that

$$\Phi_{\hat{\epsilon}}^{\pm}(Z(\hat{\epsilon})) = 0$$

It suffices to take $Z(\hat{\epsilon})$ depending analytically on $\hat{\epsilon}$ with continuous limit at $\hat{\epsilon} = 0$.

The modulus of analytic classification

The sectors $\Omega^{\pm}_{\hat\epsilon}$ intersect along two strips $\Omega^{0}_{\hat\epsilon}$ and $\Omega^{\infty}_{\hat\epsilon}$ where we can compare the Fatou coordinates



The modulus of analytic classification

We define

$$\begin{cases} \Psi_{\hat\epsilon}^{\infty} = \Phi_{\hat\epsilon}^{-} \circ (\Phi_{\hat\epsilon}^{+})^{-1} & \text{on} \quad \Omega_{\hat\epsilon}^{\infty} \\ \Psi_{\hat\epsilon}^{0} = \Phi_{\hat\epsilon}^{-} \circ (\Phi_{\hat\epsilon}^{+})^{-1} & \text{on} \quad \Omega_{\hat\epsilon}^{0} \end{cases}$$

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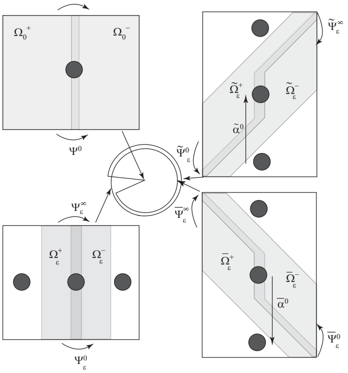

The modulus is defined as the equivalence class of

$$\left(\Psi^{0}_{\hat{\epsilon}}, \Psi^{\infty}_{\hat{\epsilon}}\right)_{\hat{\epsilon} \in V_{\delta}}$$

The equivalence relation

$$\left(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon}\right)_{\hat\epsilon\in V_\delta} \sim \left(\check\Psi^0_{\hat\epsilon}, \check\Psi^\infty_{\hat\epsilon}\right)_{\hat\epsilon\in V_\delta}$$

if and only if there exists $C_{\hat\epsilon}$ and $C'_{\hat\epsilon}$ depending analytically on $\hat\epsilon$ with continuous limit at $\hat\epsilon = 0$ such that

$$\begin{cases} \Psi^0_{\hat\epsilon} = T_{C_{\hat\epsilon}} \circ \check\Psi^0_{\hat\epsilon} \circ T_{C'_{\hat\epsilon}} \\ \Psi^\infty_{\hat\epsilon} = T_{C_{\hat\epsilon}} \circ \check\Psi^\infty_{\hat\epsilon} \circ T_{C'_{\hat\epsilon}} \end{cases}$$

Theorem. [MRR] Two germs of generic families unfolding a codimension 1 parabolic point are analytically conjugate if and only if they have the same formal invariant $a(\epsilon)$ and the same modulus

$$\left[\left(\Psi^0_{\hat\epsilon}, \Psi^\infty_{\hat\epsilon}\right)_{\hat\epsilon\in V_\delta}\right] / \sim$$

The proof

Let us suppose that two germs of prepared diffeomorphisms f_ϵ and $\tilde{f}_\epsilon$ have the same modulus. We can of course adjust the Fatou coordinates $\Phi^{\pm}_{\hat{\epsilon}}$ and $\widetilde{\Phi}^{\pm}_{\hat{\epsilon}}$ so that the representatives of the modulus be the same:

$$\Psi^{0,\infty}_{\hat{\epsilon}} = \widetilde{\Psi}^{0,\infty}_{\hat{\epsilon}}$$

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$$\Psi^{0,\infty}_{\hat{\epsilon}} = \widetilde{\Psi}^{0,\infty}_{\hat{\epsilon}}$$

Then a conjugacy $g_{\hat{\epsilon}}$ between the two systems is given by

$$g_{\hat{\epsilon}} = \begin{cases} p_\epsilon \circ (\widetilde{\Phi}^{+}_{\hat{\epsilon}})^{-1} \circ \Phi^{+}_{\hat{\epsilon}} \circ p_\epsilon^{-1} & \text{on} \quad U_+ \\ p_\epsilon \circ (\widetilde{\Phi}^{-}_{\hat{\epsilon}})^{-1} \circ \Phi^{-}_{\hat{\epsilon}} \circ p_\epsilon^{-1} & \text{on} \quad U_- \end{cases}$$

Correction to a uniform conjugacy

We consider values

$$\begin{cases} \overline{\epsilon} = \hat{\epsilon} \\ \check{\epsilon} = \hat{\epsilon} e^{2\pi i} \end{cases}$$

Then

$$h_\epsilon = \check{g}_\epsilon^{-1} \circ \overline{g}_\epsilon$$

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If f_ϵ has few symmetries then we can deduce that $h_\epsilon \equiv id$. Otherwise, we need to *correct* the family of conjugacies $g_{\hat{\epsilon}}$ to a uniform family.

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We read them in the $W = \Phi(Z)$ variable. In this variable the dynamics is given by T_1. Hence the symmetries on the image of a Fatou coordinate are given by translations.

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To be global these symmetries need to commute with the modulus components $\Psi^0_{\hat\epsilon}$ and $\Psi^\infty_{\hat\epsilon}$.

$$\begin{cases} \Psi^0_{\hat\epsilon}(W) = W + \sum_{n<0} b_n(\hat\epsilon)\exp(2\pi i n W) \\ \Psi^\infty_{\hat\epsilon}(W) = W - 2\pi i a(\epsilon) + \sum_{n>0} c_n(\hat\epsilon)\exp(2\pi i n W) \end{cases}$$

Two cases

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1. If one of b_n or c_n is not identically zero, then the symmetries are discrete (either the identity, or of the form $f_\epsilon^{\circ \frac{p}{q}}$ for some fixed q independent of ϵ). Since $h_0 = id$, then $h_\epsilon = id$.

2. If $b_n \equiv 0$ and $c_n \equiv 0$ for all n, then all symmetries are of the form $f_\epsilon^{\circ t(\epsilon)}$ for $t(\epsilon) \in \mathbb{C}$. Hence

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We correct $g_{\hat{\epsilon}}$ to

$$g_{\hat{\epsilon}} \circ f_\epsilon^{\circ \tau(\epsilon)}$$

such that

$$\tau(\check{\epsilon}) - \tau(\overline{\epsilon}) = t(\epsilon)$$

The parametric resurgence phenomenon

We prefer to present the modulus under the form

$$(\psi_{\hat{\epsilon}}^0, \psi_{\hat{\epsilon}}^\infty)$$

where

$$\psi_{\hat{\epsilon}}^{0,\infty} = E \circ \Psi_{\hat{\epsilon}}^{0,\infty} \circ E^{-1}$$

and

$$E = \exp(-2\pi i W).$$

Then

$$\begin{cases} \psi_{\hat{\epsilon}}^0 : (\mathbb{C}, 0) \to (\mathbb{C}, 0) \\ \psi_{\hat{\epsilon}}^\infty : (\mathbb{C}, \infty) \to (\mathbb{C}, \infty) \end{cases}$$

We consider values of ϵ such that $f_\epsilon'(\pm\sqrt{\epsilon}) \in \mathbb{S}^1$.

The renormalized return maps become

$$\begin{cases} h_\epsilon^0 = \psi_\epsilon^0 \circ L_\epsilon \\ h_\epsilon^\infty = \psi_\epsilon^\infty \circ L_\epsilon \end{cases}$$

We consider sequences $\{\epsilon_n\}$ of values of ϵ such that $\epsilon_n \to 0$ and

$$L_{\epsilon_n}(w) = \exp\left(\frac{2\pi i p}{q}\right) w$$

Parametric resurgence phenomenon

Then for sufficiently large n

- $h^0_{\epsilon_n}$ is non linearizable as soon as $\psi^0_0 \circ L_{\epsilon_n}$ is not linearizable.
- As a consequence f_{ϵ_n} is non linearizable at $-\sqrt{\epsilon}$ as soon as $\psi^0_0 \circ L_{\epsilon_n}$ is not linearizable.

At the other singular point

For sequences $\{\epsilon_n\}$ such that $\epsilon_n \to 0$ and

$$L_{\epsilon_n}(w)\exp(-4\pi^2 a(\epsilon)) = \exp\left(\frac{2\pi i p}{q}\right) w$$

Then for sufficiently large n

- $h^{\infty}_{\epsilon_n}$ is non linearizable as soon as $\psi^{\infty}_0 \circ L_{\epsilon_n}$ is not linearizable.
- As a consequence f_{ϵ_n} is non linearizable at $\sqrt{\epsilon}$ as soon as $\psi^{\infty}_{\infty} \circ L_{\epsilon_n}$ is not linearizable.

Interpretation of $(\psi_\epsilon^0, \psi_\epsilon^\infty)$

For values of the multiplier on the unit circle, ψ_ϵ^0 controls the dynamics at $-\sqrt{\epsilon}$ and ψ_ϵ^∞ at $+\sqrt{\epsilon}$.

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Understanding the dependence of $(\psi_\epsilon^0, \psi_\epsilon^\infty)$ would allow to understand the dynamics of points whose multiplier corresponds to an irrational rotation.

The codimension k case

The strategy

- Define a modulus for generic values of the parameters for which all fixed points are distinct.

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The strategy

- Define a modulus for generic values of the parameters for which all fixed points are distinct.
- The generic values of the parameters in a neighborhood of the origin belong to a finite union of open sets V_j, all adherent to the origin in parameter space.
- Give a description of the modulus for values of the parameters in each V_j which depends analytically on the parameters with continuous limit when $\epsilon \to 0$.

This yields a complete modulus of analytic classification

We consider two germs of prepared families of diffeomorphisms with same modulus.

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- Using the symmetries of the families, we correct to a uniform conjugacy over the generic values of the parameters in a neighborhood of the origin.

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- Using the symmetries of the families, we correct to a uniform conjugacy over the generic values of the parameters in a neighborhood of the origin.
- The conjugacies are bounded when approaching codimension 1 parameter values (one double fixed point), so they can be extended to this case.

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- The conjugacies are bounded when approaching codimension 1 parameter values (one double fixed point), so they can be extended to this case.
- We fill the holes by Hartogs' theorem.