Germs of analytic families of diffeomorphisms unfolding a parabolic point (III)

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Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier

Structure of the mini-course

- Statement of the problem (first lecture)
- The preparation of the family (first lecture)
- Construction of a modulus of analytic classification in the codimension 1 case (second lecture)
- The realization problem in the codimension 1 case (third lecture)

The classification theorem

Theorem. [MRR] Two germs of generic families unfolding a codimension 1 parabolic point are analytically conjugate if and only if they have same formal invariant $a(\epsilon)$ and same modulus

$$\left[\left(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}\right)_{\hat{\epsilon}\in V_\delta}\right] / \sim$$

The realization problem

Which $a(\epsilon)$ and modulus $\left[\left(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}}\right)_{\hat{\epsilon}\in V_\delta}\right] / \sim$ are realizable?

The strategy

1. Any $a(\epsilon)$ and $\left(\Psi_{\hat{\epsilon}}^0, \Psi_{\hat{\epsilon}}^\infty\right)$ can be realized as the modulus of a diffeomorphism $f_{\hat{\epsilon}}$. This is the *local realization*.

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1. Any $a(\epsilon)$ and $(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})$ can be realized as the modulus of a diffeomorphism $f_{\hat{\epsilon}}$. This is the *local realization*.

2. If $a(\epsilon)$ is analytic and $(\Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})$ depend analytically on $\hat{\epsilon}$, then the realization $f_{\hat{\epsilon}}$ can be made depending analytically on $\hat{\epsilon} \in V_\delta$ with uniform limit for $\hat{\epsilon} = 0$.

3. On the auto-intersection of V_δ we let

$$\begin{cases} \overline{\epsilon} = \hat{\epsilon} \\ \tilde{\epsilon} = \hat{\epsilon} e^{2\pi i} \end{cases}$$

A necessary condition for the realization by a uniform family is that $f_{\overline{\epsilon}}$ and $f_{\tilde{\epsilon}}$ be conjugate.

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4. This necessary condition, called the *compatibility condition*, is also sufficient and allows to "correct" $f_{\hat{\epsilon}}$ to a uniform family. This is the *global realization*.

The local realization for a fixed $\hat{e}$

The technique is standard: we realize on an abstract 1-dimensional complex manifold, which we recognize to be holomorphically equivalent to an open set of $\mathbb{C}$.

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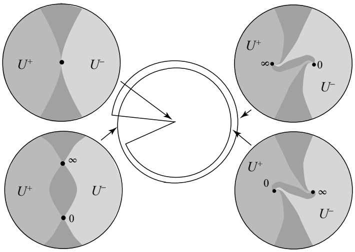
The technique is standard: we realize on an abstract 1-dimensional complex manifold, which we recognize to be holomorphically equivalent to an open set of $\mathbb{C}$.

Indeed, we consider the two sectors $U^{\pm}_{\hat{\epsilon}}$, each endowed with the *model diffeomorphism* $f^{\pm}_{\epsilon}$, i.e. the time-1 map of the vector field

$$v_\epsilon = \frac{z^2-\epsilon}{1+a(\epsilon)z}\frac{\partial}{\partial z}$$



The gluing on $U_{\hat{\epsilon}}^{+} \cap U_{\hat{\epsilon}}^{-}$

This gluing must be compatible with $f_{\epsilon}^{\pm}$ on the three parts of the intersection, $U_{\hat{\epsilon}}^{0}$, $U_{\hat{\epsilon}}^{\infty}$ and $U_{\hat{\epsilon}}^{C}$.

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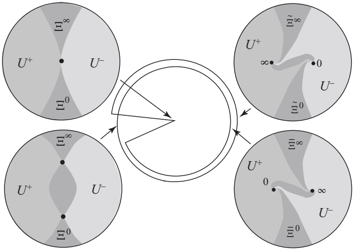
This gluing must be compatible with $f^{\pm}_{\epsilon}$ on the three parts of the intersection, $U^0_{\hat\epsilon}$, $U^\infty_{\hat\epsilon}$ and $U^C_{\hat\epsilon}$.

In the time coordinate W of v_ϵ this gluing is simply given by

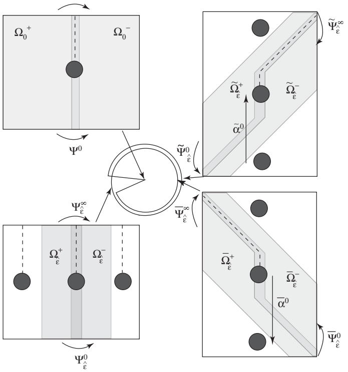
$$\begin{cases} \Psi^0_{\hat\epsilon} & \text{on} \quad U^0_{\hat\epsilon} \\ \Psi^\infty_{\hat\epsilon} & \text{on} \quad U^\infty_{\hat\epsilon} \\ \mathscr{T}_{\hat\epsilon} & \text{on} \quad U^C_{\hat\epsilon} \end{cases}$$

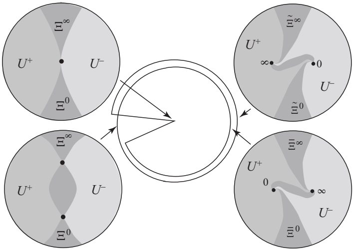
which commutes with T_1. The map $\mathscr{T}_{\hat\epsilon}$ is a translation: it is the *Lavaurs map*.



The time W of v_ϵ

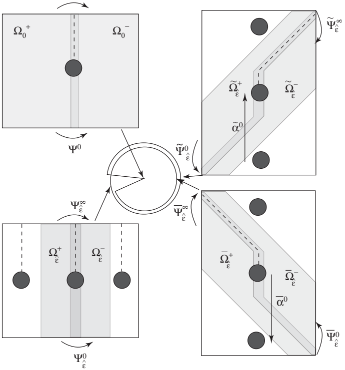
$$W = q_{\hat\epsilon}^{-1}(z) = \begin{cases} \frac{1}{2\sqrt{\hat\epsilon}} \ln \frac{z-\sqrt{\hat\epsilon}}{z+\sqrt{\hat\epsilon}} + \frac{a(\epsilon)}{2} \ln(z^2-\epsilon), & \hat\epsilon \neq 0, \\ -\frac{1}{z} + a(0)\ln(z), & \hat\epsilon = 0. \end{cases}$$





Why $\mathscr{T}_{\hat{\epsilon}}$ is a translation?

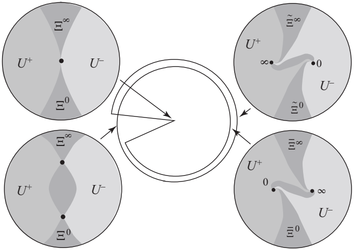
In the time coordinate W, it is a diffeomorphism commuting with T_1 on a strip of width larger then 1 going from $\operatorname{Im} W = -\infty$ to $\operatorname{Im} W = +\infty$.



The gluing in z-coordinate

In the z-coordinate, the gluing is simply given by

$$\begin{cases} \Xi^0_{\hat\epsilon} = q_{\hat\epsilon} \circ \Psi^0_{\hat\epsilon} \circ q_{\hat\epsilon}^{-1} & \text{on} \quad U^0_{\hat\epsilon} \\ \Xi^\infty_{\hat\epsilon} = q_{\hat\epsilon} \circ \Psi^\infty_{\hat\epsilon} \circ q_{\hat\epsilon}^{-1} & \text{on} \quad U^\infty_{\hat\epsilon} \\ id & \text{on} \quad U^C_{\hat\epsilon} \end{cases}$$

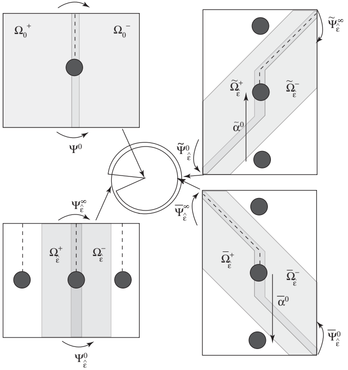


Behavior of the gluing near the fixed points

$$\Xi_{\hat\epsilon}^{0,\infty}(z) = id + \xi_{\hat\epsilon}^{0,\infty}(z) \text{ with}$$

$$\begin{cases} \left|\xi_{\hat\epsilon}^{0}(z)\right| < C(\hat\epsilon)\left|z+\sqrt{\hat\epsilon}\right|^{\frac{A}{|\sqrt{\hat\epsilon}|}} \\ \left|\xi_{\hat\epsilon}^{\infty}(z)\right| < C(\hat\epsilon)\left|z-\sqrt{\hat\epsilon}\right|^{\frac{A}{|\sqrt{\hat\epsilon}|}} \end{cases}$$

The compatibility condition



For $\hat{\epsilon}$ in the auto-intersection of V_δ we have two descriptions of the modulus. A necessary condition for realizability to a uniform family in ϵ is that they encode conjugate dynamics.

Parameter values in the auto-intersection

For these values, the fixed points are linearizable and there is an orbit from one point to the other.

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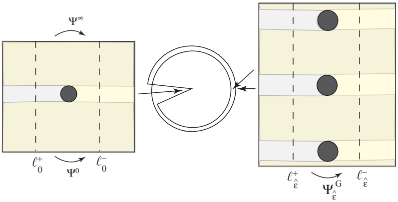
The Glutsyuk modulus is unique up to composition on the left and on the right by maps of the form v_ϵ^t.

Construction of the Fatou Glutsyuk coordinates

As before we construct Fatou Glutsyuk coordinates, Φ^l and Φ^r, but we use lines parallel to the line of holes



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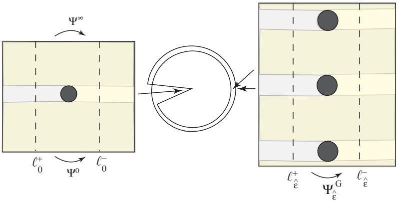


The Glutsyuk modulus is

$$\Psi^G = \Phi^r \circ (\Phi^l)^{-1}$$

It is unique up to composition on the left and on the right with translations and satisfies

$$T_{\alpha^r} \circ \Psi^G = \Psi^G \circ T_{\alpha^l}$$

How to recover the Fatou Glutsyuk coordinates?

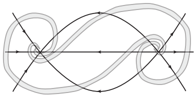
How to recover them from the modulus

$$(\hat{\epsilon}, a(\epsilon), \Psi^0_{\hat{\epsilon}}, \Psi^\infty_{\hat{\epsilon}})?$$

We describe the orbit space of F_ϵ with the help of ONE Fatou coordinate and a *renormalized return map*.

The renormalized return maps

Lavaurs point of view

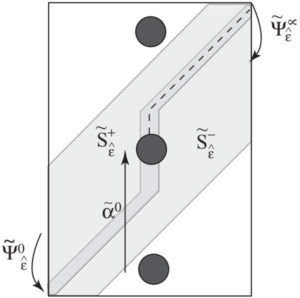


They are given by

$$\begin{cases} T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^0 \\ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^\infty \end{cases}$$

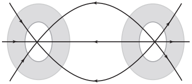
or

$$\begin{cases} \tilde{\Psi}^0 \circ T_{\tilde{\alpha}^0} \\ \tilde{\Psi}^\infty \circ T_{\tilde{\alpha}^0} \end{cases}$$



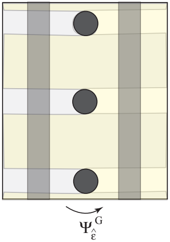
The renormalized return maps

Glutsyuk point of view



The Fatou Glutsyuk coordinates are the coordinates in which the renormalized return maps are given by

$$\begin{cases} T_{\tilde{\alpha}^0} \\ T_{\tilde{\alpha}^\infty} \end{cases}$$



The change of coordinates

The changes from Fatou (Lavaurs) coordinates to Fatou Glutsyuk coordinates are the changes of coordinates transforming

$$\begin{cases} T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^0 \\ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^\infty \end{cases} \quad \text{or} \quad \begin{cases} \tilde{\Psi}^0 \circ T_{\tilde{\alpha}^0} \\ \tilde{\Psi}^\infty \circ T_{\tilde{\alpha}^0} \end{cases}$$

to

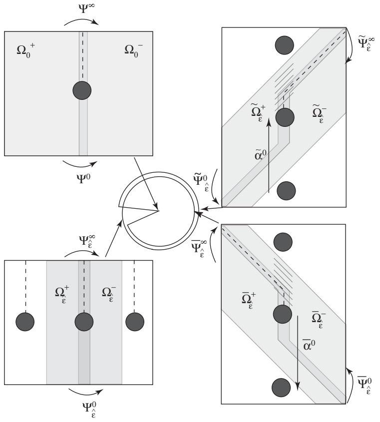
$$\begin{cases} T_{\tilde{\alpha}^0} \\ T_{\tilde{\alpha}^\infty} \end{cases}$$

Working in the upper region

There exists maps

$$\begin{cases} \tilde{H}^0 \circ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^0 = T_{\tilde{\alpha}^0} \circ \tilde{H}^0 \\ \tilde{H}^\infty \circ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^\infty = T_{\tilde{\alpha}^\infty} \circ \tilde{H}^\infty \\ \overline{H}^0 \circ \overline{\Psi}^0 \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^0} \circ \overline{H}^0 \\ \overline{H}^\infty \circ \overline{\Psi}^\infty \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^\infty} \circ \overline{H}^\infty \end{cases}$$

The maps $\tilde{H}^{0,\infty}$ and $\overline{H}^{0,\infty}$ are the changes of coordinates to Fatou Glustyuk coordinates.



The compatibility condition

It is given by:

$$\tilde{H}^\infty \circ (\tilde{H}^0)^{-1} = T_{D_\epsilon} \circ \overline{H}^0 \circ (\overline{H}^\infty)^{-1} \circ T_{D'_\epsilon}$$

It is possible to normalize the coordinates so that $D_\epsilon \equiv -2\pi i a$.

Corollary: The functions $\Psi^{0,\infty}_{\hat{\epsilon}}$ are 1-summable in $\sqrt{\hat{\epsilon}}$. The directions of non-summability are the Glutsyuk directions (real multipliers).

Theorem: The family

$$\{(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})\}_{\hat{\epsilon}\in V}$$

is realizable if and only if the compatibility condition is satisfied.

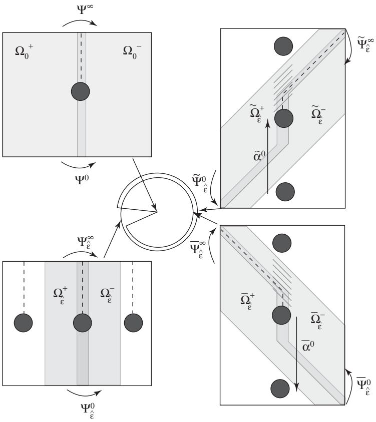
Proof of the Corollary

In upper region

$$
\begin{cases}
\tilde{H}^0 \circ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^0 = T_{\tilde{\alpha}^0} \circ \tilde{H}^0 \\
\tilde{H}^\infty \circ T_{\tilde{\alpha}^0} \circ \tilde{\Psi}^\infty = T_{\tilde{\alpha}^\infty} \circ \tilde{H}^\infty \\
\overline{H}^0 \circ \overline{\Psi}^0 \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^0} \circ \overline{H}^0 \\
\overline{H}^\infty \circ \overline{\Psi}^\infty \circ T_{\overline{\alpha}^0} = T_{\overline{\alpha}^\infty} \circ \overline{H}^\infty
\end{cases}
$$

This implies

$$
\begin{cases}
\tilde{H}^0 = id + O(\overline{C}^0) \\
\tilde{H}^\infty = T_{2\pi i a} \circ \tilde{\Psi}^\infty + O(\overline{C}^0) \\
\overline{H}^0 = id + O(\overline{C}^0) \\
(\overline{H}^\infty)^{-1} = \overline{\Psi}^\infty \circ T_{2\pi i a} + O(\overline{C}^0)
\end{cases}
$$



In lower region

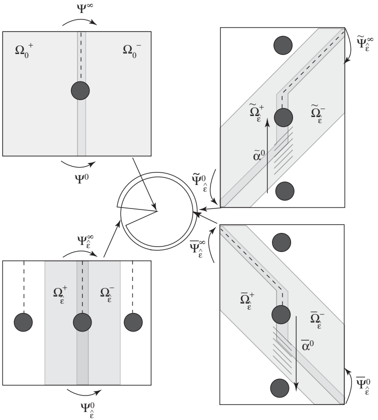
$$\begin{cases} \tilde{K}^0 \circ \tilde{\Psi}^0 \circ T_{\tilde{\alpha}^0} = T_{\tilde{\alpha}^0} \circ \tilde{K}^0 \\ \tilde{K}^\infty \circ \tilde{\Psi}^\infty \circ T_{\tilde{\alpha}^0} = T_{\tilde{\alpha}^\infty} \circ \tilde{K}^\infty \\ \overline{K}^0 \circ T_{\overline{\alpha}^0} \circ \overline{\Psi}^0 = T_{\overline{\alpha}^0} \circ \overline{K}^0 \\ \overline{K}^\infty \circ T_{\overline{\alpha}^0} \circ \overline{\Psi}^\infty = T_{\overline{\alpha}^\infty} \circ \overline{K}^\infty \end{cases}$$

The functions K are given by:

$$\begin{cases} \tilde{K}^0 = T_{-\tilde{\alpha}^0} \circ \tilde{H}^0 \circ T_{\tilde{\alpha}^0} \\ \tilde{K}^\infty = T_{-\tilde{\alpha}^0} \circ \tilde{H}^\infty \circ T_{\tilde{\alpha}^0} \\ \overline{K}^0 = T_{\overline{\alpha}^0} \circ \overline{H}^0 \circ T_{-\overline{\alpha}^0} \\ \overline{K}^\infty = T_{\overline{\alpha}^0} \circ \overline{H}^\infty \circ T_{-\overline{\alpha}^0}. \end{cases}$$

The compatibility condition becomes

$$\tilde{K}^\infty \circ (\tilde{K}^0)^{-1} = \overline{K}^0 \circ (\overline{K}^\infty)^{-1} \circ T_{2\pi i a + D'_\epsilon}$$



The 1-summability follows

In upper region:

$$\begin{cases} \tilde{H}^0 = id + O(\overline{C}^0) \\ \tilde{H}^\infty = T_{2\pi i a} \circ \tilde{\Psi}^\infty + O(\overline{C}^0) \\ \overline{H}^0 = id + O(\overline{C}^0) \\ (\overline{H}^\infty)^{-1} = \overline{\Psi}^\infty \circ T_{2\pi i a} + O(\overline{C}^0) \end{cases}$$

In lower region:

$$\begin{cases} (\tilde{K}^0)^{-1} = \tilde{\Psi}^0 + O(\overline{C}^0) \\ \tilde{K}^\infty = id + 2\pi i a + O(\overline{C}^0) \\ \overline{K}^0 = \overline{\Psi}^0 + O(\overline{C}^0) \\ (\overline{K}^\infty)^{-1} = id + 2\pi i a + O(\overline{C}^0) \end{cases}$$

Substituting in the compatibility condition:

$$\begin{cases} \tilde{H}^\infty \circ (\tilde{H}^0)^{-1} = T_{2\pi i a} \circ \overline{H}^0 \circ (\overline{H}^\infty)^{-1} \circ T_{D'_\epsilon} \\ \tilde{K}^\infty \circ (\tilde{K}^0)^{-1} = \overline{K}^0 \circ (\overline{K}^\infty)^{-1} \circ T_{2\pi i a + D'_\epsilon} \end{cases}$$

yields the existence of a constant A such that:

$$|\tilde{\Psi}^\infty - \overline{\Psi}^\infty| < A\overline{C}^0 \qquad |\tilde{\Psi}^0 - \overline{\Psi}^0| < A\overline{C}^0$$

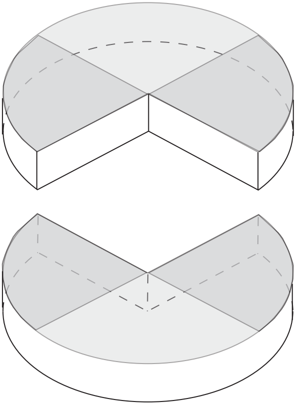
The 1-summability in $\sqrt{\epsilon}$ follows from Ramis-Sibuya's theorem since

$$|\overline{C}^0| \sim \exp\left(-\frac{2\pi}{2|\sqrt{\epsilon}|}\right)$$

The global realization

How to correct? Newlander-Nirenberg's theorem.
We construct a family over an abstract manifold by gluing

$$(\tilde{z},\tilde{\epsilon}) = \begin{cases} (g_{\overline{\epsilon}}(\overline{z}),\overline{\epsilon}) & \text{on the right} \\ (\overline{z},\overline{\epsilon}) & \text{on the left} \end{cases}$$



where

$$g_{\overline{\epsilon}} \circ \overline{f} \circ g_{\overline{\epsilon}}^{-1} = \tilde{f}$$

Adding $\epsilon = 0$ yields a C^∞ manifold. **Why?**

- $|\overline{f} - \tilde{f}| = O(\exp(-\frac{A}{\sqrt{|\epsilon|}}))$
- Hence $g_{\overline{\epsilon}} = id + O(\exp(-\frac{A}{\sqrt{|\epsilon|}}))$

End of the proof

The abstract manifold has an almost complex structure which is integrable and is a product. Hence it is a neighborhood of the origin in $\mathbb{C}^2$ with coordinates (Z, ϵ).

The Riccati case

We rather consider

$$\begin{cases} \psi^0_{\hat{\epsilon}} = E \circ \Psi^0_{\hat{\epsilon}} \circ E^{-1} \\ \psi^\infty_{\hat{\epsilon}} = E \circ \Psi^\infty_{\hat{\epsilon}} \circ E^{-1} \end{cases}$$

where

$$E(W) = \exp(-2\pi i W)$$

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The Riccati case corresponds to

$$\begin{cases} \psi^0_{\hat{\epsilon}}(w) = \frac{w}{1+A(\hat{\epsilon})w} \\ \psi^\infty_{\hat{\epsilon}}(w) = \exp(-4\pi^2 a(\epsilon))(w+B(\hat{\epsilon})) \end{cases}$$

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Then the compatibility condition is equivalent to say that there exists a presentation of the modulus with $A(\epsilon)$ and $B(\epsilon)$ analytic in ϵ.

Conjecture

If $\psi_{\hat{\epsilon}}^0$ and $\psi_{\hat{\epsilon}}^\infty$ are both nonlinear, then the only case where $\psi_{\hat{\epsilon}}^0$ and $\psi_{\hat{\epsilon}}^\infty$ can be taken depending analytically in ϵ is the Riccati case.

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If $\psi_{\hat{\epsilon}}^0$ and $\psi_{\hat{\epsilon}}^\infty$ are both nonlinear, then the only case where $\psi_{\hat{\epsilon}}^0$ and $\psi_{\hat{\epsilon}}^\infty$ can be taken depending analytically in ϵ is the Riccati case.

Otherwise, the compatibility condition is so violent that it forces non analyticity.

The end