Limits of quadratic rational maps with degenerate parabolic fixed points of multiplier $e^{2\pi i/q} \to 1$

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Degenerate parabolic fixed points

- Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map.
- A fixed point of f is *parabolic* if the multiplier is a root of unity.
- If the multiplier is $e^{2\pi i p/q}$ and ζ is a coordinate vanishing at the fixed point, then

$$\zeta \circ f^{\circ q} = e^{2\pi i p/q}\zeta \cdot (1 + \zeta^{\nu q}) + \mathcal{O}(\zeta^{\nu q+2})$$

  for some integer $\nu \geq 1$.
- The fixed point is a *degenerate* parabolic fixed point if $\nu \geq 2$.

Families of quadratic rational maps

- Consider the quadratic rational map

$$f_{a,p/q} : z \mapsto e^{2\pi i p/q} \frac{z}{1 + az + z^2}$$

which fixes 0 with multiplier $e^{2\pi i p/q}$.

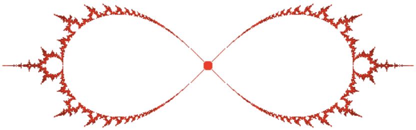
Question

What can we say regarding the set $\mathcal{A}_{p/q}$ of points $a \in \mathbb{C}$ for which $f_{a,p/q}$ has a degenerate parabolic fixed point at 0 ?

Pictures

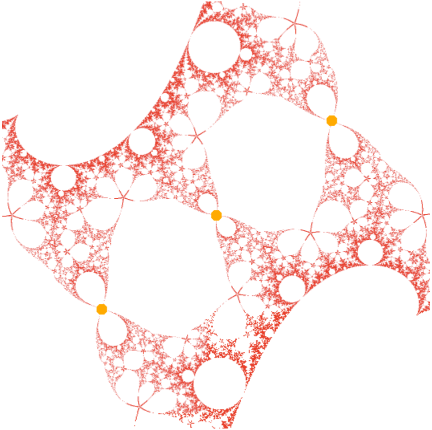
- The bifurcation locus $\mathcal{B}_{p/q}$ is the closure of the set of parameters $a \in \mathbb{C}$ for which $f_{a,p/q}$ has a parabolic cycle of period > 1.
- $\mathcal{A}_{p/q} \subset \mathcal{B}_{p/q}$.

Pictures



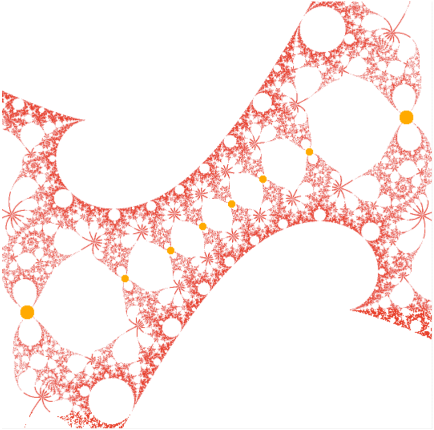
$\mathcal{B}_{0/1}$

Pictures



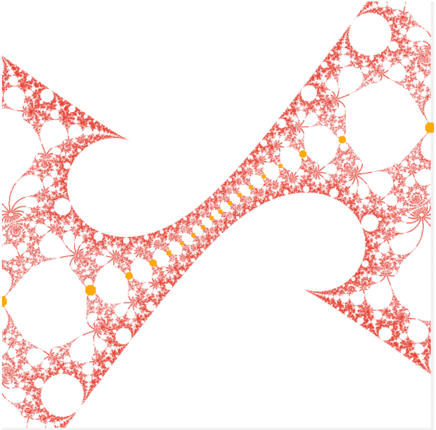
$\mathcal{B}_{1/5}$

Pictures



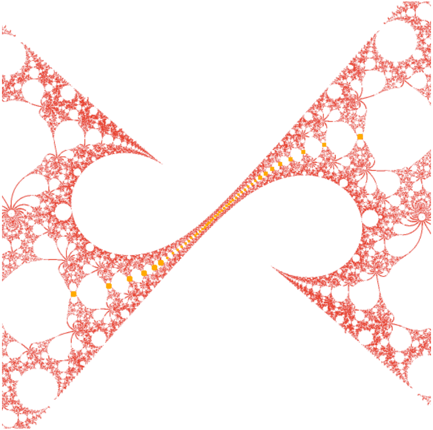
$\mathcal{B}_{1/10}$

Pictures



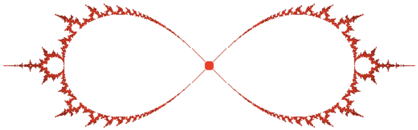
$\mathcal{B}_{1/20}$

Pictures



$\mathcal{B}_{1/50}$

Pictures



$\mathcal{B}_{0/1}$

The cardinality of $\mathcal{A}_{p/q}$

- as $z \to \infty$, we have

$$f_{a,p/q}^{\circ q}(z) = z \cdot \left(1 + C_{p/q}(a) z^q\right) + \mathcal{O}(z^{q+1}).$$

- $a \in \mathcal{A}_{p/q}$ if and only if $C_{p/q}(a) = 0$.

Proposition

$C_{p/q}$ is a polynomial of degree $q-2$ having only simple roots.

The degree $q-2$ is obtained by studying the behaviour as $a \to \infty$. The simplicity of roots is a transversality statement which we shall not study today.

Limits as $1/q \to 0$

- It is tempting to conjecture that the sets $\mathcal{B}_{1/q}$ have a Hausdorff limit in $\mathbb{C} \cup \{\infty\}$. This is still unknown.
- It is tempting to conjecture that the sets $\mathcal{A}_{1/q}$ have a Hausdorff limit in $\mathbb{C} \cup \{\infty\}$. This is almost known.

Proposition

There exists an entire function C with the following properties.

- *C has order of growth 1. More precisely, as $b \to \infty$*

$$\log|C(b)| \in \mathcal{O}(|b|\log|b|) \setminus \mathcal{O}(|b|).$$

  In particular C has infinitely many zeroes.

- *the set $\mathcal{A}$ of points $a \in \mathbb{C}$ such that $C(1/a^2) = 0$ satisfies*

$$\mathcal{A} \cup \{0\} \subseteq \liminf_{q\to\infty} \mathcal{A}_{1/q} \quad \text{and} \quad \limsup_{q\to\infty} \mathcal{A}_{1/q} \subseteq \mathcal{A} \cup \{0, \infty\}.$$

Changes of coordinates

- It is convenient to introduce the rational map

$$G_b : w \mapsto w + 1 + \frac{b}{w}.$$

- If $b = 1/a^2$, then $F_{a,0}$ is conjugate to G_b via $w = a/z$.

Ecalle-Voronin invariants

- Attracting Fatou coordinates :

$$\Phi_{b,\mathrm{att}}(w) = \lim_{n\to+\infty} G_b^{\circ n}(w) - n - b\cdot\sum_{k=1}^{n}\frac{1}{k}.$$

- Repelling Fatou parameterization :

$$\Psi_{b,\mathrm{rep}}(w) = \lim_{n\to+\infty} G_b^{\circ n}\left(w - n + b\cdot\sum_{k=1}^{n}\frac{1}{k}\right).$$

- Voronin invariants :

$$\widetilde{\mathcal{E}}_b^{\pm}(w) = \Phi_{b,\mathrm{att}}\circ\Psi_{b,\mathrm{rep}}(w).$$

The function C

- $$\widetilde{\mathcal{E}}_b^+ = \mathrm{Id} + \sum_{k\geq 0} c_k(b) e^{2\pi i k w}$$
  and
  $$\widetilde{\mathcal{E}}_b^- = \mathrm{Id} + \sum_{k\leq 0} c_k(b) e^{2\pi i k w}$$
  with c_k entire functions of b.
- The entire function C is the Fourier coefficient : $C = c_1$.

Hypertangents and multizetas

- Hypertangents :

$$\mathrm{Pe}^1 = \pi \cot(\pi w) = \sum_{k\in\mathbb{Z}} \frac{1}{k+w}$$

and

$$\mathrm{Pe}^n = \sum_{k\in\mathbb{Z}} \frac{1}{(k+w)^n}.$$

- Multizetas :

$$\zeta(s_1,\ldots,s_r) = \sum_{0<n_r<\ldots<n_2<n_1<\infty} \frac{1}{n_r^{s_r}} \cdots \frac{1}{n_2^{s_2}} \cdot \frac{1}{n_1^{s_1}}.$$

Expansion with respect to b

$$\widetilde{\mathcal{E}}_b^{\pm} = \mathrm{id} + b\mathrm{e}_1 + b^2\mathrm{e}_2 + b^3\mathrm{e}_3 + \dots$$

with

$$\begin{aligned}
\mathrm{e}_1 &= \mathrm{Pe}^1 \\
\mathrm{e}_2 &= 0 \\
\mathrm{e}_3 &= 3\zeta(3)\mathrm{Pe}^2 \\
\mathrm{e}_4 &= -\zeta(4)\mathrm{Pe}^3 + 10\zeta(5)\mathrm{Pe}^2
\end{aligned}$$

Order of growth

- $\widetilde{\mathcal{E}}_b^+$ in the upper half-plane $\Im(w) > h_b^+$ with h_b^+ comparable to $\Im(b) \log |b|$.
- $\widetilde{\mathcal{E}}_b^-$ in the lower half-plane $\Im(w) < h_b^-$ with h_b^- comparable to $\Im(b) \log |b|$.
- This is obtained by comparing the dynamics of G_b to the real flow of the vector field

$$\left(1 + \frac{b}{w}\right) \frac{d}{dw}.$$

- The Koebe 1/4-Theorem implies that

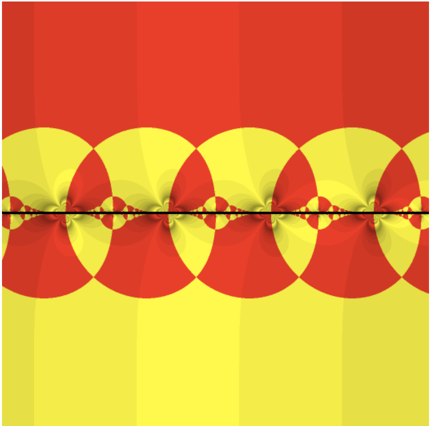
$$\log |C(b)| \leq \frac{1}{4} \cdot \frac{h_b^+}{2\pi} = \mathcal{O}(|b| \log |b|).$$

Order of growth

Assume $\Re(b) = 1/2$.

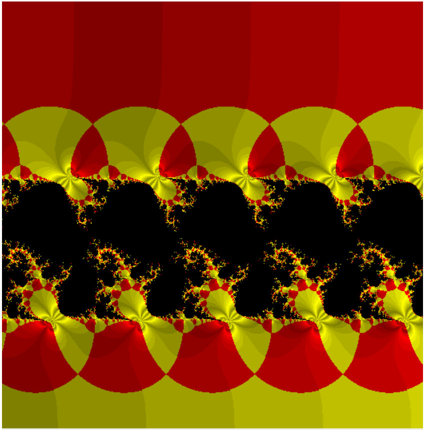
- G_b has a indifferent fixed point at $-b$ and so, the basin of ∞ only contains 1 critical point.
- There is a univalent map $\chi : \{\Im(w) > 0\} \to \{\Im(w) > h_b^-\}$ satisfying $\chi(w+1) = \chi(w) + 1$ and a translation T such that
$$\widetilde{\mathcal{E}}_{1/4} = T \circ \widetilde{\mathcal{E}}_b^+ \circ \chi.$$
- According to the Fatou-Shishikura Inequality for Finite Type Maps, $c_1(1/4) \neq 0$.
- $$\log|C(b)| \geq 2\pi h_b^- + \log\big|c_1(1/4)\big|.$$

Pictures again



$\widetilde{\mathcal{E}}^{\pm}_{1/4}$ sends each red tile univalently to a upper half-plane and each yellow tile univalently to a lower half-plane.

Pictures again



$\widetilde{\mathcal{E}}^{\pm}_{1/2+10\mathrm{i}}$ sends each red tile univalently to a upper half-plane and each yellow tile univalently to a lower half-plane.