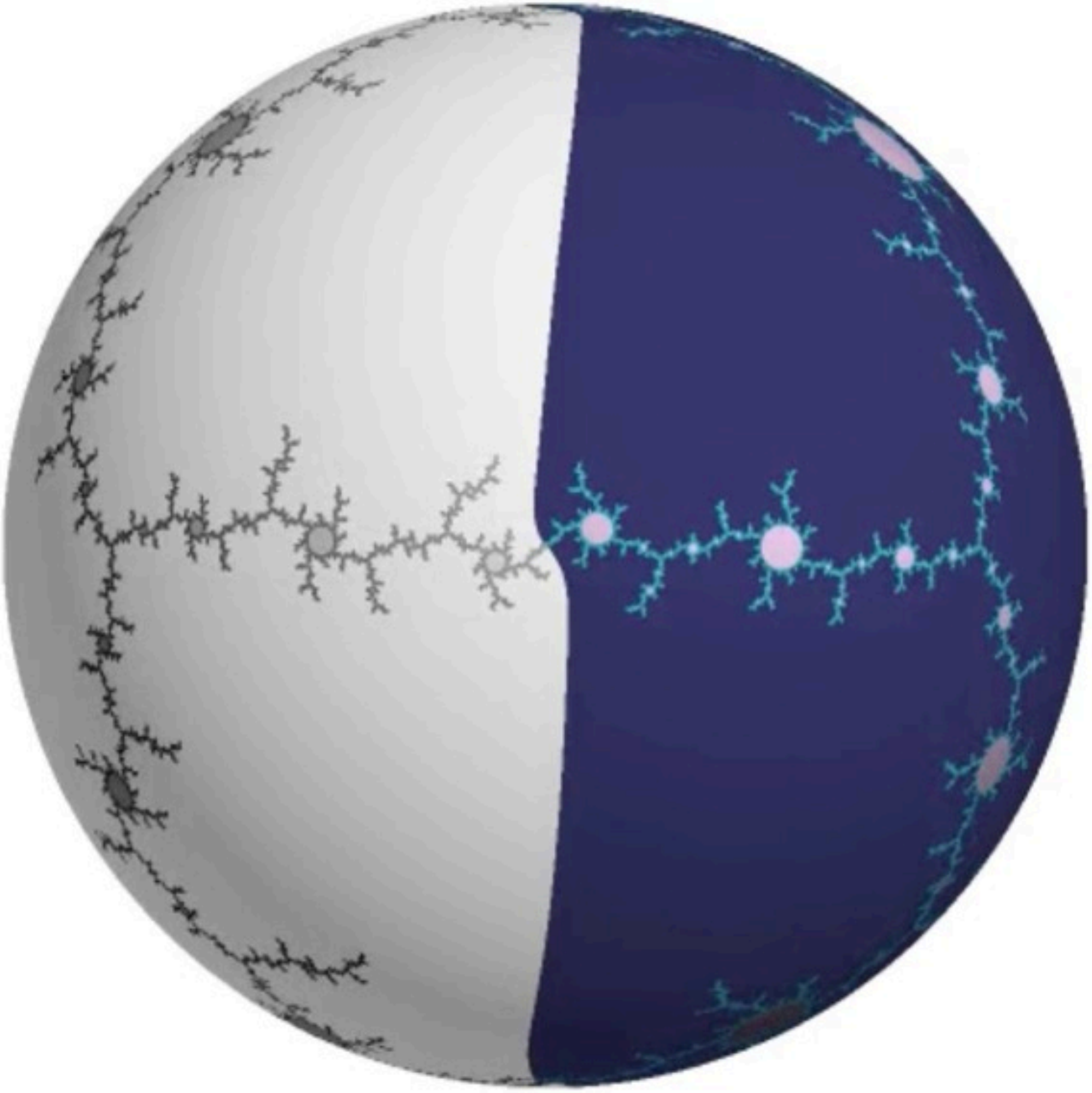
Slow Matings and Twisted Matings



SLOW MATINGS AND TWISTED MATINGS

XAVIER BUFF, ADAM L. EPSTEIN, AND SARAH KOCH

Abstract. One crucial tool for studying postcritically finite rational maps is Thurston's theorem on the topological characterization of rational functions. There, one studies the iterates of a *Thurston map* σ_f acting on Teichmüller space. This theorem has been proved to be useful for studying which mating of quadratic polynomials.

Introduction

Bartholdi and Nekrashevych, and then Koch showed that in many cases, the inverse of σ_f descends to a holomorphic map acting on moduli space. We will show that this approach can be used to study matings. We will focus on two concrete examples: the twisted matings of basilicas, and the mating of a basilica with a rabbit.

All the polynomials $P : \mathbb{C} \to \mathbb{C}$ considered in this article will be monic polynomials of degree $d \geq 2$: the coefficient of z^d is 1. The polynomials will be postcritically finite polynomials: the critical points of P have finite orbits under iteration of P. In addition, the polynomials will be hyperbolic: the orbit of any critical point eventually lands on a superattracting cycle.

The filled-in Julia set is the set

$$K(P) = \big\{z \in \mathbb{C}\ ;\ \big(P^{\circ n}(z)\big) \text{ is bounded.}\big\}.$$

The Julia set is the boundary of $K(P)$.

When P is postcritically finite, $K(P)$ and $J(P)$ are connected.[1] The complement of $K(P)$ is isomorphic to $\mathbb{C}\setminus\overline{\mathbb{D}}$, and there is an isomorphism böt : $\mathbb{C}\setminus\overline{\mathbb{D}} \to \mathbb{C}\setminus K(P)$ conjugating $z \mapsto z^d$ to P. Such an isomorphism is called a Böttcher coordinate. Since P is monic, the Böttcher coordinate can be chosen to satisfy $\text{böt}(z) = z + O(1)$ as $z \to \infty$ (there is a unique such Böttcher coordinate).

If $\theta \in \mathbb{R}/\mathbb{Z}$, the external ray $\mathcal{R}_\theta = \mathcal{R}_\theta(P)$ of angle θ is the set of points of the form $\text{böt}(\rho e^{2i\pi\theta})$ with $\rho > 1$. The polynomial P sends the external ray of angle θ to the external ray of angle $d\theta$.

0.1. **Formal mating.** We add to the complex plane $\mathbb{C}$ the *circle at infinity* which is symbolically denoted $\{\infty \cdot e^{2i\pi\theta}\ ;\ \theta \in \mathbb{R}/\mathbb{Z}\}$. We define

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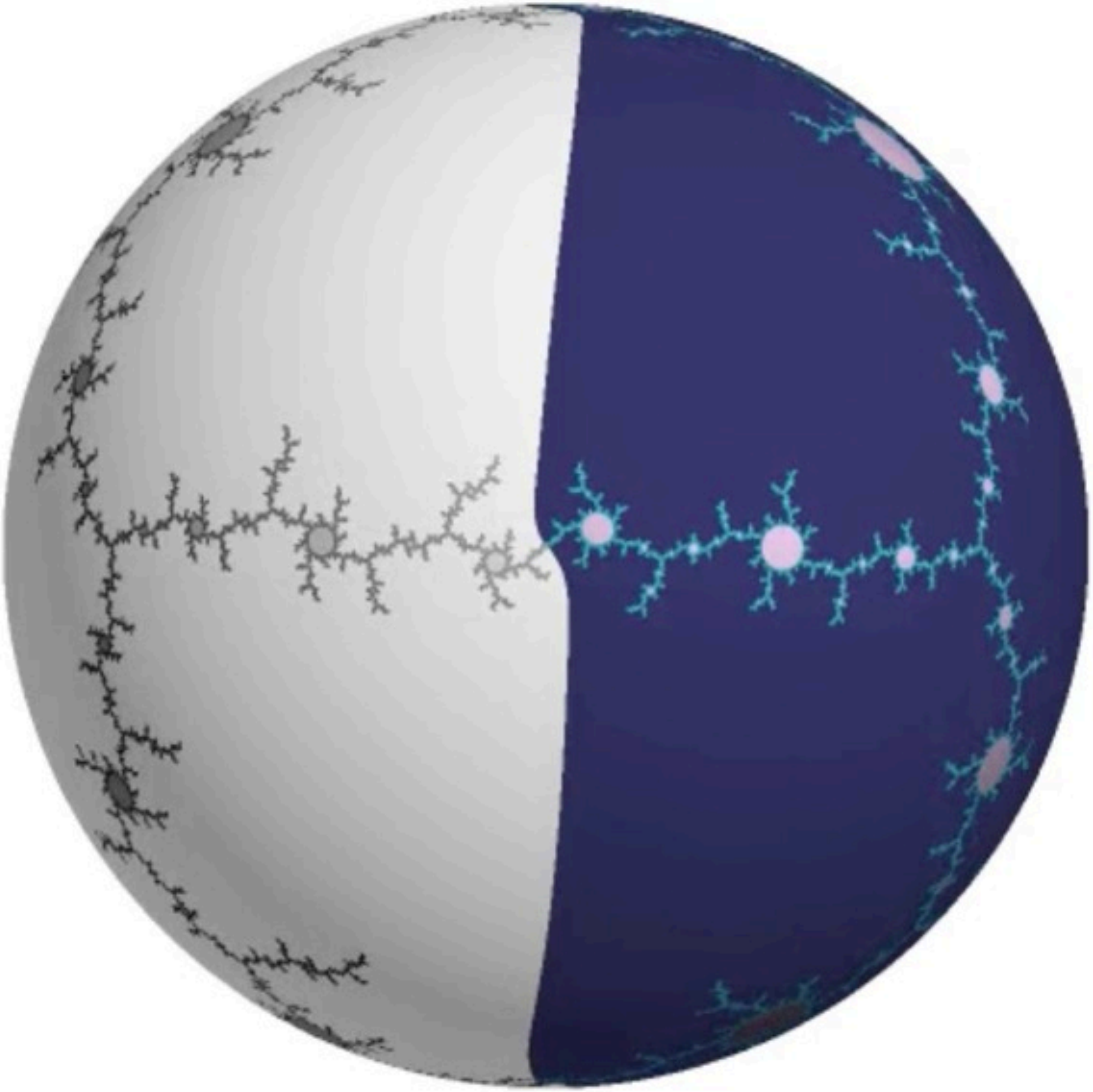
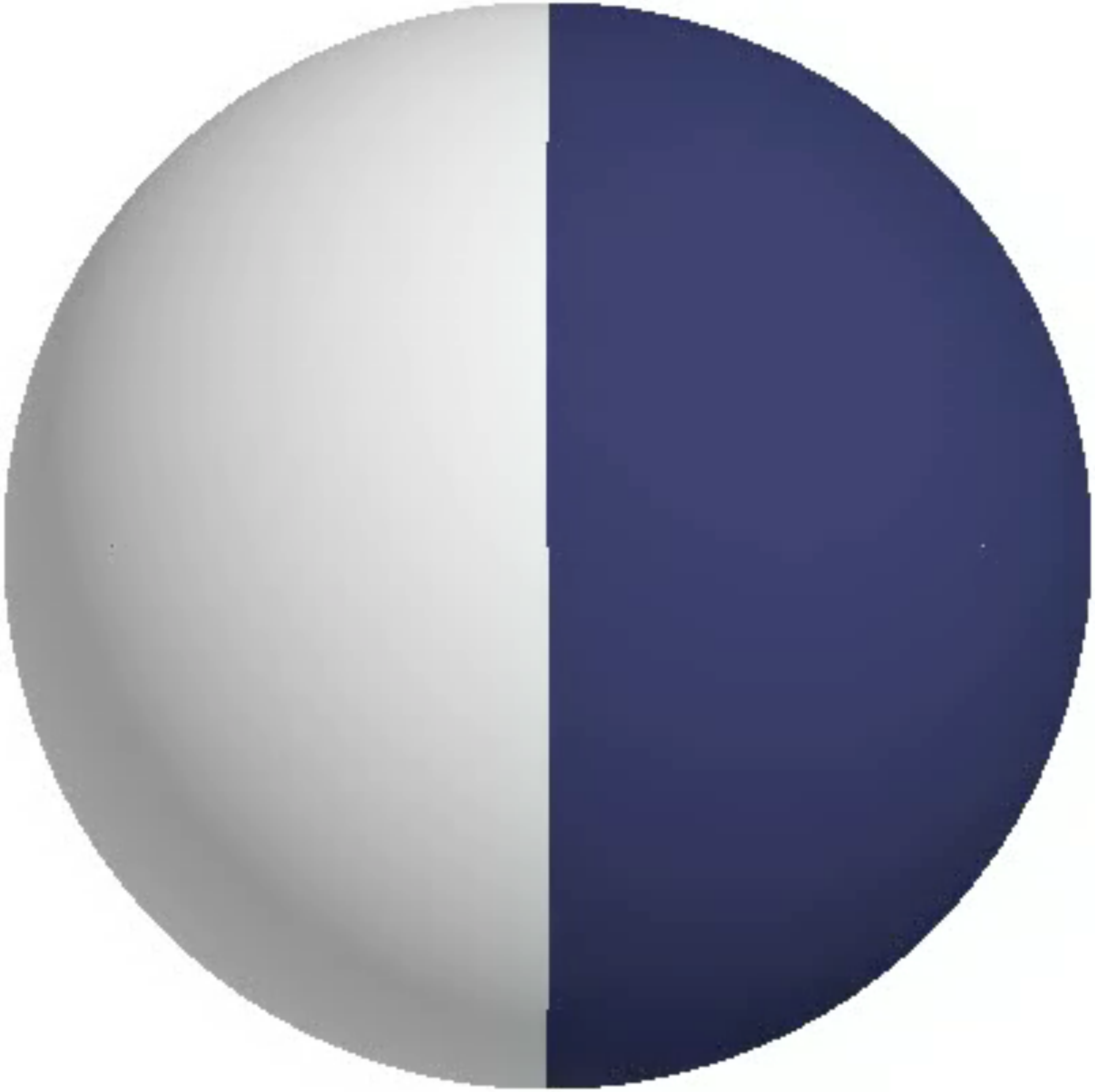
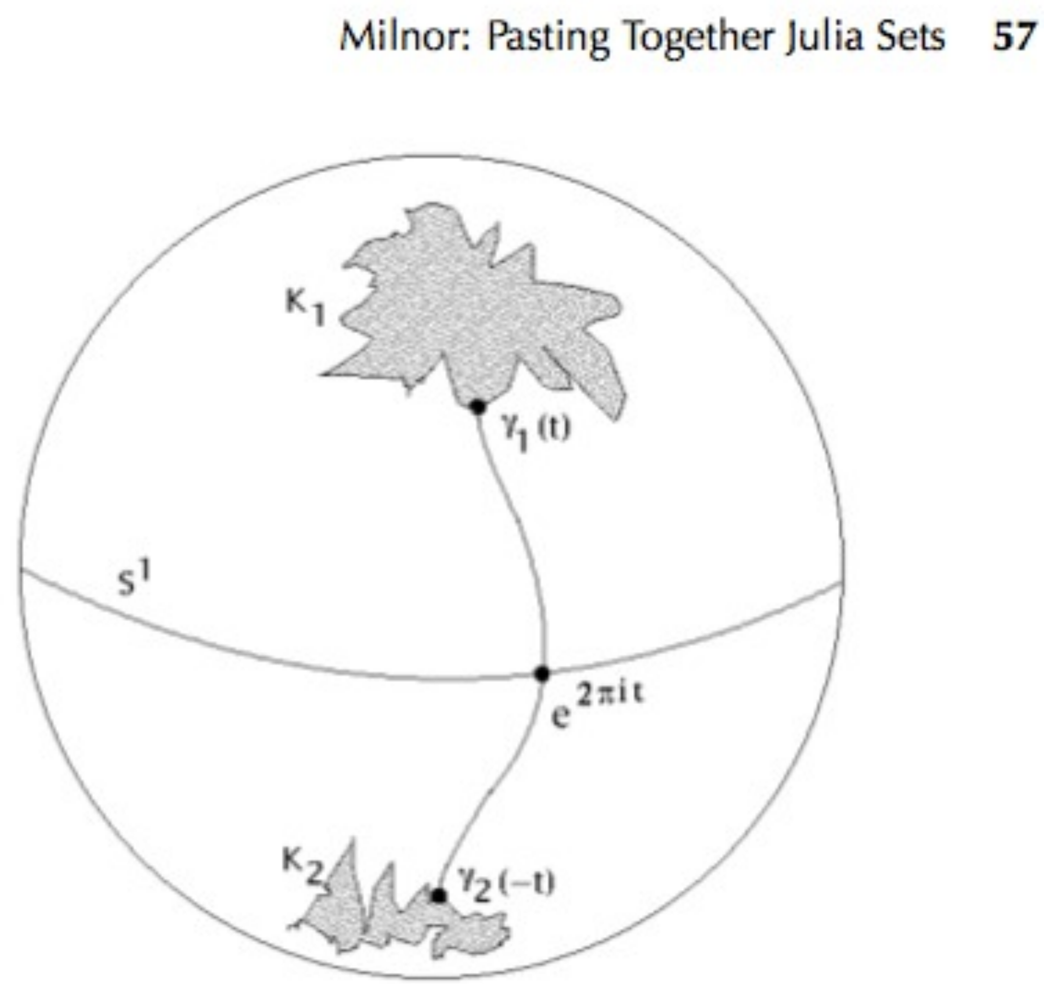


with Xavier Buff and Adam Epstein



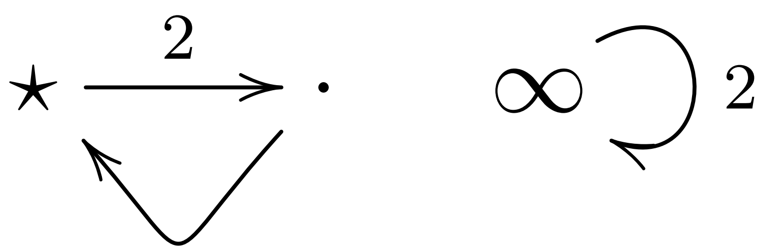
Recall...

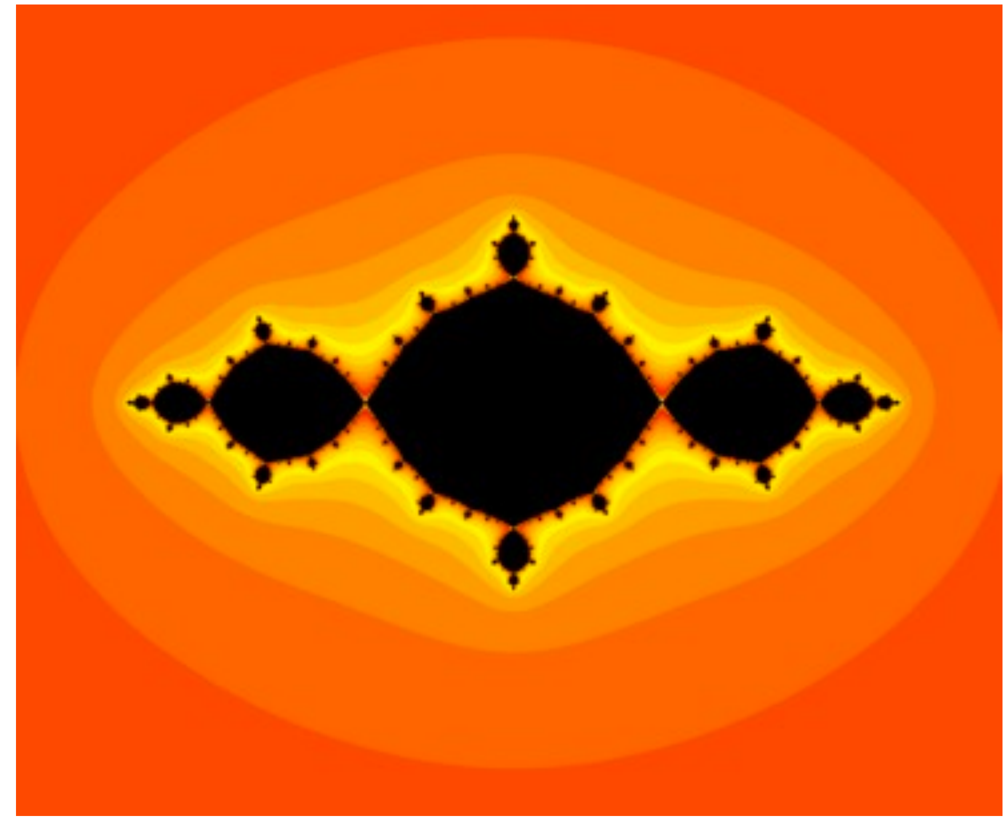
- formal mating
- topological mating
- geometric mating
- shared mating
- slow mating



work of Rees, Shishikura, Tan Lei,...

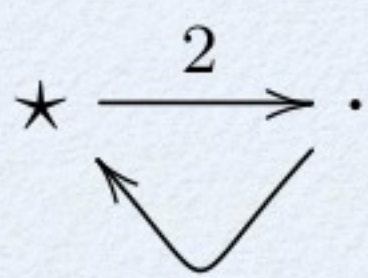
Example: Basilica mate Basilica

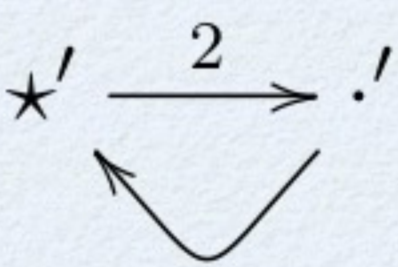




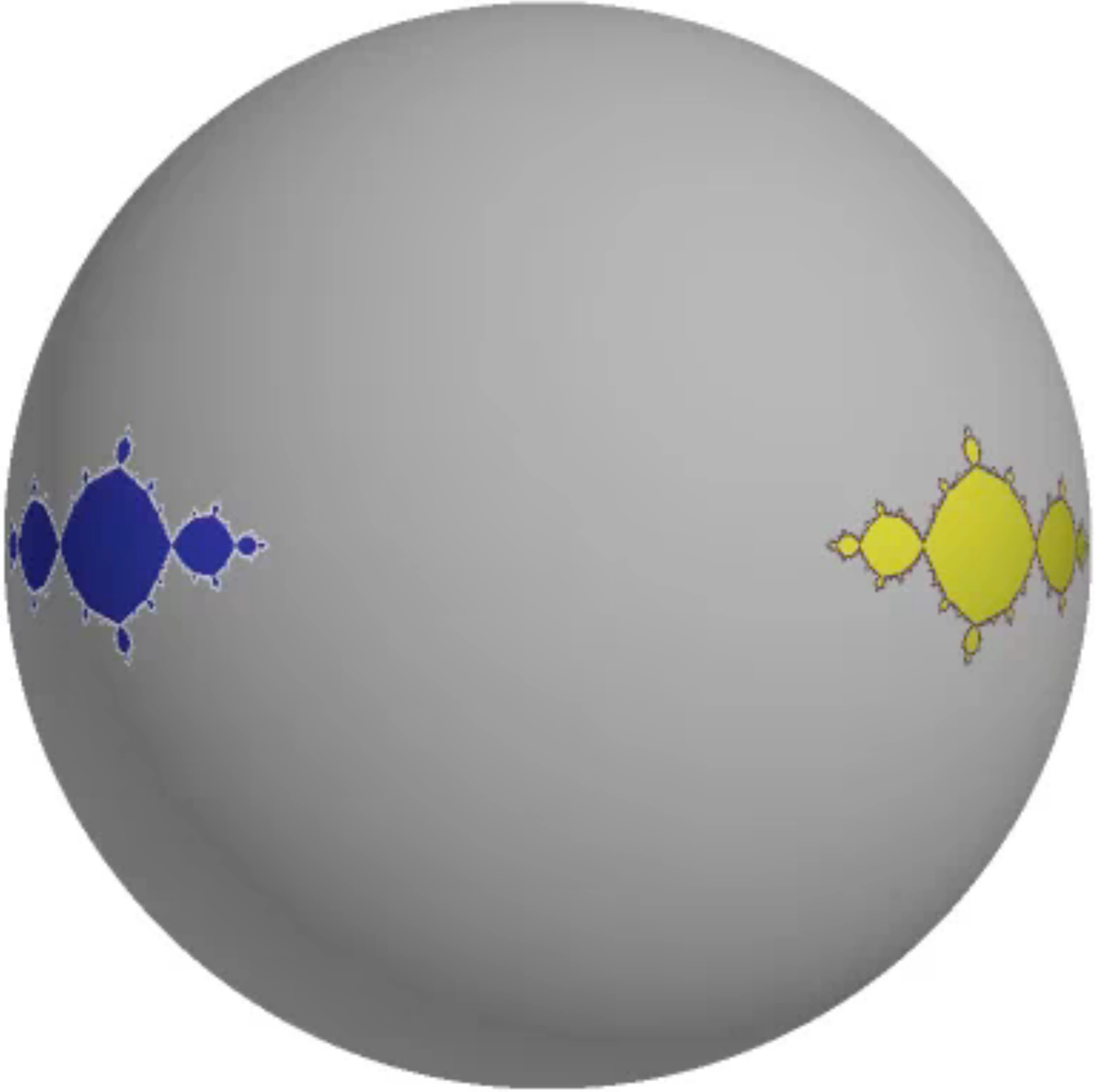
Formal mating:

$$f : (S^2, P) \to (S^2, P)$$





No geometric mating exists; this mating is *obstructed*.

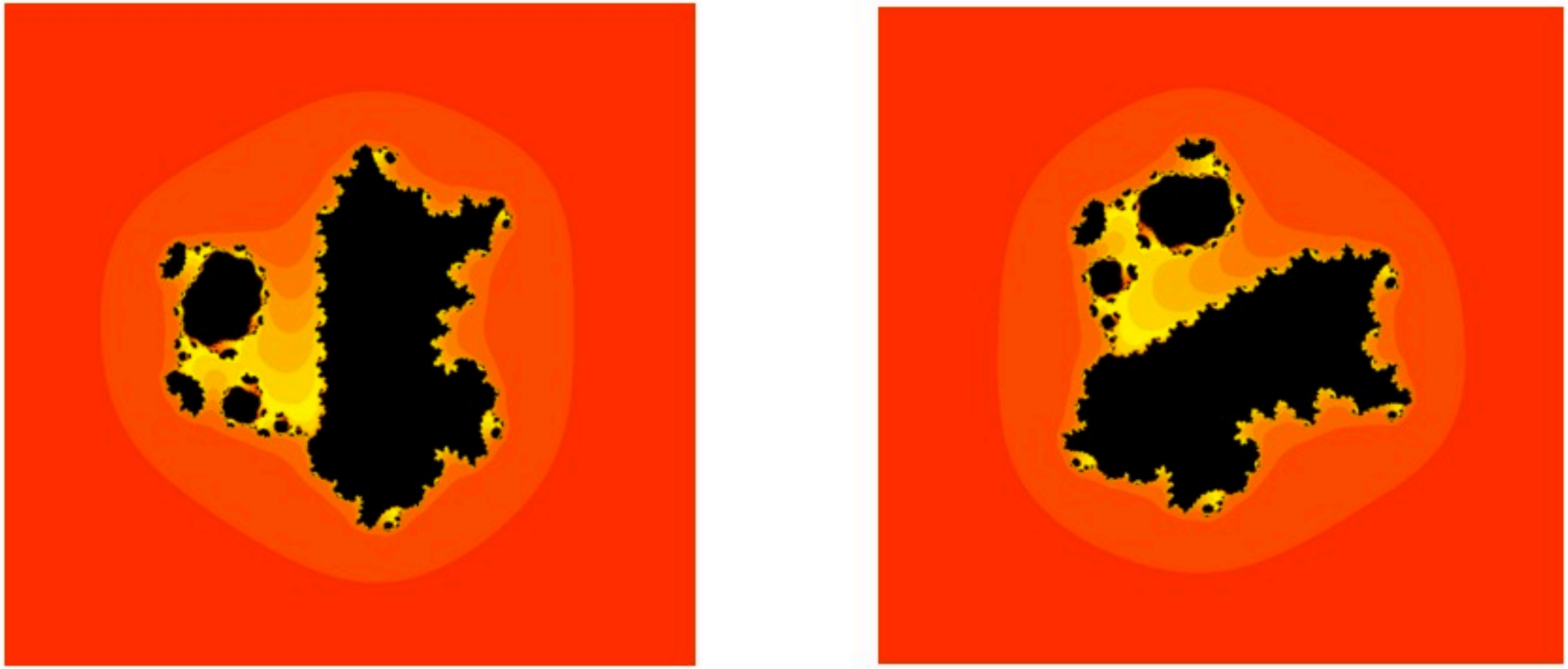


Twisted Matings

If P is a monic polynomial of degree $d \geqslant 2$, then the polynomial $T(P) : \mathbb{C} \to \mathbb{C}$ defined by

$$T(P)(z) = e^{-2\pi i/(d-1)} P(e^{2\pi i/(d-1)} z)$$

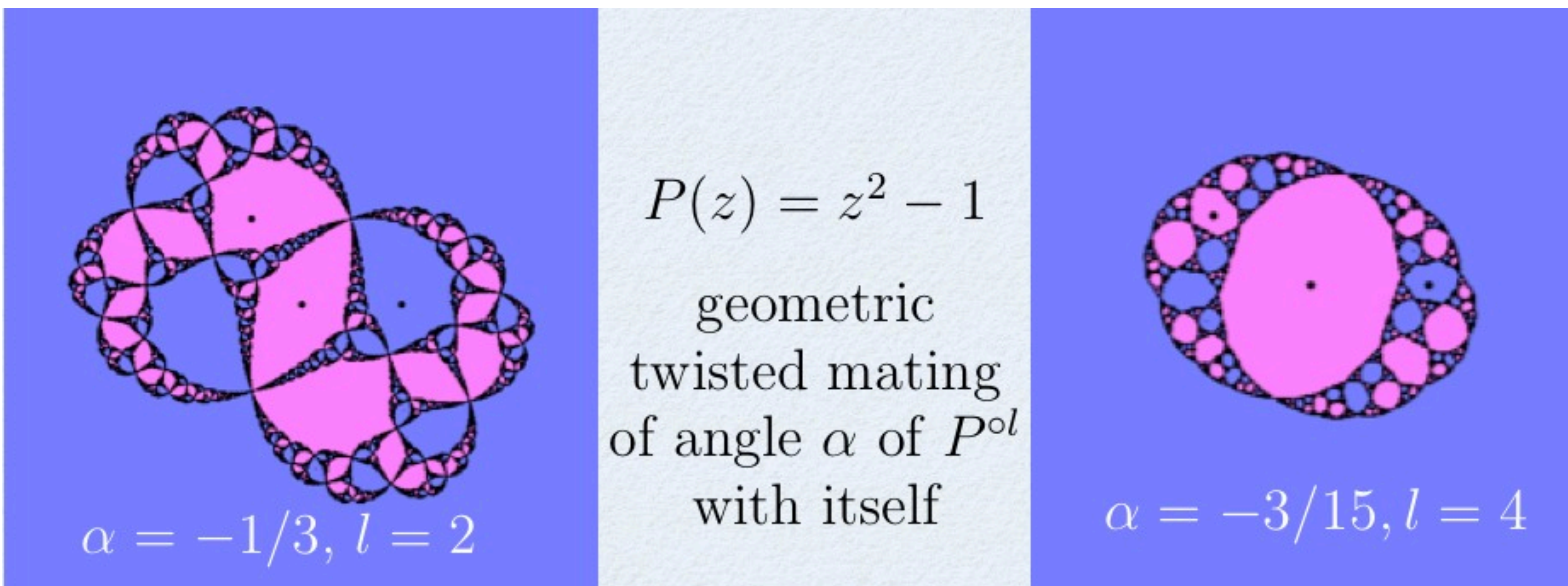
is also monic. The filled Julia set of $T(P)$ is the image of the Julia set of P by the rotation of angle $-1/(d-1)$ turns centered at 0.



$P : z \mapsto z^7 + (0.9 - 0.3i)z^4 + z + 0.2i$

Construct the formal mating $f : S^2 \to S^2$, and form $S^2/\sim$ by identifying θ and $-k/(d-1) - \theta$.

Proposition. Let P_1 and P_2 be two monic polynomials of degree $d \geqslant 2$ which are critically finite. Let $f : (S^2, \mathcal{P}_f) \to (S^2, \mathcal{P}_f)$ be the formal mating of P_1 and P_2, and let $g : (S^2, \mathcal{P}_g) \to (S^2, \mathcal{P}_g)$ be the formal mating of P_1 and $T^{\circ k}(P_2)$ (the twisted mating of angle $k/(d-1)$). Let $D : S^2 \to S^2$ be the Dehn twist around the equator of $S^2 - \mathcal{P}_f$. Then g is combinatorially equivalent to $D^{\circ k} \circ f$.



Preliminaries

Recall that if $f : (S^2, P) \to (S^2, P)$ is a critically finite branched cover, then there is an associated holomorphic endomorphism

$$\sigma_f : \mathcal{T}_P \to \mathcal{T}_P$$

where $\mathcal{T}_P$ is the *Teichmüller space* of (S^2, P):

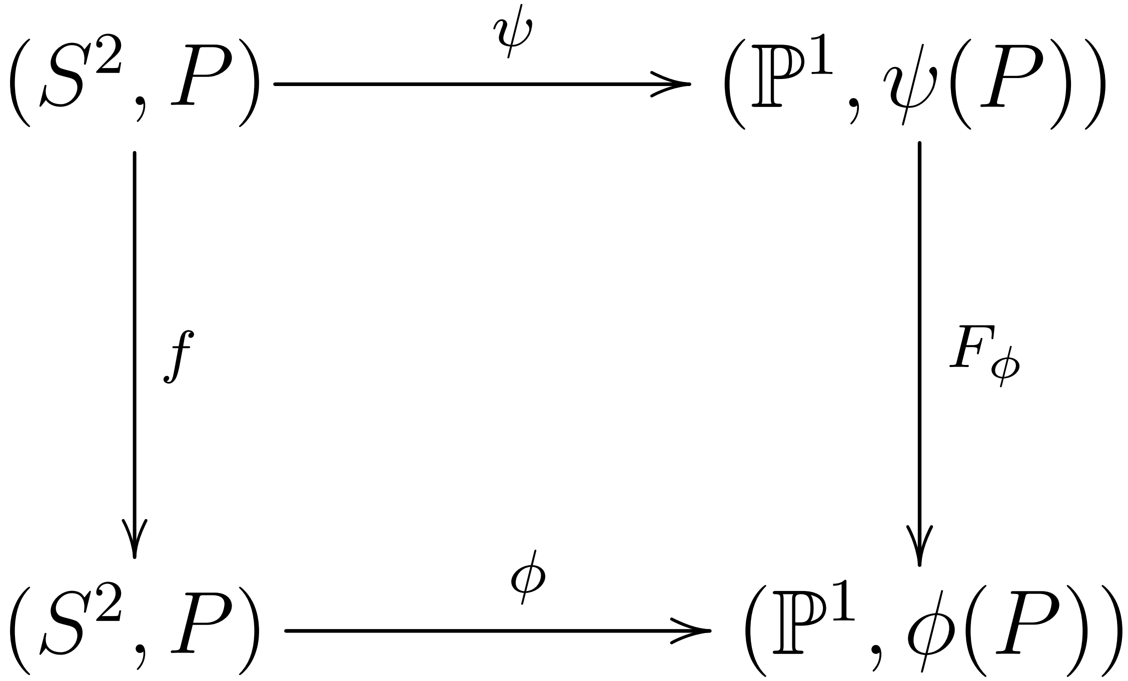
$$\phi : S^2 \to \mathbb{P}^1 \ : \ \phi_1 \sim \phi_2 \iff \exists \mu \in \mathrm{Aut}(\mathbb{P}^1) \text{ such that}$$

- $\phi_1|_P = (\mu \circ \phi_2)|_P,$ and
- ϕ_1 is isotopic to $\mu \circ \phi_2$ relative to P

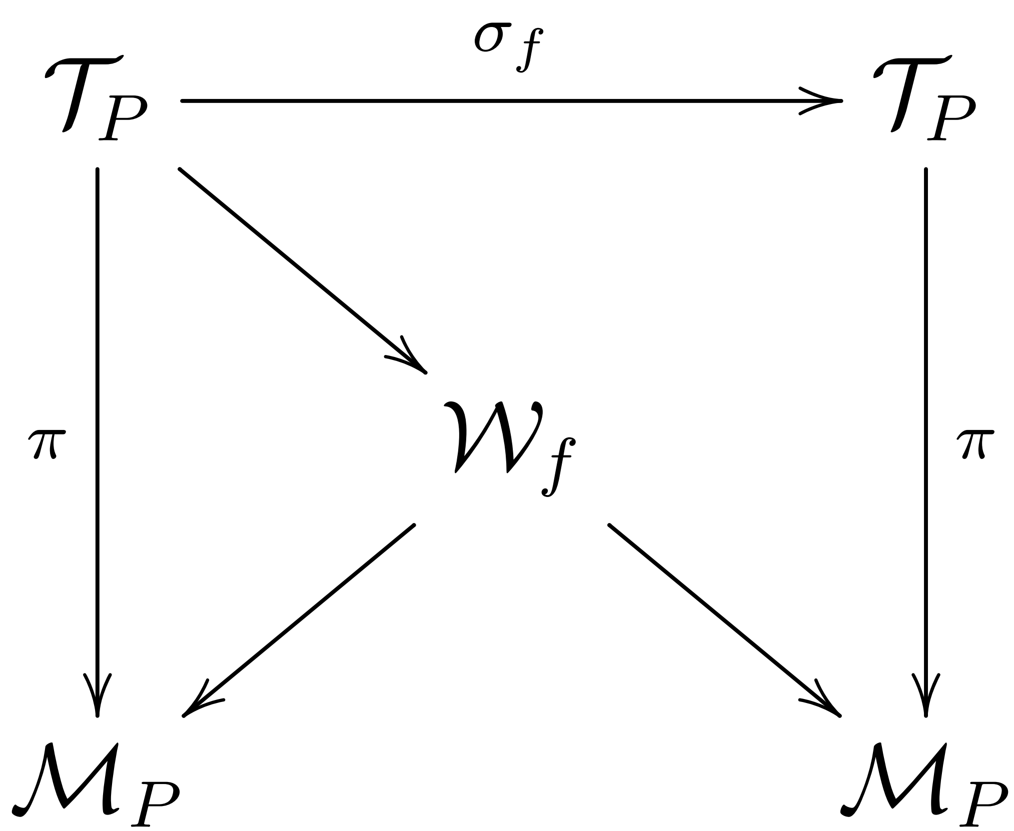
The space $\mathcal{T}_P$ is the universal cover of the *moduli space*, $\mathcal{M}_P$:

$$\{\varphi : P \hookrightarrow \mathbb{P}^1 \text{ up to postcomposition by elements of } \mathrm{Aut}(\mathbb{P}^1)\}.$$

$$\pi : \mathcal{T}_P \to \mathcal{M}_P$$

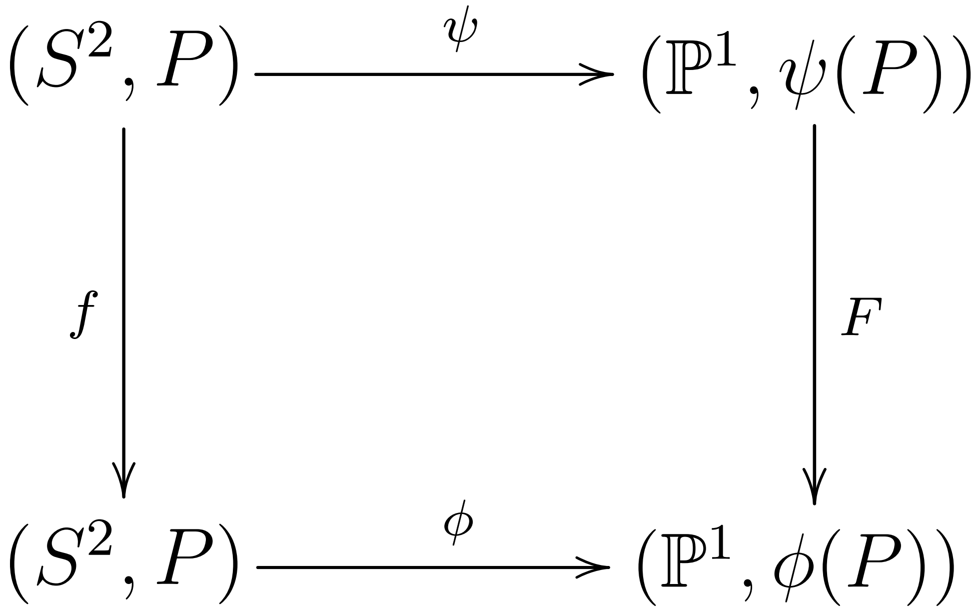


$$\sigma_f : [\phi] \mapsto [\psi]$$



$$(F_1, \alpha_1, \beta_1) \sim (F_2, \alpha_2, \beta_2) \iff \exists\, (\mu, \nu) \in \mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1) \text{ such that}$$
$$F_1 = \nu^{-1} \circ F_2 \circ \mu, \quad \alpha_2 = \mu \circ \alpha_1, \quad \text{and} \quad \beta_2 = \nu \circ \beta_1.$$

do an example....

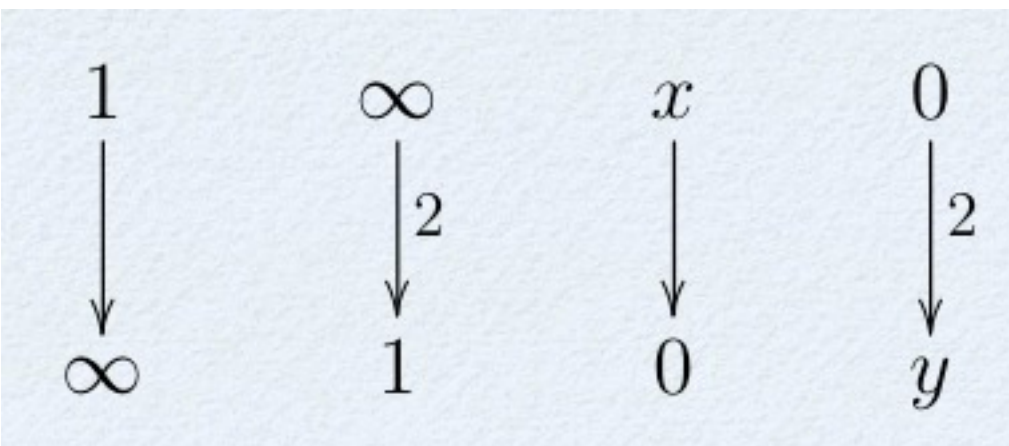


$\phi(\star) = 0$ $\quad$ $\psi(\star) = 0$

$\phi(\star) = \infty$ $\quad$ $\psi(\star) = \infty$

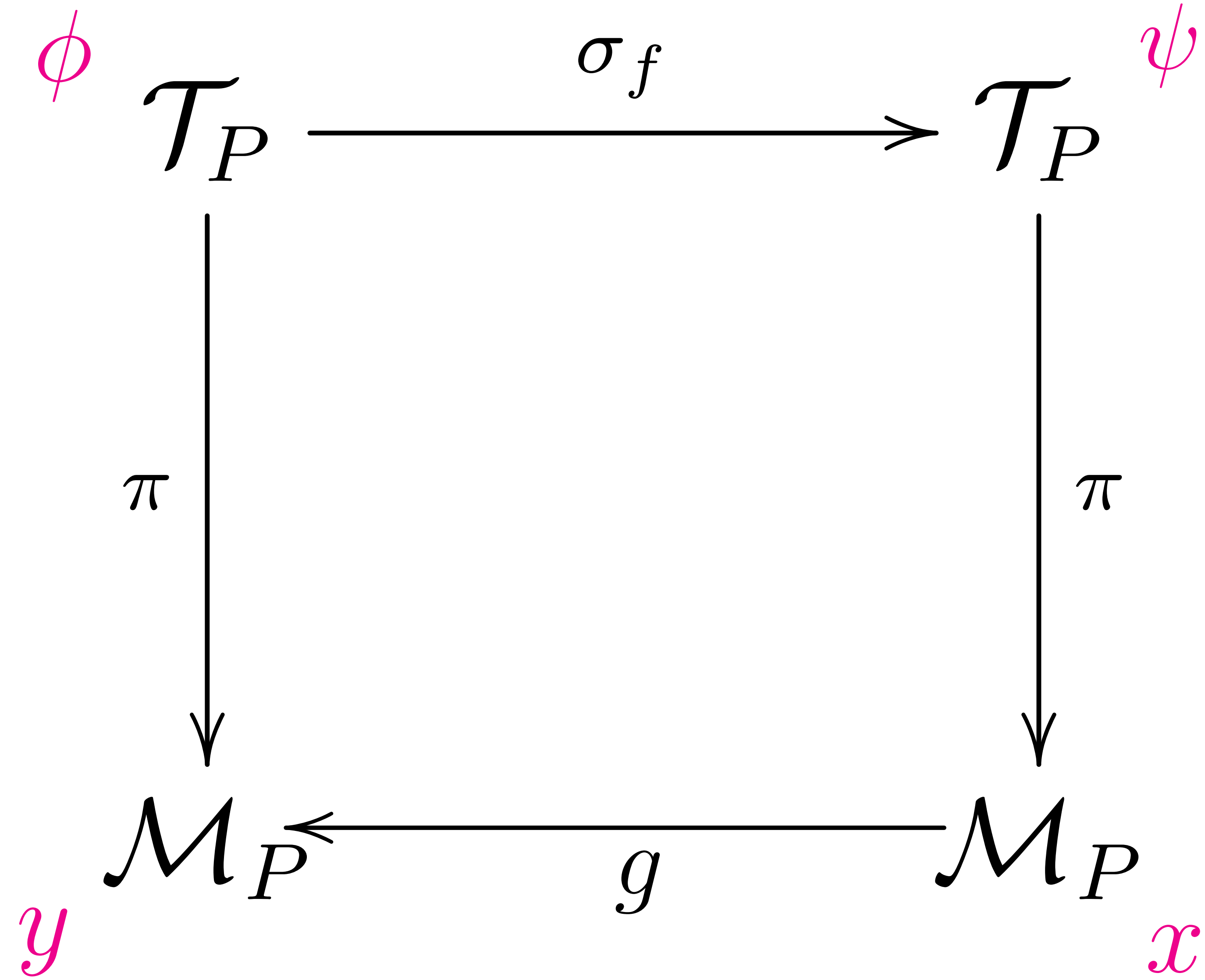
$\phi(q) = 1$ $\quad$ $\psi(q) = 1$

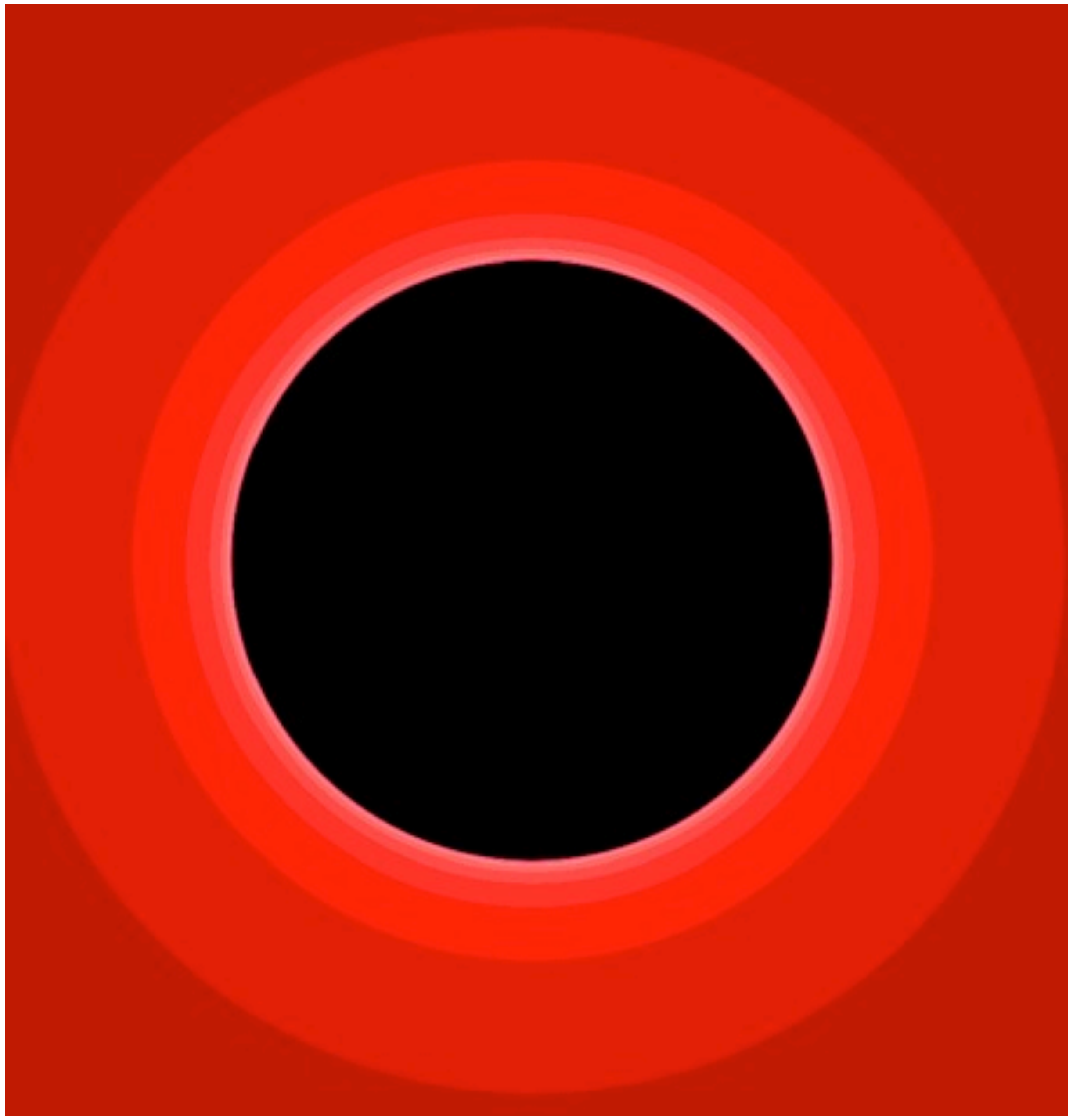
$\phi(p) = y$ $\quad$ $\psi(p) = x$



$$F(t) = \frac{t^2 - x^2}{t^2 - 1}$$

$$y = x^2$$



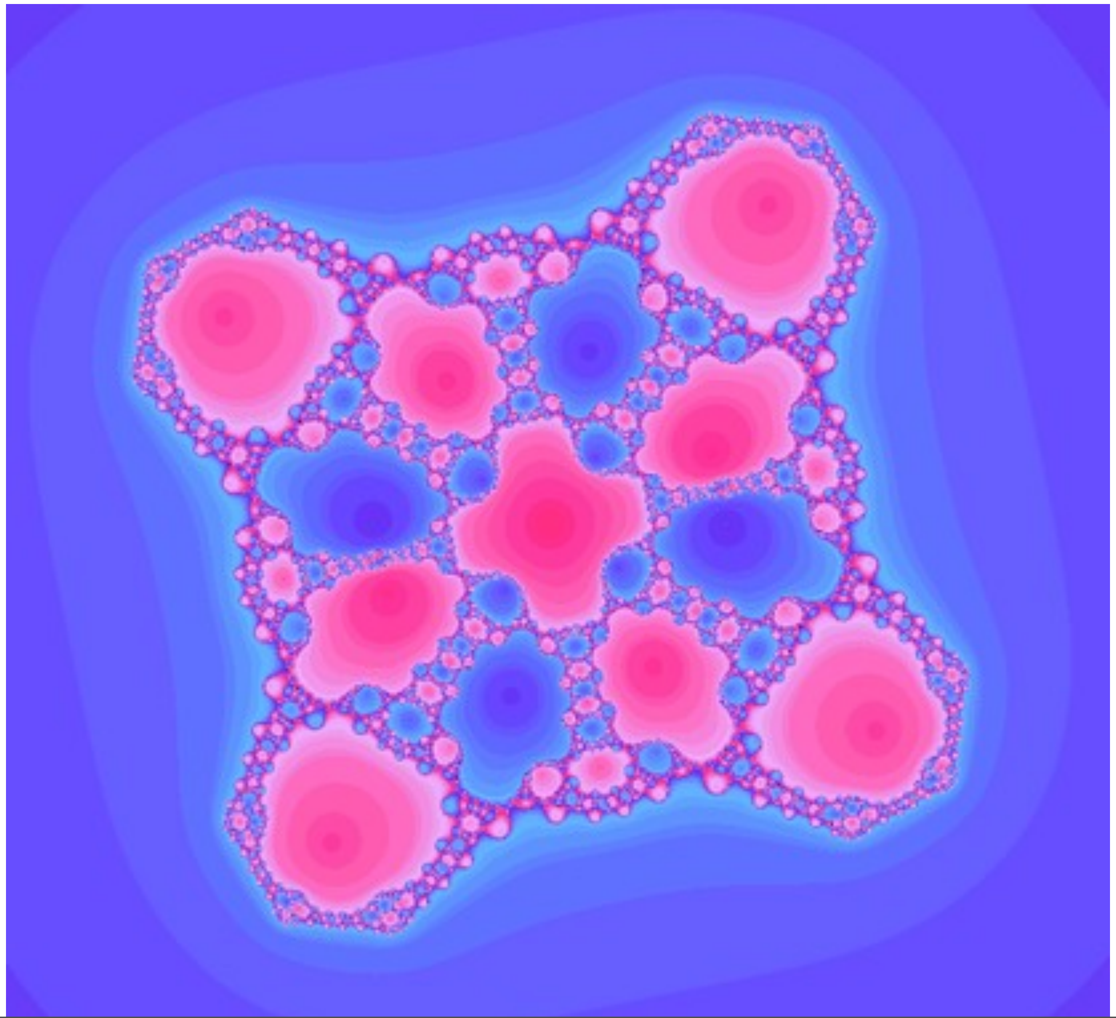


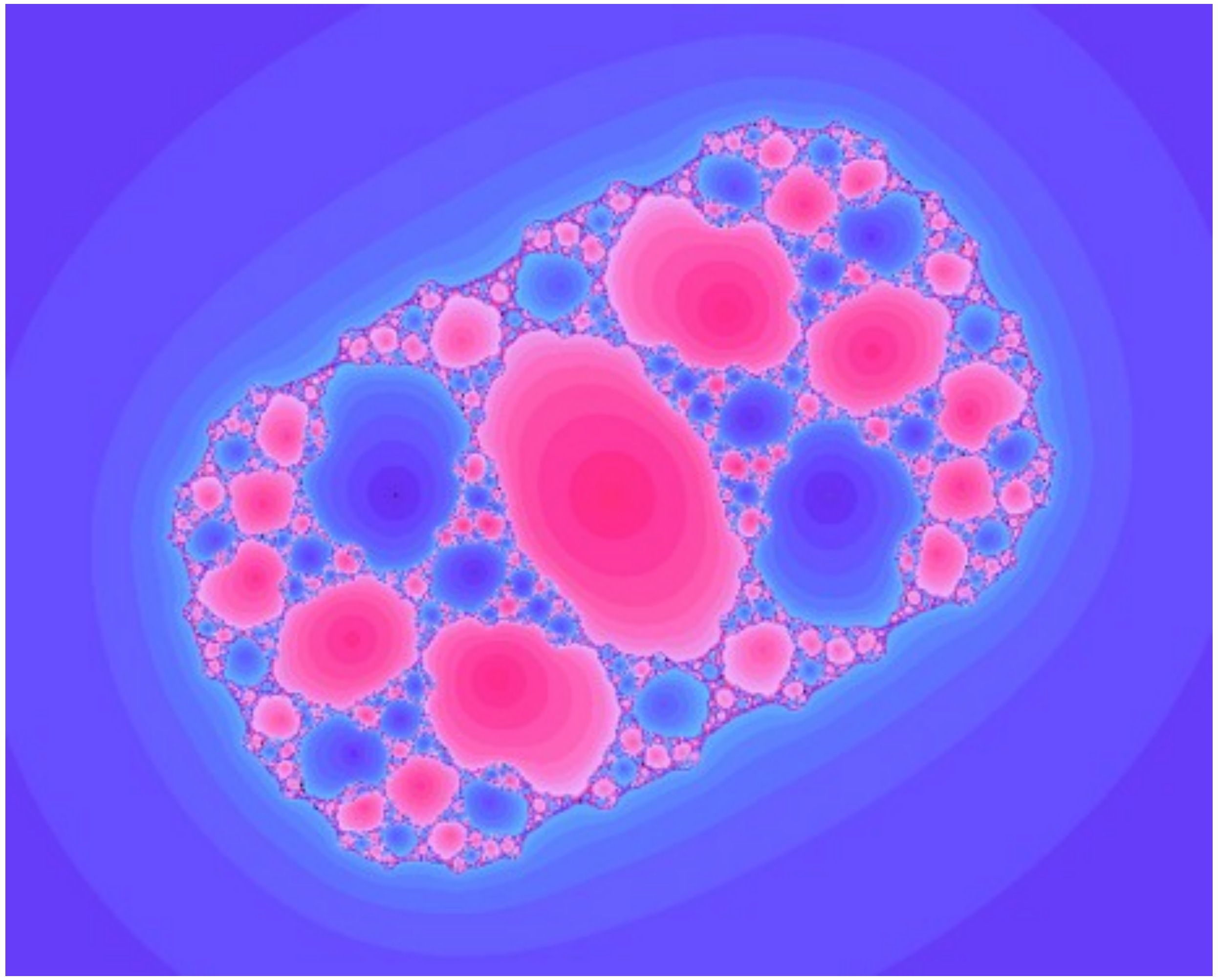
The skew product

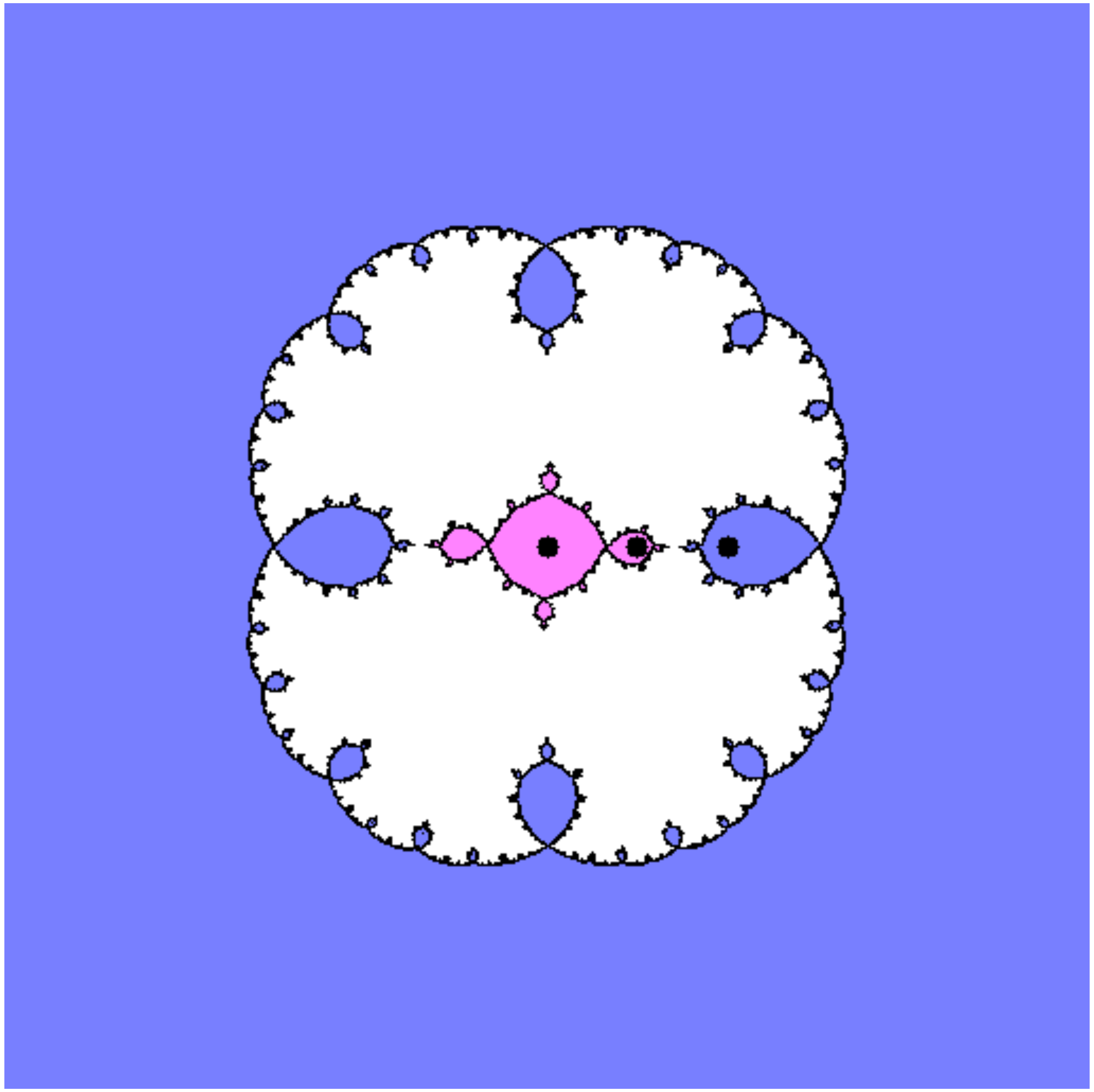
$$G : \mathbb{C}^2 \to \mathbb{C}^2 \quad \text{given by} \quad G : \begin{pmatrix} t \\ x \end{pmatrix} \mapsto \begin{pmatrix} F_x(t) \\ g(x) \end{pmatrix}$$

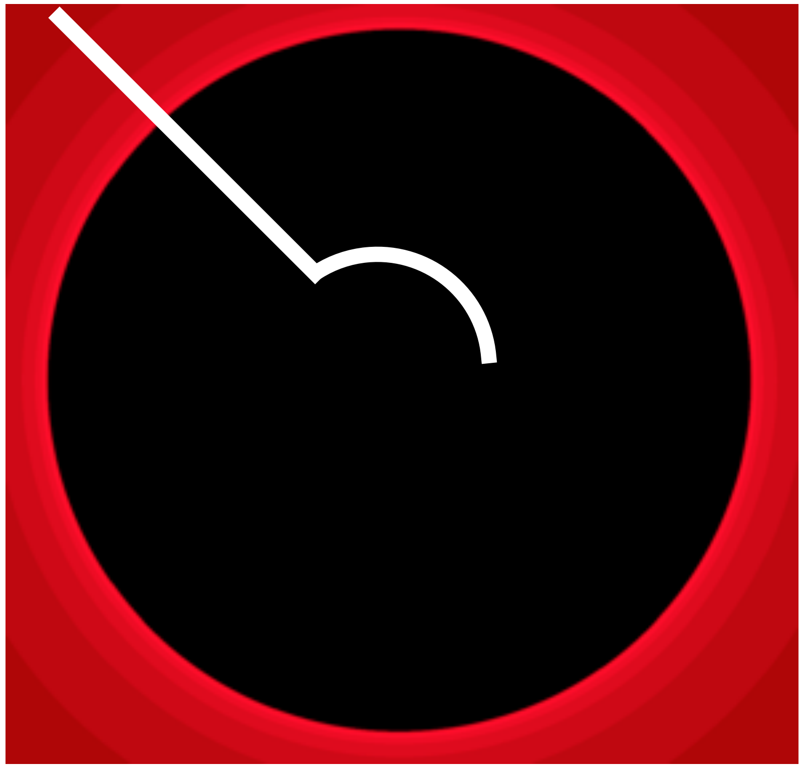
where $F_x(t) = (t^2 - x^2)/(t^2 - 1)$, and $g(x) = x^2$

Proposition. Let $\lambda = e^{2\pi i \alpha}$ be a periodic point of g, hence $\alpha = -k/(2^l - 1)$ for some l. If $k \neq 0$, the rational map $F_\lambda^{\circ l}$ is a geometric twisted mating of angle α of $P^{\circ l}$ with itself.







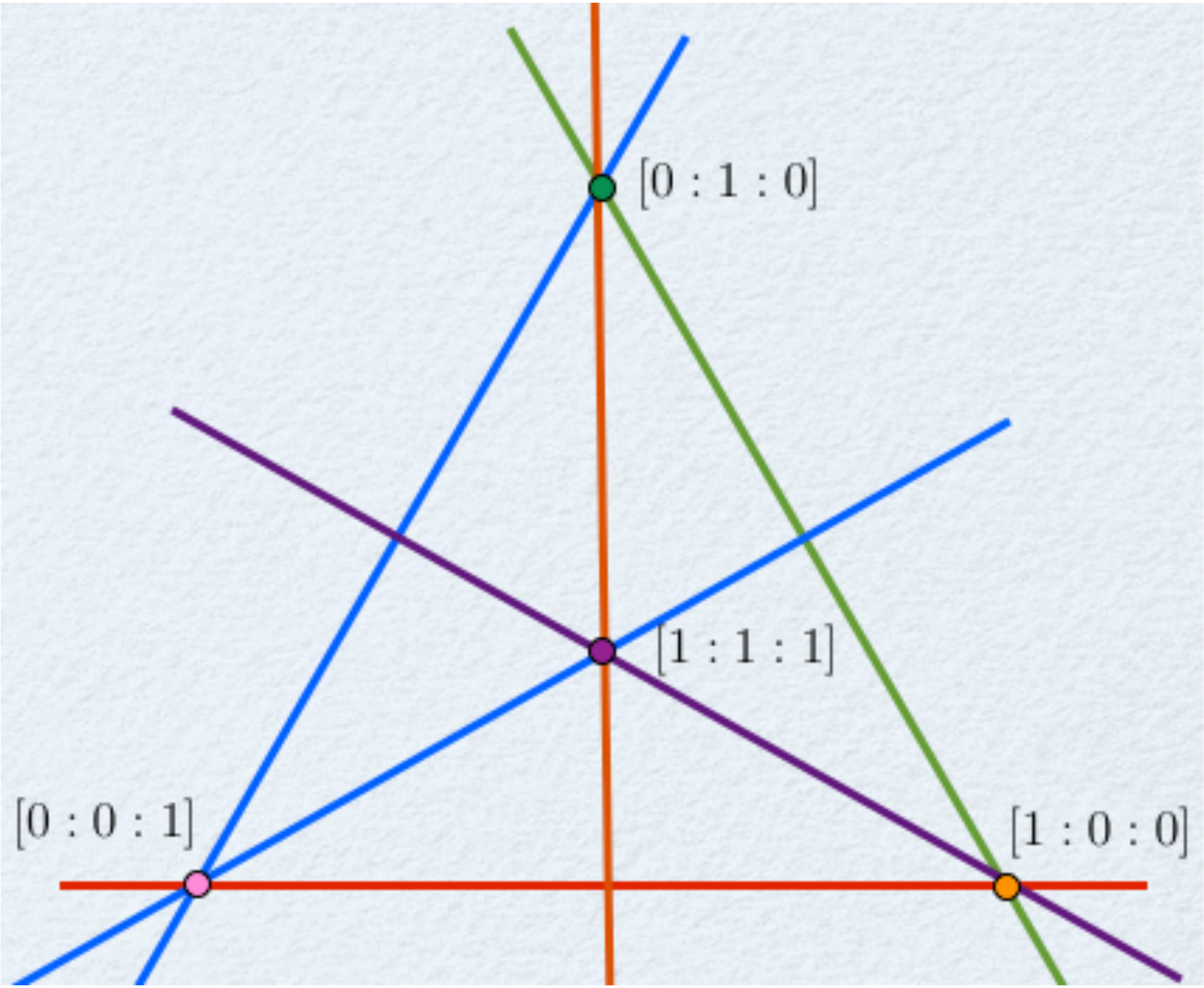


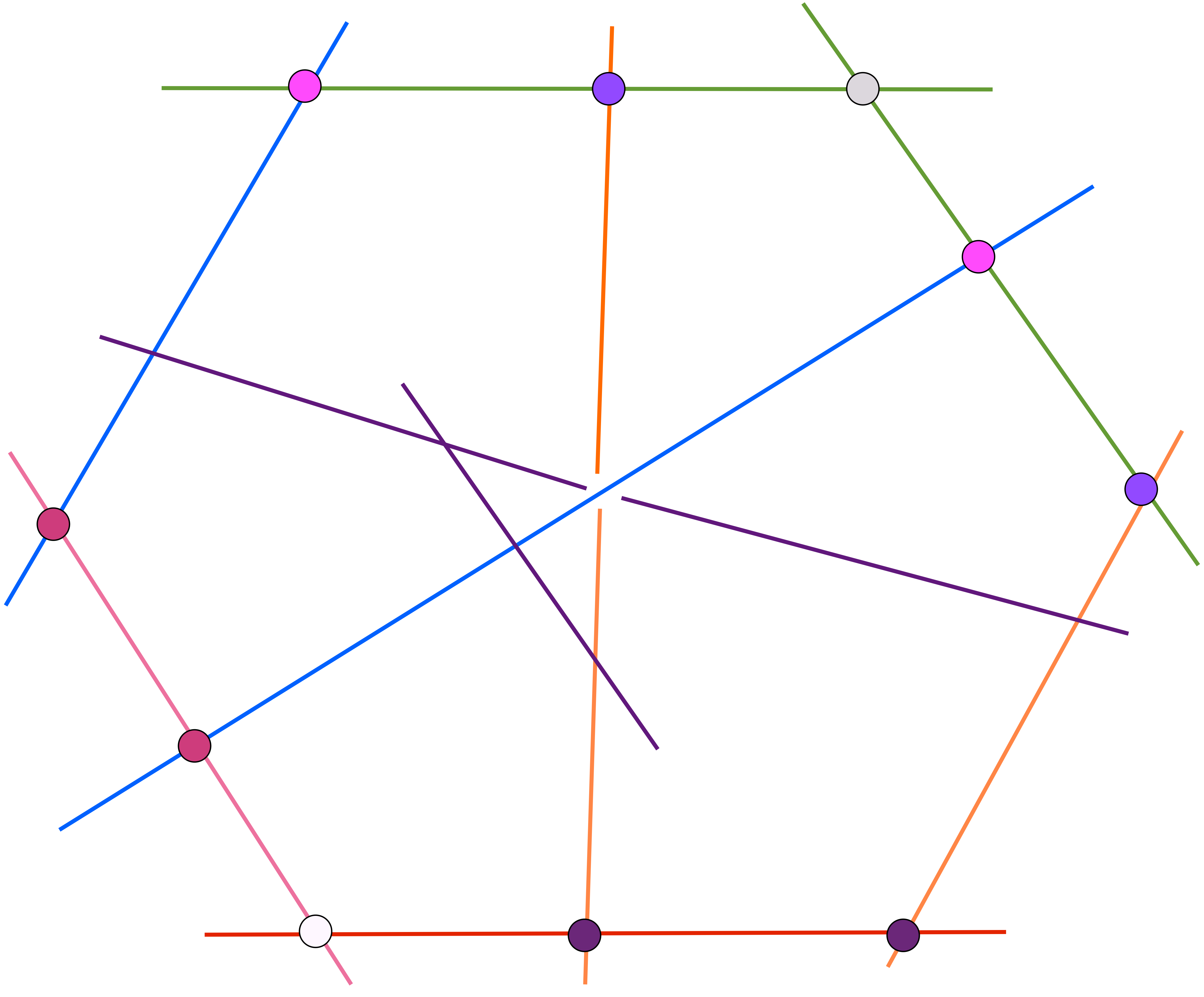
Compactifying

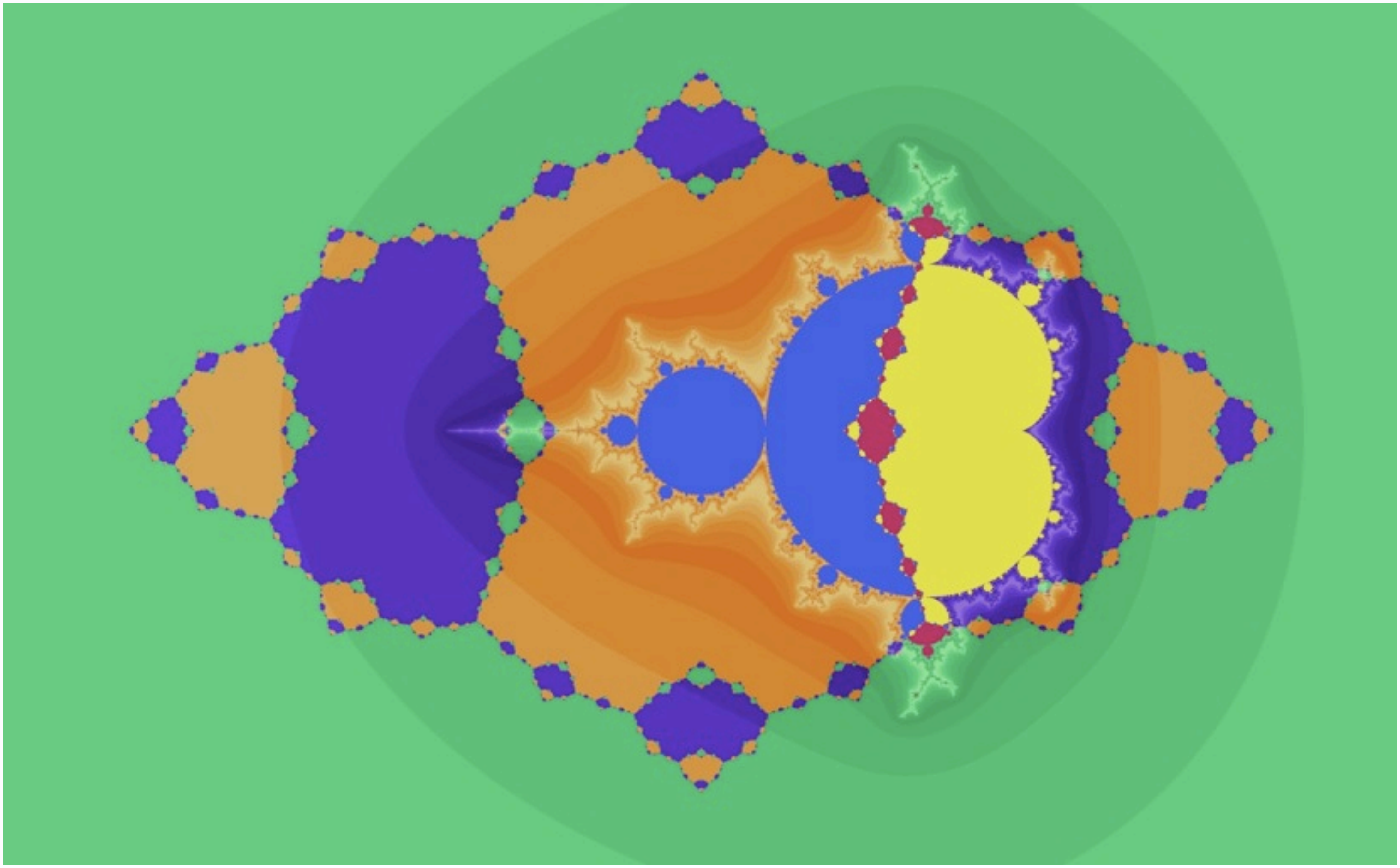
$$G : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [t : x : z] \mapsto [z^2(t^2 - x^2) : x^2(t^2 - z^2) : z^2(t^2 - z^2)]$$

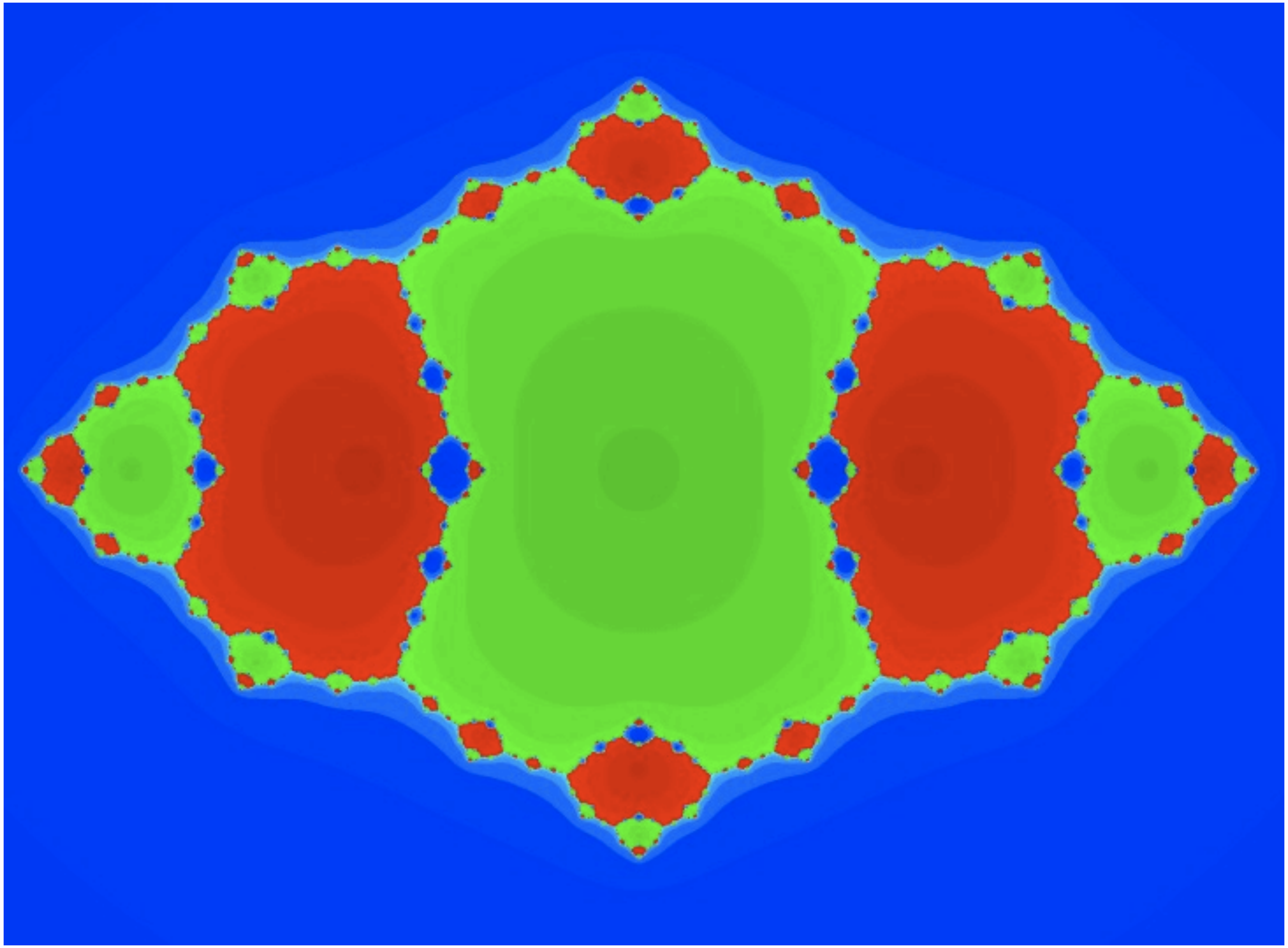
$$G = \mu \circ s \quad \text{where} \quad s : [t : x : z] \mapsto [t^2 : x^2 : z^2],$$

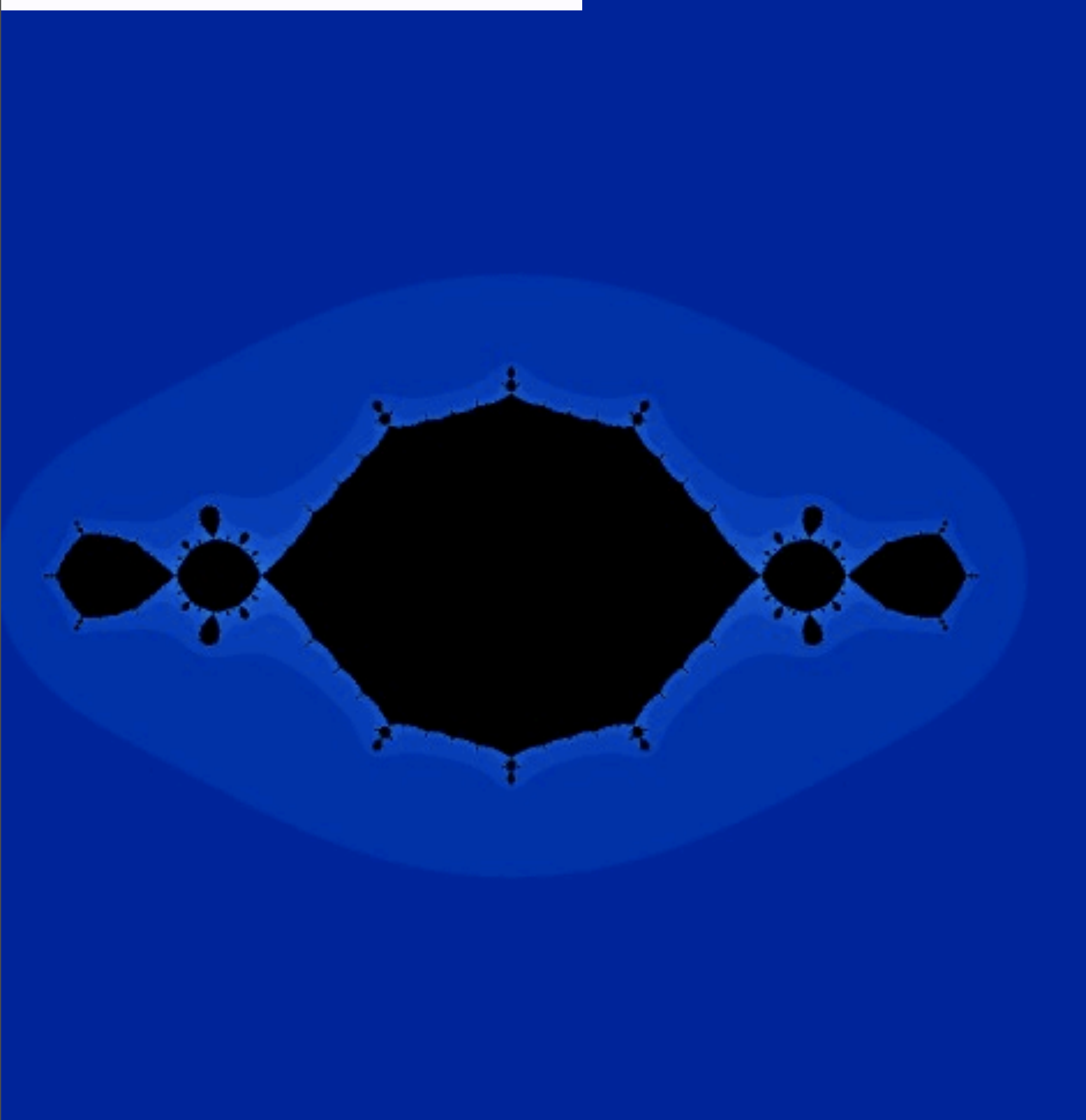
$$\mu : [t, x, z] \mapsto [z(t - x) : x(t - z) : z(t - z)]$$

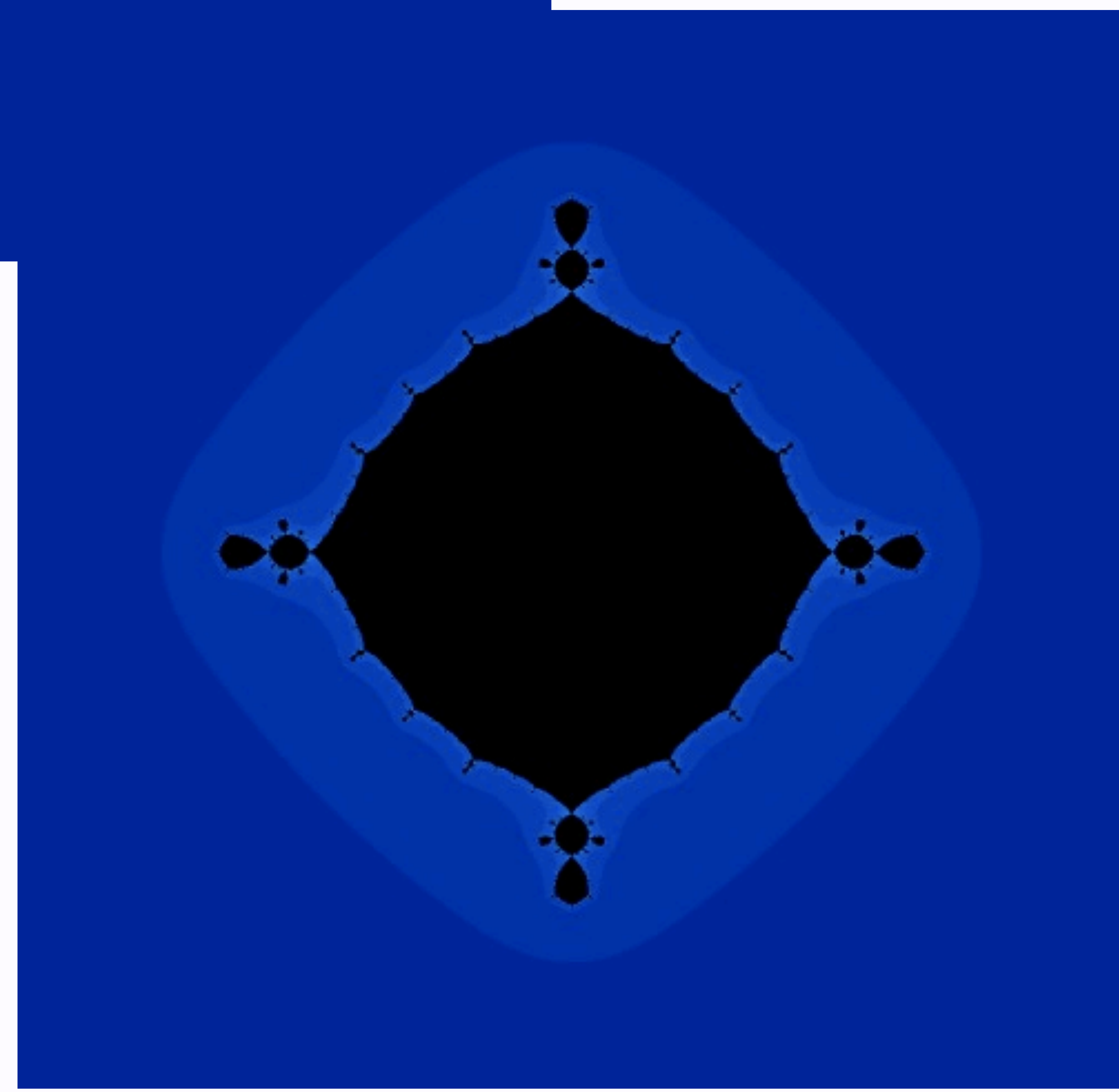


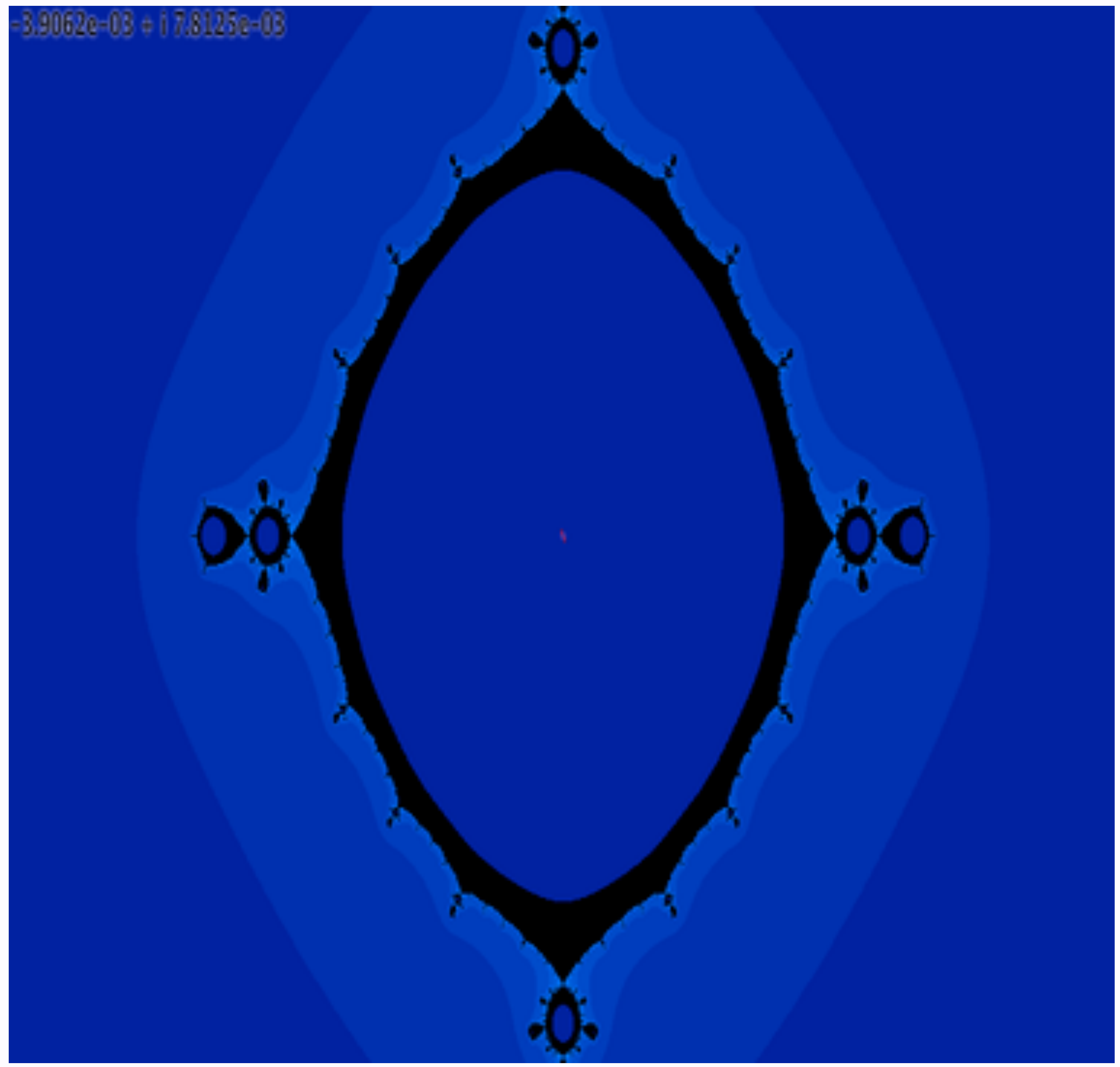








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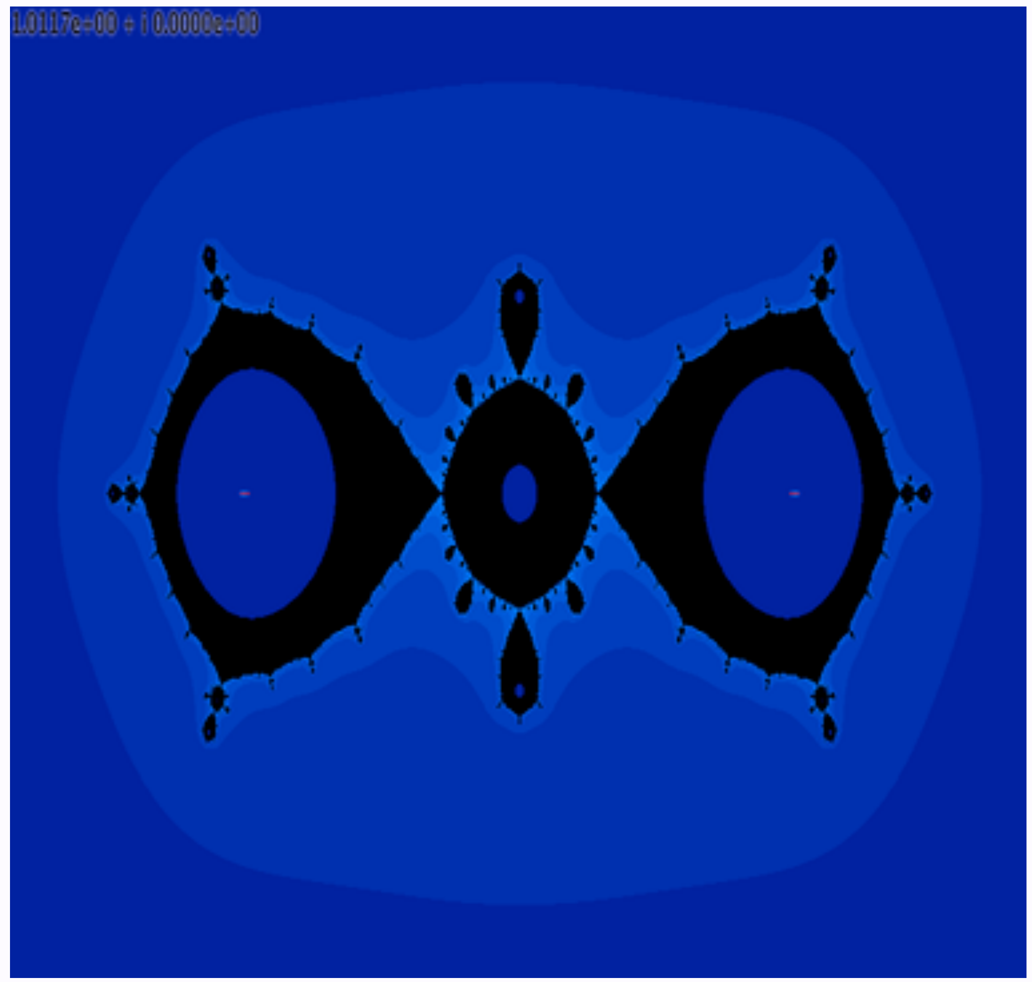
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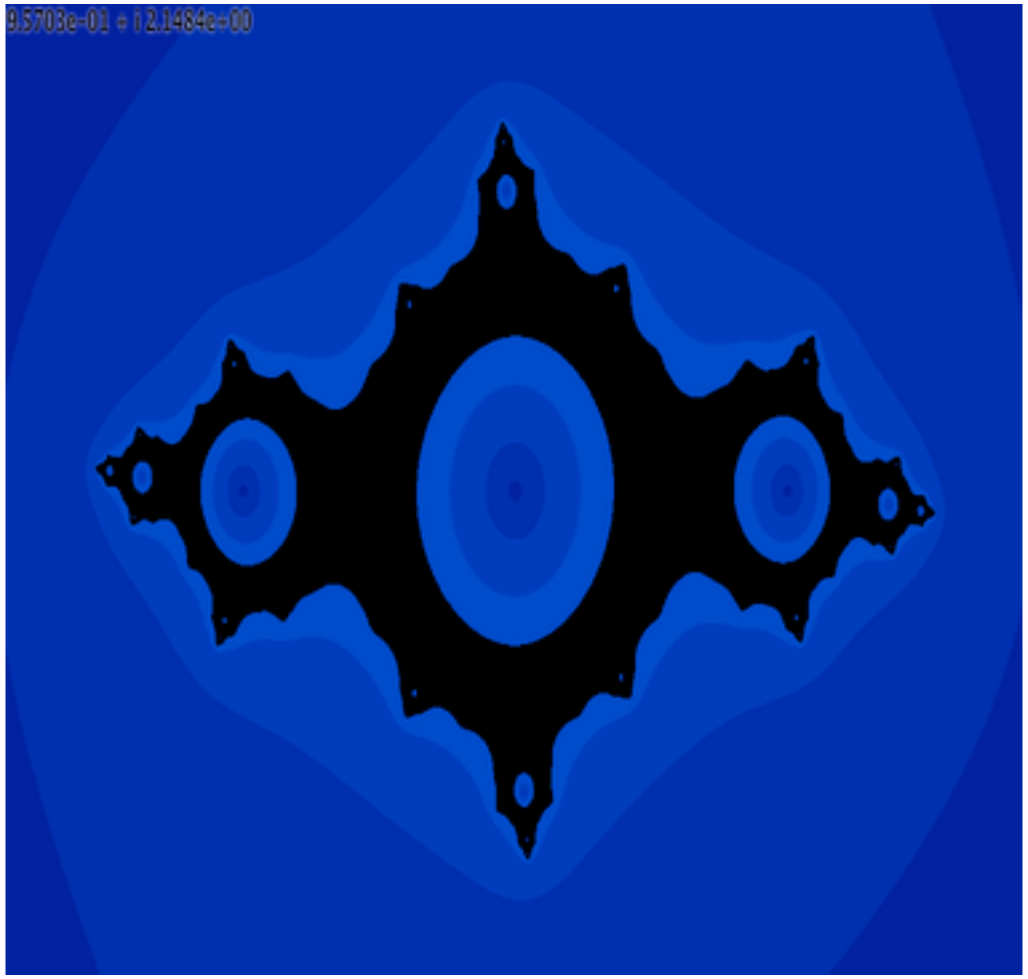

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merci :)