Beyond the Twisted Rabbit

Introduction

Goal - understand Thurston equivalence

Twisted rabbit problem

Twisted f problem: $f(z)=\frac{3z^2}{2z^3+1}$

Definitions/Notation:

Defn An orientation preserving branched cover $f: S^2 \longrightarrow S^2$ with $\deg(f) \geq 2$ is called a Thurston map if the postcritical set $P_f = \bigcup_{i>0} f^i(C_f)$ is finite

__Defn__ Two Thurston maps f, g are Thurston (combinatorially) equivalent if there are orientation-preserving homeos $h_0, h_1 : (\hat{\mathbb{C}}, P_f) \longrightarrow (\hat{\mathbb{C}}, P_g)$ so that

$$\begin{tikzcd}
(\hat{\mathbb{C}}, P_f) \arrow[r, "h_1"] \arrow[d, "f"'] \arrow[dr, phantom, "//"] & (\hat{\mathbb{C}}, P_g) \arrow[d, "g"] \\
(\hat{\mathbb{C}}, P_f) \arrow[r, "h_0"] & (\hat{\mathbb{C}}, P_g)
\end{tikzcd}$$

where $h_0 \simeq h_1$ rel P_f

__Defn__ $$\mathcal{T}_f := \left\{ \begin{matrix} \text{orientation preserving} \\ \text{homeos } \varphi : (S^2, P_f) \longrightarrow \hat{\mathbb{C}} \end{matrix} \right\} \Big/ \sim$$

where $\varphi_1 \sim \varphi_2$ if $\exists$ Möbius M so that φ_2 is isotopic to $M \circ \varphi_1$ rel P_f

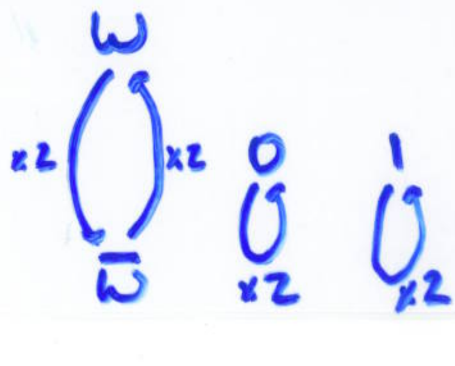
<u>Defn</u> (Thurston Pullback Map) $\sigma_f : \mathcal{T}_f \to \mathcal{T}_f$

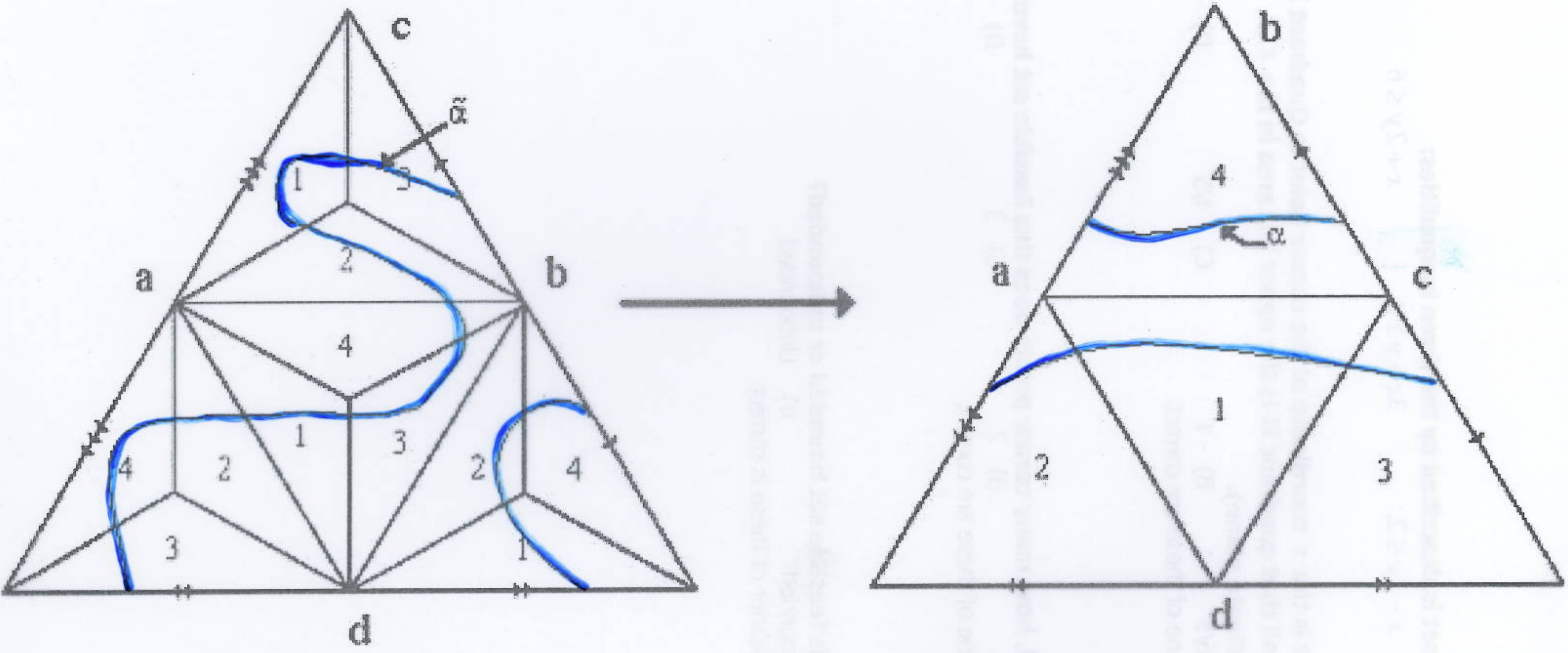
Let $[\tau] \in \mathcal{T}_f$. Then $\sigma_f([\tau]) = [\tilde{\tau}]$

$$
\begin{tikzcd}
(S^2, P_f) \arrow[r, "\tilde{\tau}"] \arrow[d, "f"] & (\hat{\mathbb{C}}, \tilde{\tau}(P_f)) \\
(S^2, P_f) \arrow[r, "\tau"] & (\hat{\mathbb{C}}, \tau(P_f))
\end{tikzcd}
$$

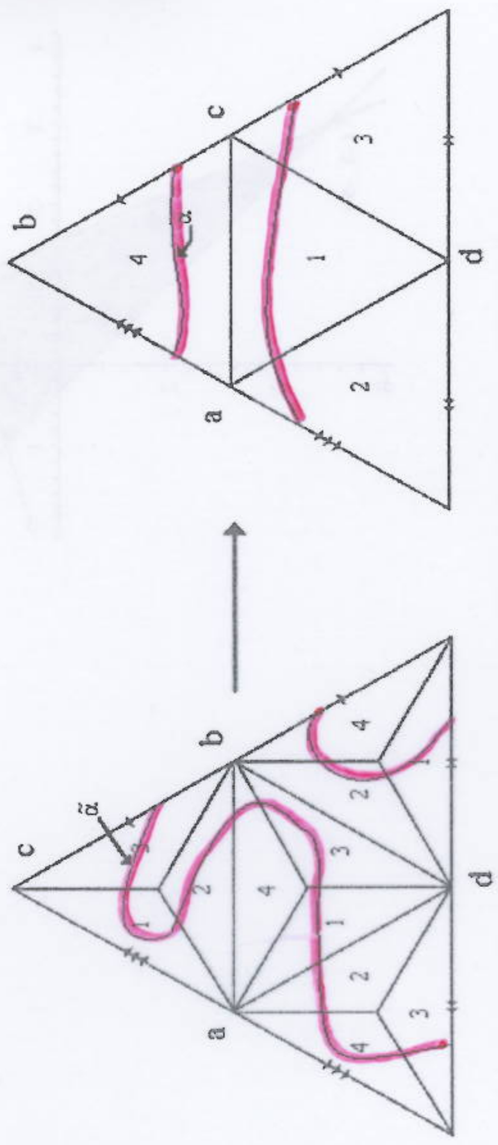
Why is $f(z) = \dfrac{3z^2}{2z^3+1}$ so interesting?

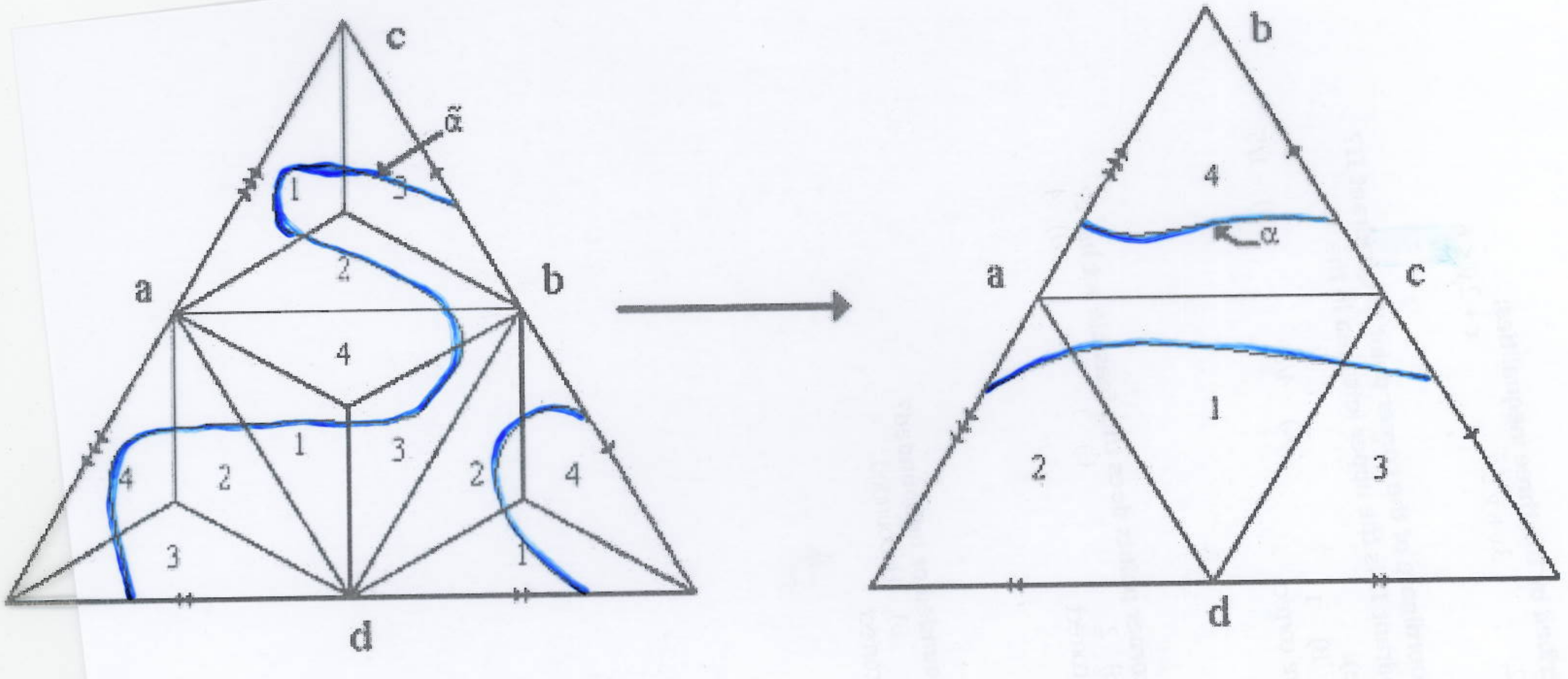
- More complicated than quadratic
- Nice triangulated model
- Studied in BEKP
  - → Picture of σ_f
  - → Selinger: $\sigma_f(S_\Gamma) \subset S_{f^{-1}(\Gamma)}$
  - → Correspondence on moduli space
  - → $P_f = \{0, 1, \omega, \bar{\omega}\}$




c
$\tilde{\alpha}$
a
b
d
1
2
3
4
b
a
c
d
α
1
2
3
4

(5)





Finite global attractor for the pullback relation

$\mathcal{C}_f = \{ \text{multicurves in } \hat{\mathbb{C}} \setminus P_f \} / \simeq$

$= \left\{ \begin{array}{l} \text{curves } \gamma \text{ so that the complement} \\ \text{of } \gamma \text{ is two twice-punctured disks} \end{array} \right\} / \simeq$

Pullback relation

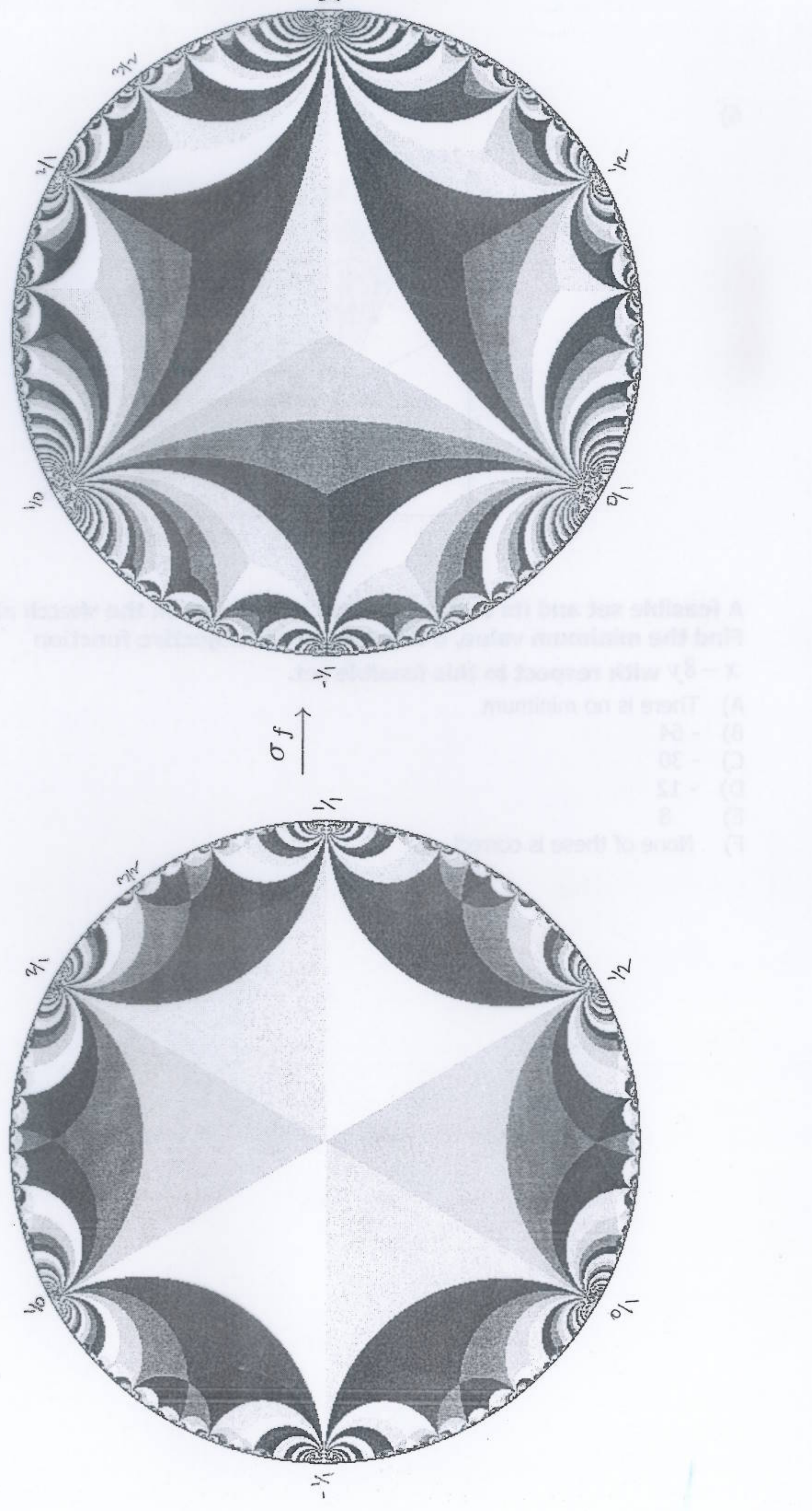
$$\mathcal{C}_f \cup \{\emptyset\} \xleftarrow{f^*} \mathcal{C}_f \cup \{\emptyset\}$$

$$f^*([\gamma]) = [f^{-1}(\gamma)]$$

Provides an invariant for Thurston equivalence.

How to compute? Virtual Endomorphisms

(7)



$$\begin{tikzcd}
\text{Curve} \arrow[d] & & \text{Curve} \\
\text{[Rig]hthand Dehn Twist} \arrow[d] & & \text{Righthand Dehn Twist} \arrow[u] \\
\text{[Para]bolic element [of] } \pi_1(\hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\}) \arrow[rr, "\varphi_f"] & & \text{Parabolic element of } \pi_1(\hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\}) \arrow[u]
\end{tikzcd}$$

[Theore]m: $\exists$ a finite global attractor for the pullback relation on $\mathcal{C}_f$. The attractor has three elements with the following mapping properties:

[Goal]: Understand φ_f under iteration

(9)

Two virtual endomorphisms:

1. $\varphi_f : H < \pi_1(\hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\}) \longrightarrow \pi_1(\hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\})$

$$\begin{tikzcd}
\mathrm{Teich}(\hat{\mathbb{C}}, P_f) \arrow[rr, "\sigma_f"] \arrow[d, "\pi"'] \arrow[dr] & & \mathrm{Teich}(\hat{\mathbb{C}}, P_f) \arrow[d, "\pi"] \\
\hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\} & \hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega},-1,-\omega,-\bar{\omega}\} \arrow[l, "\deg 4"] \arrow[r, "\deg 2"'] & \hat{\mathbb{C}} \setminus \{1,\omega,\bar{\omega}\}
\end{tikzcd}$$

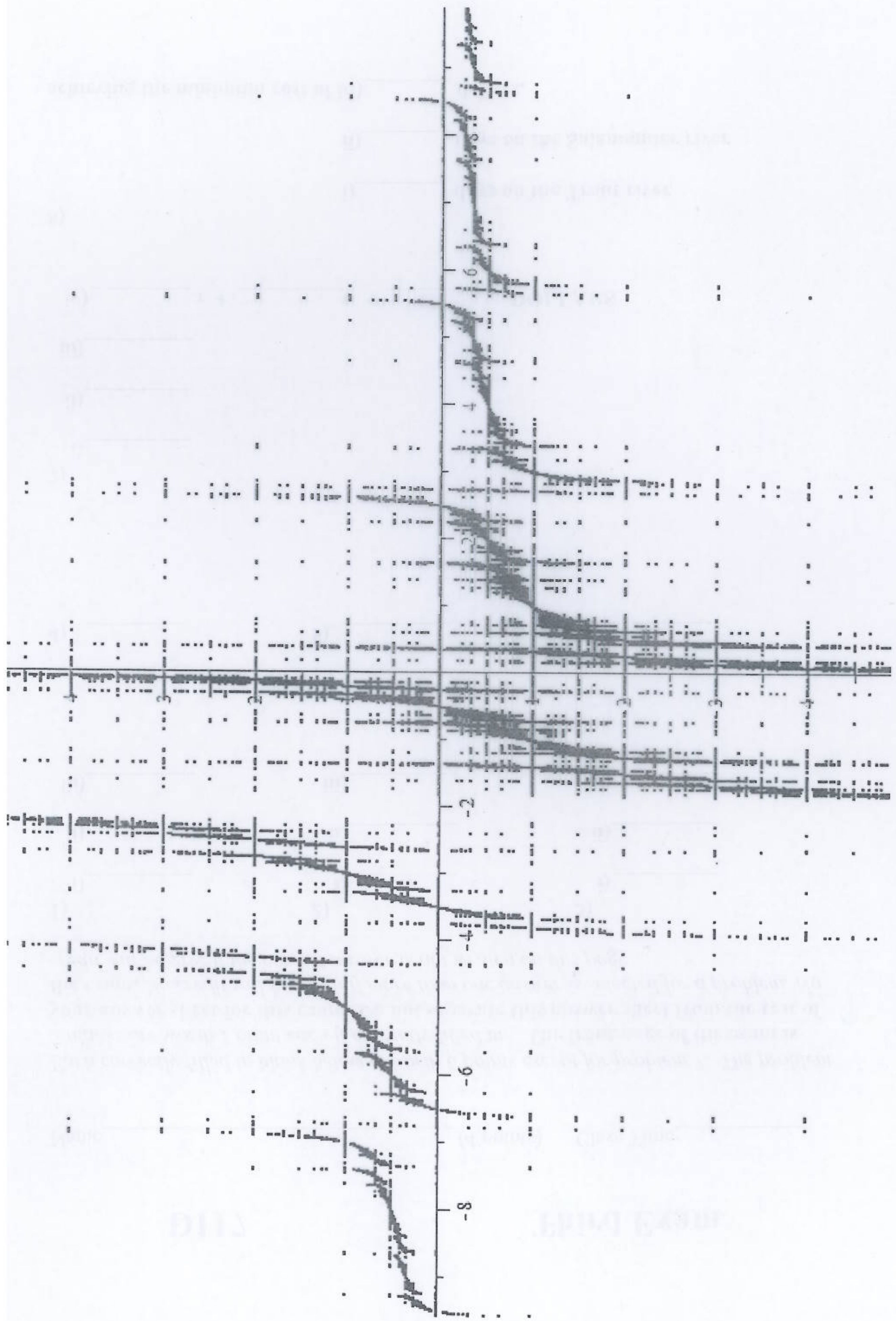
2. $\psi : H_f < \mathrm{PMCG}(\hat{\mathbb{C}}, P_f) \longrightarrow \mathrm{PMCG}(\hat{\mathbb{C}}, P_f)$

$\psi(h) = \tilde{h}$ where $h \in H_f = \{\text{elements of PMCG that lift}\}$

$$\begin{tikzcd}
(\hat{\mathbb{C}}, P_f) \arrow[r, "\tilde{h}"] \arrow[d, "f"'] & (\hat{\mathbb{C}}, P_f) \arrow[d, "f"] \\
(\hat{\mathbb{C}}, P_f) \arrow[r, "h"] & (\hat{\mathbb{C}}, P_f)
\end{tikzcd}$$

Note: If h is a Dehn twist, then $\tilde{h}$ is too for this particular f. Thus, if we understand ψ, we understand the pullback relation

(8)



Can define slope of a curve using a choice of basis for the canonical double cover of $(\hat{\mathbb{C}}, P_f)$

Example of slope computation of pullback

$$\tfrac{1}{0} \xrightarrow{f} -\tfrac{8}{3} \qquad \boxed{A(z) = \frac{z}{-2z+1} \quad B(z) = z+2}$$

$$-\tfrac{8}{3} = B^{-1}.-\tfrac{2}{3} = B^{-1}A.\tfrac{2}{1} = B^{-1}AB.\tfrac{0}{1}$$

$$= \tfrac{0}{1}.\beta\alpha\beta^{-1}$$

$$\underline{\beta\alpha^{-1}\beta^{-1}}\,\alpha^{3}\,\underline{\beta\alpha\beta^{-1}} \longrightarrow e\beta^{-1}ee\beta ee\beta e = \beta$$

$$\tfrac{0}{1}.\beta = B.\tfrac{0}{1} = \tfrac{0}{1}$$

Nice results:

Finite global attractor: $\{0/1, 1/0, -1/1\}$

$\frac{\text{Odd}}{\text{Odd}} \longrightarrow \cdots \longrightarrow -1/1$

$\frac{\text{Odd}}{\text{Even}} \longrightarrow \cdots \longrightarrow 0/1 \rightleftarrows 1/0$

$\frac{\text{Even}}{\text{Odd}} \longrightarrow \cdots \longrightarrow 0/1$

∞ to 1

surjective

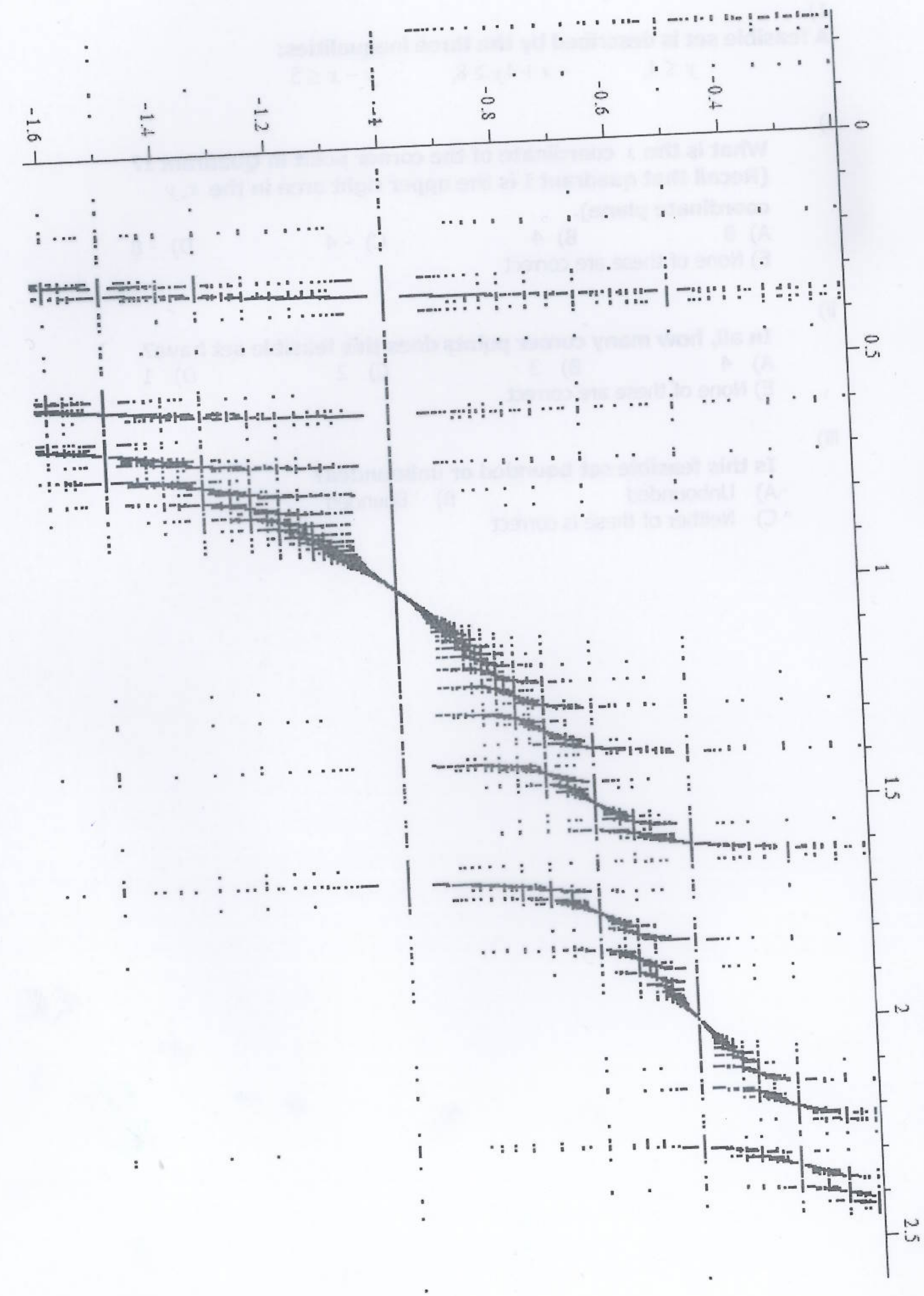
$h\left(\frac{p}{q}\right) = \max(|p|, |q|)$

Decreasing naive rational height?

Suppose $\sigma_f\left(\frac{p}{q}\right) = \frac{p'}{q'}$, $\sigma_f\left(\frac{r}{s}\right) = \frac{r'}{s'}$, and assume that $\begin{vmatrix} p & r \\ q & s \end{vmatrix} = \pm 1$. Then

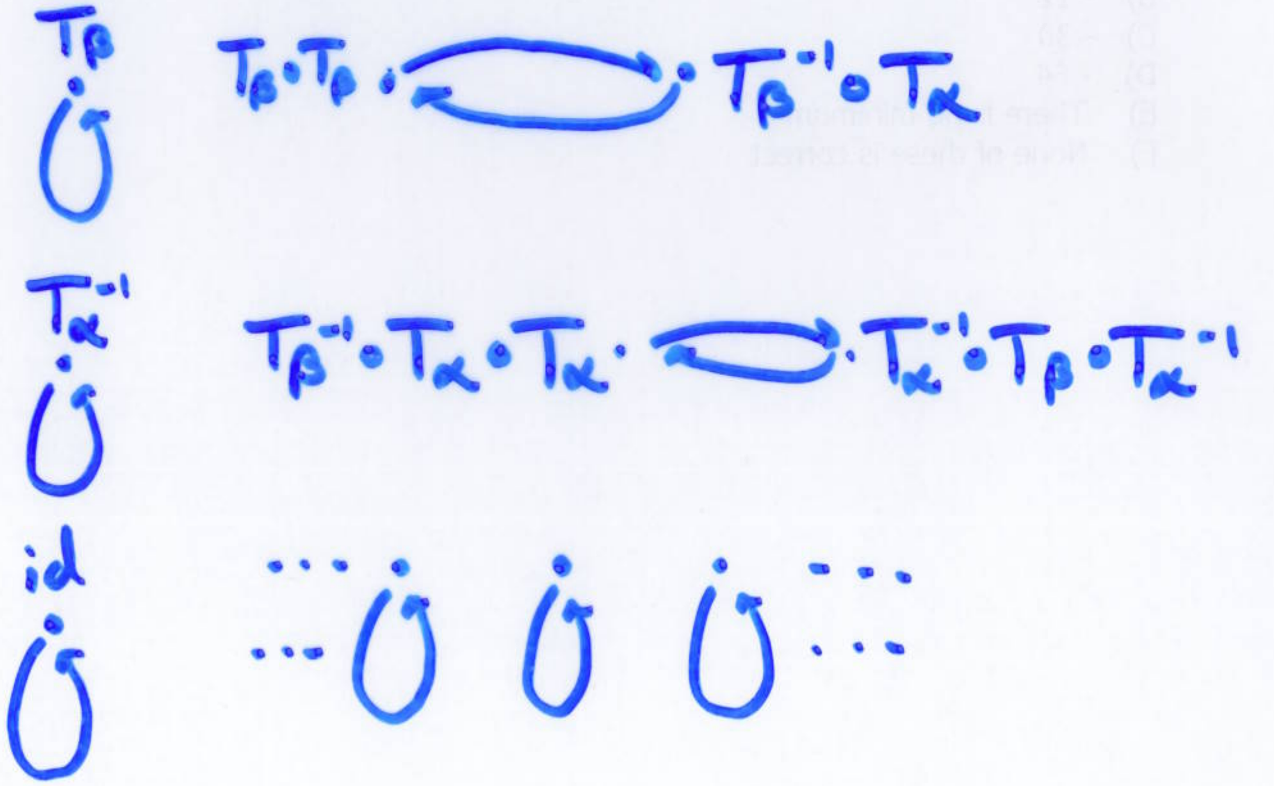
- $\begin{vmatrix} p' & r' \\ q' & s' \end{vmatrix} = \pm 1$?
- $\sigma_f\left(\underbrace{\frac{p}{q} \oplus \frac{r}{s}}_{= \frac{p+r}{q+s}}\right) = \sigma_f\left(\frac{p}{q}\right) \oplus \sigma_f\left(\frac{r}{s}\right)$?

(12)



Thm Let $g \in PMCG(\hat{\mathbb{C}}, P_f)$. Then $\exists N$ so that $\forall n > N$, $\bar{\Psi}^{\circ n}(g)$ is one of the following: T_β, $T_\beta \circ T_\beta$, $T_\beta^{-1} \circ T_\alpha$, $T_\beta^{-1} \circ T_\alpha \circ T_\alpha$, $T_\alpha^{-1} \circ T_\beta \circ T_\alpha^{-1}$, T_α^{-1}, id, or a member of a one-parameter family of obstructed maps

Pf <u>Many</u> cases



(14)

<u>Twisted-f problem</u>

Let $h \in PMCG(\hat{\mathbb{C}}, P_f) = \langle T_\alpha, T_\beta \rangle$

What is the Thurston class of $\underset{\overset{\parallel}{h \cdot f}}{f \circ h}$?

$$\text{Define } \bar{\Psi}(g) = \begin{cases} \Psi(g), & g \in H_f \\ T_\alpha \cdot \Psi(g \cdot T_\alpha), & g \cdot T_\alpha \in H_f \\ T_\alpha^{-1} \cdot \Psi(g \cdot T_\alpha^{-1}), & g \cdot T_\alpha^{-1} \in H_f \\ T_\beta \cdot \Psi(g \cdot T_\beta), & g \cdot T_\beta \in H_f \end{cases}$$

Thm $g \cdot f$ is Thurston equivalent to $\bar{\Psi}(g) \cdot f$ for all $g \in PMCG(\hat{\mathbb{C}}, P_f)$

Proof e.g. suppose $T_\alpha \circ g \in H_f$

Then $g \circ f = T_\alpha^{-1} \circ T_\alpha \circ g \circ f$

$\sim T_\alpha^{-1} \circ f \circ \Psi(T_\alpha \circ g)$

$\sim f \circ \Psi(T_\alpha \circ g) \circ T_\alpha$

(13)

E.g. $(T_\beta \circ f)^{-1}(\frac{p}{q}) = f^{-1}(T_\beta^{-1}(\frac{p}{q})) = \sigma_f(\frac{p}{q}+2)$

$_\beta \circ f)^{-1}(\frac{1}{1}) = \sigma_f(\frac{1}{1}+2) = \sigma_f(\frac{3}{1}) = -\frac{1}{1}$

$_\beta \circ f)^{-1}(-\frac{1}{1}) = \sigma_f(-\frac{1}{1}+2) = \sigma_f(\frac{1}{1}) = \frac{1}{1}$

~ //// ~

$_\beta \circ f)^{-1}(\frac{1}{0}) = \sigma_f(\frac{1}{0}+2) = \sigma_f(\frac{1}{0}) = \frac{0}{1}$

$_\beta \circ f)^{-1}(\frac{0}{1}) = \sigma_f(\frac{0}{1}+2) = \sigma_f(\frac{2}{1}) = \frac{1}{0}$

eimages of curves for rabbit,

^2+i, etc?

uestion: Will computing the pullback relation on curves be enough to solve every twisting problem?

(16)

- Each member of the one-parameter family is obstructed (since $\overline{\Psi}$ fixes the twist)

- Each of the finite list of mystery maps is unobstructed (GAP)

- Each mystery map is equivalent to either f or a second rational map g

- To determine which, note that f and g have different pullback behavior

Pullback for f: Global attractor:	Pullback for g: Global attractor contains: