How to find the polynomials of a mating

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2. *$(s_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ convergent sequences in S. Then*

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9. *For each open set* U *the set*

$$U^* := \bigcup \{[x] \mid [x] \subset U\}$$

is open.

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Then $\sim$ can be realized as the end of a pseudo-isotopy, there is a pseudo-isotopy $H\colon S^2 \times I \to S^2$ such that

$$x \sim y \text{ if and only if } H(x,1) = H(y,1),$$

for all $x, y \in S^2$.

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- $p \perp\!\!\!\perp q$ may not be equivalent to rational map. Thurston obstruction.

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There seems to be very little difference between rational maps vs. Thurston maps for the problem of deciding if map is a matings.

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- $\text{post}(f) \subset F(f) \Leftrightarrow f$ hyperbolic.

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Theorem (M)

f pcf rational map, $J(f) = \widehat{\mathbb{C}}$ (or expanding Thurston map). Then each sufficiently high iterate $F = f^n$ is a mating.

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Existence of an equator is right notion for a hyperbolic rational map to arise from a mating.

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- Any Lattès map has no equator.
- Many examples as above are matings.

A sufficient criterion for matings

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Can find the polynomials p, q, s.t. $f = p \perp\!\!\!\perp q$ by an algorithm.

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Example: $p = z^2 + i$,
external rays $R_{1/12}, R_{7/12}$ land at 0.
crit. portrait: $\{1/12, 7/12\}$.

Critical Portraits

In general: for each crit. value $a = p(c)$, let R_θ be external ray landing at a, then

$$J_c = \{\tau \mid p(R_\tau) = R_\theta, R_\tau \text{ ends at } c\}$$

$+$ compatibility assumption

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crit. portrait of p: $\{J_c \mid c \in \mathrm{crit}(p)\}$.

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- No set J_k contains more than one point from A.

Critical Portraits

Definition

$J_1, \ldots, J_m \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$ are a critical portrait, if

- the map $\mu(t) = dt \pmod 1$ maps
$$\mu(J_k) = \{\alpha_k\}$$
- $\sum_k (\#J_k - 1) = d - 1$
- $J_1, \ldots, J_m$ are non-crossing.

Let $A = \bigcup_{k,n \geq 1} \mu^n(J_k)$ (finite set).

- No set J_k contains more than one point from A.
- $\exists n_0 \in \mathbb{N}$: $\alpha, \beta \in A$ distinct, then for $m \geq n_0$ no gap of m-th order contains points from both sets $J_i \ni \alpha$, $J_k \ni \beta$.

Poirier's Theorem

Theorem (Bielefeld-Fisher-Hubbard '92, Poirier '93)

For any crit. portrait $J_1, \dots, J_m$ there is a (unique up to affine conjugacy) monic polynomial realizing it.