The Notion of Matings

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Matings Workshop

Two views on Mating

- The constructive approach:

  Mating is a procedure to construct new rational maps by combining two polynomials.
- The descriptive approach:

  Mating is a way to understand the dynamics of certain rational maps in terms of pairs of polynomials.

Background Definitions

- Let $R(z) = p(z)/q(z) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map, where the polynomials p and q are without common factors.
- The degree of R, i.e. the maximum of the degrees of p and q, will be assumed to be at least 2.
- We consider the dynamical system given by iteration of R, i.e. with orbits:

$$z_0, z_1, \ldots, z_n = R(z_{n-1}), \ldots.$$

- A point $z \in \overline{\mathbb{C}}$ is periodic if $R^k(z) = z$ for some $k \geq 1$.
- The multiplier or eigenvalue of a k-periodic point z_0 is the complex number $\lambda = DR^k(z) = (R^k)(z)$.
- Periodic orbits are classified according to their multiplier, super attracting, attracting, neutral and repelling.

The Fatou and Julia set

- The Fatou set F_R is the open set of points in $\overline{\mathbb{C}}$, for which the family of iterates $\{R^n\}_n$ form a normal family in the sense of Montel in some neighbourhood of the point.
- The Julia set J_R is the compact complement. Alternatively the Julia set is the closure of the set of repelling periodic points.
- For a polynomial

$$f(z) = a_d z^d + a_{d-1} z^{d-1} + ... a_1 z + a_0$$

  the point ∞ is a super attracting fixed point. Consequently the Julia set is a compact subset of $\mathbb{C}$.
- Infact:

Theorem (Böttcher)

Given a monic polynomial

$$f(z) = z^d + a_{d-1}z^{d-1} + ...a_1 z + a_0$$

There exists a unique germ of a holomorphic map

$$\phi = \phi_f : (\overline{\mathbb{C}}, \infty) \to (\overline{\mathbb{C}}, \infty), \qquad \phi(z) = z + a_{d-1}/d + \mathcal{O}(1/z)$$

such that:

$$\begin{array}{ccc} (\overline{\mathbb{C}}, \infty) & \xrightarrow{f} & (\overline{\mathbb{C}}, \infty) \\ \phi \downarrow & & \downarrow \phi \\ (\overline{\mathbb{C}}, \infty) & \xrightarrow{z^d} & (\overline{\mathbb{C}}, \infty) \end{array}$$

The filled-in Julia set K_f.

- The set

  $$K_f = \{z \in \mathbb{C} | f^n(z) \not\to \infty, \text{as } n \to \infty\}$$

  is called the filled-in Julia set.
- $J_f = \partial K_f$.
- J_f and hence K_f is connected precisely, if no critical point escapes or iterates to ∞.
  In this case the Böttcher-coordinate at infinity extends to a biholomorphic map

  $$\phi : \overline{\mathbb{C}} \setminus K_f \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \quad \text{with inverse} \quad \psi : \overline{\mathbb{C}} \setminus \overline{\mathbb{D}} \to \overline{\mathbb{C}} \setminus K_f.$$

External rays

In the following we shall assume that f is a monic polynomial and that K_f is connected.

Definition

The external ray $R(\theta)$ of argument $\theta \in \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is the arc

$$R(\theta)(t) = R_f(\theta)(t) = \psi(e^{t+i2\pi\theta}), \qquad t > 0.$$

Here parametrized by the value of the Greens function:

$$g(z) = g_f(z) = \log|\phi(z)|.$$

Theorem (Caratheodory)

A univalent map $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ *has a continuous extension to* $\mathbb{S}^1 = \partial\mathbb{D}$ *if and only if* $\partial\phi(\mathbb{D})$ *is locally connected*

Corollary

The Böttcher parameter ψ_f*, at* ∞ *extends to a continuous map*

$$\psi : \overline{\mathbb{C}} \setminus \mathbb{D} \to \overline{\mathbb{C}} \setminus \mathring{K}_f,$$

if and only if J_f *is locally connected.*

The Caratheodory loop

Definition

When J_f is the locally connected, the continuous map $\gamma : \mathbb{T} \longrightarrow J_f$

$$\gamma(\theta) = \gamma_f(\theta) = \psi_f(\mathrm{e}^{i2\pi\theta}) = \lim_{t \to 0^+} R_f(\theta)(t)$$

is called the Caratheodory loop or Caratheodory semi-conjugacy.

It evidently satisfies

$$f(\gamma(\theta)) = \gamma(d \cdot \theta).$$

Mating Definitions

In the following I shall discuss several definitions of Matings

- Topological Mating
- Formal Mating
- Intermediate forms of Matings, a la Buff-Cheritat
- Geometric or Conformal mating
- The Zakeri-Yampolsky definition of Conformal Mating

Set Up for the Rest of this talk

- $f_1, f_2 : \mathbb{C} \to \mathbb{C}$ are monic degree $d > 1$ polynomials with connected and locally connected Julia sets.
- K_1 and K_2 are their filled-in Julia sets and
- $\gamma_i : \mathbb{T} \to J_i$, $i = 1, 2$ are the Caratheodory loops.
- $\sim_T$ denotes the smallest equivalence relation on the disjoint union $K_1 \sqcup K_2$ for which :

$$\forall \theta \in \mathbb{T} : \gamma_1(\theta) \sim_T \gamma_2(-\theta).$$

The Topological Mating

Definition

- Let $K_1 \perp\!\!\!\perp K_2 = (K_1 \sqcup K_2)/\sim$.
- Let $\pi_T : K_1 \sqcup K_2 \longrightarrow K_1 \perp\!\!\!\perp K_2$ denote the natural projection.
- Equip $K_1 \perp\!\!\!\perp K_2$ with the quotient topology.
- Define $f_1 \perp\!\!\!\perp f_2 : K_1 \perp\!\!\!\perp K_2 \longrightarrow K_1 \perp\!\!\!\perp K_2$ by

$$f_1 \perp\!\!\!\perp f_2(z) = \begin{cases} \pi_T(f_1(w)), & \text{if } w \in K_1 \text{ and } \pi_T(w) = z, \\ \pi_T(f_2(w)), & \text{if } w \in K_2 \text{ and } \pi_T(w) = z. \end{cases}$$

Questions I

Natural questions are

- When is the topological space $K_1 \amalg K_2$ homeomorphic to the two sphere $\mathbb{S}^2$?
- In case there is a homeomorphism $h : K_1 \amalg K_2 \longrightarrow \mathbb{S}^2$, when is the conjugate map

$$h \circ f_1 \amalg f_2 \circ h^{-1} : \mathbb{S}^2 \to \mathbb{S}^2$$

  a branched covering?
- equivalent to a rational map $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}} \sim \mathbb{S}^2$?
- What does equivalence mean?

Equivalences

Assuming for the moment that $K_1 \amalg K_2 \simeq \mathbb{S}^2$. Then there are essentially two notions of equivalence in use.

- A weak form called Thurston equivalence and
- A stronger form called Conformal or Geometric Mating.

Even the Conformal Mating definition comes in various strengths.

Definition (Conformal/Geometric Mating I)

Two degree $d > 1$ polynomials f_1, f_2 with connected and locally connected filled-in Julia sets K_1, K_2 are conformally/geometrically mateable if there exists a degree d rational map $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ and a homeomorphism

$$h : K_1 \amalg K_2 \to \overline{\mathbb{C}}$$

conformal on $\pi_T(\mathring{K}_1 \cup \mathring{K}_2)$ and such that

$$\begin{array}{ccc} K_1 \amalg K_2 & \xrightarrow{f_1 \amalg f_2} & K_1 \amalg K_2 \\ h \big\downarrow & & \big\downarrow h \\ \overline{\mathbb{C}} & \xrightarrow{R} & \overline{\mathbb{C}}. \end{array}$$

Questions II

Before we proceed to the other definitions let me formulate some more questions:

- If there exists a conformal mating of two polynomials, is it then unique up to Möbius conjugacy?
- How many ways can a rational map be obtained as a mating of two polynomials?
- When does the mating depend continuously/measureably/??? on input data?

Branched Coverings

Definition

A branched covering $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ is a map such that: For all $x \in \mathbb{S}^2$ there exists local coordinates $\eta : \omega(x) \longrightarrow \mathbb{C}$ and $\zeta : \omega(F(x)) \longrightarrow \mathbb{C}$ and $d \geq 1$ such that

$$\zeta \circ F \circ \eta^{-1}(z) = z^d.$$

When $d > 1$ above the point x is called a critical point. The set of critical points for F is denoted Ω_F.
The branched covering F is called post critically finite (PCF) if the post critical set

$$P_f = \{F^n(x) | x \in \Omega_f, n > 0\}$$

is finite.

Thurston Equivalence

Definition (Thurston Equivalence)

Two PCF branched coverings $F_1, F_2 : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ are said to be Thurston equivalent if and only if there exists a pair of homeomorphisms $\Phi_1, \Phi_2 : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ isotopic relative to the post critical set of F_1 such that

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{F_1} & \mathbb{S}^2 \\ \Phi_1 \downarrow & & \downarrow \Phi_2 \\ \mathbb{S}^2 & \xrightarrow{F_2} & \mathbb{S}^2. \end{array}$$

The fine Print

- The topological mating though very intuitive is not very operational in terms of proving theorems.
- A more successful definition in this respect is the notion of formal mating.

Definition (Formal Mating)

- Denote by $\widehat{\mathbb{C}} = \mathbb{C} \cup \{(\infty, z) | z \in \mathbb{S}^1\}$ the compactification of $\mathbb{C}$ obtained by adjoining a circle at infinity.
- Let $\widehat{\mathbb{C}}_i$ denote the compactifications as above of the dynamical planes $\mathbb{C}_i$ for each polynomial f_i, $i = 1, 2$.
- Define

$$\widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2 = (\widehat{\mathbb{C}}_1 \sqcup \widehat{\mathbb{C}}_2)/(\infty_1, z) \sim (\infty_2, \bar{z})$$

- Then the Formal Mating $f_1 \uplus f_2 : \widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2 \longrightarrow \widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2$ is

$$f_1 \uplus f_2(z) = \begin{cases} f_1(z), & \text{if } z \in \mathbb{C}_1, \\ f_2(z), & \text{if } z \in \mathbb{C}_2, \\ (\infty, z^d), & \text{for } (\infty_i, z). \end{cases}$$

The Formal Mating II

- Then $\widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2$ is homeomorphic to $\mathbb{S}^2$ and $f_1 \uplus f_2$ is a branched covering.
- If we identify $\mathbb{S}^2$ with $\overline{\mathbb{C}}$ and if both polynomials f_1, f_2 are PCF. Then we may ask if $f_1 \uplus f_2$ is Thurston equivalent to a rational map?
- Moreover we can reconstruct the topological mating in a way, which is more ameanable to proving theorems:

Relation to Topological Mating

- Let $\sim_F$ denote the smallest equivalence relation on $\widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2$ for which $\forall\, \theta \in \mathbb{T}$ the connected set

$$R_1(\theta) \cup \{(\infty_1, e^{i2\pi\theta})\} \cup R_2(-\theta)$$

  is contained in one equivalence class.
- Let $K_1 \perp\!\!\!\perp_F K_2 = (\widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2)/\sim_F$.
- Let $\pi_F : \widehat{\mathbb{C}}_1 \uplus \widehat{\mathbb{C}}_2 \longrightarrow K_1 \perp\!\!\!\perp_F K_2$ denote the natural projection.
- Equip $K_1 \perp\!\!\!\perp_F K_2$ with the quotient topology.
- Let $f_1 \perp\!\!\!\perp_F f_2 : K_1 \perp\!\!\!\perp_F K_2 \longrightarrow K_1 \perp\!\!\!\perp_F K_2$ be the mapping induced by $f_1 \uplus f_2$

- Then there is a natural homeomorphism

  $\chi : K_1 \amalg_F K_2 \longrightarrow K_1 \amalg_T K_2$ conjugating dynamics, i.e.

$$\begin{array}{ccc} K_1 \amalg_F K_2 & \xrightarrow{f_1 \amalg_F f_2} & K_1 \amalg_F K_2 \\ \chi \downarrow & & \downarrow \chi \\ K_1 \amalg K_2 & \xrightarrow{f_1 \amalg f_2} & K_1 \amalg K_2. \end{array}$$

The Formal Advantage

The Formal Mating is more operational, because:

Theorem (R.L. Moore)

Let $\sim$ *be any topologically closed equivalence relation on* $\mathbb{S}^2$*, with more than one equivalence class and with only connected equivalence classes. Then* $\mathbb{S}^2/\sim$ *is homeomorphic to* $\mathbb{S}^2$ *if and only if each equivalence class is non separating.*
Moreover let $\pi : \mathbb{S}^2 \longrightarrow \mathbb{S}^2/\sim$ *denote the natural projection. In the positive case above we may choose the homeomorphism* $h : \mathbb{S}^2/\sim \longrightarrow \mathbb{S}^2$ *such that the composite map* $h \circ \pi$ *is a uniform limit of homeomorphisms.*

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In the positive case above we may choose the homeomorphism $h : \mathbb{S}^2/\sim \longrightarrow \mathbb{S}^2$ such that the composite map $h \circ \pi$ is a uniform limit of homeomorphisms.

Conformal mating revisited

Definition (Conformal/Geometric Mating II)

Two degree $d > 1$ polynomials f_1, f_2 with connected and locally connected filled-in Julia sets K_1, K_2 are conformally mateable if there exists a degree d rational map $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ and a homeomorphism

$$h : K_1 \amalg K_2 \to \overline{\mathbb{C}}$$

such that $h \circ \pi_F$ is a uniform limit of homeomorphisms which are conformal on $(\mathring{K}_1 \cup \mathring{K}_2)$ and such that

$$\begin{array}{ccc} K_1 \amalg K_2 & \xrightarrow{f_1 \amalg f_2} & K_1 \amalg K_2 \\ h \downarrow & & \downarrow h \\ \overline{\mathbb{C}} & \xrightarrow{R} & \overline{\mathbb{C}}. \end{array}$$

Conformal mating according to Zakeri-Yampolsky

Definition (Conformal/Geometric Mating Ia)

Two degree $d > 1$ polynomials f_1, f_2 with connected and locally connected filled-in Julia sets K_1, K_2 are conformally mateable if there exists a degree d rational map $R : \overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ and two continuous semi-conjugacies

$$\phi_i : K_i \to \overline{\mathbb{C}}, \quad \text{with} \quad \phi_i \circ f_i = R \circ \phi_i,$$

conformal in the interior of the filled Julia sets, with $\phi_1(K_1) \cup \phi_2(K_2) = \overline{\mathbb{C}}$ and with $\phi_i(z) = \phi_j(w)$ for $i, j \in \{1, 2\}$ if and only if $z \sim_T w$.

Mating Questions 3

- When is the equivalence relation $\sim_F$ closed?
- When are all equivalence classes non-separating?
- If $K_1 \perp\!\!\!\perp K_2$ is homeomorphic to $\mathbb{S}^2 \sim \overline{\mathbb{C}}$, when is there then a homeomorphism which cojugates $f_1 \perp\!\!\!\perp f_2$ to a rational map?
- Are the equivalence classes of $\sim_T$ always finite sets?
- If bounded can they be of arbitrary size? (This is the question of long ray-connections)

Existence of Matings

Theorem (Tan Lei, Rees, Shishikura)

Let $f_1(z) = z^2 + c_1$ and $f_2(z) = z^2 + c_2$ be two post critically finite quadratic polynomials. Then f_1 and f_2 are conformally mateable (in the strong sense) if and only if c_1 and c_2 does not belong to conjugate limbs of the Mandelbrot set. Moreover if mateable the resulting rational map is unique up to Mbius conjugacy.

Let there Be FILMS

Multicurves

Let $P \subset \mathbb{S}^2$ be a finite set.

- A simple closed curve $\gamma : \mathbb{S}^1 \longrightarrow \mathbb{S}^2 \setminus P$ is called peritheral if one of the complementary components $\mathbb{S}^2 \setminus \gamma$ contains at most one point of P.
- A multi curve Γ in $\mathbb{S}^2 \setminus P$ is a set or collection of mutually non homotopic, non-peritheral simple closed curves in $\mathbb{S}^2 \setminus P$.
- Note that a multi curve has at most $\#P - 3$ elements.

Thurston matrices

- Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be a PCF branched covering with post critical set P, $\#P > 3$
- A multicurve $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ in $\mathbb{S}^2 \setminus P$ is F-stable if for every j and every connected component δ of $F^{-1}(\gamma_j)$, the simple closed curve δ is either homotopic to some γ_i or peritheral in $\mathbb{S}^2 \setminus P$.
- The Thurston Matrix of F with respect to the F-stable multicurve Γ is the non negative $n \times n$ matrix $A = A_{i,j}$ given by

$$A_{i,j} = \sum_{\delta} 1/\deg(F : \delta \to \gamma_j)$$

where the sum is over all connected components δ of $F^{-1}(\gamma_j)$ homotopic to γ_i relative to P, i.i. in $\mathbb{S}^2 \setminus P$.

Thurston obstructions

- Having only non negative entries, the Thurston matrix A has a positive leading eigenvalue, i.e. eigenvalue of maximal modulus.
- A Thurston obstruction to F is an F-stable multicurve Γ with leading eigenvalue of modulus at least 1.

Theorem (Thurston)

Let $F : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be a PCF branched covering with post critical set P, and hyperbolic orbifold. Then F is Thurston equivalent to a rational map if and only if F has no Thurston obstruction.

The Orbifold of F

- The orbifold $\mathcal{O}_F$ associated to F is the topological orbifold $(\mathbb{S}^2, \nu)$ with underlying space $\mathbb{S}^2$ and whose weight $\nu(x)$ at x is the least common multiple of the local degree of F^n over all iterated preimages $F^{-n}(x)$ of x.
- The orbifold $\mathcal{O}_F$ is said to be hyperbolic if its Euler characteristic $\chi(\mathcal{O}_F)$ is negative, that is if:

$$\chi(\mathcal{O}_F) := 2 - \sum_{x \in P}(1 - \frac{1}{\nu(x)}) < 0.$$

J. Milnor, *Pasting together Julia sets: Aworked out example.* Exp. Math. Vol 13 (2004) No. 1.