A set of Wittner captures

Mary Rees

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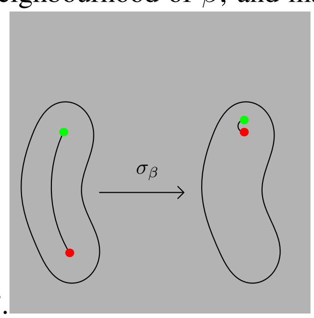
Definition of σ_β

Given a directed arc β on the sphere, the homeomorphism σ_β is defined to be the identity outside a suitably small disc neighbourhood of β, and maps the first endpoint of β to the second endpoint of β.

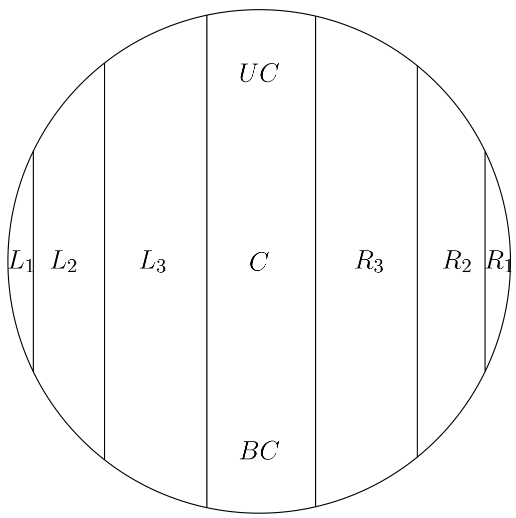


It is sometimes convenient to work with the *lamination maps* which are Thurston equivalent to hyperbolic quadratic polynomials, rather than the polynomials themselves, in order to make use of the symbolic dynamics arising from Markov partitions of the unit disc whose boundaries are lamination leaves. For any hyperbolic quadratic polynomial f, the associated lamination map $s = s_L$ is equal to z^2 outside the unit disc, and preserves a *quadratic invariant lamination* L inside the unit disc. It therefore descends to a map $[s]$ on the quotient space $\overline{\mathbb{C}}/\sim_L$, where $\sim_L$ is the equivalence relation on $\overline{\mathbb{C}}$ whose equivalence classes are the closures of leaves or finite-sided gaps of L.

The maps f and $[s]$ are topologically conjugate, under a homeomorphism which maps the Julia set of f to $S^1/\sim_L$. The maps f and s are Thurston equivalent.

The aeroplane

Here are some leaves of the lamination for the aeroplane polynomial, which give the boundary of a Markov partition which we will use.



The different sets of the partiition are mapped as follows

$$UC,\ BC\ \to\ L_1,$$

$$L_3,\ R_3\ \to L_2 \cup L_3,$$

$$L_2,\ R_2\ \to BC \cup UC \cup R_3,$$

$$L_1,\ R_1 \to R_1 \cup R_2$$

Wittner Captures: Definition

- A (degree two preperiodic) *capture* is then described by a critically periodic quadratic polynomial f, and an arc β (up to suitable homotopy) in the dynamical plane of the equivalent lamination map $s = s_L$ which we call the *capture path*.

- The arc β starts from the fixed critical point ∞ of s, and ends at a nonperiodic point $x \in \cup_{n\geq 2} s^{-n}(0)$ and *crosses the* $S^1 \cup L$ *exactly once*.

- The (Wittner) capture (map) associated to f and β is then the branched covering

$$\sigma_\beta \circ s.$$

- Every type III map has such a form, up to *Thurston equivalence* but for a path β *which may not be a capture path.*

Description up to topological conjugacy

The topological dynamics of the rational map which is Thurston equivalent to $\sigma_\beta \circ s$ can be recovered from s and β.

This is slightly easier to describe for Wittner captures than in general.

Replace the full orbit of the point $\beta \cap S^1$ by an arc. in the boundary of the gap of the modified lamination on the unit disc which is in the boundary of the gap containing $\beta(1)$. Extend the lamination L, by adding in a *doubled diagonal lamination* for which the external minor leaf is the arc of S^1 crossed by β.

Equivalence to rational maps

By the result of Tan Lei's thesis, (or an analogue for captures) a capture $\sigma_\beta \circ s_L$ is Thurston equivalent to a rational map if and only if the endpoint x of β is not in the *forbidden limb* which contains the minor leaf of L (and the critical value of s_L).

This forbidden limb is bounded by the leaf joining points on the circle with arguments $\frac{1}{3}$ and $\frac{2}{3}$.

For a capture map $\sigma_\beta \circ s$, it is natural to conjecture that there is a one-to-one correspondence between $[\sigma_\beta \circ s]$ and the endpoint $\beta(1)$ of β.

In some regions this is true, and in some it is not true.

In any region where it is true, it is possible to associate a set of rational maps which are Thurston equivalent to Wittner captures with a subset of the dynamical plane of s.

I am interested in regions where it is *not* true. My studies are mainly confined to the aeroplane polynomial and the set of capture paths crossing the unit circle at points with arguments in $(\frac{2}{7}, \frac{9}{28})$.

I shall denote this set of capture paths (up to natural equivalence) by $\mathcal{Z}$.

Simpler lack of correspondence

- For the aeroplane polynomial, there can be two capture paths β_1 and β_2 with the *same* second endpoint such that $\sigma_{\beta_1} \circ sf$ and $\sigma_{\beta_2} \circ s$ are *not* Thurston equivalent.
- But there are never more than two, and two happens relatively rarely with density tending to 0 as preperiod increases, for *some* second endpoints on the real line, where the crossing point of β_1 has argument in $(\frac{2}{7}, \frac{1}{3})$ and the crossing point of β_2 has argument in $(\frac{2}{3}, \frac{5}{7})$.

- There are also examples of paths β_1 and β_2 with crossing points as above, such that $\beta_1(1) \neq \beta_2(1)$ and yet $[\sigma_{\beta_1} \circ s] \neq [\sigma_{\beta_2} \circ s]$. In fact this probably happens with positive density.

- Counting is sufficient to show that the map $[\beta] \mapsto [\sigma_\beta \circ s]$ is not surjective.

Non-injective

Theorem 1. *(R, ETDS 2010)*
For each N there are N captures by s, for paths which all have different endpoints, but all with crossing points with arguments in $(\frac{2}{7}, \frac{9}{28})$, which are all Thurston equivalent. Moreover this happens for the preperiod of ∞ being $O(2^N)$.

The capture paths in this theorem are described by their endpoints, because of the restriction on S^1 crossing point. We use the symbolic dynamics to describe the endpoints.

Let

$$a = L_3(L_2R_3)^2, \;\; b = L_3^5, \;\; c = L_3^3L_2C, \;\; d = L_3^3(L_2R_3)^2 = L_3^2a$$

We have
$$a < d < c < b$$
meaning that the labelled set $D(a)$, which is bounded by vertical leaves, is to the right of the other three, and so on.

We write

$$v_0 = L_3L_2R_3a, \;\; w_0 = L_3L_2R_3b, \;\; t_0 = L_3L_2R_3d, \;\; u_0 = L_3L_2R_3c,$$

$$v_{\ell+1} = v_\ell t_\ell a, \;\; t_{\ell+1} = v_\ell t_\ell d, \;\; u_{\ell+1} = v_\ell t_\ell c, \;\; x_\ell = v_\ell u_\ell$$

Let $\alpha \in \{a, b\}^{\mathbb{N}}$.

We generalise the previous definitions.

$$v_{0,\alpha} = L_3L_2R_3\alpha(0), \;\; t_{0,\alpha} = t_0,$$

$$v_{\ell+1,\alpha} = v_{\ell,\alpha} t_{\ell,\alpha} \alpha(\ell+1), \;\; t_{\ell+1,\alpha} = v_{\ell,\alpha} t_{\ell,\alpha} d,$$
$$u_{\ell+1,\alpha} = v_{\ell,\alpha} t_{\ell,\alpha} c, \;\; x_{\ell,\alpha} = v_{\ell,\alpha} u_{\ell,\alpha}$$

Of course, $v_{\ell,\alpha}$ and $x_{\ell,\alpha}$ depend only on $\alpha(i)$ for $i \leq \ell$.
We define $\alpha^{\ell} \in\in \{a, b\}^{\mathbb{N}}$ by

$$\alpha^{\ell}(i) = \begin{cases} a \text{ if } i < \ell, \\ b \text{ if } i \geq \ell \end{cases}$$

Then for any $k > \ell$,

$$v_{\ell} = v_{\ell,\alpha^k}, \;\; t_{\ell} = t_{\ell,\alpha^k},$$
$$u_{\ell} = u_{\ell,\alpha^k}, \;\; x_{\ell} = x_{\ell,\alpha^k}$$

We define $\gamma^{r,\alpha}$ to be the capture path in $\mathcal{Z}$ with endpoint coded by $x_{r,\alpha}$,and $\gamma^{r,\ell} = \gamma^{r,\alpha^{\ell}}$.

The theorem of ETDS 2010 can then be stated as

Theorem 2. *$\sigma_{\gamma^{r,\ell}} \circ s$ are Thurston equivalent for $0 \leq \ell \leq r$*

The set of captures which I want to discuss is

$$\sigma_{\gamma^{r,\alpha}} \circ s$$

This set has 2^{r+1} elements (because 2^{r+1} different endpoints) and all endpoints have the same preperiod, which is $O(2^r)$.

Conjecture 1. *$[\sigma_{\gamma^{r,\alpha}} \circ s] = [\sigma_{\gamma^{r,\omega}} \circ s] \Leftrightarrow$ there are $0 \leq \ell_1 \leq \ell_2 \leq r$ such that 1 to 3 hold.*

1. *$\alpha(i) = \omega(i)$ for $i < \ell_1$.*
2. *$\ell_1 = r$ or $\alpha(i) = a$ and $\omega(i) = b$ for $\ell_1 \leq i < \ell_2$ (without loss of generality).*
3. *$\alpha(i) = \omega(i) = b$ for $\ell_2 \leq i \leq r - 1$.*

Corollary 3. *There are precisely* $2r+2$ *captures that are equivalent to* $\sigma_{\gamma^{r,0}} \circ s$*. As well as* $\sigma_{\gamma^{r,\ell}} \circ s$ *for* $0 \leq r+1$ *there are also* $\sigma_{\gamma^{r,\omega}}$ *for* $\omega = \omega^{\ell}$ *for* $0 \leq \ell \leq r-1$ *where*

$$\omega^{\ell}(i) = \begin{cases} \alpha^{\ell}(i) \text{ for } i < r \\ a \text{ for } i = r \end{cases}$$

Conjecture 2. *This is for all captures, not just those of the form* $\sigma_{\gamma^{r,\omega}} \circ s$*.*

Corollary 4. *There are precisely* $2(r - \ell_1 + 1)$ *captures equivalent to* $\sigma_{\gamma^{r,\alpha}} \circ s$*, where* $\alpha(i) = b$ *for* $0 < i < \ell_1$*, and there is* $\ell_2 \leq r$ *such that* $\alpha(i) = a$ *for* $\ell_1 \leq i < \ell_2$ *and* $\alpha(i) = b$ *for* $\ell_2 \leq i \leq r-1$*.*

Consequently there are 2^{r-1} *inequivalent captures among the set* $\sigma_{\gamma^{r,\alpha}} \circ s$*.*