# The Space of Matings between Quadratic Polynomials and the Modular Group

Shaun Bullett + Andrew Curtis
Queen Mary, University of London

Toulouse
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# 3 types of holomorphic dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

## ① Kleinian groups

Discrete subgroups of $PSL_2(\mathbb{C})$ acting as Möbius transformations

$$z \longrightarrow \frac{az+b}{cz+d}$$

(example: $PSL_2(\mathbb{Z})$ )

## ② Rational maps

$$z \longrightarrow \frac{p(z)}{q(z)}$$ (p & q polynomials)

## ③ Holomorphic correspondences

Multivalued maps

$$z \xrightarrow{\mathcal{F}} w$$

determined by a polynomial relation

$$P_{\mathcal{F}}(z,w) = 0$$

Given a parameterised family of correspondences, we may ask for what set of parameter values the correspondence acts "discretely" on $\hat{\mathbb{C}}$.

## How should we define "discrete"?

- $\Gamma \subset PSL_2(\mathbb{C})$ is discrete
  - $\Leftrightarrow$ discrete as a set of matrices
  - $\Leftrightarrow$ every grand orbit $\Gamma \underline{x}$ is a discrete subset of the Poincaré disc $D^3$.

Passing to the action of $\Gamma$ on $\hat{\mathbb{C}}$, the boundary of $D^3$, a <u>sufficient</u> condition for $\Gamma$ to be discrete is that $\exists$ open $U \subset \hat{\mathbb{C}}$ such that $\Gamma z$ is a discrete subset of $\Gamma U$ for each $z \in U$.

- For the duration of this talk we adopt the following terminology for a correspondence $\mathcal{F}$

  $\mathcal{F}$ is "discrete" $\Leftrightarrow$ $\exists$ open $U \subset \hat{\mathbb{C}}$ such that the grand orbit $\mathcal{F}z$ is a discrete subset of $\mathcal{F}U$ $\forall z \in U$.

  $\mathcal{F}$ is "chaotic" $\Leftrightarrow$ $\mathcal{F}z$ is dense in $\hat{\mathbb{C}}$, $\forall z \in \hat{\mathbb{C}}$.

# The family of matings between quadratic polynomials and the modular group

$$\boxed{\mathcal{F}_a : z \to w \quad \text{where } z^2 + z(Jw) + (Jw)^2 = 3}$$

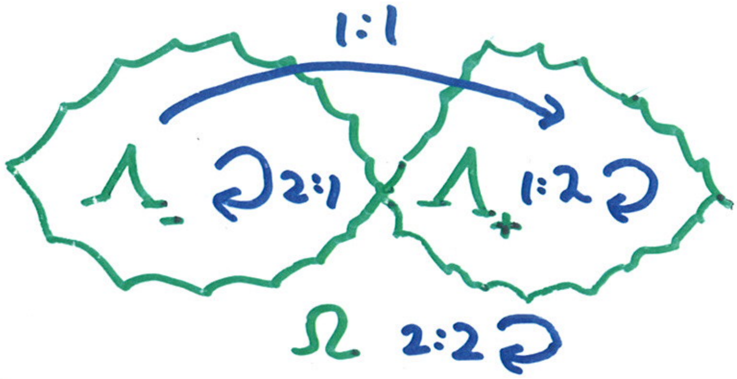
Here $J$ denotes the Möbius involution of $\hat{\mathbb{C}}$ which has fixed points $1$ and $a$.

$$\boxed{J(z) = \frac{(a+1)z - 2a}{2z - (a+1)}}$$

For certain values of the parameter $a \in \mathbb{C}$, the correspondence $\mathcal{F}_a$ behaves dynamically as a <u>mating</u> between a quadratic polynomial $q_c : z \to z^2 + c$ and the modular group $PSL_2(\mathbb{Z})$.

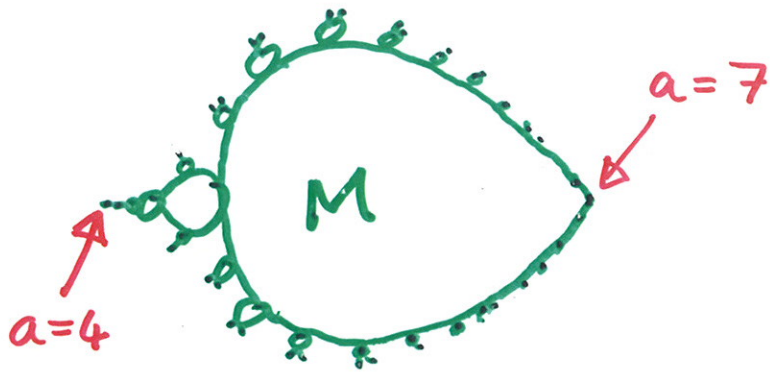

(SB + C. Penrose 1994
SB + W. Harvey 2000
SB + P. Haïssinsky 2007)

Dynamics of a mating $\mathcal{F}_a$




- $\mathcal{F}_a|_{\Lambda_- \to \Lambda_-}$ is conjugate to $q_c$ on $K(q_c)$
- $\mathcal{F}_a|_{\Lambda_+ \to \Lambda_+}$ " " " $q_c^{-1}$ " $K(q_c)$
- $\mathcal{F}_a|_{\Omega \to \Omega}$ " " " $z \nearrow z+1$, $z \searrow \dfrac{z}{z+1}$ on $H^+$

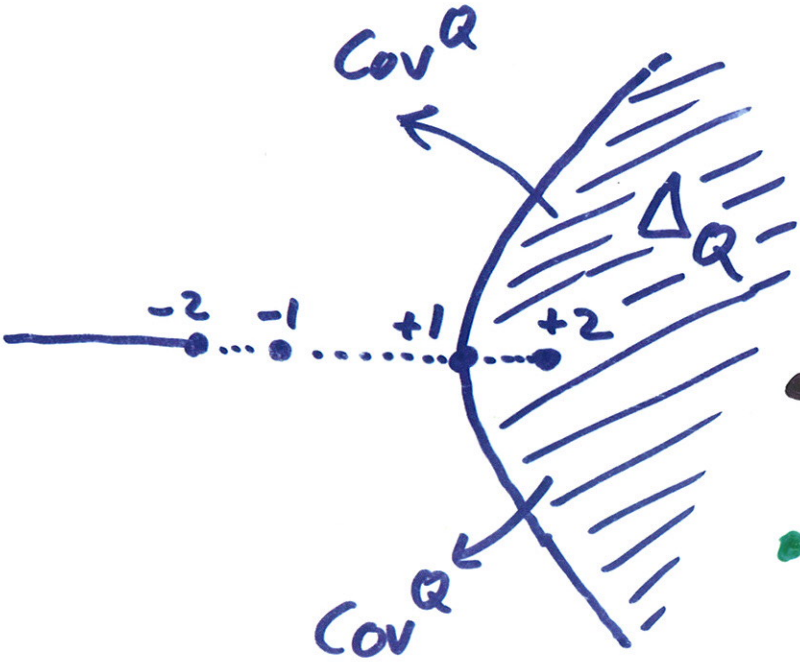
Parameter space

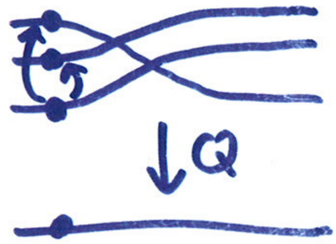



Conjecture: $M \approx$ Mandelbrot set

$\mathcal{F}_a$ is the composite $J \circ Cov^Q$ of the covering correspondence $Cov^Q$ of $Q(z)=z^3-3z$ followed by the involution $J$.

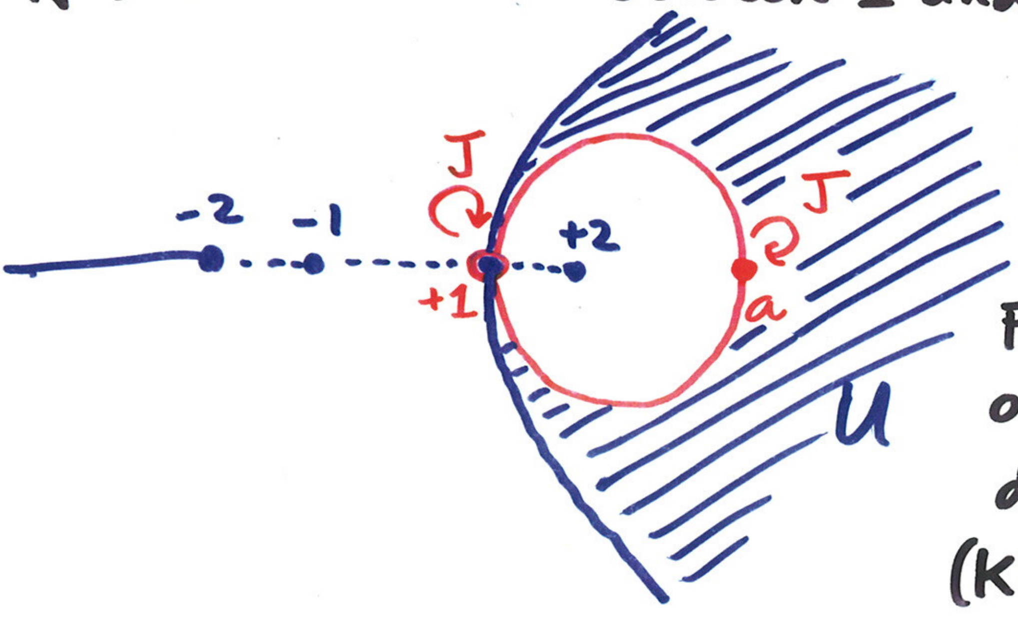
$$Cov^Q : z \to w \text{ where } \frac{Q(w)-Q(z)}{w-z}=0$$



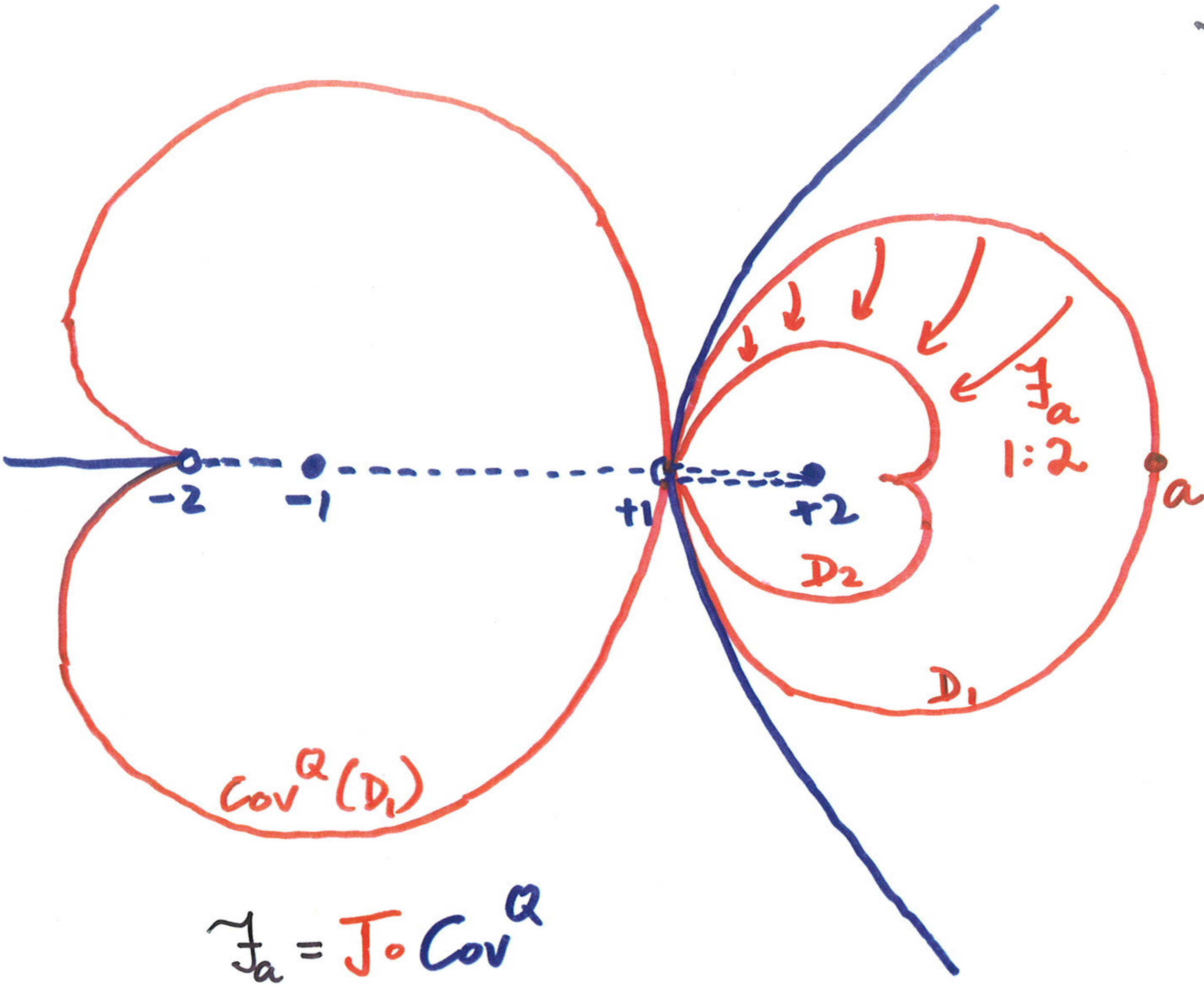

- $\Delta_Q$ is a fundamental domain for $Cov^Q$
- $+2$ and $-2$ are singular points (each has just one image)

If $a$ is real and between 1 and 7:




$U = \Delta_Q \cap \Delta_J$

For $z \in U$, grand orbit $\mathcal{F}_a z$ is discrete (Klein Combination Theorem)




$$\mathcal{F}_a = J \circ Cov^Q$$

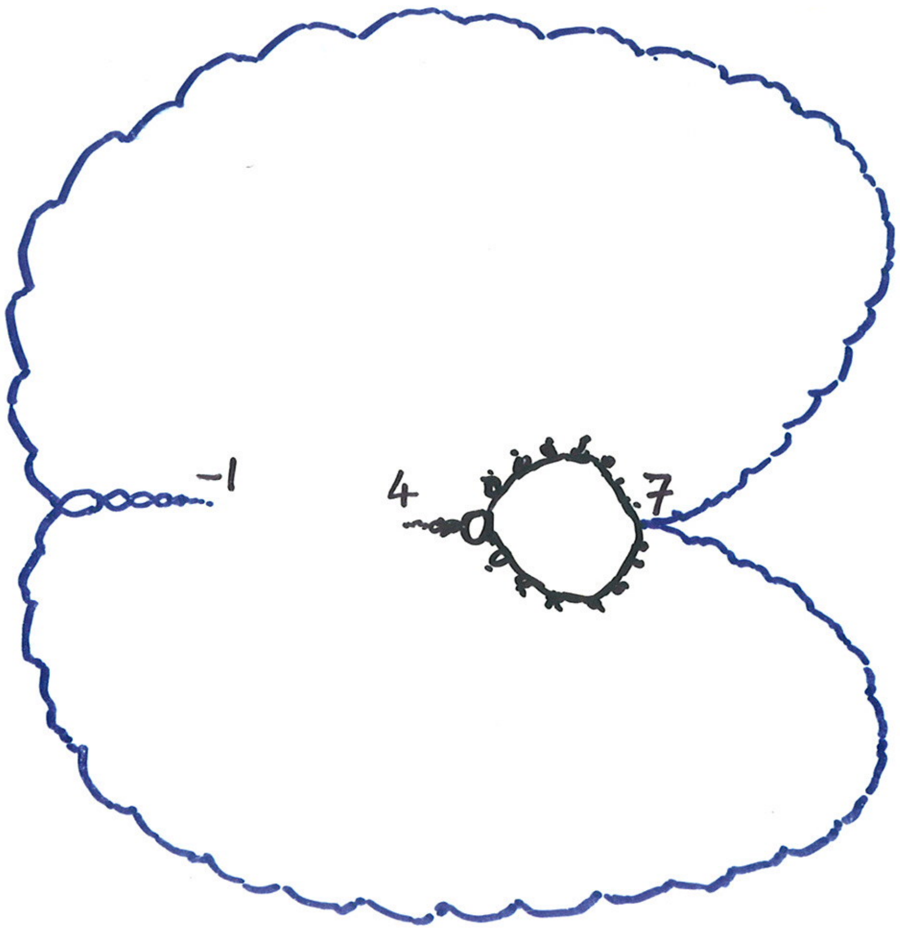
$\mathcal{F}_a^{-1} : D_2 \to D_1$ is a (pinched) quadratic-like 2 to 1 holomorphic map

Set $\Lambda_+ = \bigcap_{n \geq 0} \mathcal{F}_a^n(D_1)$

- $+2 \in \Lambda_+ \iff \Lambda_+$ is connected
- $+2 \notin \Lambda_+ \iff \Lambda_+$ is a Cantor set

# Locus of Discretion (in Parameter Space)

Experimentally




- for values of the parameter $a$ in the region bounded by the blue curve the correspondence $\mathcal{F}_a$ is discrete: outside this curve $\mathcal{F}_a$ is chaotic.

- within this region but outside $M$ the limit set of $\mathcal{F}_a$ is a Cantor set: except for a countable set of isolated values of $a$, all these $\mathcal{F}_a$ form a single quasiconformal conjugacy class.

# Fundamental domain for $\mathcal{F}_a$ when $-1<a<0$

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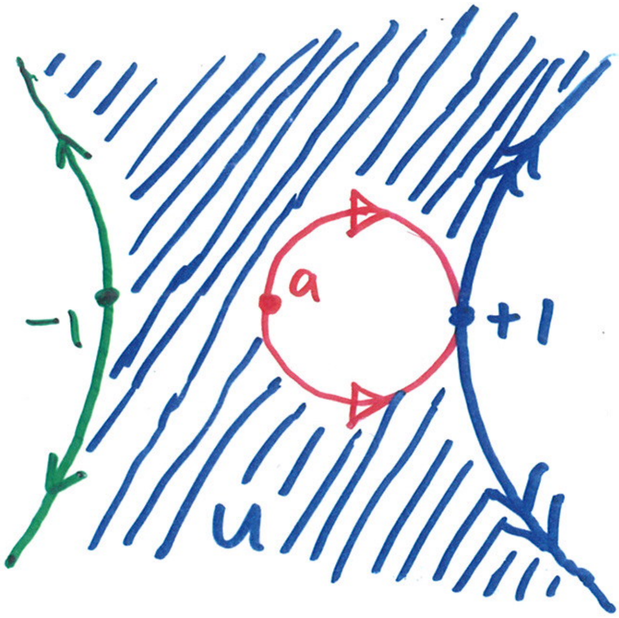
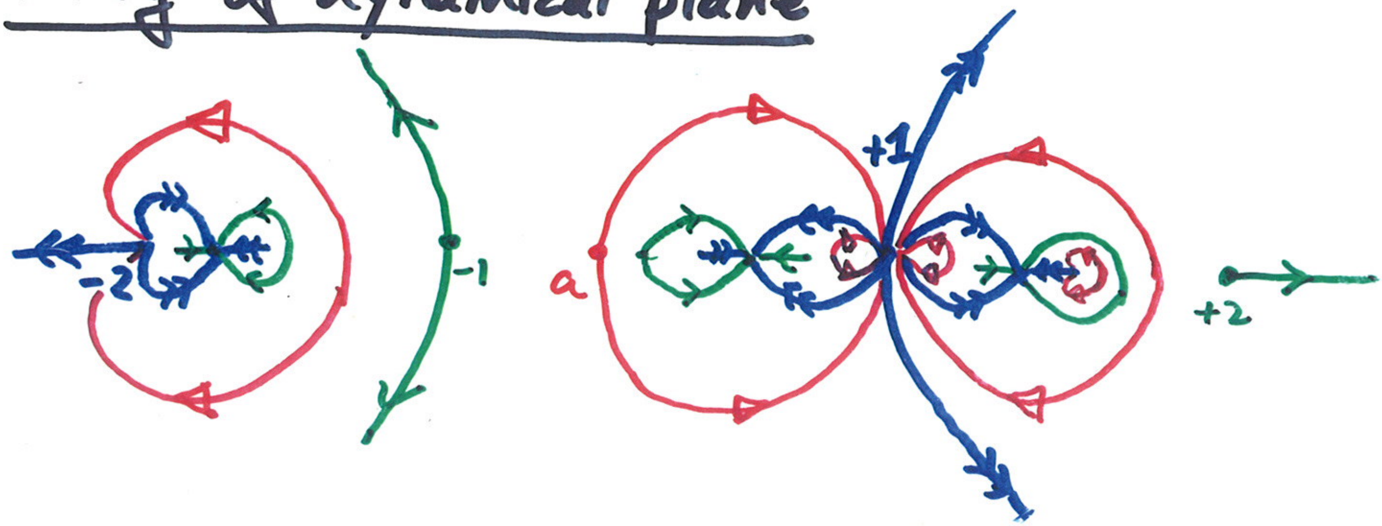
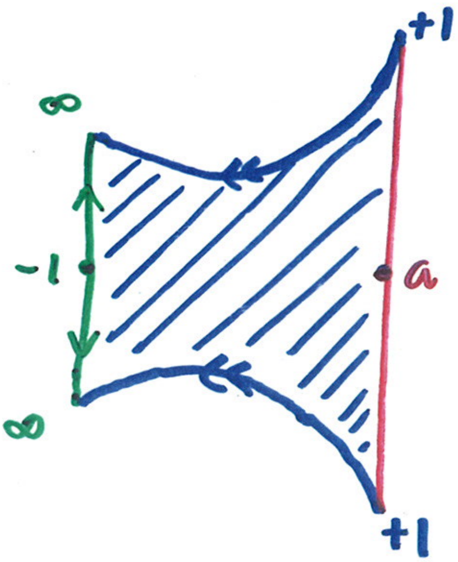



# "Tiling" of dynamical plane




# The grand orbit orbifold $\mathcal{O}_{\mathcal{F}_a}$

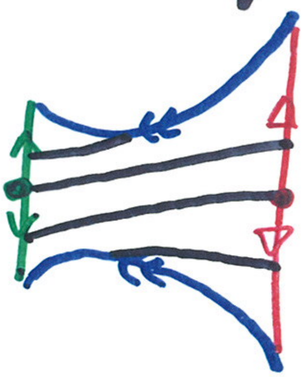



$\mathcal{O}_{\mathcal{F}_a}$ is a sphere with

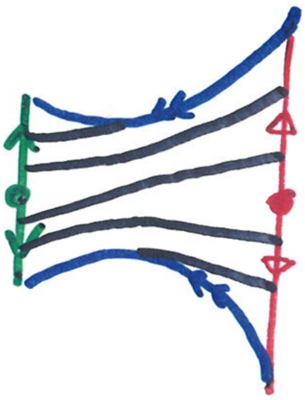
- one puncture point ($z=1$)
- one $\frac{2\pi}{3}$ cone point ($z=\infty$)
- two $\pi$ cone points ($z=-1, a$)

- Moving the parameter $a$ around in the "ocean of discretion" corresponds to deforming the complex structure on $\mathcal{O}_{\mathcal{J}_a}$.

- For each rational $p/q$ there is a geodesic $\gamma_{p/q}$ on $\mathcal{O}_{\mathcal{J}_a}$.



$\gamma_{1/3}$

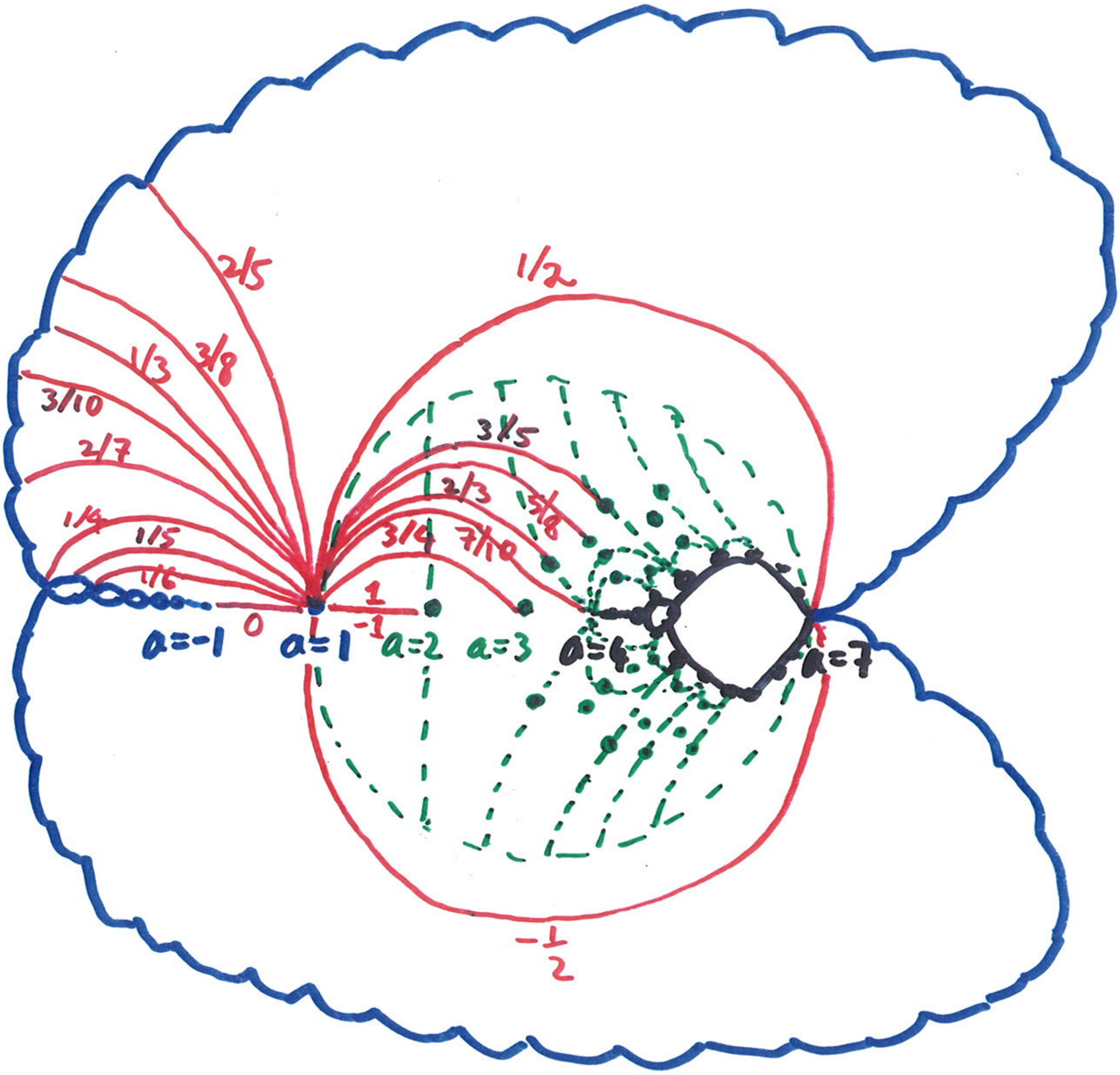


$\gamma_{1/4}$

- These geodesics lift to unions of arcs on the dynamical plane.

- Deforming the complex structure on $\mathcal{O}_{\mathcal{J}_a}$ by contracting $\gamma_{p/q}$ corresponds to approaching a point on the shore of the ocean.

# The Boundary of Discretion

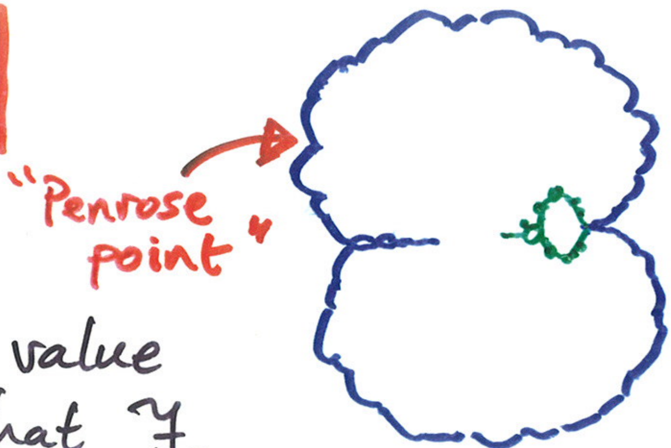



## Notes

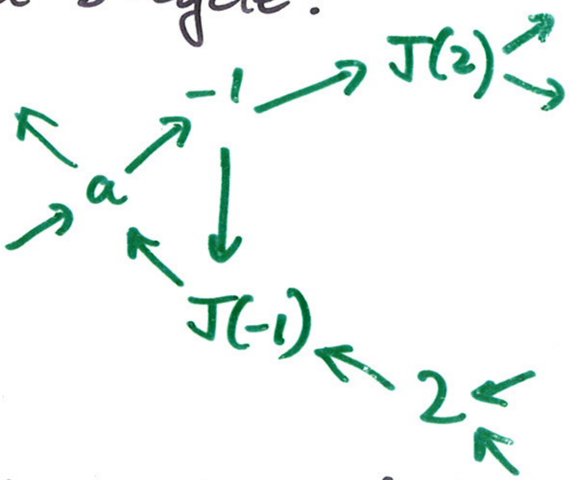
- a = 1 is a puncture point in the parameter space.
- the points marked in green are isolated parameter values where grand orbits of two of the "marked points" −2, +2, ∞, a, coincide.
- moving along the red p/q ray away from a = 1 corresponds to contracting $\gamma_{p/q}$.

## Examples on the boundary of discretion

1) $\boxed{a = -\frac{9}{2} + \frac{\sqrt{23}}{2} i}$




We compute this value of $a$ by asking that $\mathcal{F}_a$ map the point $a$ to the point $-1$. Then $\mathcal{F}_a$ has a 3-cycle:




It may be verified that for this value of $a$ the correspondence $\mathcal{F}_a$ has the dynamics obtained from $\mathcal{F}_{a_0}$ by contracting $\gamma_{1/3}$ to a point.

For $a = -\frac{9}{2} + \frac{\sqrt{23}}{2} i$ we can construct an explicit fundamental domain and show that the correspondence acts "discretely".

2) $\boxed{a = -2{\cdot}464}$ : this example, and others on the real axis, will be discussed by Andrew Curtis.

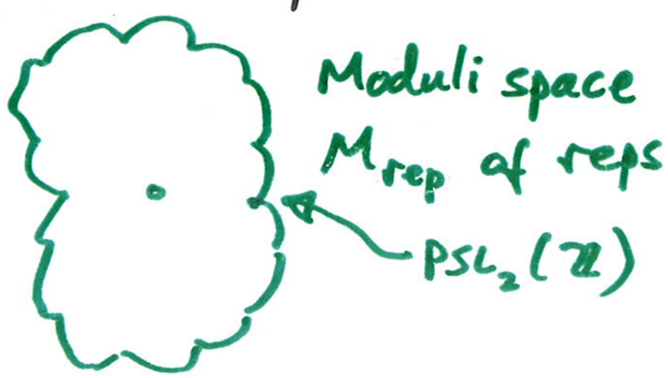
# Conjectures

- the boundary of discretion is locally connected;
- every point of the boundary of discretion is accessible by contracting a geodesic $\gamma_{p/q}$ or a geodesic lamination $\gamma_\nu$ ($\nu \in \mathbb{R} - \mathbb{Q}$);
- the boundary of discretion is a quotient of a simple Jordan curve, and is obtained from such a curve by identifying the points corresponding to $\gamma_{1/2n}$ and $\gamma_{-1/2n}$ for each $n \in \mathbb{N}$;
- outside the boundary of discretion every grand orbit of the correspondence is dense on the Riemann sphere.

# Matings between representations of $C_2 * C_3$ in $PSL_2(\mathbb{C})$ and quadratic polynomials

Let $C_2$ be the cyclic group of order 2, generator $\sigma$, and $C_3$ be the cyclic group of order 3, generator $\rho$.

The moduli space of faithful discrete reps. of the free product $C_2 * C_3$ in $PSL_2(\mathbb{C})$ is parameterised by the cross-ratio of the fixed points of $\sigma$ with those of $\rho$, so has complex dimension 1.


Each rep. $r$ in the interior of $M_{rep}$ has limit set $\Lambda(r)$ a Cantor set, and ordinary set $\Omega(r)$ a connected set. We define a <u>mating</u> between such an $r$, and a quadratic polynomial $z \to z^2+c$ to be a 2:2 correspondence $\mathcal{F}$ which partitions $\hat{\mathbb{C}}$ into 3 regions


where ....

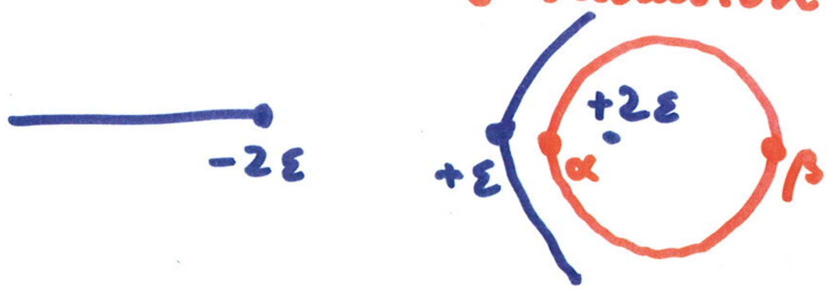
- $\mathcal{F}|_{\Lambda_- \to \Lambda_-}$ is conj. to $q_c$ on $K(q_c)$
- $\mathcal{F}|_{\Lambda_+ \to \Lambda_+}$ " " " $q_c^{-1}$ " "
- $\mathcal{F}|_{\Omega \to \Omega}$ has grand orbit space $\approx \dfrac{\Omega(r)}{\langle r \rangle}$

**Theorem** (SB + W. Harvey, 2000) For every $r \in \mathring{M}_{rep}$ and $c \in \mathcal{M}$ (Mandelbrot set) there is a mating between $r$ and $q_c$.

The proof is by surgery. Moreover all these matings have representatives in the family of correspondences

$$\mathcal{F} = J \circ \mathrm{Cov}^{Q_\varepsilon} \qquad Q_\varepsilon : z \to z^3 - 3\varepsilon^2 z$$

$J$ involution



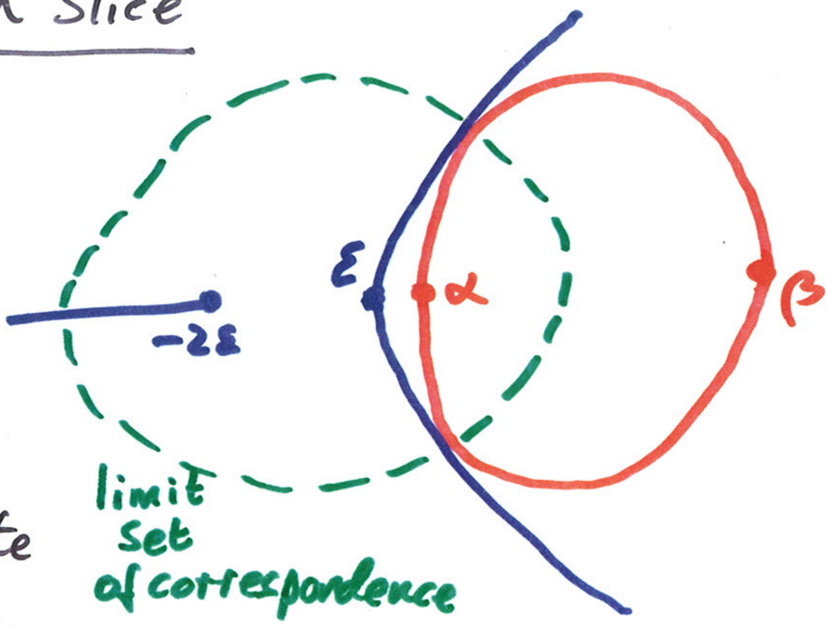

After scaling, this is a $2\mathbb{C}$ family of 2:2 correspondences.

The family of matings between $PSL(2,\mathbb{Z})$ and quadratic polynomials is a $1\mathbb{C}$ slice (obtained by setting $\alpha = \varepsilon$).

There are other interesting slices. For example:

## ① The Quasifuchsian Slice

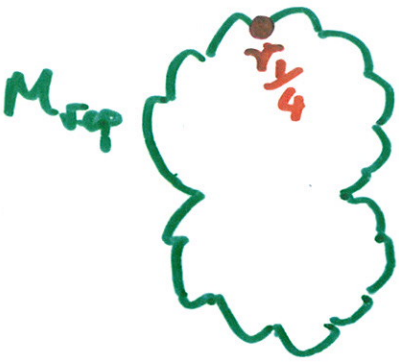
M. Samarasinghe
SB

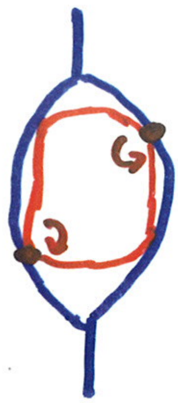



When $\varepsilon = 0$ the correspondence $J \circ Cov^{Q_\varepsilon}$ is conjugate to $PSL_2(\mathbb{Z})$ on $\hat{\mathbb{C}}$.

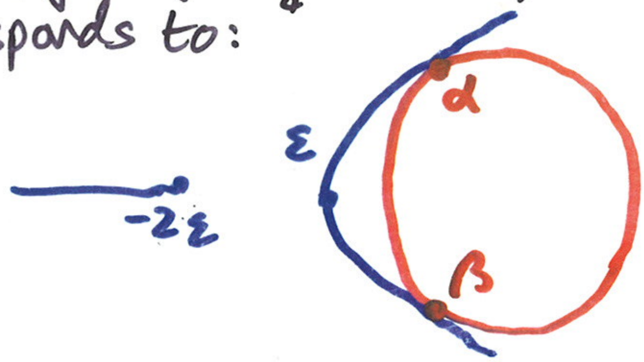
The "contact condition" keeps the limit set a quasicircle when $\varepsilon$ is deformed from $0$.

## ② Spaces of matings of circle-packing reps of $C_2 * C_3$ with quadratic maps



The space of matings of $r_{\frac{1}{4}}$ with quadratic polynomials corresponds to:



Experiment indicates that:

- it is possible to mate $r_{\frac{1}{4}}$ with $q_c$ for any $c \in \mathcal{M} - \{\frac{1}{2}\text{ limb}\}$
- and that the parameter space picture is a copy of $\mathcal{M} - \{\frac{1}{2}\text{ limb}\}$ mated with the basilica, enclosed by an outer "boundary of discretion".