Trees and the combinatorics of Rational/Thurston maps

Mitsuhiro Shishikura
(Kyoto University)

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For polynomials, there are established combinatorial way of describing the dynamics using external rays, critical portraits or Hubbard trees.

There are no unified way to describe the dynamics for rational maps (or branched coverings) which are not polynomials. The mating is an approach in this direction. In this, we present an alternative approach to such a description using trees and piecewise linear map on them.

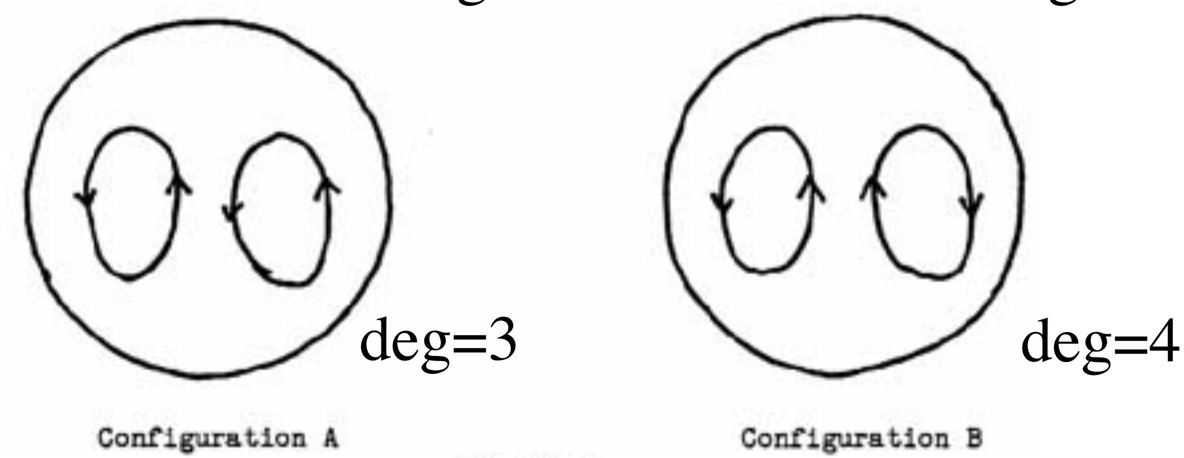
- Trees describing the "configuration" of Fatou components (except parabolic basins)
- Trees describing the "configuration" of invariant multicurve of branched covering of the 2-sphere

Relation to mating:

- Levy cycles theorem: top. polynomials (Levy), degree 2 (Rees, Tan)
- An example of a Thurston obstruction which is not a Levy cycle (with Tan Lei; cf. Chéritat's talk)

A method of showing non-existence of Thurston obstructions using the intersection theory (cf. Pilgrim's talk)

Original motivation: Configuration of Herman rings



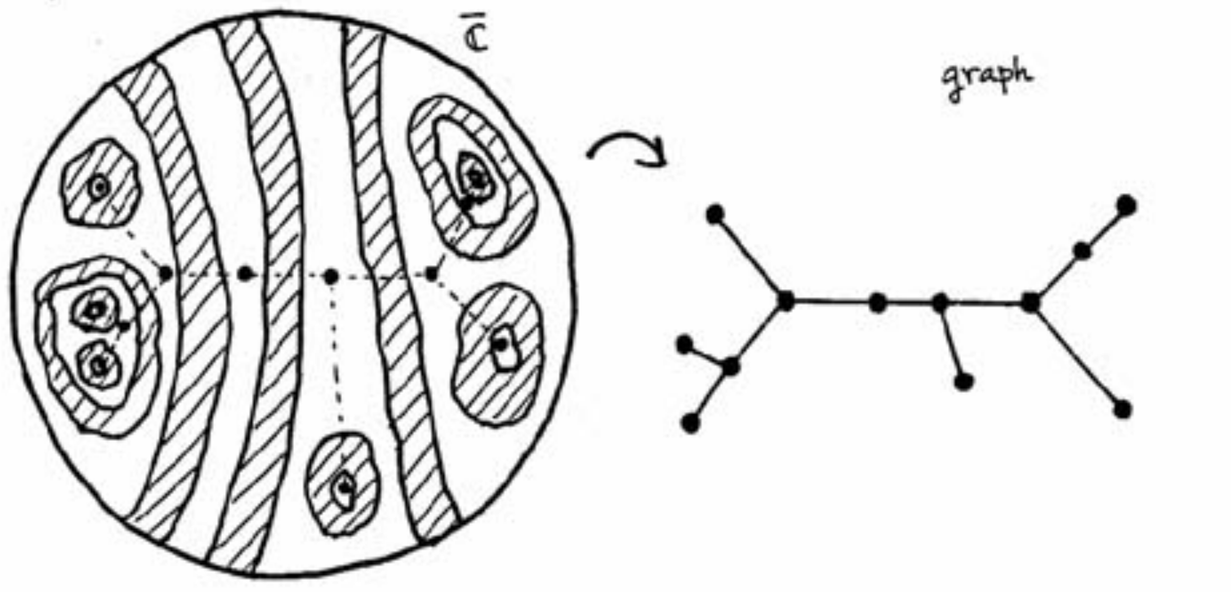
Inverse images are indispensable part of the dynamics.

Some constraints on the degree from conformal modulus.

Certain configurations force the complexity (=degree) to be high.

Other topics:
the connectivity of the Julia set for Newton's method
finitely connected (preperiodic) Fatou component (Beardon's question)
asymptotics for the limit of qc-deformation
(cf. Kiwi's work on the dynamics over the field of puiseux series)
construction of a rational map from a branched covering with an invariant multicurve
(cf. Cui-Tan's work)

Key idea



Definition. Let $\mathcal{A}$ be a collection on *disjoint* annuli of $\widehat{\mathbb{C}}$. Each annulus A is canonically foliated by topological circles. For $x, y \in \widehat{\mathbb{C}}$, let $A[x,y]$ be the union of leaves which separate x and y. So $A[x,y]$ is either a subannulus of A or an emptyset. Define $d : \widehat{\mathbb{C}} \times \widehat{\mathbb{C}} \to [0, +\infty]$ by

$$d(x,y) = \sum_{A \in \mathcal{A}} mod\ A[x,y].$$

Let

$$T = T_{\mathcal{A}} = \widehat{\mathbb{C}}/_{\sim_{\mathcal{A}}},$$

where $x \sim_{\mathcal{A}} y$ if and only if $d(x,y) = 0$. Then T is a tree and d induces a (generalized) metric, which may take value ∞.

A tree and a piecewise linear map for a rational map

Let f be a rational map with a periodic Fatou component, which is not a parabolic basin. In each periodic Fatou component other than parabolic basin, i.e., Attracting basin, superattracting basin, Siegel disk or Herman ring, there is a canonical foliation by circles. (Use linearization, Böttcher etc.) Remove the leaves containing the grand orbits of critical points, and take all the inverse images. We obtain a collection of disjoint annuli $\mathcal{A}_f$ on $\widehat{\mathbb{C}}$. From $\mathcal{A}_f$, the tree $T = T_f = T_{\mathcal{A}_f}$ is defined.

The map f defined a continuous map $F : T \to T$ and the degree function $\deg F : T \smallsetminus Sing(T) \to \mathbb{N}$ with the following properties:

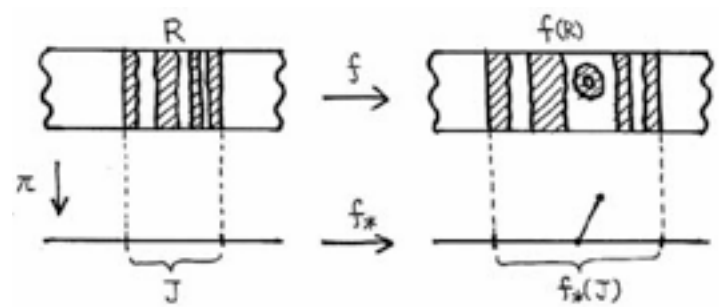
if $J = [x, y]$ is an arc in T joining x and y such that $F|_J$ is injective and $\deg F$ is constant on J, then $d(F(x), F(y)) = (\deg F|_J)\, d(x, y)$.

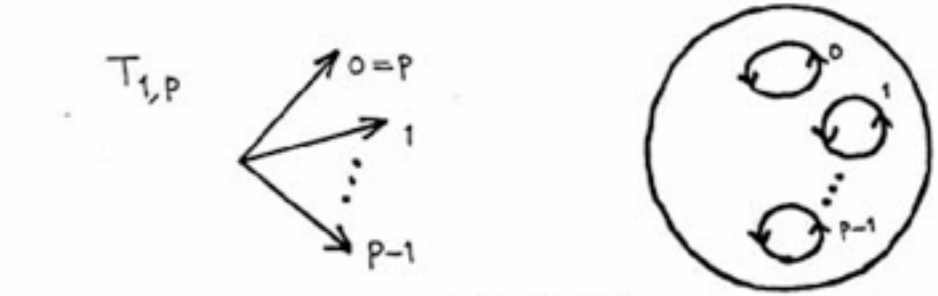
There is a dense open set in T such that any orbit from this set eventually lands on a periodic arcs on which the return map is identity (Siegel disks, Herman rings), a translation or an expansion towards a point at infinity (attracting or superattracting basin).

finite skeleton

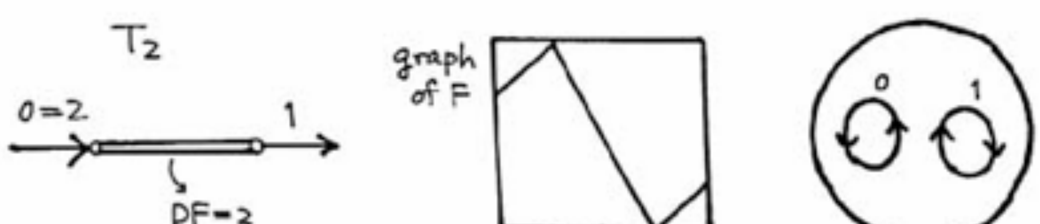
It is more convenient to extract a finite tree. Let X a forward invariant set for f such that X does not intersect the annuli from $\mathcal{A}_f$ and has only finitely many connected components. The canonical choice is the collection of all non-repelling periodic points and the boundaries of Siegel disks and Herman rings. Let $\mathcal{A}_{f,X}$ be the collection of annuli from $\mathcal{A}_f$ separating X. $T = T_{f,X}$ is the tree defined from $\mathcal{A}_{f,X}$ and this is a finite tree. The map $F : T \to T$ and $\deg F : T \setminus Sing(T) \to \mathbb{N}$ can be defined similarly, but $Sing(T)$ is now a finite set consisting of vertices, branch points, discontinuity of $\deg F$ and the projection of critical points.

examples



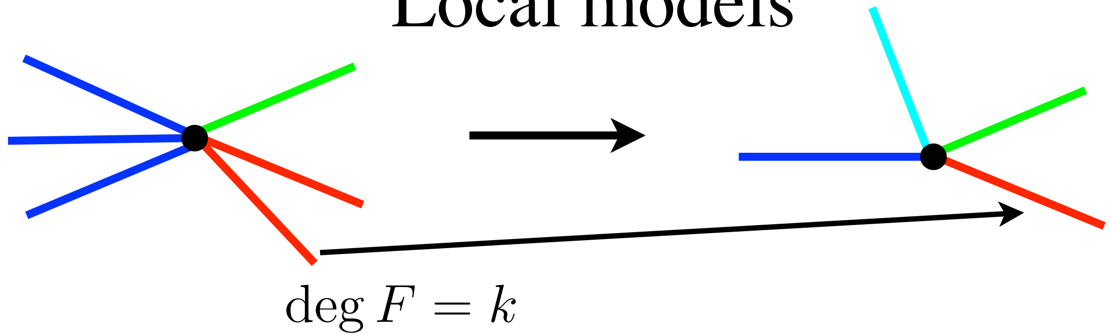


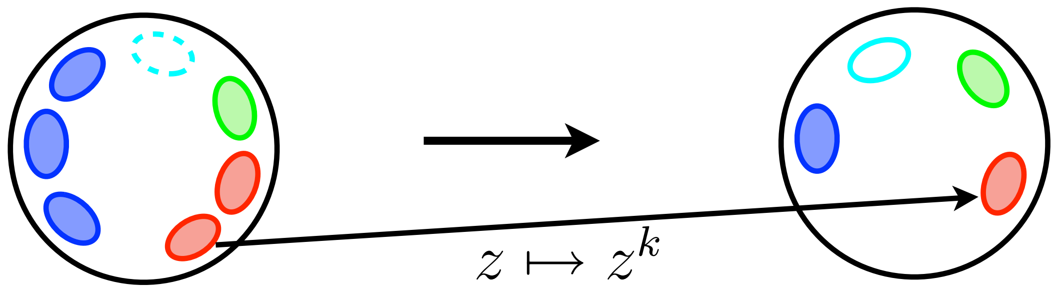
Example 1(p)



Example 2

Local models

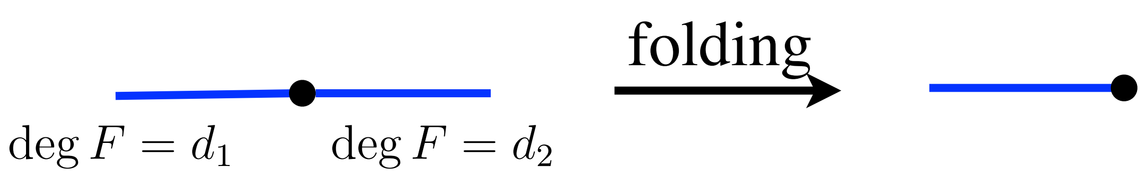




If the point on the tree is periodic, a quasiconformal surgery can be carried out so that periodic isometric branches correspond to Siegel disks and periodic expanding branches correspond to superattracting basin.

A periodic orbit not intersecting $Sing(T)$ corresponds to a quasicircle.

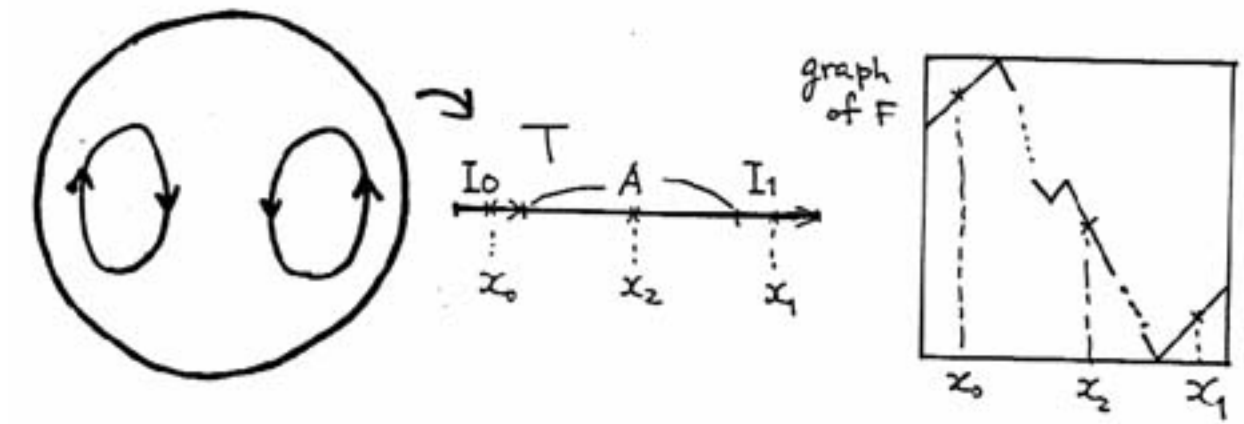
An estimate on the number of critical points



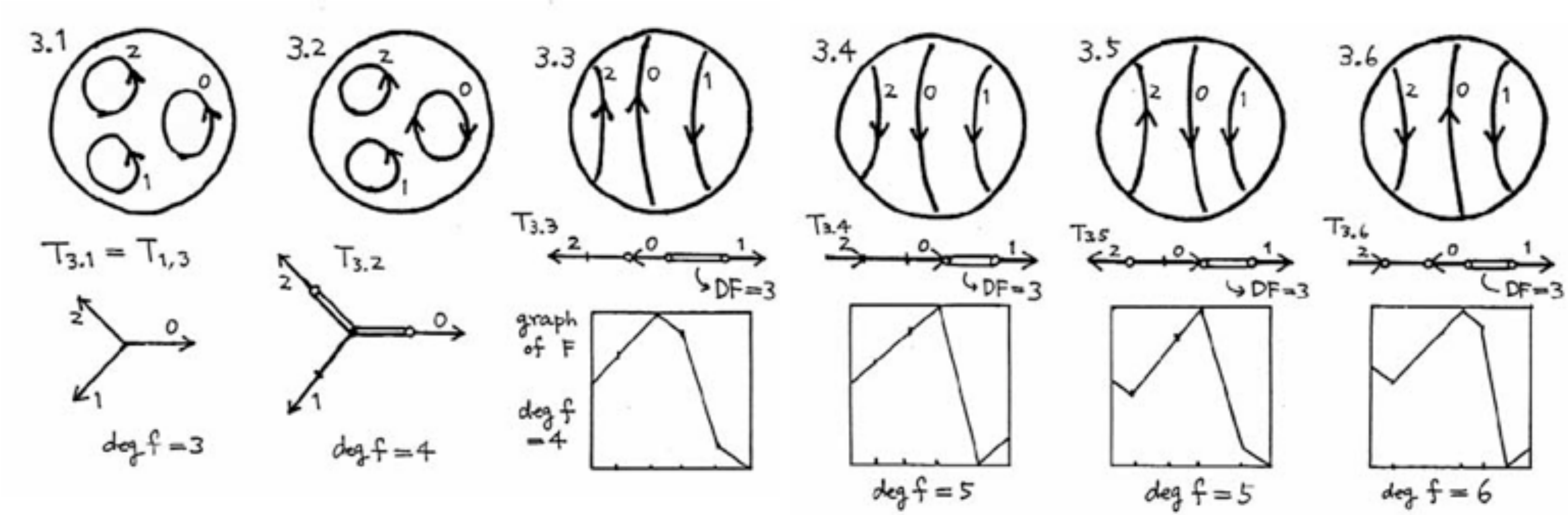
local degree $\geq d_1 + d_2$

$$\# \text{ of crit pts} \geq 2(d_1+d_2) - 2 - (d_1 - 1) - (d_2 - 1) = d_1 + d_2$$

estimate on degree for Herman rings of period $2, 3$



$\deg f \geq 4$



Another type of complexity: weakly repelling fixed point

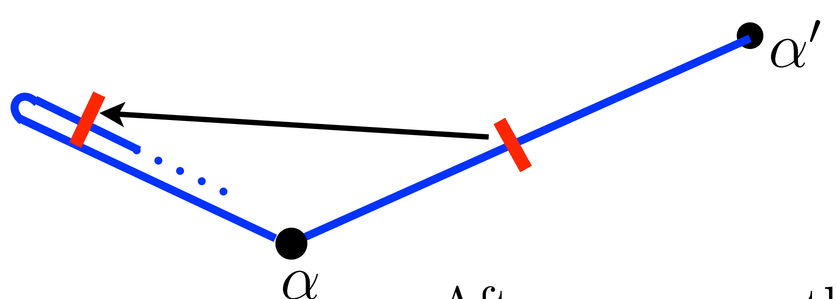
Theorem (Fatou): Every rational map of degree ≥ 1 has a weakly repelling fixed point (repelling or parabolic with multiplier 1).

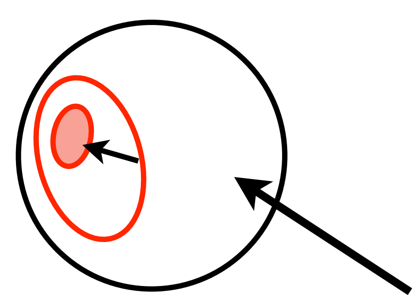
Theorem (S.) The Julia set of the Newton's method of a polynomial is connected. More generally if a Julia set of a rational map is not connected, there are two weakly repelling fixed points which are separated by a Fatou component.

Typical argument in the proof:

Suppose the Julia set is not connected. There exists a fixed point α of F, which is the projection of a weakly repelling fixed point and α' an inverse image of α, $\alpha' \neq \alpha$.

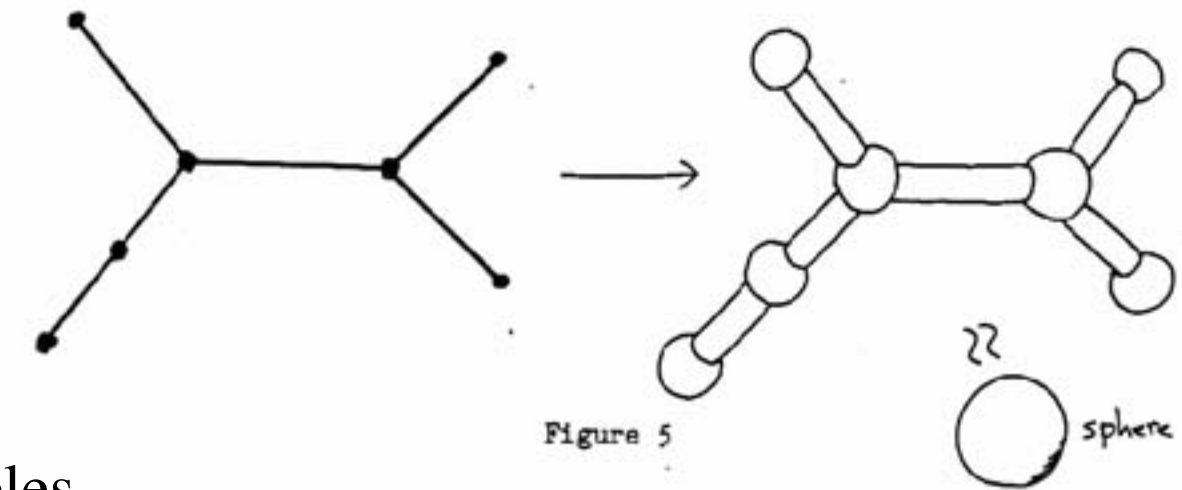
In the case where the branch of $[\alpha, \alpha']$ at α is not fixed ...





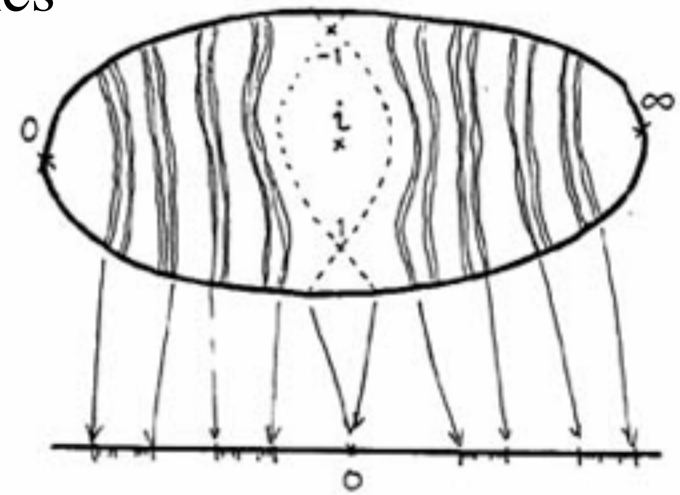
After a surgery, there must be another weakly repelling fixed point separated by a Fatou component.

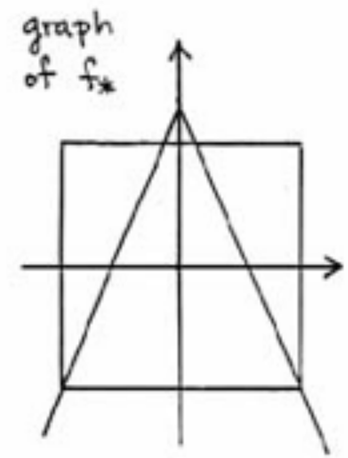
Construction of a rational map from a tree map and local models



If $Sing(T)$ has finite orbit, a surgery can be carried out.

Examples



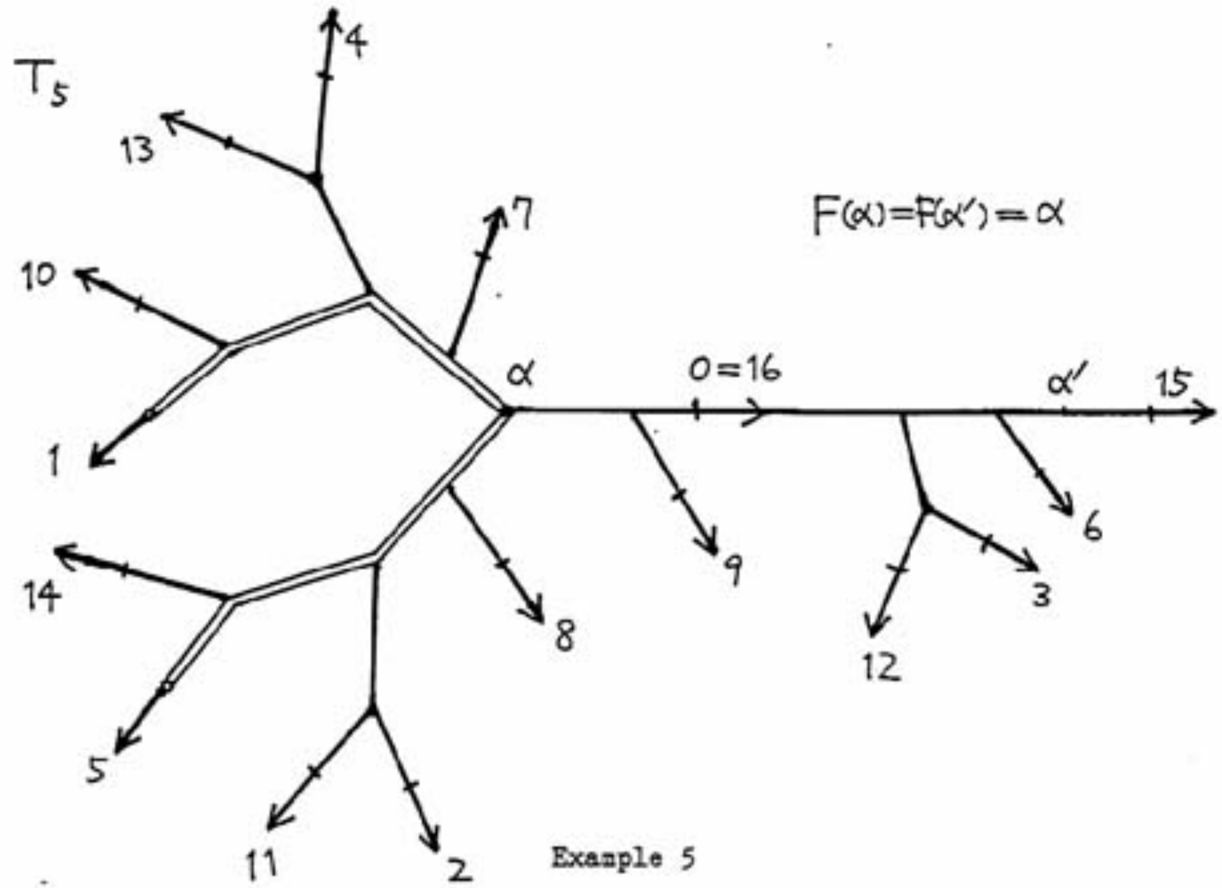


$J(f) = \text{Cantor set} \times S^1$

$f(z) = c\left(\frac{z}{1+z^2}\right)^3$, with $|c| > 67$.

For any $p \geq 2$, there exists f such that f has a prefixed Fatou component U such that $\widehat{\mathbb{C}} \setminus U$ has p connected component and $f(U)$ is a simply connected (super)attracting basin of period 1. (Beardon's question)

(With M. Kisaka) There exists an entire function which has wandering annuli. (Baker's problem)



Can be realized in degree 3

Asymptotics: the limit of qc-deformation

By stretching the annuli in $\mathcal{A}_f$ simultaneously, we obtain a one parameter family of qc-deformation $f_t = \varphi_t \circ f \circ \varphi_t^{-1}$. (When $t \to 0$, the annuli become fatter.) The corresponding tree will be simply stretched.

In general f_t degenerates when $t \to 0$. The choice of the normalization is important. If φ_t is normalized so that $\varphi_t^{-1}(0)$, $\varphi_t^{-1}(1)$, $\varphi_t^{-1}(\infty)$ correspond to three different branches around a fixed point α of F (or two different branches and a point in $\pi^{-1}(\alpha)$), then we obtain the local model for α as the limit.

In order to obtain the local models for a periodic orbit $\beta, F(\beta), \dots, F^{p-1}(\beta)$ of period p, we need to consider *moving frames* $h_t^{(i)}$ $(i = 0, \dots, p-1)$, which satisfy a similar condition around $F^i(\beta)$. Then $h_t^{(i+1)} \circ f_t \circ (h_t^{(i)})^{-1}$ converge to local models. The scaling rate will be given by the distance on the tree.

In fact, one can determine the asymptotic form of the rational maps obtained by the surgery by imposing the limit condition along the moving frames.

Beardon's example. DeMarco-Pilgrim?

Tree associated to an invariant multicurve of a Thurston map

Given a multicurve Γ on S^2, one can associate a tree $T = T_\Gamma$ so that each connected component of $S^2 \setminus \cup\Gamma$ corresponds to a vertex of T; each γ corresponds to an edge of T which connects the two vertices corresponding to components of $S^2 \setminus \cup\Gamma$ sharing γ as boundary.

Given a positive vector $v \in \mathbb{R}^\Gamma = \{\sum_{\gamma\in\Gamma} m_\gamma \gamma : m_\gamma \in \mathbb{R}\}$, define a metric on T so that the length on an edge corresponding to γ is m_γ.

For a postcritically finite branched covering $f : S^2 \to S^2$, define $\tilde{f}_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^{f^{-1}(\Gamma)}$ by

$$\tilde{f}_\Gamma(\gamma) = \sum_{\gamma' \subset f^{-1}(\Gamma)} \frac{1}{\deg(f : \gamma' \to \gamma)} \gamma' \quad \text{for } \gamma \in \Gamma$$

and $\pi_\Gamma : \mathbb{R}^{f^{-1}(\Gamma)} \to \mathbb{R}^\Gamma$ by assigning the homotopic element of Γ if there is one, 0 otherwise.

Thurston's linear transformation is $f_\Gamma = \pi_\Gamma \circ \tilde{f}_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$.

There is a natural piecewise linear map $\tilde{F} : (T_{f^{-1}(\Gamma)}, \tilde{f}_\Gamma(v)) \to (T_\Gamma, v)$, with expansion constant equal to $\deg(f : \gamma' \to \gamma)$ on the edge corresponding to γ'.

Suppose Γ is an invariant multicurve for f and take a non-negative eigenvector $v = \sum m_\gamma \gamma$ of f_Γ with eigenvalue λ. If $m_\gamma = 0$, eliminate γ from Γ. Define $\iota : T_\Gamma \to T_{f^{-1}(\Gamma)}$ assigning edges corresponding to homotopic curves. ι should expand the metric my factor λ. Then $F = \tilde{F} \circ \iota : T \to T$ is a piecewise linear map such that on each subedge corresponding to $\gamma' \in f^{-1}(\Gamma)$, the expansion rate is $\lambda \deg(f : \gamma' \to f(\gamma'))$.

If one does not eliminate γ with $m_\gamma = 0$, the map is piecewise linear, but λ is not a constant (depends on edges).

Theorem. *A branched covering decomposes into a tree map with $\lambda \geq 1$ corresponding to all non-intersecting Thurston obstructions and local models $g_x : S^2_x \to S^2_{F(x)}$ such that periodic part of local models are equivalent to rational maps, homeomorphisms, or branched coverings with non-hyperbolic orbifolds. The intersecting obstructions should come from pseudo-Anosov homeomorphisims or Lattès maps. (cf. Pilgrim's canonical decomposition)*

In fact, one can decompose by non-intersecting Thurston obstructions

Application of the tree

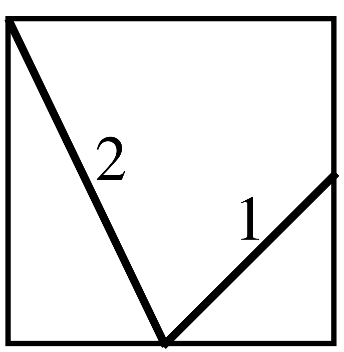
Levy cycles Theorem (Levy) If f is a postcritically topological polynomial, then any Thurston obstruction is a degenerate (removable) Levy cycle, i.e. bounds disks on which the map is homeomorphic.

Levy cycles Theorem (Rees) If f is a postcritically branched covering with only two critical points, then any Thurston obstruction is either a degenerate Levy cycle or an essential Levy cycle, i.e. the Levy cycle bounds a single component on which the map is homeomorphic.

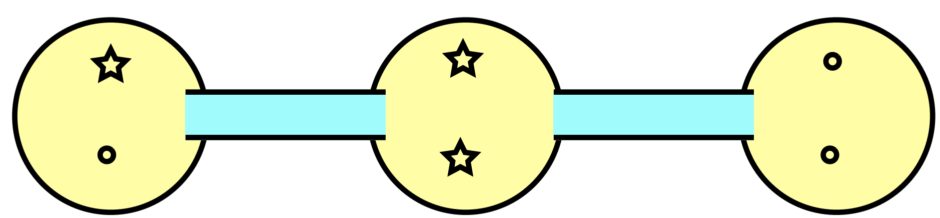
Construction of Thurston obstruction

One can construct a Thurston obstruction by giving a tree map and adding an information on local models. When the number of branches is small, it is easy to create appropriate local models.

Theorem (S.-Tan). *There exists a mating of cubic polynomials such that it has a Thurston obstruction, but has no Levy cycle.*



$\lambda = 1$



An example for which one can conclude non-existence of Thurston obstruction

Geometric intersection number

Definition. Let α and β be non-peripheral simple closed curves in $S^2 \setminus P_f$. Define the geometric intersection number to be

$$\alpha \cdot \beta = \min\{\#(\alpha' \cap \beta') | \alpha' \sim \alpha, \beta' \sim \beta\},$$

where the minimum is always attained (for example by hyperbolic geodesics in the homotopy classes). Obviously this number can also be defined for the homotopy classes of simple closed curves, and naturally extends bilinearly to $\mathbb{R}^{\underline{\alpha}} \times \mathbb{R}^{\underline{\beta}}$ for multicurves. $\underline{\alpha}, \underline{\beta}$ Instead of simple closed curves, one can take one of α and β to be simple arcs in $S^2 \setminus P_f$ joining points of P_f.

Lemma. *Let α and β be non-peripheral simple closed curves in $S^2 \setminus P_f$. Let α' be a connected component of $f^{-1}(\alpha)$ such that $f : \alpha' \to \alpha$ is a covering of degree k. Then we have*

$$\alpha' \cdot f^{-1}(\beta) \leq k\alpha \cdot \beta.$$

Definition (Unweighted Thurston matrix and μ_Γ)**.** Let us define the unweighted Thurston operator $f_\Gamma^\#$ by

$$f_\Gamma^\#(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma.$$

Denote the leading eigenvalue of $f_\Gamma^\#$ by μ_Γ.

Remark. It is obvious from the definition that $\lambda_\Gamma \leq \mu_\Gamma$.

Definition (Reduced multicurve)**.** An invariant multicurve Γ is called *reduced* if all the coefficients of the eigenvector of Thurston operator are positive. From any invariant multicurve, one can extract a reduced with the same eigenvalue.

Theorem. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\lambda_{\underline{\alpha}}\, \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

Theorem. *Let $\underline{\beta}$ be reduced invariant multicurve and $\underline{\alpha}$ a Levy cycle (or a simple cycle or arcs joining points in P_f) for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\mu_{\underline{\beta}} \leq 1.$$

In particular, either $\underline{\beta}$ is not a Thurston obstruction, or it contains a Levy cycle. (Head, S.-Tan, Pilgrim-Tan)

Proof. Let $u_{\underline{\alpha}}, v_{\underline{\beta}}$ be positive eigenvectors for $f_{\underline{\alpha}}$ and $f^{\#}_{\underline{\beta}}$, hence $f_{\underline{\alpha}}(u_{\underline{\alpha}}) = \lambda_{\underline{\alpha}} u_{\underline{\alpha}}$ and $f^{\#}_{\underline{\beta}}(v_{\underline{\beta}}) = \mu_{\underline{\beta}} v_{\underline{\beta}}$ Lemma 5 applied to f^n implies that (note that $P_{f^n} = P_f$) for each component $\alpha' \subset f^{-n}(\alpha)$, we have

$$\alpha' \cdot f^{-n}(\beta) \leq \deg(f^n : \alpha' \to \alpha)\alpha \cdot \beta.$$

Hence

$$\mu^n_{\underline{\beta}} \frac{1}{\deg(f^n : \alpha' \to \alpha)} \alpha' \cdot v_{\underline{\beta}} \leq \alpha \cdot v_{\underline{\beta}}.$$

Now denote N_n be the maximum number of non-peripheral components of $f^{-n}(\alpha)$ for $\alpha \in \underline{\alpha}$. By multiplying the coefficients of $u_{\underline{\alpha}}$ and adding (**??**) for all components $\alpha' \subset f^{-n}(\alpha)$ and $\alpha \in \underline{\alpha}$, we obtain

$$\lambda^n_{\underline{\alpha}} \mu^n_{\underline{\beta}} u_{\underline{\alpha}} \cdot v_{\underline{\beta}} \leq N_n u_{\underline{\alpha}} \cdot v_{\underline{\beta}}.$$

By Perron-Frobenius Theorem, we have $N_n \leq C\mu^n_{\underline{\alpha}}$ for some $C > 0$. Hence $\lambda^n_{\underline{\alpha}} \mu^n_{\underline{\beta}} \leq N_n \leq C\mu^n_{\underline{\alpha}}$. Taking n-th root and the limit, we conclude that

$$\lambda_{\underline{\alpha}} \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

□

Definition. Let α be a non-peripheral simple closed curve in $S^2 \setminus P_f$. Let α' be a connected component of $f^{-1}(\alpha)$. The *effective degree* eff-deg$(f : \alpha' \to \alpha)$ is the smallest $k \geq 1$ such that for any non-peripheral simple closed curve β in $S^2 \setminus P_f$, the following holds:

$$\alpha' \cdot f^{-1}(\beta) \leq k\alpha \cdot \beta.$$

Example. Suppose α and $\alpha'(\subset f^{-1}(\alpha))$ bound disks D_1 and D_0 such that $f(D_0) = D_1$ and f has only one critical point ω in D_0. If $P_f \cap D_1 = \{f(\omega), y\}$ and $\#(P_f \cap f^{-1}(y)) = k$, then

$$\text{eff-deg}(f : \alpha' \to \alpha) \leq k.$$

Definition (Effective Thurston matrix and μ_Γ)**.** Let us define the effective Thurston operator $f_\Gamma^\$$ by

$$f_\Gamma^\$(\gamma) = \sum_{\gamma' \subset f^{-1}(\gamma)} \frac{1}{\text{eff-deg}(f : \gamma' \to \gamma)} [\gamma']_\Gamma \quad \text{for } \gamma \in \Gamma.$$

Denote the leading eigenvalue of $f_\Gamma^\$$ by ν_Γ.

Remark. It is obvious from the definition that

$$1 \leq \text{eff-deg}(f : \gamma' \to \gamma) \leq \deg(f : \gamma' \to \gamma) \text{ and } \lambda_\Gamma \leq \nu_\Gamma \leq \mu_\Gamma.$$

Thurston effective unweighted

As before one can prove:

Theorem. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\nu_{\underline{\alpha}} \, \mu_{\underline{\beta}} \leq \mu_{\underline{\alpha}}.$$

Corollary. *Let $\underline{\alpha}$ and $\underline{\beta}$ be reduced invariant multicurves for f such that $\underline{\alpha} \cdot \underline{\beta} > 0$. Then we have*

$$\nu_{\underline{\alpha}} \, \nu_{\underline{\beta}} \leq 1.$$

Since $\lambda_{\underline{\beta}} \leq \nu_{\underline{\beta}}$, we have

Theorem. *Let $\underline{\alpha}$ be a reduced invariant multicurve for f such that $(\lambda_{\underline{\alpha}} <) 1 < \nu_{\underline{\alpha}}$. Then f has no Thurston obstruction intersecting $\underline{\alpha}$.*

If a branched covering is constructed from a tree map $F : T \to T$ with $\lambda < 1$ but with the effective eigenvalue $\nu_F > 1$ and the local models are rational maps, then it has no Thurston obstruction, hence is equivalent to a rational map.

Merci!