Matings, captures and regluings

Vladlen Timorin*

*Faculty of Mathematics
National Research University Higher School of Economics,
Moscow

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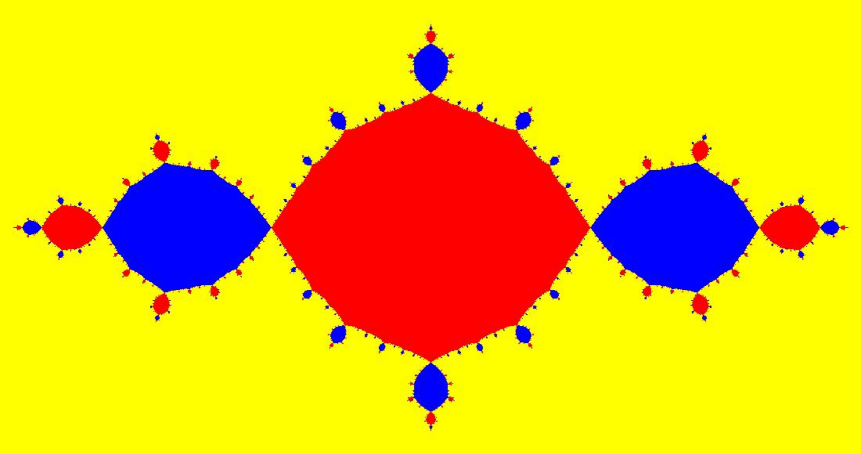
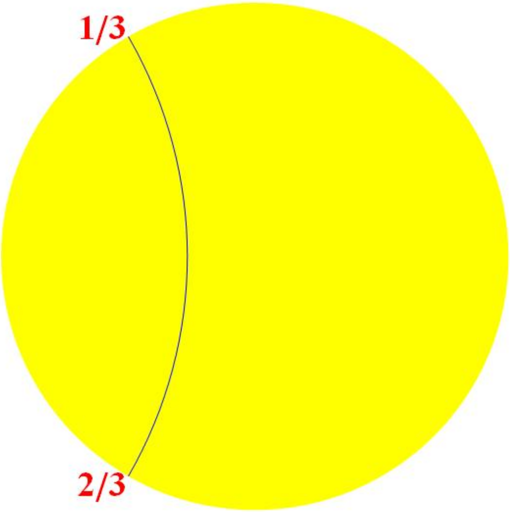
Geodesic laminations

Laminations provide topological models for polynomials, say, $f_c(z) = z^2 + c$, with connected and locally connected Julia sets. E.g. consider the basilica $f(z) = z^2 - 1$.



Lamination for the basilica

The Julia set of f can be modeled as follows.



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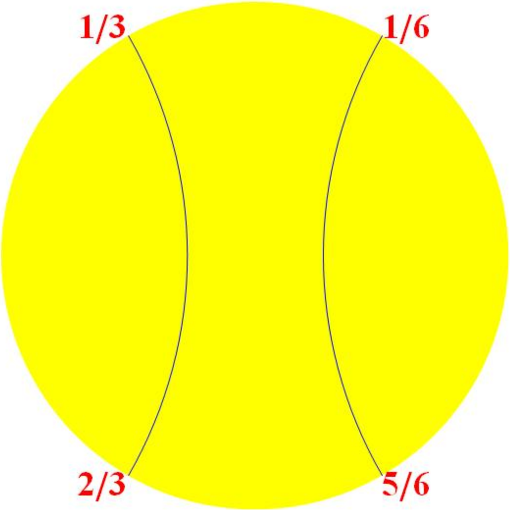

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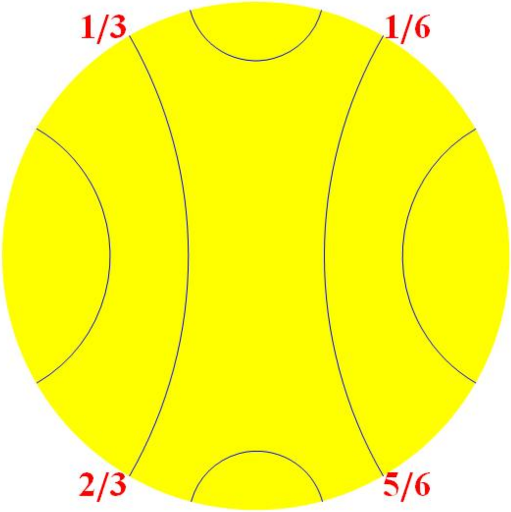

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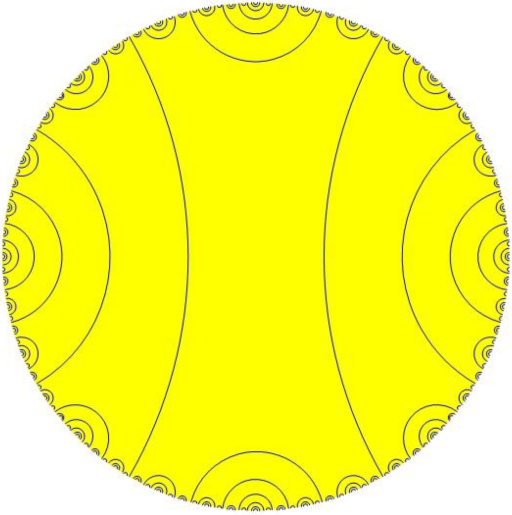
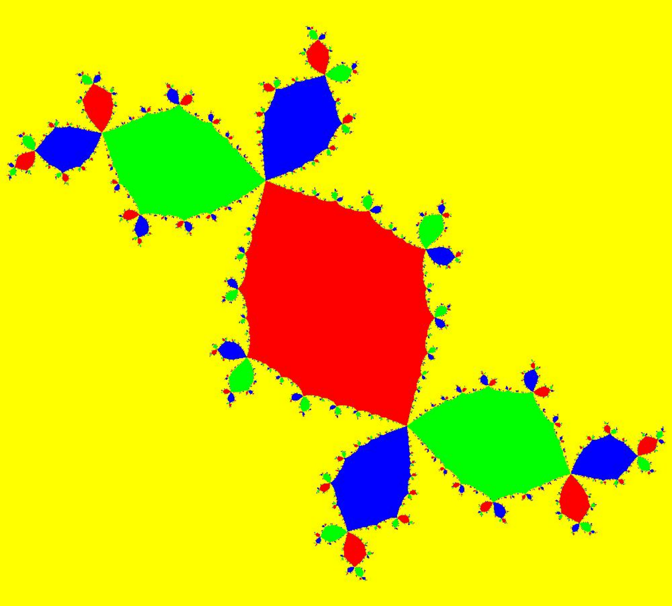


Rabbit



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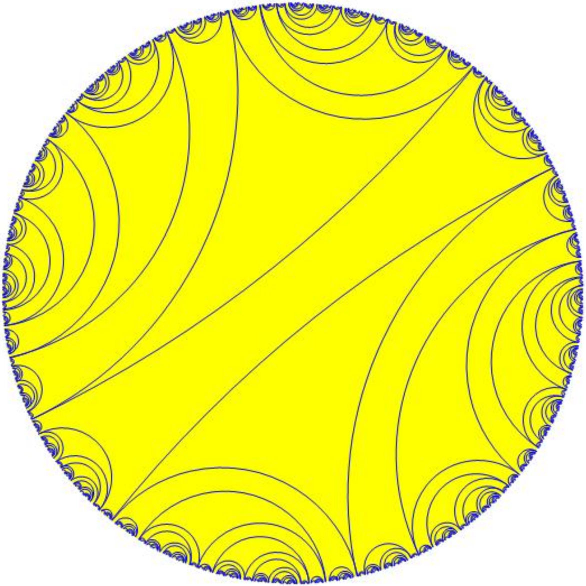
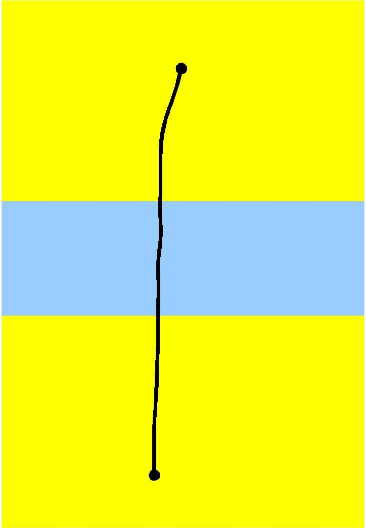


Topological mating $f_1 \amalg f_2$

- Let f_1 and f_2 be two quadratic polynomials with locally connected Julia sets.
- Consider the corresponding laminations L_1 and L_2.
- Draw L_1 in the closed unit disk, and L_2 in the complement to the open unit disk.
- Collapse leaves and polygons of both laminations.

Path homeomorphisms

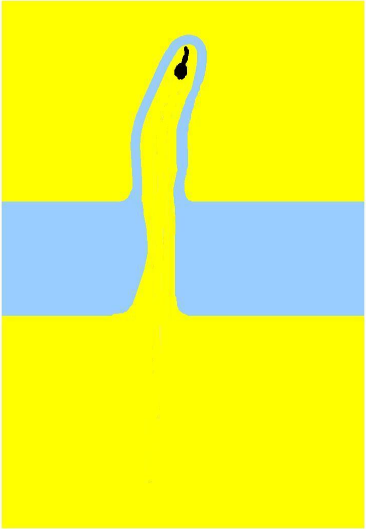


Definition

Let $\beta : [0,1] \longrightarrow S^2$ be a simple path. Define a path homeomorphism $\sigma_\beta : S^2 \rightarrow S^2$ as a homeomorphism such that

- $\sigma_\beta(\beta(0)) = \beta(1)$,
- $\sigma_\beta(x) = x$ except in a narrow tube around $\beta[0,1]$.

Path homeomorphisms



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Formal capture

Suppose that $f(z) = z^2 + c$ is such that $f^{\circ k}(0) = 0$. Let a be a strictly preperiodic point that is eventually mapped to 0; denote by U the Fatou component of f containing a. Choose β as the union of

- external ray landing at some point $b \in \partial U$,
- the point b,
- internal ray of U connecting b with a.

Then $\sigma_\beta \circ f$ is a formal capture of f at a.

Conformal capture

The formal capture $\sigma_\beta \circ f$ is a Thurston map with critical points 0 and ∞. Moreover, 0 is periodic of period k, and ∞ gets eventually mapped to 0.

Suppose that $\sigma_\beta \circ f$ is Thurston equivalent to a rational map. Call it the conformal capture.

Hyperbolic rational functions

Definition

A rational function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is called hyperbolic if it is expanding with respect to some Riemannian metric on a neighborhood of the Julia set.

The topological dynamics of hyperbolic rational functions is stable and in many cases easy to understand.

A conformal capture is a hyperbolic map.

The slices $Per_k(0) = V_k$

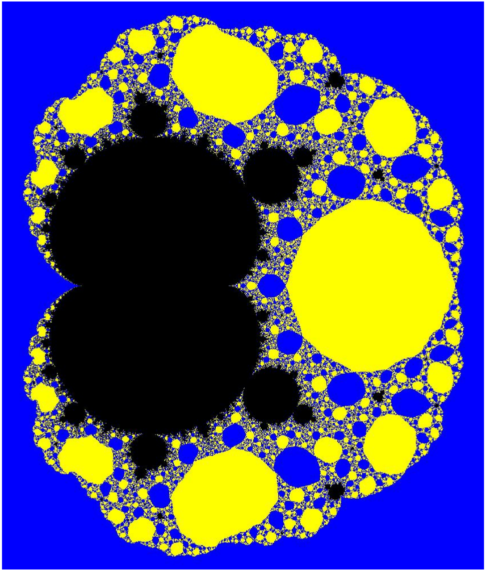
$Per_k(0)$ is the set of (Möbius conjugacy classes of) rational maps of degree 2 with marked critical points c_1, c_2 such that c_1 is periodic of period k. E.g.

- $Per_1(0) = \{z^2 + c\}$,
- $Per_2(0) = \{\frac{1}{z^2}\} \cup \{\frac{c}{z^2+2z}\}$.

Capture components

Let $R \in Per_k(0)$ be a conformal capture. Consider the hyperbolic component in $Per_k(0)$ (i.e. component of the set of hyperbolic maps in $Per_k(0)$) containing R. This component is called the capture component.

The parameter plane of maps $c/(z^2 + 2z)$



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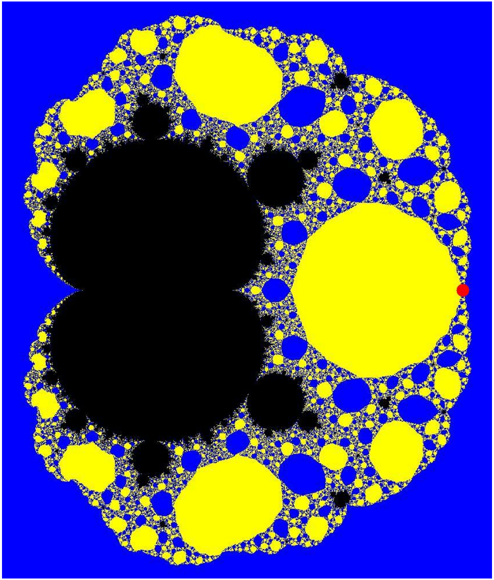

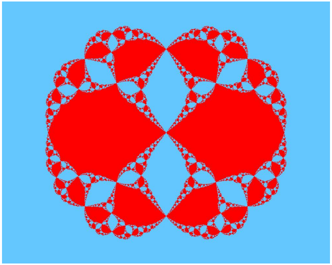

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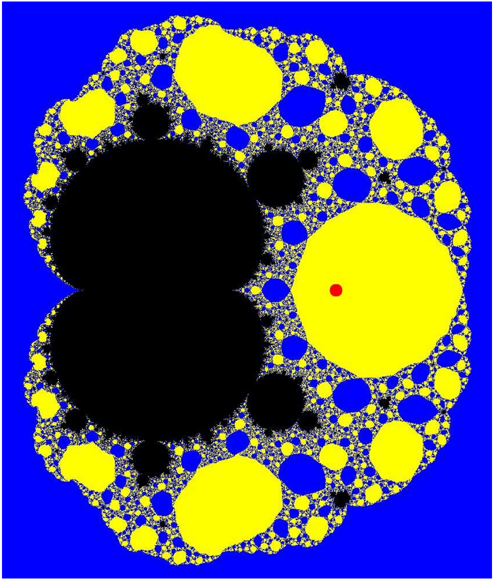

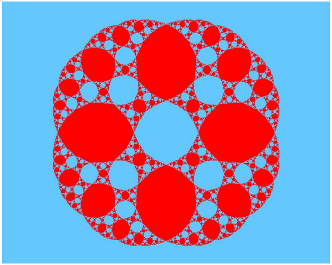

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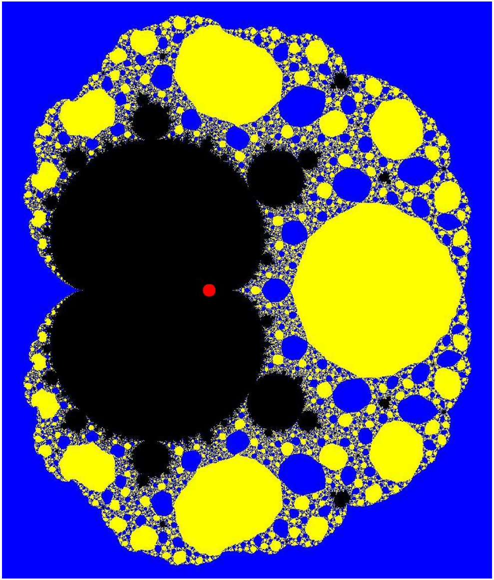

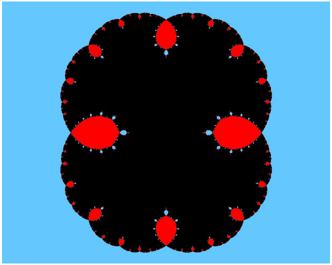

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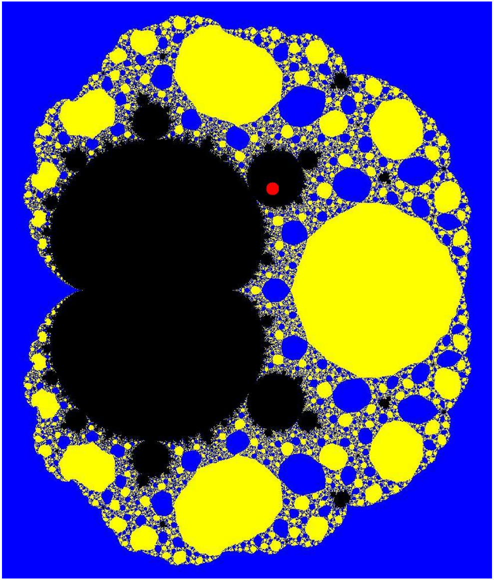
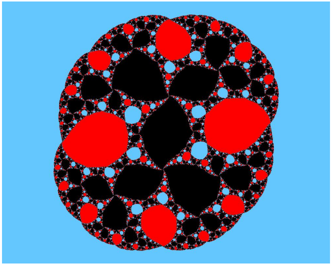
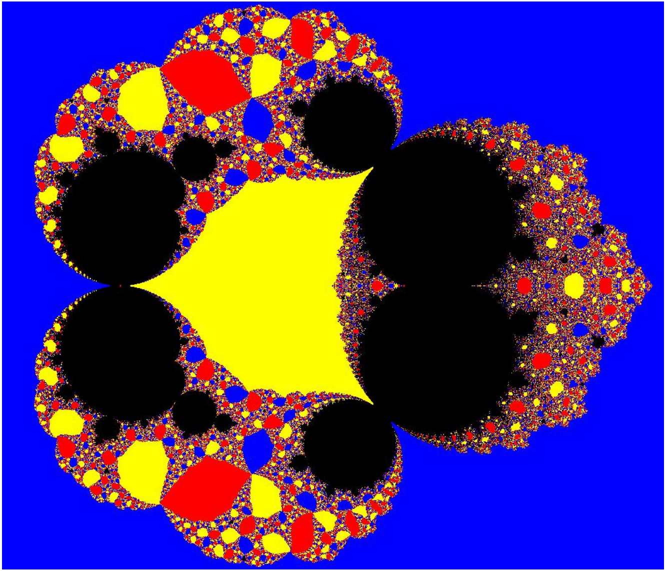
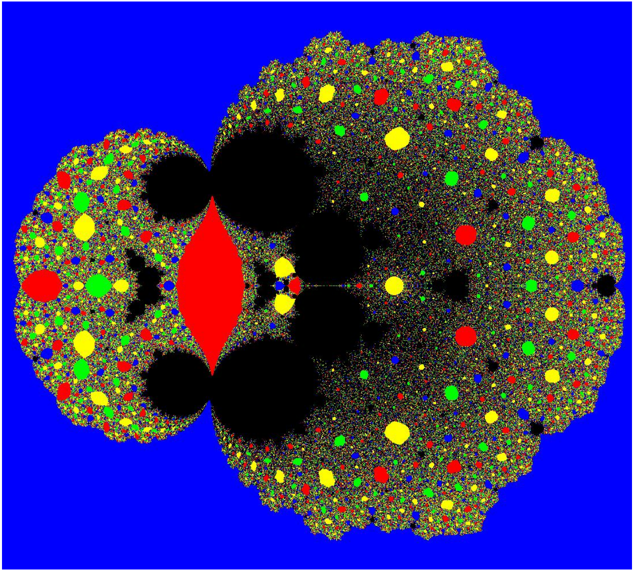




Parameter plane $Per_3(0)$



Parameter plane $Per_4(0)$



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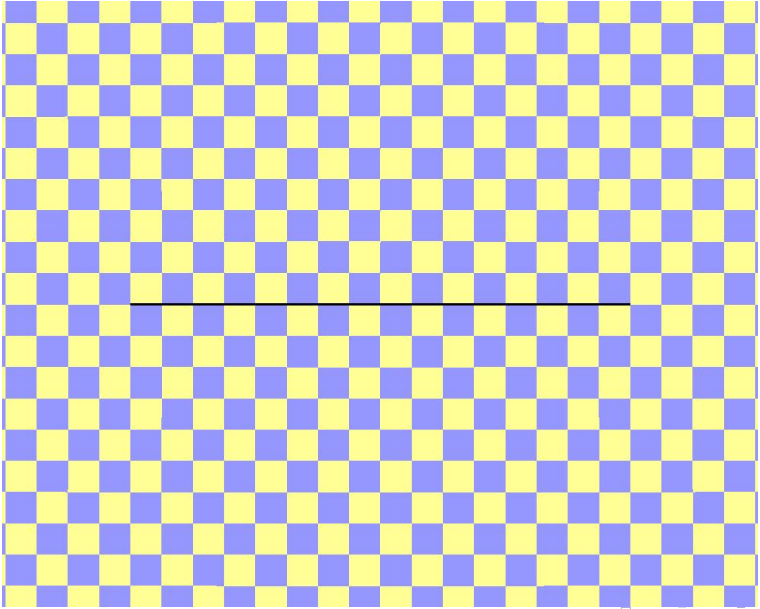
Capture vs mating

Fix β as above, and consider its pullbacks under $\sigma_\beta \circ f$. The intersections of these pullbacks with $\overline{\mathbb{C}} - K_f$ can be straightened. This yields a geodesic lamination L in $\overline{\mathbb{C}} - K_f$. If we put L into $\mathbb{D}$, then it will correspond to a polynomial p. Thus a fixed capture path β gives rise to both the capture R and the mating $f \amalg p$.

The mating p lies on the boundary of the capture component of R.

A regluing



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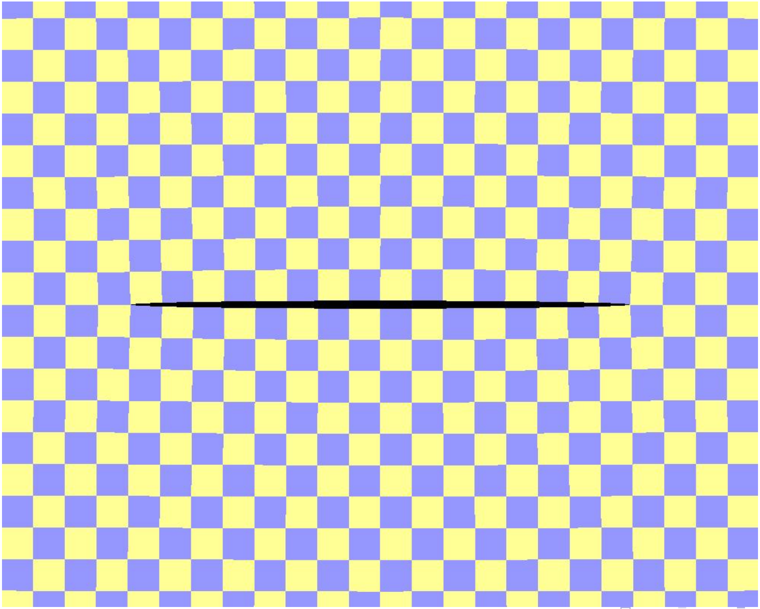

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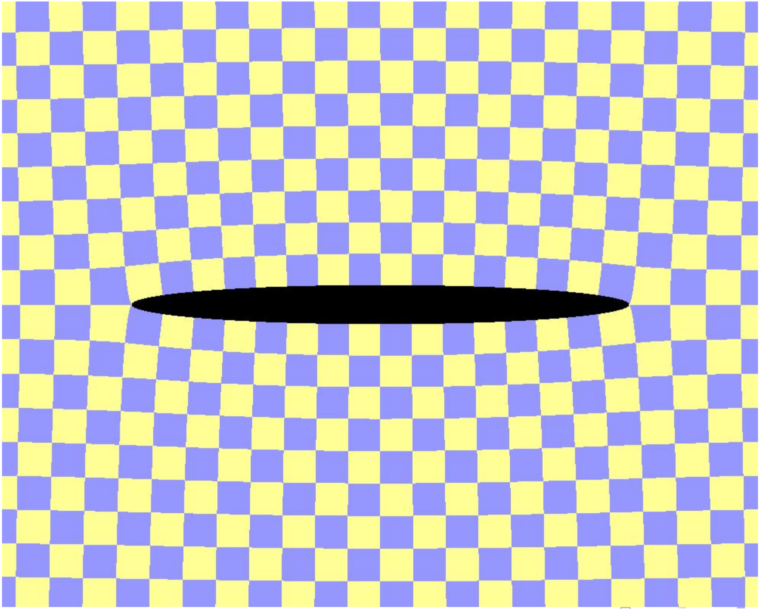

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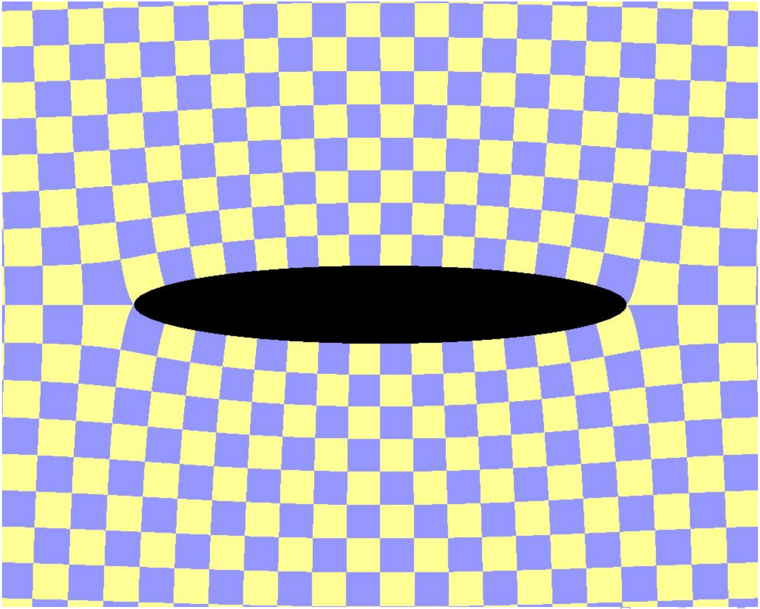

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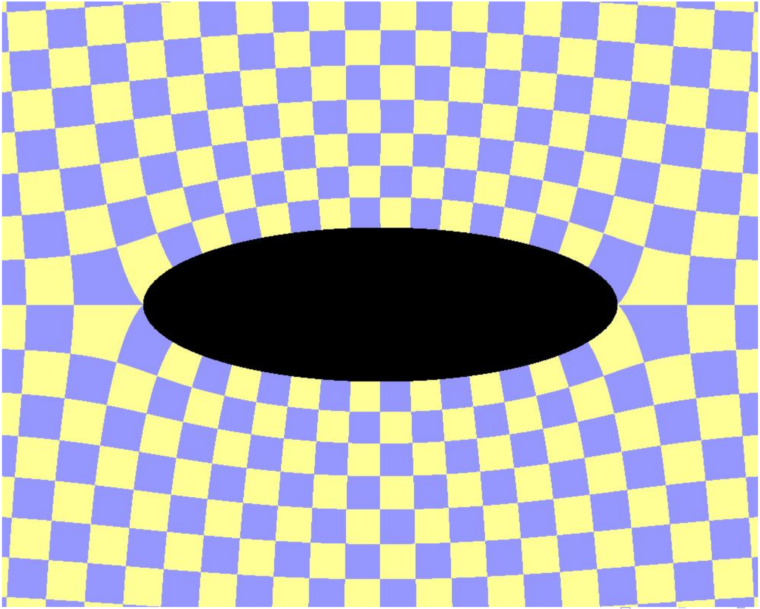

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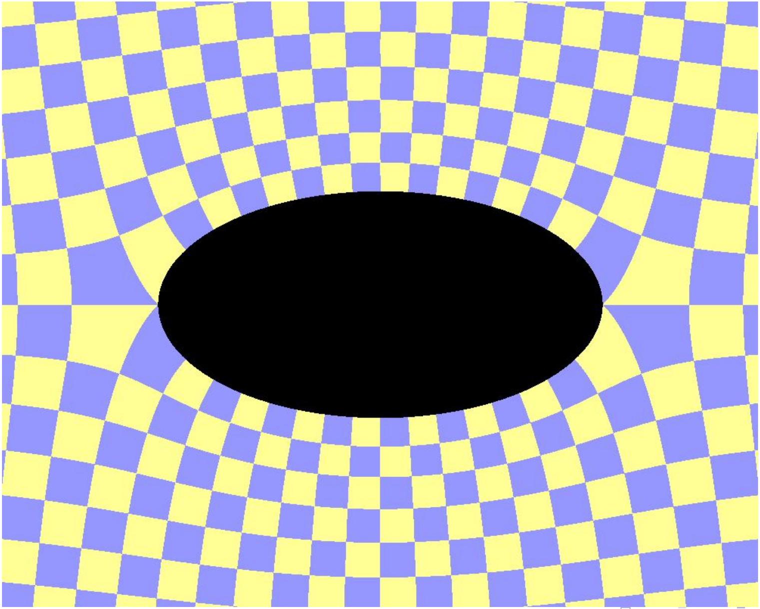

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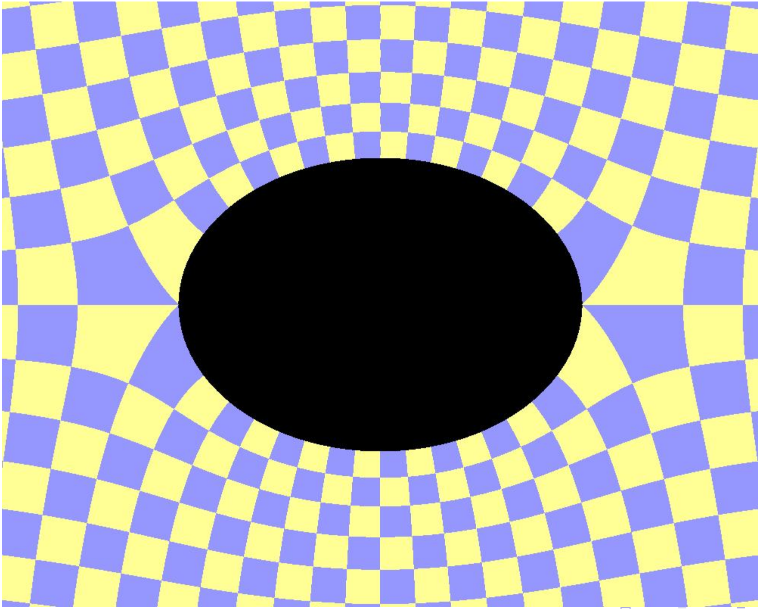

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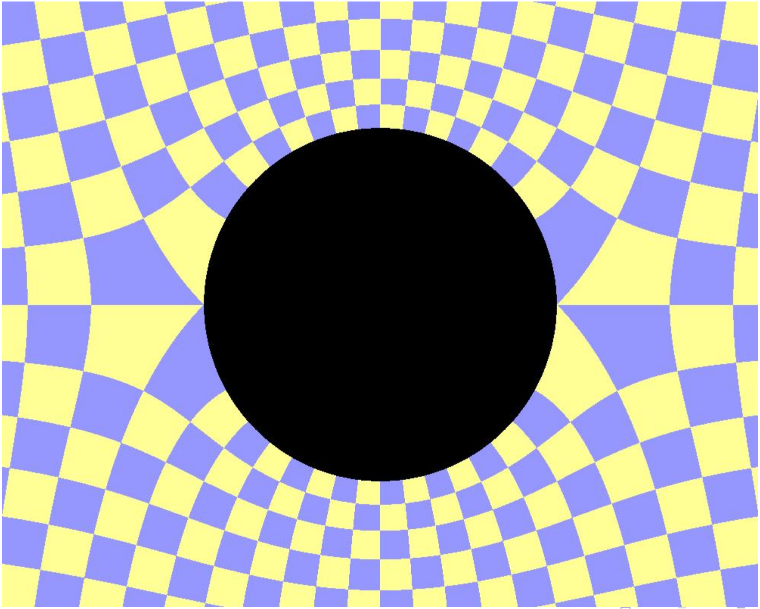

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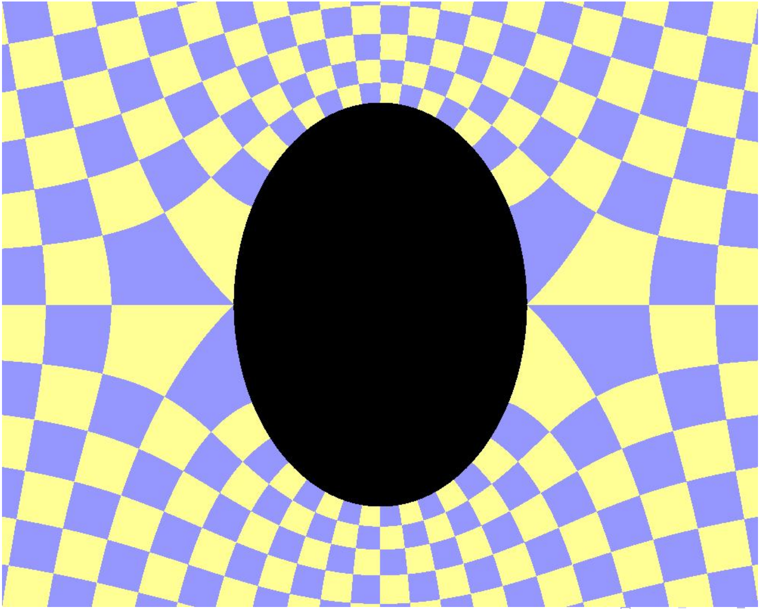

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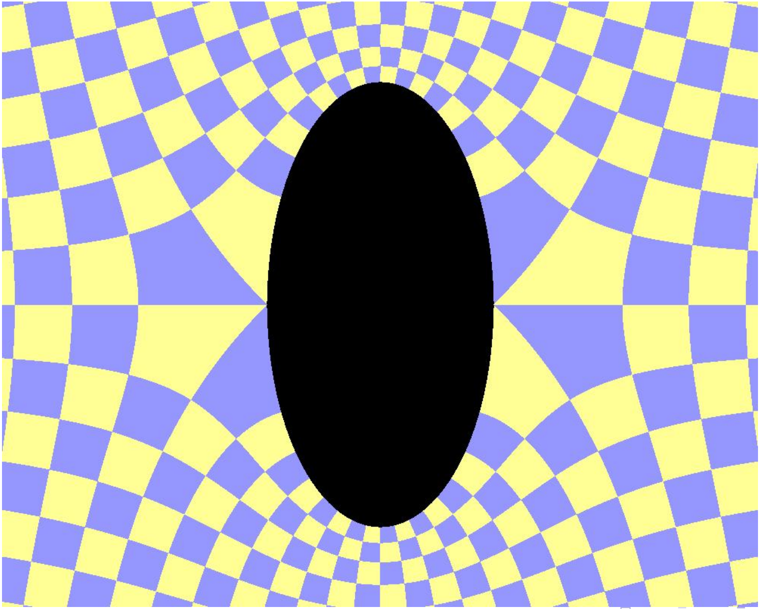

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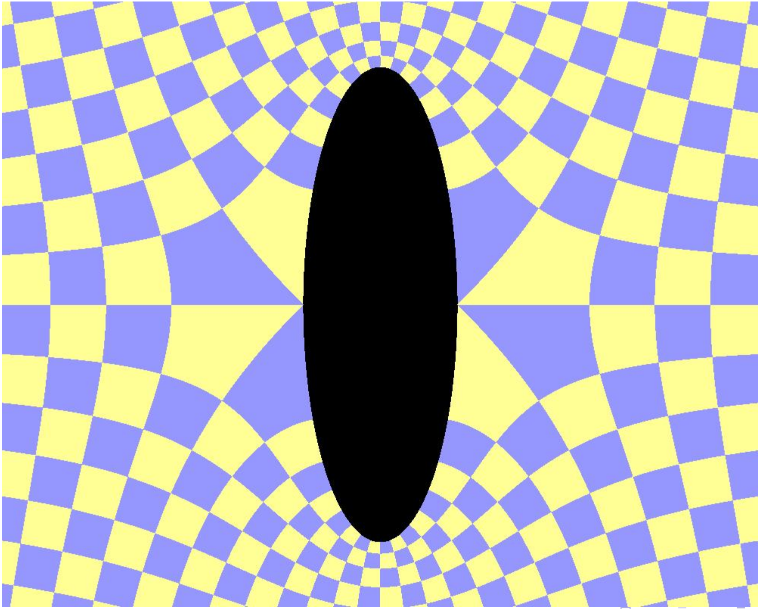

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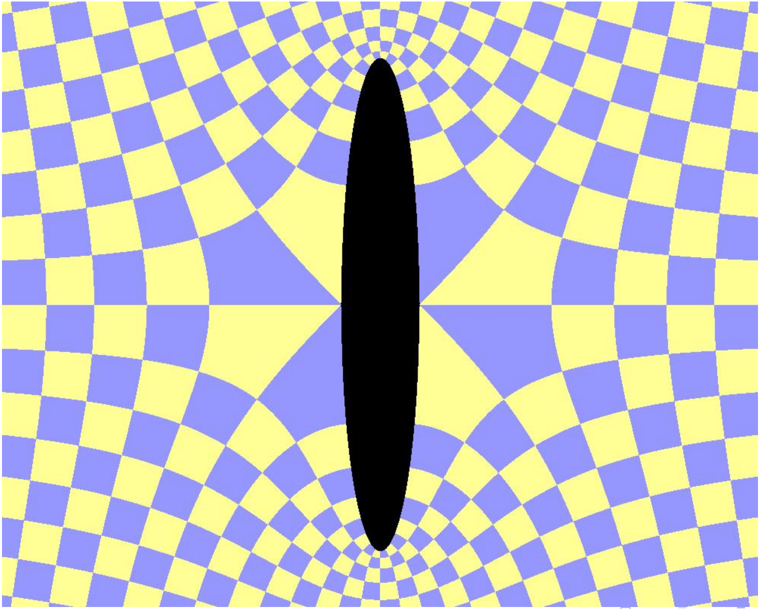

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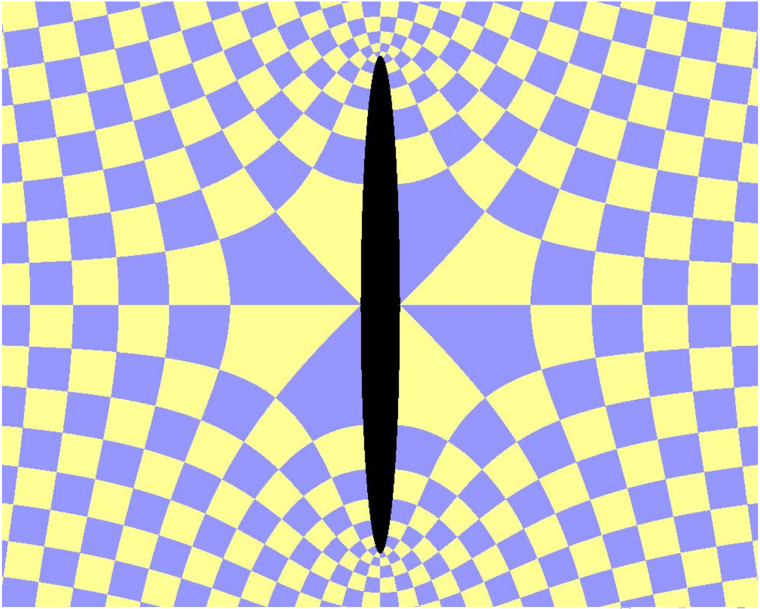

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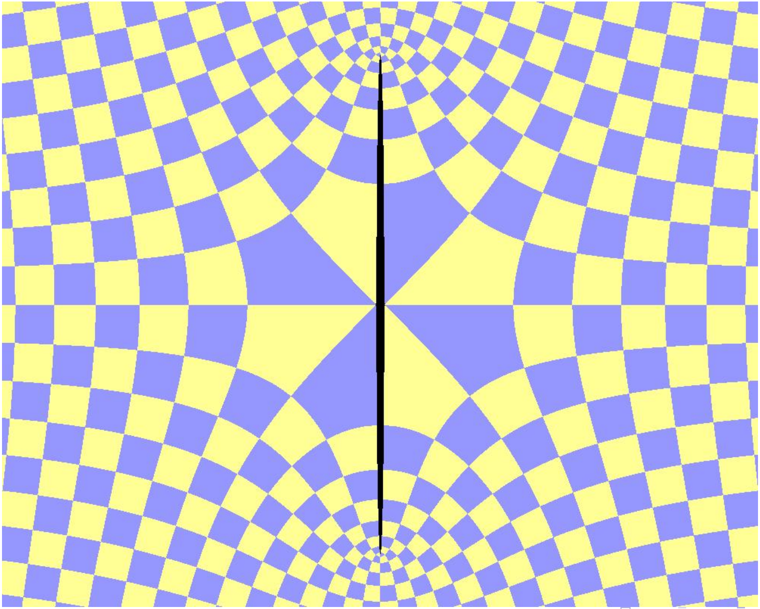

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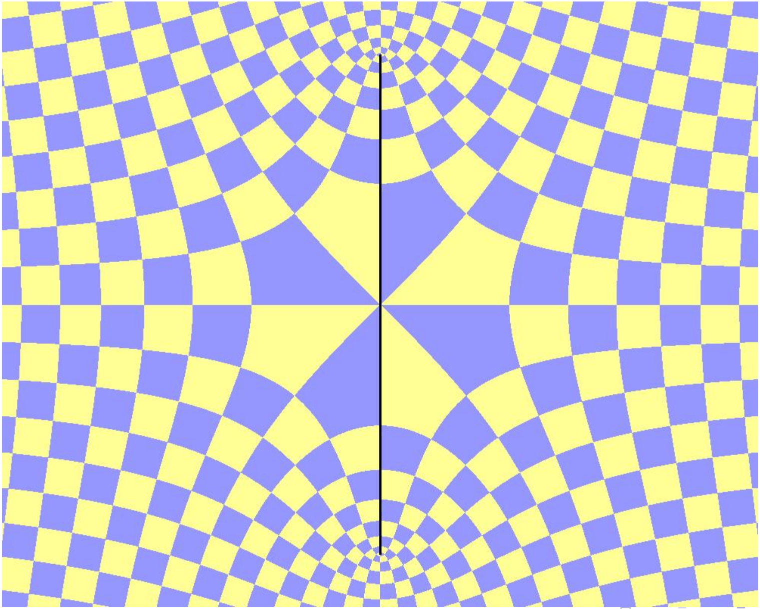
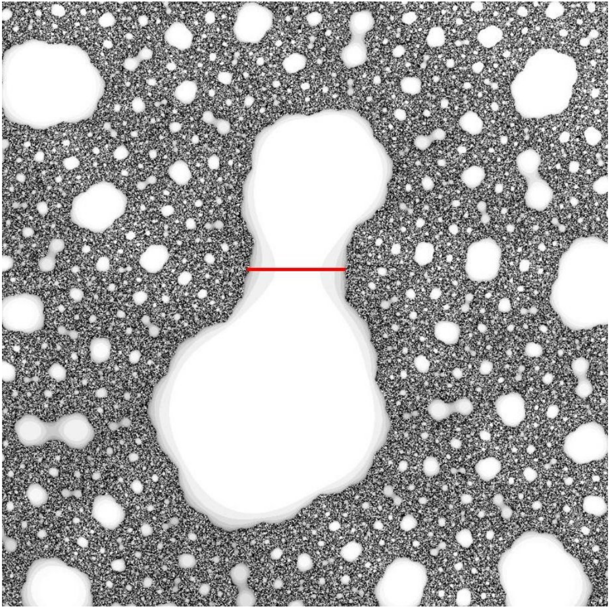


Capture vs mating: regluing



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