KINETIC MODELLING OF STRONGLY MAGNETIZED TOKAMAK PLASMAS WITH MASS DISPARATE PARTICLES. THE ELECTRON BOLTZMANN RELATION.

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ABSTRACT. The present work aims to justify on a formal level the obtention of the electron Boltzmann relation from a fully kinetic description of magnetically confined fusion plasmas, performing a suitable asymptotic limit. The obtained asymptotic limit model consists of the electron Boltzmann-equilibrium along the magnetic field lines, completed with a non-trivial dynamics perpendicular to these field lines. In the same asymptotic limit, the ions behave kinetically or reach either a gyrokinetic or a hydrodynamic regime. The Boltzmann approximation for the electrons is widely used in numerical simulations in the aim to drastically reduce the computational burden. It is thus crucial to understand how to obtain this reduced model from modelling assumptions and asymptotic considerations, starting from a microscopic description of the plasma.

Keywords: Singularly-perturbed problem, Anisotropic BGK-equation, Asymptotic Analysis, Modelling of plasma dynamics, Strong magnetic fields, Mass disparate particles, Electron Boltzmann relation.

1. INTRODUCTION

The subject matter of the present paper is the formal obtention of the electron Boltzmann relation, from the underlying kinetic description of strongly confined tokamak plasmas with mass disparate particles. At the same time, we also present the asymptotic limit models obtained during this procedure for the ions, namely the kinetic, gyrokinetic or hydrodynamic ion models.

Fusion plasmas are weakly collisional, due to high temperatures and low densities, such that the kinetic framework is the appropriate approach for their detailed description. In particular this amounts to solve a coupled system of two Vlasov or Boltzmann equations for the ion/electron distribution functions $f_{i,e}$ together with the Poisson equation for the electrostatic potential ϕ (or Maxwell's equations in the electromagnetic case). The difficulty with a fully kinetic treatment is however the high-dimensionality of the phase-space ($6D$). Furthermore the presence of multiple spatio-temporal scales makes the problem

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even more complicated, rendering it inaccessible for numerical simulations. To mention only some examples, the temporary scales extend from the fast electron plasma frequency ω_p, to the fast Larmor gyromotion ω_c, further to the collisional frequencies $\nu_{i,e}$ and finally to the confinement time τ_E . Concerning the spatial scales, they range from the small Debye length λ_D, to the electron Larmor radius ρ_e, further to the mean free path of the particles and finally to the spatial extent of the tokamak L. These various scales impose the use of very small time and space steps in numerical simulations in order to follow all the microscopic motions. In our particular case, apart the strong magnetic field, it is the small mass ratio $m_e/m_i \approx 10^{-4}$ of the particles which induces disparate scales and hence difficulties; in particular for a typical tokamak plasma with similar electron and ion temperatures the electron dynamics is faster than the ion dynamics, the ratio of the thermal velocities being given by $v_{th,e}/v_{th,i} = \sqrt{\frac{m_i}{m_e}} \approx 10^2$. This fact poses rather restrictive time-step constraints related to the fast electron motion, when a standard discretization of the bi-kinetic system is used, meaning that the numerical stability requires a CFL-condition of the type $v_{th,e}\Delta t \leq \Delta x$.

The fully-kinetic system contains however too many irrelevant spatio-temporal scales for the study of many interesting plasma processes. To redress this situation, a more macroscopic approach has to be adopted, eliminating the unnecessary fast dynamics and keeping the complete low-frequency physics. Such a macroscopic or reduced model is obtained via an asymptotic analysis, letting some specific parameters tend towards zero. In our particular case, we are only interested in following the plasma evolution on the large ion time-scales. At these time-scales the electrons attain a certain macroscopic thermal equilibrium, namely the electron Boltzmann-regime. In some words, this Boltzmann relation is obtained by assuming zero electron inertia ($m_e \to 0$) and zero viscosity in the "parallel" electron equation of motion ("parallel" with respect to the strong magnetic field), leading to the relation

$$\nabla_{||} p_e = -q\, n_e\, \mathbf{E}_{||}\,, \qquad \mathbf{E} = -\nabla \phi\,. \tag{1.1}$$

This relation indicates that the pressure-gradient and the electrostatic forces acting on the electrons (parallel to the magnetic field) are in balance. Moreover, rapid parallel thermal conduction assures that $\nabla_{||} T_e \sim 0$, such that with the thermodynamic equation of state $p_e = n_e\, k_B\, T_e$, one obtains

$$n_e(t,\mathbf{x}) = c(t,\mathbf{x}_\perp)\, \exp\left(\frac{q\,\phi(t,\mathbf{x})}{k_B\, T_e(t,\mathbf{x}_\perp)}\right)\,, \quad \mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_{||} \in \mathbb{R}^3\,,\ \ t \in \mathbb{R}^+\,. \tag{1.2}$$

Equation (1.2) is the so-called Boltzmann relation or adiabatic response, relating the electron density to the electric potential. Here $c(t,\mathbf{x}_\perp)$ and $T_e(t,\mathbf{x}_\perp)$ are functions to be determined from the remaining transport equations as well as initial and boundary conditions; they do not depend on the parallel coordinate $\mathbf{x}_{||}$. Once c and T_e are known, the relation

(1.2) can be inserted into the Poisson equation for the electrostatic potential, which can then be coupled to a model for the ion dynamics (kinetic or fluid). Such a procedure is extensively applied in plasma simulations [14, 18, 27, 36], with the purpose to study the ITG (ion-temperature-gradient) and TEM (trapped electron mode) micro-instabilities and associated turbulences, and this because the adiabatic electron response leads to large reductions in computational costs. Its practical use is due to the time-scale separation between ion and electron dynamics.

Our main aim in the here presented work is to understand how to obtain from a kinetic electron description the Boltzmann relation (1.2) in the so-called adiabatic limit. This asymptotic passage from the kinetic to the adiabatic regime can be very interesting for simulations in regions where the electron adiabatic response is violated, for example near the edge of the tokamak [24]. There, one has to go back to the more precise kinetic electron description, where the need to find the way how to couple these two regimes via a suitable limit procedure. The first difficulty in this asymptotic study is to find the adequate scaling of the starting kinetic equation, in particular to identify small parameters permitting to obtain the desired asymptotic limit model. The second arduous task is to close the asymptotic limit model, meaning to be able to find a well-posed macroscopic limit model containing the relation (1.2). The mathematical literature in this field is not so abundand, we refer the interested reader for example to the works [2, 4, 19, 23, 32, 34, 39] and references therein. Other works on asymptotic regimes for strongly magnetized plasmas where collisions are taken into account are mentioned here for completeness [3, 7, 12]. The collision-less case has been studied quite extensively, for example in [8, 9, 25, 26, 31]. The two works [21, 22] investigate binary gaz mixtures, where resembling considerations (to the here presented ones) are explored.

The present paper is structured in the following manner. Section 2 presents the physical scaling and identifies the small parameter ε permitting to capture the electron Boltzmann relation in the limit $\varepsilon \to 0$. Section 3 deals with the rigorous mathematical study of the dominant gyrokinetic-operator as well as of the advection operator along the magnetic field lines. Finally in Section 4 we formally prove that the chosen scaling permits indeed to get the adiabatic limit regime for the electrons. An Appendix is attached, condensing some important relations and computations, permitting thus to keep the presentation as simple as possible.

2. The fully kinetic model and its scaling

Starting point of our study is a kinetic description of the plasma dynamics in a tokamak, given by the following Boltzmann equations for the two species of charged particles, ions and electrons, *i.e.*

$$\left\{ \begin{array}{l} \partial_t f_i + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \dfrac{q}{m_i} \, (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_i = Q_{ii}(f_i) + Q_{ie}(f_i, f_e) \\ \partial_t f_e + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e - \dfrac{q}{m_e} \, (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_e = Q_{ee}(f_e) + Q_{ei}(f_e, f_i) \, , \end{array} \right. \tag{2.3}$$

coupled to the Poisson equation for the electrostatic potential

$$-\Delta \phi = \frac{q}{\varepsilon_0} (n_i - n_e) \, , \quad \mathbf{E} = -\nabla \phi \, , \tag{2.4}$$

where q is the elementary charge, $m_{e,i}$ the electron and ion masses, ε_0 the vacuum permittivity, whereas the electron and ion densities n_e, n_i are given by

$$n_{e,i}(t, \mathbf{x}) := \int_{\mathbb{R}^d} f_{e,i}(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v}. \tag{2.5}$$

In plasmas, the adequate collision operators are of the Fokker-Planck-Landau type, however we shall choose here for simplicity reasons to deal with the artificial BGK-operators; important for our present study are only the properties of conservation, entropy-dissipation and thermal equilibria. These BGK-operators read

$$Q_{ee,ii}(f_{e,i})(\mathbf{v}) := \nu_{ee,ii} \left(\mathcal{M}_{n_{e,i}, \mathbf{u}_{e,i}, T_{e,i}} - f_{e,i} \right) , \tag{2.6}$$

$$Q_{ei}(f_e, f_i)(\mathbf{v}) := \nu_{ei} \left(\mathcal{M}^e_{n_e, \mathbf{u}_i, T_i} - f_e \right) , \quad Q_{ie}(f_i, f_e)(\mathbf{v}) := \nu_{ie} \left(\mathcal{M}^i_{n_i, \mathbf{u}_e, T_e} - f_i \right) , \tag{2.7}$$

where $\nu_{ee}, \nu_{ei}, \nu_{ie}, \nu_{ii}$ are the relaxation frequencies of the distribution functions towards the Maxwellian distributions, given by

$$\mathcal{M}_{n_{e,i}, \mathbf{u}_{e,i}, T_{e,i}}(t, \mathbf{x}, \mathbf{v}) = n_{e,i}(t, \mathbf{x}) \left(\frac{m_{e,i}}{2\pi k_B T_{e,i}} \right)^{d/2} \exp \left(- m_{e,i} \frac{|\mathbf{u}_{e,i}(t, \mathbf{x}) - \mathbf{v}|^2}{2 k_B T_{e,i}} \right) . \tag{2.8}$$

Here k_B is the Planck constant, $\mathbf{u}_{e,i}$ resp. $T_{e,i}$ are the particle mean momenta resp. temperatures defined as

$$n_{e,i} \mathbf{u}_{e,i}(t, \mathbf{x}) := \int_{\mathbb{R}^d} \mathbf{v} \, f_{e,i}(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v} \, , \quad \frac{d}{2} k_B n_{e,i} T_{e,i} := \frac{m_{e,i}}{2} \int_{\mathbb{R}^d} |\mathbf{v} - \mathbf{u}_{e,i}|^2 \, f_{e,i}(t, \mathbf{x}, \mathbf{v}) \, d\mathbf{v} \, . \tag{2.9}$$

To have the classical mass, momentum and energy conservation properties of the collision operators, we have to suppose in the following the relation

$$m_i n_i \nu_{ie} = m_e n_e \nu_{ei} \, .$$

Remark here also the notation $\mathcal{M}^e_{n_e, \mathbf{u}_i, T_i}$ resp. $\mathcal{M}^i_{n_i, \mathbf{u}_e, T_e}$ in the definition of the inter-species collision operators, the upper index permitting to clarify which mass to take. These inter-species collision operators have been taken from the NRL Plasma Formulary [40].

System (2.3)-(2.4) together with the collision operators (2.6)-(2.7) describes the motion of charged particles in an externally given magnetic field and a self-consistent electric field.

It permits the study of micro-turbulences arising in fusion plasma dynamics, such as the ion-temperature-gradient (ITG) and the trapped electron mode (TEM) micro-instabilities. Micro-turbulence is nowadays one of the most important topics in fusion plasma studies as it is believed to be responsible for the anomalous transport and thus for the confinement performances of the device [37].

2.1. **Characteristic scales and regime of interest.** Let us now identify some small parameters, characterizing the different regimes of the fusion plasma dynamics. This shall be done by firstly introducing the orders of magnitude of the quantities involved in the description of the phenomenon we are interested in, in our particular case phenomena occurring at the ion spatio-temporal scales. The characteristic scales are summarized here:

- Temperature (hot plasma) (Parameter: γ)

$$T_i \sim T_e\,, \quad T_i = \bar{T}\,T_i'\,, \quad T_e = \bar{T}\,T_e'\,, \quad \gamma := \frac{q\bar{\phi}}{k_B\bar{T}}\,.$$

- Disparate masses (Parameter: ε)

$$\varepsilon^2 := \frac{m_e}{m_i} \ll 1\,.$$

- Microscopic velocity scale

$$\bar{v}_i := v_{th,i} = \sqrt{\frac{k_B\bar{T}}{m_i}}\,, \quad \bar{v}_e := v_{th,e} = \sqrt{\frac{k_B\bar{T}}{m_e}} = \frac{1}{\varepsilon}\,\bar{v}_i\,.$$

- Microscopic time and length scale (Parameter: η)
  $\rightarrow$ related to the strong magnetic field:

$$\omega_{ci} := \frac{q\bar{B}}{m_i}\,, \quad \tau_{ci} := \frac{1}{\omega_{ci}} \quad \text{(ion cyclotron frequency)}\,,$$

$$\rho_L := \frac{\bar{v}_i}{\omega_{ci}} = \bar{v}_i\,\tau_{ci} \quad \text{(ion Larmor radius)}\,,$$

  $\rightarrow$ related to the ionic collision process:

$$\tau_c := \tau_{ii} \quad \text{(elapsed time between 2 ionic collisions)}\,,$$

$$l_c := \bar{v}_i\,\tau_c \quad \text{(mean free path between 2 ionic collisions)}\,,$$

$$\eta := \frac{\tau_{ci}}{\tau_c} \ll 1\,.$$

- Macroscopic velocity scale (Parameter: α)

$$u_i \sim u_e\,, \quad \bar{u}\,\bar{B} = \bar{E} \quad \text{(Electric drift relation)}\,,$$

$$\alpha := \frac{\bar{u}}{\bar{v}_i}\,.$$

- Macroscopic time and space scale (Dependent parameters: δ, τ, β)

$$\bar{x} \quad \text{(distance of interest)}\,, \quad \delta := \frac{l_c}{\bar{x}}\,,$$

$$\bar{t} := \frac{\bar{x}}{\bar{u}} \quad \text{(observation time)}\,, \qquad \tau := \frac{\tau_c}{\bar{t}}\,, \qquad \beta := \frac{\tau_{ci}}{\bar{t}}\,,$$

$\rightarrow$Relations between these parameters:

$$\beta = \frac{\alpha^2}{\gamma}\,, \quad \tau = \frac{\beta}{\eta}\,, \quad \delta = \frac{\tau}{\alpha}\,.$$

- Collision operators, distribution functions

$$n_i \sim n_e\,, \quad n_i = \bar{n}\, n_i'\,; \quad n_e = \bar{n}\, n_e'\,, \quad \bar{f}_e = \frac{\bar{n}}{\bar{v}_e^d}\,, \quad \bar{f}_i = \frac{\bar{n}}{\bar{v}_i^d}\,,$$

$$\bar{Q}_{ee} = \nu_{ee}\bar{f}_e\,, \quad \bar{Q}_{ii} = \nu_{ii}\bar{f}_i\,, \quad \bar{Q}_{ei} = \nu_{ei}\bar{f}_e\,, \quad \bar{Q}_{ie} = \nu_{ie}\bar{f}_i\,.$$

- Collision frequencies [28]

$$\nu_{ii} = \varepsilon\nu_{ee}\,, \quad \nu_{ie} = \varepsilon^2\nu_{ei}\,, \quad \nu_{ie} = \varepsilon\nu_{ii}\,, \quad \nu_{ee} = \nu_{ei}\,, \quad \tau_{ee} = \tau_{ei} = \varepsilon\tau_c\,, \quad \tau_{ie} = \frac{\tau_c}{\varepsilon}\,.$$

- Debye length (Parameter: λ)

$$\lambda_D := \sqrt{\frac{\varepsilon_0\, k_B\, \bar{T}}{\bar{n}_i\, q^2}} = \frac{v_{th,i}}{\omega_p}\,, \quad \omega_p := \sqrt{\frac{\bar{n} q^2}{\varepsilon_0\, m_i}}\,, \quad \tau_p := 1/\omega_p \quad \text{(plasma frequency)}\,,$$

$$\lambda := \frac{\lambda_D}{\bar{x}} = \frac{\bar{v}_i}{\bar{x}}\,\frac{1}{\omega_p} = \frac{\bar{v}_i}{\bar{u}}\,\frac{\tau_p}{\bar{t}} = \frac{1}{\alpha}\,\frac{\tau_p}{\bar{t}}\,.$$

The units or scales chosen here are adapted to the plasma regimes we want to study (electron Boltzmann regime, gyrokinetic or hydrodynamic ion regimes). The reader not so familiar with the physics of tokamak fusion plasmas is referred to the introductory books [17, 28, 29, 33]. Let us also remark that we have a set of 5 independent parameters, for ex. $(\gamma, \varepsilon, \eta, \alpha, \lambda)$ or $(\beta, \varepsilon, \eta, \alpha, \lambda)$.

2.2. **Non-dimensional kinetic system.** With the just defined characteristic scales, we shall perform now the following changes of variables

$$\mathbf{x} = \bar{x}\mathbf{x}'\,, \quad t = \bar{t}t'\,, \quad \mathbf{v}_i = \bar{v}_i\mathbf{v}'\,, \quad \mathbf{v}_e = \bar{v}_e\mathbf{v}'\,,$$

$$E(t,\mathbf{x}) = \bar{E}E'(t',\mathbf{x}')\,, \quad B(t,\mathbf{x}) = \bar{B}B'(t',\mathbf{x}')\,, \quad f_{e,i}(t,\mathbf{x},\mathbf{v}_{e,i}) = \bar{f}_{e,i}f'_{e,i}(t',\mathbf{x}',\mathbf{v}')\,,$$

which leads to the non-dimensional system (the primes were omitted for simplicity reasons)

$$\begin{cases} \partial_t f_i + \dfrac{1}{\alpha}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_i + \dfrac{\alpha}{\beta}\left(\mathbf{E} + \dfrac{1}{\alpha}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_i = \dfrac{\eta}{\beta}\left[Q_{ii}(f_i) + \varepsilon Q_{ie}(f_i,f_e)\right] \\ \partial_t f_e + \dfrac{1}{\varepsilon\,\alpha}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_e - \dfrac{\alpha}{\varepsilon\beta}\left(\mathbf{E} + \dfrac{1}{\varepsilon\,\alpha}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_e = \dfrac{\eta}{\varepsilon\beta}\left[Q_{ee}(f_e) + Q_{ei}(f_e,f_i)\right]\,, \end{cases} \tag{2.10}$$

coupled to the rescaled Poisson equation

$$-\lambda^2 \Delta \phi = n_i - n_e\,, \quad n_{e,i}(t,\mathbf{x}) := \int_{\mathbb{R}^d} f_{e,i}(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,, \quad \mathbf{E} := -\nabla\,\phi\,. \tag{2.11}$$

Here the BGK collision operators have now the forms (in the new variables)

$$Q_{ee}(f_e)(\mathbf{v}) = \mathcal{M}^s_{n_e,\varepsilon\alpha\mathbf{u}_e,T_e}(t,\mathbf{x},\mathbf{v}) - f_e\,, \quad Q_{ii}(f_i)(\mathbf{v}) = \mathcal{M}^s_{n_i,\alpha\mathbf{u}_i,T_i}(t,\mathbf{x},\mathbf{v}) - f_i\,.$$

$$Q_{ei}(f_e,f_i)(\mathbf{v}) = \mathcal{M}^s_{n_e,\varepsilon\alpha\mathbf{u}_i,T_i}(t,\mathbf{x},\mathbf{v}) - f_e\,, \quad Q_{ie}(f_i,f_e)(\mathbf{v}) = \mathcal{M}^s_{n_i,\alpha\mathbf{u}_e,T_e}(t,\mathbf{x},\mathbf{v}) - f_i\,,$$

where $\mathcal{M}^s$ stands for the rescaled Maxwellian ($m_{e,i} = 1$ and $k_B = 1$), *i.e.*

$$\mathcal{M}^s_{n,\mathbf{u},T}(t,\mathbf{x},\mathbf{v}) = n(t,\mathbf{x}) \left(\frac{1}{2\pi T}\right)^{d/2} \exp\left(-\frac{|\mathbf{u}(t,\mathbf{x}) - \mathbf{v}|^2}{2T}\right).$$

Several asymptotic regimes can be studied starting from (2.10), permitting the description of various plasma phenomena, some of them being presented in Section 2.4. For the moment let us write down the corresponding, not-closed fluid equations.

2.3. **The corresponding non-dimensional fluid model.** In the aim to take the moments of the just obtained non-dimensional kinetic system (2.10), let us introduce the rescaled macroscopic quantities.

$$n_{e,i}(t,\mathbf{x}) := \int_{\mathbb{R}^d} f_{e,i}(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,,$$

$$\varepsilon\alpha\, n_e\mathbf{u}_e(t,\mathbf{x}) := \int_{\mathbb{R}^d} \mathbf{v}\, f_e(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,, \quad \alpha\, n_i\mathbf{u}_i(t,\mathbf{x}) := \int_{\mathbb{R}^d} \mathbf{v}\, f_i(t,\mathbf{x},\mathbf{v})\, d\mathbf{v}\,,$$

$$w_e(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^d} |\mathbf{v}|^2\, f_e(t,\mathbf{x},\mathbf{v})\, d\mathbf{v} = \frac{1}{2}\varepsilon^2\alpha^2\, n_e|\mathbf{u}_e|^2 + \frac{d}{2}n_eT_e\,,$$

$$w_i(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^d} |\mathbf{v}|^2\, f_i(t,\mathbf{x},\mathbf{v})\, d\mathbf{v} = \frac{1}{2}\alpha^2\, n_i|\mathbf{u}_i|^2 + \frac{d}{2}n_iT_i\,,$$

$$\mathbb{P}_e(t,\mathbf{x}) := \int_{\mathbb{R}^d} (\mathbf{v} - \varepsilon\alpha\,\mathbf{u}_e)\otimes(\mathbf{v} - \varepsilon\alpha\,\mathbf{u}_e) f_e\, d\mathbf{v}\,, \quad \mathbb{P}_i(t,\mathbf{x}) := \int_{\mathbb{R}^d} (\mathbf{v} - \alpha\,\mathbf{u}_i)\otimes(\mathbf{v} - \alpha\,\mathbf{u}_i) f_i\, d\mathbf{v}\,,$$

$$\mathbf{q}_e(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{v} - \varepsilon\alpha\,\mathbf{u}_e)|\mathbf{v} - \varepsilon\alpha\,\mathbf{u}_e|^2 f_e\, d\mathbf{v}\,, \quad \mathbf{q}_i(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^d} (\mathbf{v} - \alpha\,\mathbf{u}_i)|\mathbf{v} - \alpha\,\mathbf{u}_i|^2 f_i\, d\mathbf{v}\,.$$

Taking now the moments of (2.10) leads to the following systems of macroscopic equations

$$\begin{cases} \partial_t\, n_i + \nabla_{\mathbf{x}}\cdot(n_i\mathbf{u}_i) = 0\,, \\ \partial_t\,(n_i\mathbf{u}_i) + \nabla_{\mathbf{x}}\cdot(n_i\mathbf{u}_i\otimes\mathbf{u}_i) + \dfrac{1}{\alpha^2}\nabla_{\mathbf{x}}\cdot\mathbb{P}_i - \dfrac{1}{\beta}\, n_i(\mathbf{E} + \mathbf{u}_i\times\mathbf{B}) = \varepsilon\,\dfrac{\eta}{\beta}\, n_i(\mathbf{u}_e - \mathbf{u}_i) \\ \partial_t w_i + \nabla_{\mathbf{x}}\cdot(w_i\mathbf{u}_i + \mathbb{P}_i\mathbf{u}_i + \dfrac{1}{\alpha}\,\mathbf{q}_i) - \dfrac{\alpha^2}{\beta}\, n_i\mathbf{E}\cdot\mathbf{u}_i = \varepsilon\,\dfrac{\eta}{\beta}\, n_i\left[\dfrac{\alpha^2}{2}\,(|\mathbf{u}_e|^2 - |\mathbf{u}_i|^2) + \dfrac{d}{2}(T_e - T_i)\right]\,, \end{cases} \tag{2.12}$$

$$\begin{cases} \partial_t n_e + \nabla_{\mathbf{x}} \cdot (n_e \mathbf{u}_e) = 0\,, \\ \partial_t (n_e \mathbf{u}_e) + \nabla_{\mathbf{x}} \cdot (n_e \mathbf{u}_e \otimes \mathbf{u}_e) + \dfrac{1}{\varepsilon^2 \alpha^2} \nabla_{\mathbf{x}} \cdot \mathbb{P}_e + \dfrac{1}{\varepsilon^2 \beta} n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) = \dfrac{\eta}{\varepsilon \beta} n_e (\mathbf{u}_i - \mathbf{u}_e) \\ \partial_t w_e + \nabla_{\mathbf{x}} \cdot (w_e \mathbf{u}_e + \mathbb{P}_e \mathbf{u}_e + \dfrac{1}{\varepsilon \alpha} \mathbf{q}_e) + \dfrac{\alpha^2}{\beta} n_e \mathbf{E} \cdot \mathbf{u}_e = \dfrac{\eta}{\varepsilon \beta} n_e \left[\dfrac{\varepsilon^2 \alpha^2}{2} (|\mathbf{u}_i|^2 - |\mathbf{u}_e|^2) \right. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. + \dfrac{d}{2} (T_i - T_e) \right]\,, \end{cases} \tag{2.13}$$

coupled to the Poisson equation

$$-\lambda^2 \Delta \phi = n_i - n_e\,, \quad \mathbf{E} := -\nabla\, \phi\,. \tag{2.14}$$

This system of equations (2.12)-(2.13) is not closed, it is coupled to the kinetic system (2.10) via the pressure $\mathbb{P}_i\,, \mathbb{P}_e$ and the heat flux $\mathbf{q}_i\,, \mathbf{q}_e$ quantities. To close this system one has to identify a small parameter leading (in the vanishing limit) towards a macroscopic equilibrium, to be characterized in detail.

2.4. **Review of different asymptotic models.** As mentioned earlier, a variety of regimes can be studied starting from (2.10), depending on the particular choice of the various involved parameters. Let us mention here some examples and introduce the regime which is adequate to get the Boltzmann relation. Let us suppose in the following that $\gamma = 1$ (meaning that the electric energy is of the same order as the thermal energy), which implies $\beta = \alpha^2$, and assume ε small, thus we shall start from

$$\begin{cases} \partial_t f_i + \dfrac{1}{\alpha} \mathbf{v} \cdot \nabla_{\mathbf{x}} f_i + \dfrac{1}{\alpha} \left(\mathbf{E} + \dfrac{1}{\alpha} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f_i = \dfrac{\eta}{\alpha^2} [Q_{ii}(f_i) + \varepsilon Q_{ie}(f_i, f_e)] \\ \partial_t f_e + \dfrac{1}{\varepsilon\, \alpha} \mathbf{v} \cdot \nabla_{\mathbf{x}} f_e - \dfrac{1}{\varepsilon\, \alpha} \left(\mathbf{E} + \dfrac{1}{\varepsilon \alpha} \mathbf{v} \times \mathbf{B} \right) \cdot \nabla_{\mathbf{v}} f_e = \dfrac{\eta}{\varepsilon \alpha^2} [Q_{ee}(f_e) + Q_{ei}(f_e, f_i)]\,. \end{cases} \tag{2.15}$$

As one can observe from this system, the dynamics of a plasma gaz is characterized by several time scales, coming among others from the specificity of plasmas of having very disparate masses. Observe first the scaling between the various collision operators (inter- and intra-collision operators). Due to the small mass ratio, the ion-electron collision term is negligible in the relaxation process of the ions towards their thermodynamic equilibrium. Besides, both collision operators Q_{ee} and Q_{ei} act on the same time-scale and contribute together to the thermodynamic relaxation of the electrons. Secondly, observe also the scaling of the transport parts, which in some situations will be of hydrodynamic type for the ions (heavy species) and of diffusion type for the electrons (light species). The ions relax much slower than the electrons towards their corresponding equilibrium, namely $\sqrt{m_e/m_i}$-slower, such that the electron fluid equations are established in advance to the ion hydrodynamic ones.

System (2.15) gives rise to a variety of hybrid ion/electron models, relevant for the description of a fusion plasma gaz. We can identify for example the following regimes:

- $\eta = \alpha = 1$, $\varepsilon \ll 1$:
  $\to$ ions: kinetic regime
  $\to$ electrons: $\star$ Drift-Diffusion (if Q_{ei} not negligible) [35, 38, 41]
  $\star$ or adiabatic regime (electron Boltzmann relation, if Q_{ei} negligible – no friction);
- $\eta = 1$, $\alpha \sim 1$, $\alpha^2 \ll 1$, $\varepsilon\alpha \ll 1$:
  $\to$ ions: hydrodynamic scaling [1, 20, 30];
  $\to$ electrons: Drift-Diffusion or adiabatic regime ;
- $\eta = \alpha$, $\alpha \sim 1$, $\alpha^2 \ll 1$, $\varepsilon\alpha \ll 1$:
  $\to$ ions: gyrokinetic regime [5, 6, 31];
  $\to$ electrons: Drift-Diffusion or adiabatic regime;
- $\eta = \alpha$, $\alpha \ll 1$, $\varepsilon \ll 1$: long time asymptotics:
  $\to$ ions: adiabatic regime (ion Boltzmann relation);
  $\to$ electrons: Drift-Diffusion or adiabatic regime.

Several other possible asymptotic regimes can be identified. The regime we shall investigate in the sequel is the electron Boltzmann relation regime, since it is still badly understood up to now, but often used in numerical simulations. Thus we shall examine in detail the ε-behaviour of the following rescaled kinetic equation

$$\partial_t f_e^\varepsilon + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f_e^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_e^\varepsilon = \frac{1}{\varepsilon} Q_{ee}^\varepsilon(f_e^\varepsilon)\,. \tag{2.16}$$

This scaling highlights the delicate competition between transport, oscillating (due to the strong magnetic field) and dissipative (due to the collision term) effects. As we shall see later on, the assumption of the electron adiabatic response is based on the assertion that these particles move very fast along the magnetic field lines, in such a way that they reach rapidly a thermal equilibrium. However, one has to underline here that this electron thermal equilibrium (Maxwellian distribution function) is maintained by electron collisions. When the electrons do not undergo collisions, the Boltzmann equilibrium can not be reached or even maintained and the approximation becomes controversial.

3. Mathematical study of the dominant operator

In the following we shall consider that we are living in the phase-space $\Omega \times \mathbb{R}^3$ (from now on $d = 3$), with regular and bounded space domain $\Omega \subset \mathbb{R}^3$ describing a flat torus (see Section 4), which is a simplification of a real tokamak. The study of the asymptotic limit $\varepsilon \to 0$ of the electron kinetic equation (2.16), naturally leads to the investigation of the properties of two operators, namely the dominant operator $\mathcal{T} := (\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}$ as well as the transport operator $\mathcal{A} := \mathbf{B} \cdot \nabla_x$. This shall be done in the present section, starting by supposing for the rest of this paper the following hypothesis concerning the magnetic field:

Hypothesis A *Let us assume that the magnetic field* $\mathbf{B}$ *is given, time-independent and sufficiently smooth, with direction* $\mathbf{b}(\mathbf{x}) := \frac{\mathbf{B}(\mathbf{x})}{|\mathbf{B}(\mathbf{x})|}$ *and magnitude* $\mathfrak{M}(\mathbf{x}) := |\mathbf{B}(\mathbf{x})|$, *satisfying* $\inf_{\mathbf{x}\in\Omega} \mathfrak{M}(\mathbf{x}) = \gamma > 0$ *for some constant* γ. *Furthermore we suppose that* $\nabla \cdot \mathbf{B} = 0$.

With respect to this magnetic field, we shall use in the sequel often the following notation for a vector $\mathbf{h} \in \mathbb{R}^3$, a scalar function $\phi(\mathbf{x})$ or a vector field $\mathbf{v}(\mathbf{x})$:

$$
\begin{aligned}
&h_{\mathbf{b}} := \mathbf{h} \cdot \mathbf{b}\,, \quad \mathbf{h}_{||} := (\mathbf{b} \otimes \mathbf{b})\mathbf{h} = h_{\mathbf{b}}\, \mathbf{b}\,, \qquad \mathbf{h}_{\perp} := (\mathbb{I} - \mathbf{b} \otimes \mathbf{b})\mathbf{h}\,, \qquad \mathbf{h}^{\perp} := \mathbf{h} \times \mathbf{b}\,.\\
&\nabla_{||}\phi := (\mathbf{b} \otimes \mathbf{b})\, \nabla\phi = (\partial_{\mathbf{b}}\phi)\, \mathbf{b}\,, \quad \nabla_{\perp}\phi := (\mathbb{I} - \mathbf{b} \otimes \mathbf{b})\nabla\phi\,, \quad \text{so that} \quad \nabla\phi = \nabla_{||}\phi + \nabla_{\perp}\phi\,,\\
&\nabla_{||} \cdot \mathbf{v} := \nabla \cdot \mathbf{v}_{||}\,, \qquad \nabla_{\perp} \cdot \mathbf{v} := \nabla \cdot \mathbf{v}_{\perp}\,, \qquad \text{so that} \quad \nabla \cdot \mathbf{v} = \nabla_{||} \cdot \mathbf{v} + \nabla_{\perp} \cdot \mathbf{v}\,,
\end{aligned}
\tag{3.17}
$$

where we denoted by $\otimes$ the vector tensor product. Let us also remark that in some situations it will be helpful to pass to cylindrical coordinates in velocity space, and this with respect to a fixed $\mathbf{b}(\mathbf{x})$. In more details, each $\mathbf{v} \in \mathbb{R}^3$ can be characterized by the coordinates $(v_{\mathbf{b}}, r, \varpi) \in \mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^1_{\mathbf{b}}$, the identification being specified by the decomposition

$$
\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp} = v_{\mathbf{b}}\, \mathbf{b} + r\, \varpi\,, \quad r := |\mathbf{v}_{\perp}|\,, \quad \varpi := \frac{\mathbf{v}_{\perp}}{|\mathbf{v}_{\perp}|} \in \mathbb{S}^1_{\mathbf{b}}\,,
$$

where $\mathbb{S}^1_{\mathbf{b}}$ is the one-dimensional set of unitary vectors, perpendicular to $\mathbf{b}$, namely

$$
\mathbb{S}^1_{\mathbf{b}} := \{\varpi \in \mathbb{R}^3 \ / \ |\varpi| = 1\,, \ \varpi \cdot \mathbf{b} = 0\}\,.
$$

Figure 1 represents schematically this cylindrical coordinate system with respect to $\mathbf{b}$. Let us finally also note that one has $d\mathbf{v} = r\, dv_{\mathbf{b}}\, dr\, d\varpi$.
Coming now to the study of the dominant operator, we can show that $\mathcal{T} = (\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}$ is a well-defined, linear operator defined on the following spaces

$$
\mathcal{T} : D(\mathcal{T}) \subset L^2(\Omega\times\mathbb{R}^3) \to L^2(\Omega\times\mathbb{R}^3)\,, \qquad D(\mathcal{T}) := \left\{ f \in L^2(\Omega \times \mathbb{R}^3) \ / \ \mathcal{T}f \in L^2(\Omega \times \mathbb{R}^3) \right\}\,.
\tag{3.18}
$$

The characteristics $(X(s), V(s))$ associated with this transport operator and passing through the point $(\mathbf{x}, \mathbf{v})$, are defined by the ODE

$$
\left\{
\begin{aligned}
&\frac{dX}{ds} = 0\,,\\
&\frac{dV}{ds} = \mathfrak{M}(X(s))\, V(s) \times \mathbf{b}(X(s))\,,
\end{aligned}
\right.
\tag{3.19}
$$

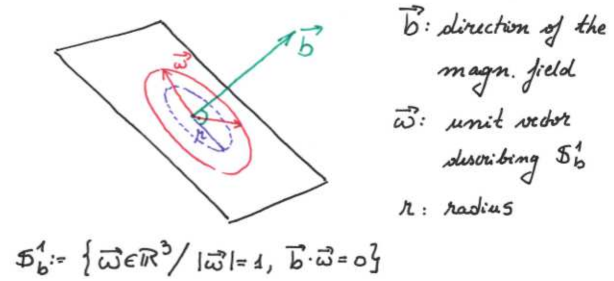

FIGURE 1. The cylindrical coordinate system with respect to a unitary, fixed vector $\mathbf{b} \in \mathbb{R}^3$.

with initial condition $(X(0), V(0)) = (\mathbf{x}, \mathbf{v})$. We shall denote them $(X(s; \mathbf{x}, \mathbf{v}), V(s; \mathbf{x}, \mathbf{v}))$. One can show immediately that for fixed $(\mathbf{x}, \mathbf{v}) \in \Omega \times \mathbb{R}^3$, these trajectories have the explicit form

$$X(s; \mathbf{x}, \mathbf{v}) = \mathbf{x}\,, \quad V(s; \mathbf{x}, \mathbf{v}) = \cos(\mathfrak{M}(\mathbf{x})\, s)\, \mathbf{v}_\perp + \sin(\mathfrak{M}(\mathbf{x})\, s)\, {}^\perp\mathbf{v} + \mathbf{v}_{||}\,, \quad \forall s \in \mathbb{R}\,,$$

and are periodic in s, with period $T_c(\mathbf{x}) := \frac{2\pi}{\mathfrak{M}(\mathbf{x})}$. The functions in the kernel of the operator $\mathcal{T}$ are nothing else than the functions which are constant along these characteristics. Remark also that there are three invariants of this characteristic flow, namely $\mathbf{x}$, $|\mathbf{v}_\perp|$ and $\mathbf{v}_{||}$. Finally let us introduce the gyro-average operator, corresponding to this periodic flow, *i.e.*

$$\mathcal{J}(\cdot) : L^2(\Omega \times \mathbb{R}^3) \to \ker(\mathcal{T})\,, \tag{3.20}$$

defined for $f \in L^2(\Omega \times \mathbb{R}^3)$ as the average over the trajectories, *i.e.*

$$\begin{aligned} \mathcal{J}(f)(\mathbf{x}, \mathbf{v}) &:= \frac{1}{T_c(\mathbf{x})} \int_0^{T_c(\mathbf{x})} f(X(s; \mathbf{x}, \mathbf{v}), V(s; \mathbf{x}, \mathbf{v}))\, ds \\ &= \frac{1}{2\pi} \int_{\mathbb{S}^1_\mathbf{b}} f(\mathbf{x}, v_\mathbf{b}\, \mathbf{b} + |\mathbf{v}_\perp|\, \varpi)\, d\varpi\,. \end{aligned}$$

Let us summarize now the properties of these operators. For the proof of the following Proposition we refer the interested reader to the work [13].

Proposition 1. *The dominant operator $\mathcal{T}$ and gyro-average operator $\mathcal{J}$, defined respectively in* (3.18) *and* (3.20)*, satisfy the following properties:*
(i) The transport operator $\mathcal{T}$ is a linear, continuous operator $\mathcal{T} \in \mathcal{L}(D(\mathcal{T}), L^2)$, if we endow the definition domain $D(\mathcal{T})$ with the norm

$$||f||_{D(\mathcal{T})} := ||f||_{L^2} + ||(\mathbf{v} \times \mathbf{B}) \cdot \nabla_\mathbf{v} f||_{L^2}\,, \quad \forall f \in D(\mathcal{T})\,.$$

(ii) The kernel of $\mathcal{T}$ *corresponds to the functions which are constant along the characteristics* (3.19)*, i.e.*

$$\ker \mathcal{T} := \{f \in L^2(\Omega \times \mathbb{R}^3) \ / \ \exists\, g : \Omega \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \text{ such that } f(\mathbf{x},\mathbf{v}) = g(\mathbf{x}, v_{\mathbf{b}}, |\mathbf{v}_\perp|)\}\,.$$

(iii) The gyro-average operator $\mathcal{J} : L^2 \to \ker(\mathcal{T})$ *is linear, continuous and corresponds to the orthogonal projection onto* $\ker \mathcal{T}$*, i.e.*

$$\int_\Omega \int_{\mathbb{R}^3} (f - \mathcal{J}(f))\varphi \, d\mathbf{v} d\mathbf{x} = 0\,, \quad \forall \varphi \in \ker \mathcal{T}\,.$$

(iv) The L^2*-space can be decomposed, by means of the orthogonal projector* $\mathcal{J}$*, as follows*

$$L^2(\Omega \times \mathbb{R}^3) = \ker(\mathcal{T}) \oplus (\ker(\mathcal{T}))^\perp = \ker(\mathcal{T}) \oplus^\perp \ker(\mathcal{J}) = \ker(\mathcal{T}) \oplus^\perp \overline{\mathcal{I}m(\mathcal{T})}\,,$$

where

$$f = \mathcal{J}(f) + f'\,, \quad f' := (Id - \mathcal{J}) f \in \ker(\mathcal{J})\,.$$

(v) As $\mathfrak{M}(\mathbf{x}) \geq \gamma > 0$ *for all* $\mathbf{x} \in \Omega$*, then* $\mathcal{I}m(\mathcal{T})$ *is closed and one has the one-to-one mapping*

$$\mathcal{T} : D(\mathcal{T}) \cap \ker(\mathcal{J}) \to \ker(\mathcal{J})\,, \tag{3.21}$$

with inverse belonging to $\mathcal{L}(\ker(\mathcal{J}), \ker(\mathcal{J}))$*. Furthermore the Poincaré-like inequality holds*

$$||u||_{L^2} \leq \frac{2\pi}{\gamma}\, ||\mathcal{T}u||_{L^2}\,, \quad \forall u \in D(\mathcal{T}) \cap \ker(\mathcal{J})\,.$$

We shall also need in the following to take the average of some quantities along the magnetic field lines. This average will be related to the following transport operator

$$\mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)\,, \qquad \mathcal{A}u := \mathbf{B} \cdot \nabla_x u\,, \tag{3.22}$$

with definition domain given by

$$D(\mathcal{A}) := \{u \in L^2(\Omega) \ / \ \mathbf{B} \cdot \nabla_x u \in L^2(\Omega)\}\,.$$

The characteristics $Z(s; \mathbf{x})$ associated with this operator and passing through the point $\mathbf{x} \in \Omega$ are defined as the solutions to the ODE

$$\begin{cases} \dfrac{dZ(s)}{ds} = \mathbf{B}(Z(s))\,, \quad \forall s \in \mathbb{R}\,, \\ Z(0) = \mathbf{x}\,. \end{cases} \tag{3.23}$$

We shall consider in this work a tokamak plasma configuration, meaning that Ω and $\mathbf{B}$ are given so that $Z(s, \Omega) = \Omega$ for all $s \in \mathbb{R}$, hence the magnetic field lines do not leave the domain and fill the whole Ω. Now, let us define the average $\langle a \rangle$ of a scalar function $a : \Omega \to \mathbb{R}$ along the $\mathbf{B}$-field lines, called field-line average, by the formula

$$\langle a \rangle(\mathbf{x}) := \lim_{L_s \to \infty} \frac{1}{L_s} \int_0^{L_s} a(Z(s; \mathbf{x}))\, ds\,, \qquad \langle \cdot \rangle : L^2(\Omega) \to \ker(\mathcal{A})\,. \tag{3.24}$$

Lemma 2. [10, 11] *For any $a \in L^2(\Omega)$ the sequence $\{\langle a\rangle_L\}_{L>0}$, with $\langle a\rangle_L := \frac{1}{L}\int_0^L a(Z(s;\mathbf{x}))\,ds$, converges strongly in $L^2(\Omega)$ as $L \to 0$ towards a function $\langle a\rangle \in \ker\mathcal{A}$, corresponding to the orthogonal projection of a on $\ker\mathcal{A}$.*

The following proposition summarizes the properties of these two new operators. For the proof, we refer to [10, 11].

Proposition 3. *The operator $\mathcal{A}$ and its average-operator $\langle\cdot\rangle$, defined respectively in (3.22) and (3.24), satisfy the following properties:*
(i) The operator $\mathcal{A}$ is a linear, continuous operator $\mathcal{A} \in \mathcal{L}(D(\mathcal{A}), L^2(\Omega))$, if we endow the definition domain $D(\mathcal{A})$ with the norm

$$||u||_{D(\mathcal{A})} := ||u||_{L^2(\Omega)} + ||\mathbf{B}\cdot\nabla_x u||_{L^2(\Omega)}\,, \quad \forall u \in D(\mathcal{A})\,.$$

(ii) The kernel of $\mathcal{A}$ corresponds to the functions which are constant along the characteristics (3.23), i.e.

$$\ker\mathcal{A} := \{u \in L^2(\Omega)\ /\ \mathbf{B}\cdot\nabla_x u = 0\} = \{u \in L^2(\Omega)\ /\ u(\mathbf{x}) = u(Z(s;\mathbf{x}))\ \forall (s,\mathbf{x}) \in \mathbb{R}\times\Omega\}\,.$$

(iii) The average-operator $\langle\cdot\rangle : L^2(\Omega) \to \ker(\mathcal{A})$ is well-defined, linear, continuous and corresponds to the orthogonal projection onto $\ker\mathcal{A}$, i.e.

$$\int_\Omega (u - \langle u\rangle)\varphi\,d\mathbf{x} = 0\,, \quad \forall \varphi \in \ker\mathcal{A}\,.$$

(iv) The L^2-space can be decomposed, by means of the orthogonal projector $\langle\cdot\rangle$, as follows

$$L^2(\Omega) = \ker\mathcal{A} \oplus (\ker\mathcal{A})^\perp = \ker\mathcal{A} \oplus^\perp \ker\langle\cdot\rangle = \ker\mathcal{A} \oplus^\perp \overline{\mathcal{I}m\mathcal{A}}\,,$$

where each scalar quantity is decomposed into its mean and fluctuation part along $\mathbf{B}$

$$u = \langle u\rangle + u^*\,, \quad u^* := u - \langle u\rangle \in \ker\langle\cdot\rangle\,. \tag{3.25}$$

Remark 3.1. *One can argue that the definition of the field-line average (3.24) is not the usual curve-integral average along the trajectory of $\mathbf{B}$, which is rather given by*

$$\langle\langle a\rangle\rangle(\mathbf{x}) := \lim_{L_s\to\infty}\frac{1}{L_s}\int_0^{L_s} a(Z(s;\mathbf{x}))\left|\frac{dZ(s;\mathbf{x})}{ds}\right|ds = \lim_{L_s\to\infty}\frac{1}{L_s}\int_0^{L_s} a(Z(s;\mathbf{x}))\,|\mathbf{B}(Z(s;\mathbf{x}))|\,ds\,, \tag{3.26}$$

$$\langle\langle\cdot\rangle\rangle : L^2(\Omega) \to \ker(\mathcal{A})\,.$$

Indeed, $\langle\langle\cdot\rangle\rangle$ is also a projection on $\ker(\mathcal{A})$, however it is not an orthogonal projection with respect to the $L^2(\Omega)$ scalar product. One could use in the following the projection $\langle\langle\cdot\rangle\rangle$ however by changing the scalar product in $L^2(\Omega)$, including a weight-function.

Remark 3.2. *Let us also remark that when $|\mathbf{B}(Z(s;\mathbf{x}))|$ is constant along the characteristics, as it will be in the particular cases we shall consider here, namely for the helical magnetic fields introduced below, one has*

$$\langle\langle a\rangle\rangle(\mathbf{x}) = |\mathbf{B}(\mathbf{x})|\,\langle a\rangle(\mathbf{x})\,,$$

such that the two projections differ only by a constant.

Remark 3.3. *Finally, it will be useful in the sequel to take also the average along* $\mathbf{B}$ *of a field-aligned vector function. What we mean with this is the following: For a given vector-field* $\Gamma : \Omega \to \mathbb{R}^3$*, for which we have the decomposition* $\Gamma = \Gamma_{||} + \Gamma_{\perp}$*, with* $\Gamma_{||} = \Gamma_{\mathbf{b}}\, \mathbf{b} \in \mathbb{R}^3$ *we shall mean with "average" and "fluctuation" part of* $\Gamma_{||}$ *the following quantities:*

$$\langle \Gamma_{||} \rangle := \langle \Gamma_{\mathbf{b}} \rangle\, \mathbf{b}\,, \qquad \Gamma^*_{||} := \Gamma^*_{\mathbf{b}}\, \mathbf{b}\,, \quad \textit{where } \Gamma_{\mathbf{b}} = \langle \Gamma_{\mathbf{b}} \rangle + \Gamma^*_{\mathbf{b}} \in \mathbb{R}\,. \tag{3.27}$$

4. Obtention of the Boltzmann-relation limit $\varepsilon \to 0$

The aim of this section is to obtain from the following rescaled electron kinetic equation

$$\partial_t f^\varepsilon + \frac{1}{\varepsilon}\mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon - \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v} \times \mathbf{B}\right) \cdot \nabla_{\mathbf{v}} f^\varepsilon = \frac{1}{\varepsilon} Q^\varepsilon_{BGK}(f^\varepsilon)\,, \tag{4.28}$$

the Boltzmann relation in the limit $\varepsilon \to 0$ and this for a helical magnetic field, given by

$$\mathbf{B} := B_z\, \mathbf{e}_z + B_\theta(r)\, \mathbf{e}_\theta\,, \quad \mathbf{e}_\theta := r\,(\sin(\theta), -\cos(\theta), 0)^t\,,$$

where $B_z \in \mathbb{R}$, $(r, \theta) \in \mathbb{R}^+ \times [0, 2\,\pi)$ and $B_\theta : \mathbb{R}^+ \to \mathbb{R}$. Figure 2 is representing this magnetic field, which is curving around the so-called "magnetic surfaces". The field lines are either closed and thus periodic, or open and fill the whole magnetic surface. Remark that in this case the direction $\mathbf{b}(\mathbf{x}) := \mathbf{B}(\mathbf{x})/|\mathbf{B}(\mathbf{x})|$ is verifying also $\nabla_{\mathbf{x}} \cdot \mathbf{b} = 0$. A simple example of such a helical magnetic field is

$$\mathbf{B}(\mathbf{x}) = \begin{pmatrix} y \\ -x \\ 1 \end{pmatrix}.$$

Equation (4.28) describes a long-time scaling of the so-called gyrokinetic regime of the BGK-equation. Remark that this equation corresponds to a situation where the magnetic force is stronger than the collision term. However the collision term appears at the same order as the transport terms $\mathbf{v} \cdot \nabla_{\mathbf{x}} f^\varepsilon$ and $\mathbf{E} \cdot \nabla_{\mathbf{v}} f^\varepsilon$, fact which will bring some difficulties in the mathematical study. A competition will be installed between the transport term, transporting the particles, the magnetic-field force term, confining the particles around the filed lines, and the collision term, introducing some diffusion in the velocity space. This competition is very delicate and the magnitude of the three different terms is very important for the obtention of a well-defined equilibrium in the limit $\varepsilon \to 0$.

The boundary conditions we associate with (4.28) correspond to the situation of a periodic flat torus and are the following:

- The spatial domain $\Omega := S \times (0, L_z) \subset \mathbb{R}^3$ with circular base S and boundary $\partial\Omega := \{\partial S \times (0, L_z)\} \cup \{S \times \{0, L_z\}\}$ is considered periodic in $z \in [0, L_z]$, meaning

$$f(t, \mathbf{x}, \mathbf{v}) = f(t, \mathbf{x} + L_z\, \mathbf{e}_z, \mathbf{v})\,, \quad \forall (t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3\,. \tag{4.29}$$

Furthermore, specular reflection conditions are imposed on the mantle of the torus, namely

$$f(t,\mathbf{x},\mathbf{v}) = f(t,\mathbf{x},\mathbf{v}-2(\mathbf{v}\cdot\nu(\mathbf{x}))\,\nu(\mathbf{x}))\,, \quad \forall \mathbf{x}\in\partial S\times(0,L_z) \text{ and } \nu(\mathbf{x})\cdot\mathbf{v}<0\,, \tag{4.30}$$

where $\nu(\mathbf{x})$ denotes the outward normal vector to $\partial\Omega$.

- The velocity domain is the whole space $\mathbb{R}^3$ with vanishing boundary conditions, namely

$$f(t,\mathbf{x},\mathbf{v}) \to_{|\mathbf{v}|\to\pm\infty} = 0\,, \quad \forall (t,\mathbf{x})\in\mathbb{R}^+\times\Omega\,. \tag{4.31}$$

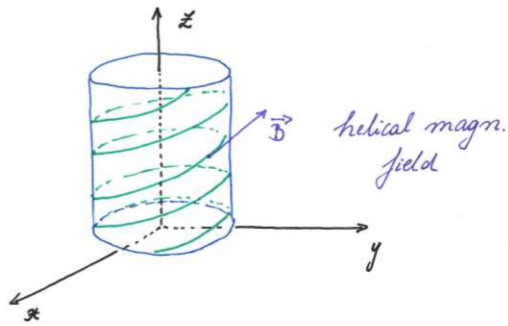

FIGURE 2. The toroidal magnetic-field configuration (flat torus).

To explicit the equation (4.28) further, recall that the collision operator has the form

$$Q^\varepsilon_{BGK}(f^\varepsilon) := \mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon} - f^\varepsilon\,, \qquad \mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon} = \frac{n^\varepsilon}{(2\pi T^\varepsilon)^{3/2}}\exp\left(-\frac{|\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2}{2T^\varepsilon}\right)\,, \tag{4.32}$$

where $\mathcal{M}_{n^\varepsilon,\varepsilon\mathbf{u}^\varepsilon,T^\varepsilon}$ stands for the local Maxwellian function with the same moments as the distribution function f^ε, in particular we have

$$n^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3} f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{4.33a}$$

$$\varepsilon\, n^\varepsilon(t,\mathbf{x})\mathbf{u}^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3} \mathbf{v} f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{4.33b}$$

$$w^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v} = \frac{3}{2}\,n^\varepsilon T^\varepsilon + \varepsilon^2\,\frac{n^\varepsilon|\mathbf{u}^\varepsilon|^2}{2}\,, \tag{4.33c}$$

$$\frac{3}{2}\,n^\varepsilon(t,\mathbf{x})T^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2 f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,. \tag{4.33d}$$

Let us recall, for clarity reasons, the definitions of the stress tensor and heat flux

$$\mathbb{P}^\varepsilon(t,\mathbf{x}) := \int_{\mathbb{R}^3} (\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon)\otimes(\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon) f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{4.34}$$

$$\mathbf{q}^\varepsilon(t,\mathbf{x}) := \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon|^2(\mathbf{v}-\varepsilon\mathbf{u}^\varepsilon) f^\varepsilon(t,\mathbf{x},\mathbf{v})d\mathbf{v}\,, \tag{4.35}$$

and denote by $p^\varepsilon := n^\varepsilon T^\varepsilon$ the scalar pressure. The (not-closed) fluid model associated to the kinetic equation (4.28) is obtained by taking the moments of the BGK-equation and reads

$$\begin{cases} \partial_t n^\varepsilon + \nabla \cdot (n^\varepsilon \mathbf{u}^\varepsilon) = 0\,, \\[2mm] \partial_t (n^\varepsilon \mathbf{u}^\varepsilon) + \nabla \cdot (n^\varepsilon \mathbf{u}^\varepsilon \otimes \mathbf{u}^\varepsilon) + \dfrac{1}{\varepsilon^2} \nabla \cdot \mathbb{P}^\varepsilon + \dfrac{1}{\varepsilon^2} n^\varepsilon (\mathbf{E} + \mathbf{u}^\varepsilon \times \mathbf{B}) = 0\,, \\[2mm] \partial_t w^\varepsilon + \nabla \cdot (w^\varepsilon\, \mathbf{u}^\varepsilon + \mathbb{P}^\varepsilon \cdot \mathbf{u}^\varepsilon) + \dfrac{1}{\varepsilon} \nabla \cdot \mathbf{q}^\varepsilon + n^\varepsilon \mathbf{E} \cdot \mathbf{u}^\varepsilon = 0\,. \end{cases} \tag{4.36}$$

We shall now remain on a formal level and suppose that the kinetic equation is well-posed.

The main theorem of this paper is the following:

Theorem 4. ***(Limit-model)*** *Under Hypothesis A and assuming a given smooth electric field* $\mathbf{E} = -\nabla \phi$, *let us suppose that* f^ε, *solution of the kinetic equation* (4.28) *with boundary conditions* (4.29)-(4.31), *is sufficiently regular. Then, in the limit* $\varepsilon \to 0$, f^ε *tends formally towards a local Maxwellian of the form*

$$f_0(t, \mathbf{x}, \mathbf{v}) = \mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0} \right)\,,$$

where the temperature T_0 *is independent on the field-coordinate, i.e.* $\nabla_{||} T_0 \equiv 0$, *and the density* n_0 *has the form*

$$n_0(t, \mathbf{x}) = c(t, \mathbf{x}_\perp)\, \exp\left(\frac{\phi(t, \mathbf{x})}{T_0(t, \mathbf{x}_\perp)} \right)\,, \quad \forall \mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_{||} \in \mathbb{R}^3\,,\ \ \forall t \in \mathbb{R}\,. \tag{4.37}$$

The functions $c(t, \mathbf{x}_\perp)$ *and* $T_0(t, \mathbf{x}_\perp)$ *are determined by the following Limit-model*

$$(L) \quad \begin{cases} \partial_t n_0 + \nabla_\perp \cdot (n_0 \mathbf{u}_0)_\perp + \nabla_{||} \cdot (n_0 \mathbf{u}_0)^*_{||} = 0\,, \qquad \langle (n_0 \mathbf{u}_0)^*_{||} \rangle = 0\,, \\[2mm] \partial_t \langle n_0 T_0 \rangle + \dfrac{5}{3} \langle \nabla_\perp \cdot (n_0 T_0 \mathbf{u}_{\perp,0}) \rangle + \dfrac{5}{3} \langle \nabla_\perp \cdot (n_0 T_0 \dfrac{\nabla T_0 \times \mathbf{B}}{|\mathbf{B}|^2}) \rangle \\[2mm] \qquad\qquad + \dfrac{2}{3} \langle (n_0 \mathbf{u}_{\perp,0}) \cdot \mathbf{E}_\perp \rangle - \dfrac{2}{3} \langle (n_0 \mathbf{u}_0)^*_{||} \cdot \nabla_{||} \phi \rangle = 0\,. \end{cases} \tag{4.38}$$

where $\mathbf{u}_{\perp,0} := \mathbf{u}_E + \mathbf{u}_D$ *with* $\mathbf{u}_E := \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2}$ *the* $\mathbf{E} \times \mathbf{B}$*-drift velocity and* $\mathbf{u}_D := \frac{1}{n_0} \frac{\nabla p_0 \times \mathbf{B}}{|\mathbf{B}|^2}$ *the diamagnetic drift velocity. We used the decomposition* (3.25) *for a quantity in the mean and fluctuation part along the magnetic field lines, as well as the notation* (3.27).

Remark 4.1. *The relation* (4.37) *is the so-called electron Boltzmann relation. Its physical meaning is that the electrons, being very light and hence mobile, accelerate to high energies very quickly, leaving behind them a region of large ion charges, which creates a retarding electric field. An equilibrium is hence achieved between the two antagonist forces, the pressure-gradient and the electric force.*

Remark 4.2. *The Limit-model* (4.38) *describes the non-trivial electron dynamics perpendicular to the magnetic field. The well-posedness of this problem (existence, uniqueness and stability results) is an interesting issue and is left for further works.*

Remark 4.3. *In the specific case of an axis-aligned magnetic field* $\mathbf{B} := \mathbf{e}_z$*, the Limit model takes the simpler form*

$$(L) \quad \begin{cases} \partial_t n_0 + \mathbf{u}_E \cdot \nabla_\perp n_0 + \nabla_{||} \cdot \Gamma^*_{||,0} = 0\,, \qquad \langle \Gamma^*_{||,0} \rangle = 0\,, \\ \partial_t \langle n_0 T_0 \rangle + \langle \mathbf{u}_E \cdot \nabla_\perp (n_0 T_0) \rangle - \dfrac{2}{3} \langle \Gamma^*_{||,0} \cdot \nabla_{||} \phi \rangle = 0\,, \end{cases} \tag{4.39}$$

where for simplicity reasons we denoted by $\Gamma^*_{||,0} := (n_0 \mathbf{u}_0)^*_{||}$ *the electron momentum along the* $\mathbf{B}$*-direction. Remark that the diamagnetic drift velocity* $\mathbf{u}_D$ *disappeared in this axis-aligned, not-physical configuration.*

Formal proof of Theorem 4:

The formal obtention of the Limit-model is based on the following Hilbert-expansion

$$f^\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \mathcal{O}(\varepsilon^3)\,, \tag{4.40}$$

and the properties of the dominant operator $\mathcal{T} := (\mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}}$, investigated in detail in the last section. Indeed, inserting the Hilbert-Ansatz (4.40) into the kinetic equation (4.28) and identifying the terms of the equal power in ε, yields the infinite hierarchy

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} f_0 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f_0 - \mathcal{T} f_1 = \mathcal{M}_0 - f_0\,, \tag{4.41a}$$

$$\partial_t f_0 + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_1 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f_1 - \mathcal{T} f_2 = \mathcal{M}_1 - f_1 \tag{4.41b}$$

$$\partial_t f_1 + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_2 - \mathbf{E} \cdot \nabla_{\mathbf{v}} f_2 - \mathcal{T} f_3 = \mathcal{M}_2 - f_2\,, \quad \cdots \tag{4.41c}$$

where the Maxwellians $\mathcal{M}_i$ are given in Appendix 6.1. The fluid variables $(n^\varepsilon, \mathbf{u}^\varepsilon, T^\varepsilon)$ associated to f^ε are expanded according to (6.55) which yields a corresponding fluid hierarchy (6.56) or (6.60). The expansions of the stress tensor $\mathbb{P}^\varepsilon$ and of the heat flux $\mathbf{q}^\varepsilon$ occurring therein are given in (6.58) and in (6.59), respectively. The resolution of this kinetic hierarchy together with the handling of the corresponding fluid hierarchy (6.60) shall permit to obtain some information about f_0 and acquire in this manner step by step the Limit model (4.38). Let us show how this is achieved.

Step 1: H-theorem. The first information we can get, comes from the H-theorem, which states that the asymptotic limit of the sequence $\{f^\varepsilon\}_{\varepsilon>0}$, as $\varepsilon \to 0$, is a Maxwellian with zero mean velocity, *i.e.*

$$f_0(t, \mathbf{x}, \mathbf{v}) = \mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0} \right)\,, \tag{4.42}$$

such that zeroth order stress tensor and heat flux are given by $\mathbb{P}_0 = n_0\, T_0\, \mathbb{I}$ and $\mathbf{q}_0 = 0$.

Indeed, firstly for the BGK-collision operator one can show the non-increase of the H-functional in time, *i.e.*

$$\int_{\mathbb{R}^3\times\Omega} Q^\varepsilon_{BGK}(f^\varepsilon)\, \ln(f^\varepsilon)\, d\mathbf{v}\, d\mathbf{x} \le 0 \qquad \text{(H-theorem)}\,.$$

Secondly, multiplying the kinetic equation (4.28) by $\ln(f^\varepsilon)$ and integrating over the phase-space, yields

$$\varepsilon \int_{\mathbb{R}^3\times\Omega} \partial_t f^\varepsilon\, \ln(f^\varepsilon)\, d\mathbf{v} d\mathbf{x} = \int_{\mathbb{R}^3\times\Omega} Q^\varepsilon_{BGK}(f^\varepsilon)\, \ln(f^\varepsilon)\, d\mathbf{v} d\mathbf{x}\,,$$

which gives in the limit $\varepsilon \to 0$ even the equality

$$\int_{\mathbb{R}^3\times\Omega} Q^0_{BGK}(f^0)\, \ln(f^0)\, d\mathbf{v}\, d\mathbf{x} = 0\,.$$

The H-theorem [15, 16, 42] permits then to establish that f_0 is a local Maxwellian, *i.e.* $f_0 = \mathcal{M}_{n_0,0,T_0}$.

Remains now to find the evolution equations permitting to compute the density n_0 and the temperature T_0. This shall be done by going forth and back between the kinetic hierarchy (4.41) and the corresponding fluid hierarchy (6.60), in order to obtain as much information as possible.

Step 2: Drift velocities. Let us start by writing down the first three conservation laws of the fluid hierarchy (6.60). Taking $i = 0$ and expliciting the energy conservation law in terms of the pressure $p_0 = n_0 T_0$, yields

$$\begin{cases} \partial_t n_0 + \nabla\cdot(n_0 \mathbf{u}_0) = 0\,, \\ \nabla p_0 + n_0 \mathbf{E} + (n_0 \mathbf{u}_0)\times \mathbf{B} = 0\,, \\ \partial_t p_0 + \dfrac{5}{3}\nabla\cdot(p_0 \mathbf{u}_0) + \dfrac{2}{3}\nabla\cdot \mathbf{q}_1 + \dfrac{2}{3} n_0\, \mathbf{u}_0\cdot \mathbf{E} = 0\,. \end{cases} \tag{4.43}$$

It is this system which shall permit, after some rearrangements, to get the desired evolution equations for (n_0, T_0) (namely the Limit-model (4.38)), allowing thus to determine completely the limit distribution function f_0 via (4.42). This system is however only closed if one finds a manner to express $\mathbf{q}_1$ in terms of the unknowns $(n_0, \mathbf{u}_0, T_0)$, which will be done, by trying to find some information about f_1.

A further, useful information we get from (4.43) is the expression of $\mathbf{u}_{\perp,0}$ in terms of (n_0, T_0). Indeed, taking the cross product of the momentum conservation law with $\mathbf{B}$, yields for the mean velocity

$$\mathbf{u}_{\perp,0} = \frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{B}|^2} + \frac{1}{n_0}\frac{\nabla p_0 \times \mathbf{B}}{|\mathbf{B}|^2} =: \mathbf{u}_E + \mathbf{u}_D\,, \tag{4.44}$$

where $\mathbf{u}_E$ denotes the electric-field drift velocity and $\mathbf{u}_D$ stands for the diamagnetic drift velocity [29, 33]. The parallel part of the momentum equation implies a balance between the pressure force and the electric force, *i.e.*

$$\nabla_{||}(n_0\, T_0) + n_0\, \mathbf{E}_{||} = 0\,.$$

For the moment this equation does not permit to get the Boltzmann relation.

Step 3: Boltzmann relation. Going now back to the kinetic hierarchy and taking the gyro-average (3.20) over the Eqs. (4.41), eliminates the transport operator $\mathcal{T}$ and leads first to

$$\mathcal{J}(\mathbf{v}\cdot\nabla_{\mathbf{x}} f_0) - \mathcal{J}(\mathbf{E}\cdot\nabla_{\mathbf{v}} f_0) = 0\,, \tag{4.45a}$$

$$\partial_t \mathcal{J}(f_0) + \mathcal{J}(\mathbf{v}\cdot\nabla_{\mathbf{x}} f_1) - \mathcal{J}(\mathbf{E}\cdot\nabla_{\mathbf{v}} f_1) = \mathcal{J}(\mathcal{M}_1) - \mathcal{J}(f_1)\,. \tag{4.45b}$$

In order to determine the equations for the fluctuations f_i' we subtract (4.45) from (4.41) and obtain

$$(\mathbf{v}\cdot\nabla_{\mathbf{x}} f_0)' - (\mathbf{E}\cdot\nabla_{\mathbf{v}} f_0)' - \mathcal{T} f_1' = 0\,, \tag{4.46a}$$

$$\partial_t f_0' + (\mathbf{v}\cdot\nabla_{\mathbf{x}} f_1)' - (\mathbf{E}\cdot\nabla_{\mathbf{v}} f_1)' - \mathcal{T} f_2' = \mathcal{M}_1' - f_1'\,. \tag{4.46b}$$

For our further development, namely the obtention of the Limit model (4.38), we shall only need to solve (4.45a) and (4.46a), as will be seen in the following.

Let us thus start with the resolution of (4.45a). Knowing that $f_0 = \mathcal{M}_0$, one can immediately compute

$$\begin{aligned} g_0 &:= \mathbf{v}\cdot\nabla_{\mathbf{x}} f_0 - \mathbf{E}\cdot\nabla_{\mathbf{v}} f_0 \\ &= \mathbf{v}\cdot\left[\left(\frac{\nabla n_0}{n_0} - \frac{3}{2}\frac{\nabla T_0}{T_0}\right) + \frac{|\mathbf{v}|^2}{2\,T_0}\frac{\nabla T_0}{T_0} + \frac{\mathbf{E}}{T_0}\right]\mathcal{M}_{n_0,0,T_0}\,, \end{aligned}$$

and using the definition of the gyro-average operator (3.20), one gets

$$\mathcal{J}(g_0)(t,\mathbf{x},\mathbf{v}) = \mathbf{v}_{||}\cdot\left[\left(\frac{\nabla_{||} n_0}{n_0} - \frac{3}{2}\frac{\nabla_{||} T_0}{T_0}\right) + \frac{|\mathbf{v}|^2}{2\,T_0}\frac{\nabla_{||} T_0}{T_0} + \frac{\mathbf{E}_{||}}{T_0}\right]\mathcal{M}_{n_0,0,T_0}\,.$$

Hence, the first equation in the hierarchy (4.45a), yields the relation

$$\mathbf{v}_{||}\cdot\left[\left(\frac{\nabla_{||} n_0}{n_0} - \frac{3}{2}\frac{\nabla_{||} T_0}{T_0}\right) + \frac{|\mathbf{v}_{||}|^2 + |\mathbf{v}_\perp|^2}{2\,T_0}\frac{\nabla_{||} T_0}{T_0} + \frac{\mathbf{E}_{||}}{T_0}\right]\mathcal{M}_{n_0,0,T_0} = 0\,, \quad \forall \mathbf{v}\in\mathbb{R}^3\,,$$

which permits immediately to show, comparing the terms of equal power in $\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_\perp$, that

$$\frac{\nabla_{||} T_0}{T_0} \equiv 0 \quad \Rightarrow \nabla_{||} T_0 \equiv 0 \quad \text{and} \quad \frac{\nabla_{||} n_0}{n_0} - \frac{\nabla_{||}\phi}{T_0} \equiv 0\,,$$

recovering the expected Boltzmann relation

$$n_0(t,\mathbf{x}) = c(t,\mathbf{x}_\perp)\, \exp\left(\frac{\phi(t,\mathbf{x})}{T_0(t,\mathbf{x}_\perp)}\right), \quad \forall \mathbf{x} = \mathbf{x}_\perp + \mathbf{x}_{||} \in \mathbb{R}^3\,, \ \forall t \in \mathbb{R}\,. \tag{4.47}$$

Remains to find the evolution equations for the quantities $c(t, \mathbf{x}_\perp)$ and $T_0(t, \mathbf{x}_\perp)$, which shall emerge from (4.43).

Step 4: Heat flux. After having computed f_0, one can solve immediately (4.46a) for f_1', using the property that the dominant operator $\mathcal{T}$ is bijective, see (3.21). Recalling that $\mathcal{J}(g_0) = 0$ (see (4.45a)), equation (4.46a) can be uniquely solved in $\ker(\mathcal{J})$, *i.e.*

$$\mathcal{T} f_1' = g_0\,, \qquad f_1' \in \ker(\mathcal{J})\,.$$

The unique solution is obtained by passing to cylindrical coordinates in velocity and is given by

$$f_1'(t,\mathbf{x},\mathbf{v}) = \mathbf{v}_\perp \cdot \Big[\frac{1}{\mathfrak{m}(\mathbf{x})}\Big(\frac{|\mathbf{v}|^2}{2T_0} - \frac{5}{2}\Big)\frac{\nabla^\perp T_0}{T_0} + \frac{\mathbf{u}_{\perp,0}}{T_0}\Big]\, \mathcal{M}_0\,, \tag{4.48}$$

which is easily verified.

As we are just interested in obtaining the limit model as $\varepsilon \to 0$, it is not necessary to go further and compute the full distribution function f_1. Indeed, one finally only needs $\mathbf{q}_{\perp,1}$, as will be shown Step 5. Let us for the moment compute this quantity.

According to Eq. (6.59c) the perpendicular heat flux is computed via

$$\mathbf{q}_{\perp,1} = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp \mathcal{J}(f_1)\, d\mathbf{v} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp f_1'\, d\mathbf{v} - \frac{5}{2}\mathbf{u}_{\perp,0}\, n_0 T_0\,. \tag{4.49}$$

Firstly, one remarks that, due to the fact that $\mathcal{J}(f_1)$ does not depend on $\varpi \in \mathbb{S}^1_{\mathbf{b}}$, we have

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp \mathcal{J}(f_1)\, d\mathbf{v} = 0\,.$$

The second term on the right-hand-side in (4.49) is computed finally via (4.48) and gives

$$\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 \mathbf{v}_\perp f_1'\, d\mathbf{v} = \frac{5}{2}\, n_0 T_0\, \frac{\nabla T_0 \times \mathbf{B}}{|\mathbf{B}|^2} + \frac{5}{2}\mathbf{u}_{\perp,0}\, n_0 T_0\,.$$

Therefore, $\mathbf{q}_{\perp,1}$ has been entirely determined, and one has

$$\frac{2}{3}\,\nabla_\perp \cdot \mathbf{q}_{\perp,1} = \frac{5}{3}\,\nabla_\perp \cdot \Big(n_0 T_0\, \frac{\nabla T_0 \times \mathbf{B}}{|\mathbf{B}|^2}\Big)\,.$$

With all these information, let us finally explain, how to obtain, starting from the fluid system (4.43), the asymptotic Limit-model (4.38) corresponding to the kinetic equation (4.28) and permitting to obtain the remaining quantities $c(t, \mathbf{x}_\perp)$ and $T_0(t, \mathbf{x}_\perp)$. Recall for this the decomposition and notation (3.25), (3.27) for the mean and the fluctuation part of a function along the $\mathbf{B}$-field.

Step 5: Limit model. The equation for the Boltzmann-relation "constant" $c(t, \mathbf{x}_\perp)$ is obtained by taking the field-line average of the particle conservation law. Indeed, one can rewrite the particle conservation law as

$$\partial_t n_0 + \nabla_\perp \cdot (n_0 \mathbf{u}_0)_\perp + \nabla_{||} \cdot (n_0 \mathbf{u}_0)_{||} = 0\,. \tag{4.50}$$

Taking now the field-line mean leads to

$$\partial_t \langle n_0 \rangle + \langle \nabla_\perp \cdot (n_0 \mathbf{u}_0)_\perp \rangle = 0\,, \tag{4.51}$$

where we used the fact that $\langle \nabla_{||} \cdot \Gamma_{||,0} \rangle = 0$, with Γ_0 standing for $n_0 \mathbf{u}_0$ and $\Gamma_{\mathbf{b},0} := (n_0 \mathbf{u}_0)_{\mathbf{b}}$. Clearly, for given T_0 this equation can be written as an evolution equation for the function $c(t, \mathbf{x}_\perp)$.

The evolution of the temperature $T_0(t, \mathbf{x}_\perp)$ is then obtained by averaging the energy equation along $\mathbf{B}$, *i.e.*

$$\partial_t \langle n_0 T_0 \rangle + \frac{5}{3} \langle \nabla_\perp \cdot (n_0 T_0 \mathbf{u}_{\perp,0}) \rangle + \frac{5}{3} \langle \nabla_\perp \cdot (n_0 T_0 \frac{\nabla T_0 \times \mathbf{B}}{|\mathbf{B}|^2}) \rangle + \frac{2}{3} \langle (n_0 \mathbf{u}_{\perp,0}) \cdot \mathbf{E}_\perp \rangle - \frac{2}{3} \langle \Gamma^*_{||,0} \cdot \nabla_{||} \phi \rangle = 0\,. \tag{4.52}$$

We used the fact that $\Gamma_{||,0} = \left[\langle \Gamma_{\mathbf{b},0} \rangle + \Gamma^*_{\mathbf{b},0} \right] \mathbf{b}$ as well as $\langle \partial_{\mathbf{b}} \phi \rangle = 0$, permitting thus the elimination of the mean part $\langle \Gamma_{\mathbf{b},0} \rangle \, \mathbf{b}$ in the system, part which we are in the impossibility to compute. In what concerns the fluctuation part $\Gamma^*_{||,0} = \Gamma^*_{\mathbf{b},0} \, \mathbf{b}$, we shall use the remaining information from the particle continuity equation for its computation. Indeed, subtracting the average (4.51) from the particle conservation law yields

$$\nabla_{||} \cdot \Gamma^*_{||,0} = -\partial_t n_0^* - (\nabla_\perp \cdot (n_0 \mathbf{u}_0)_\perp)^* = 0\,, \qquad \langle \Gamma^*_{||,0} \rangle = 0\,.$$

For given n_0 and $\mathbf{u}_{\perp,0}$ this last equation is readily solved by integrating along the field lines of $\mathbf{B}$; the constant of integration is then determined via the integral constraint $\langle \Gamma^*_{||,0} \rangle = 0$, which guarantees the uniqueness of the solution $\Gamma^*_{||,0}$. Summarizing, we obtain the Limit-model (4.38) for the unknowns $(n_0, (n_0\mathbf{u}_0)^*_{||}, T_0)$, *i.e*

$$
(L) \quad \begin{cases} \partial_t n_0 + \nabla_\perp \cdot (n_0\mathbf{u}_0)_\perp + \nabla_{||} \cdot (n_0\mathbf{u}_0)^*_{||} = 0\,, \qquad \langle (n_0\mathbf{u}_0)^*_{||}\rangle = 0\,, \\[2ex] \partial_t \langle n_0 T_0\rangle + \dfrac{5}{3}\,\langle \nabla_\perp \cdot (n_0 T_0 \mathbf{u}_{\perp,0})\rangle + \dfrac{5}{3}\langle \nabla_\perp \cdot (n_0 T_0\, \dfrac{\nabla T_0 \times \mathbf{B}}{|\mathbf{B}|^2})\rangle \\[2ex] \qquad\qquad + \dfrac{2}{3}\langle (n_0\mathbf{u}_{\perp,0}) \cdot \mathbf{E}_\perp\rangle - \dfrac{2}{3}\langle (n_0\mathbf{u}_0)^*_{||} \cdot \nabla_{||}\phi\rangle = 0\,. \end{cases} \tag{4.53}
$$

The study of the well-posedness of this Limit-model is a rather hard task and is left for further works.

5. Conclusions and prospects

In this work we proved how to obtain on a formal level the electron Boltzmann relation from a suitable scaling of the underlying electron kinetic equation. In this scaling the magnetic force term is the dominant operator, whereas the collision term appears at the same order as the transport terms. It is a scaling which corresponds to a long-time asymptotics (or zero electron mass, low Mach-number regime) with negligible friction.
Starting from this work several directions can now be considered in future works. Firstly, the rigorous mathematical study of this asymptotic limit as well as the well-posedness of the obtained Limit-model is an interesting and hard point. Remark that there exists no mathematical rigorous justification of the formal development presented in the present work.
Secondly, the construction of an Asymptotic-Preserving scheme permitting to solve numerically the electron kinetic equation in all regimes $\varepsilon \in [0,1]$ (microscopic as well as adiabatic regimes) with no huge computational costs (ε-independent accuracy, stability and grids) would be very helpful for plasma studies. Indeed, the Boltzmann relation is often used in numerical simulations, however it is not valid all over the tokamak device. In particular near the edge of the device this approximation is violated, so that one has to go back to a kinetic electron description in order to be accurate enough. This points out the need for a scheme permitting to mimic at the discrete level, what has been performed in the present paper at the continuous level. Some first steps in this direction have been performed in the recent work [19].

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6. Appendix

The author decided to regroup in this Appendix some cumbersome computations and expansions, in order to keep the presentation of this work as clear as possible.

6.1. Expansion of the Maxwellian in powers of ε. The series expansion of the Maxwellian (4.32) in powers of ε reads

$$\mathcal{M}^\varepsilon = \mathcal{M}_0 + \varepsilon \mathcal{M}_1 + \varepsilon^2 \mathcal{M}_2 + \mathcal{O}(\varepsilon^3)\,,$$

where

$$\mathcal{M}_0 = \frac{n_0}{(2\pi T_0)^{3/2}} \exp\left(-\frac{|\mathbf{v}|^2}{2T_0}\right)\,, \tag{6.54a}$$

$$\mathcal{M}_1 = \mathcal{M}_0 \left[\frac{n_1}{n_0} + \frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{T_1}{T_0}\Big(\frac{3}{2} - \frac{|\mathbf{v}|^2}{2T_0}\Big)\right]\,, \tag{6.54b}$$

$$\begin{aligned}\mathcal{M}_2 = \mathcal{M}_0 \Bigg[&\frac{n_2}{n_0} + \frac{\mathbf{v}\cdot\mathbf{u}_1}{T_0} + \frac{n_1}{n_0}\frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{5}{2}\frac{T_1}{T_0}\frac{\mathbf{v}\cdot\mathbf{u}_0}{T_0} - \frac{3}{2}\frac{T_2}{T_0} - \frac{3}{2}\frac{n_1 T_1}{n_0 T_0} + \frac{15}{8}\frac{T_1^2}{T_0^2} \\ &-\frac{|\mathbf{u}_0|^2}{2T_0} + \frac{T_2}{2T_0^2}|\mathbf{v}|^2 + \frac{n_1}{n_0}\frac{T_1}{2T_0^2}|\mathbf{v}|^2 - \frac{5}{4}\frac{T_1^2}{T_0^3}|\mathbf{v}|^2 + \frac{(\mathbf{v}\cdot\mathbf{u}_0)^2}{2T_0} + \frac{T_1}{2T_0^3}|\mathbf{v}|^2(\mathbf{v}\cdot\mathbf{u}_0) \\ &+\frac{1}{8}\frac{T_1^2}{T_0^4}|\mathbf{v}|^4\Bigg]\,,\end{aligned} \tag{6.54c}$$

6.2. Expansion of the fluid variables in powers of ε. The fluid variables $(n^\varepsilon, \mathbf{u}^\varepsilon, T^\varepsilon)$, defined in (4.33), have to be expanded in powers of ε as well,

$$n^\varepsilon = n_0 + \varepsilon n_1 + \varepsilon^2 n_2 + \mathcal{O}(\varepsilon^3)\,, \tag{6.55a}$$

$$\mathbf{u}^\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \mathcal{O}(\varepsilon^3)\,, \tag{6.55b}$$

$$T^\varepsilon = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \mathcal{O}(\varepsilon^3)\,. \tag{6.55c}$$

The expansion coefficients of products $a^\varepsilon b^\varepsilon$ are defined as

$$(a^\varepsilon b^\varepsilon)_0 := a_0 b_0\,, \quad (a^\varepsilon b^\varepsilon)_1 := a_1 b_0 + a_0 b_1\,, \quad (a^\varepsilon b^\varepsilon)_2 := a_2 b_0 + a_1 b_1 + a_0 b_2\,, \quad \text{etc.}$$

Inserting the expansion of the macroscopic quantities (6.55) into Eqs. (4.36) permits to get the corresponding infinite fluid hierarchy, for $i \geq 0$,

$$\begin{cases} \partial_t n_i + \nabla\cdot(n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-2} + \nabla\cdot(n\mathbf{u}\otimes\mathbf{u})_{i-2} + \nabla\cdot\mathbb{P}_i = -n_i\mathbf{E} - (n\mathbf{u})_i\times\mathbf{B}\,, \\ \partial_t w_{i-1} + \nabla\cdot\left[(w\,\mathbf{u})_{i-1} + (\mathbb{P}\cdot\mathbf{u})_{i-1}\right] + \nabla\cdot\mathbf{q}_i = -(n\mathbf{u})_{i-1}\cdot\mathbf{E}\,, \end{cases} \tag{6.56}$$

where the energy at order i is given by

$$w_i = \frac{3}{2}p_i + (n|\mathbf{u}|^2/2)_{i-2} \quad \Longrightarrow \quad w_0 = \frac{3}{2}p_0\,, \quad w_1 = \frac{3}{2}p_1\,, \quad w_2 = \frac{3}{2}p_2 + (n|\mathbf{u}|^2/2)_0\,,$$

and where we employed the convention that quantities with a negative index are not taken into account.

6.3. **Expansion of the pressure and heat flux in powers of ε.** Writing the power series of the stress tensor (4.34) and the heat flux (4.35) as

$$\mathbb{P}^\varepsilon = \mathbb{P}_0 + \varepsilon\mathbb{P}_1 + \varepsilon^2\mathbb{P}_2 + \mathcal{O}(\varepsilon^3)\,, \qquad \mathbf{q}^\varepsilon = \mathbf{q}_0 + \varepsilon\mathbf{q}_1 + \varepsilon^2\mathbf{q}_2 + \mathcal{O}(\varepsilon^3)\,, \tag{6.57}$$

one obtains using (4.40) and $f_0 = \mathcal{M}_0$ that

$$\mathbb{P}_0 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} = p_0\mathbb{I}\,, \tag{6.58a}$$

$$\mathbb{P}_1 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_0 d\mathbf{v} = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,, \tag{6.58b}$$

$$\mathbb{P}_2 = \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_1 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_1\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_1) f_0 d\mathbf{v} \tag{6.58c}$$

$$+ \mathbf{u}_0\otimes\mathbf{u}_0 \int_{\mathbb{R}^3} f_0 d\mathbf{v}$$

$$= \int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_2 d\mathbf{v} - \int_{\mathbb{R}^3} (\mathbf{u}_0\otimes\mathbf{v} + \mathbf{v}\otimes\mathbf{u}_0) f_1 d\mathbf{v} + n_0\,\mathbf{u}_0\otimes\mathbf{u}_0\,, \tag{6.58d}$$

as well as

$$\mathbf{q}_0 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_0 d\mathbf{v} = 0\,, \tag{6.59a}$$

$$\mathbf{q}_1 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} - \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} \tag{6.59b}$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_1 d\mathbf{v} - \frac{5}{2}\mathbf{u}_0\, p_0\,, \tag{6.59c}$$

$$\mathbf{q}_2 = \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_2 d\mathbf{v} - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_1\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_0 d\mathbf{v} \tag{6.59d}$$

$$- \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v} - \mathbf{u}_1\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_0 d\mathbf{v} + |\mathbf{u}_0|^2\frac{1}{2}\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v} + \mathbf{u}_0\otimes\mathbf{u}_0\int_{\mathbb{R}^3} \mathbf{v} f_0 d\mathbf{v} \tag{6.59e}$$

$$= \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2\mathbf{v} f_2 d\mathbf{v} - \frac{5}{2}\mathbf{u}_1 p_0 - \mathbf{u}_0\frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{v}|^2 f_1 d\mathbf{v} - \mathbf{u}_0\cdot\int_{\mathbb{R}^3} \mathbf{v}\otimes\mathbf{v} f_1 d\mathbf{v}\,.$$

6.4. **Shifted fluid hierarchy.** The fact that $\mathbf{q}_0 = 0$ entails that the energy equation in (6.56) at order $i = 0$ contains no information for the fluid variables (it is identically zero). We thus shift the index in the energy equation, $i \to i + 1$, such that a more meaningful fluid hierarchy reads

$$\begin{cases} \partial_t n_i + \nabla \cdot (n\mathbf{u})_i = 0\,, \\ \partial_t (n\mathbf{u})_{i-2} + \nabla \cdot (n\mathbf{u} \otimes \mathbf{u})_{i-2} + \nabla \cdot \mathbb{P}_i = -n_i \mathbf{E} - (n\mathbf{u})_i \times \mathbf{B}\,, \\ \partial_t w_i + \nabla \cdot [(w\,\mathbf{u})_i + (\mathbb{P} \cdot \mathbf{u})_i] + \nabla \cdot \mathbf{q}_{i+1} = -(n\mathbf{u})_i \cdot \mathbf{E}\,. \end{cases} \tag{6.60}$$

For a given $i \geq 0$ we recall that terms with negative subscripts are not taken into account.

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