UE Asymptotic Statistics
Exam
December 2019

Recommendations

- Duration 3 hours.
- Documents are allowed.
- You can use the results from the lectures without proving them.
- All the answers must be justified.
- The notation will take the clarity of the examination paper into account.

Exercize 1

1) Let $X_1, \ldots, X_n$ be i.i.d. random variables with uniform distribution on $[-\pi, \pi]$. Consider the sequence of random variables defined by

$$\frac{1}{n^{2/5}} \sum_{i=1}^{n} \cos(X_i).$$

Is this sequence of random variables tight?

2) Let $Y_1, \ldots, Y_n$ be i.i.d. random variables with distribution $\mathcal{N}(0,1)$. Consider the sequence of random variables defined by

$$\sqrt{n}\left(\exp\left(1 + \frac{1}{n}\sum_{i=1}^{n} Y_i^2\right) - \exp(2)\right).$$

Show that this sequence of random variables converges in distribution and find the limit distribution.

Indication: You can use that $\mathrm{Var}(Y_1^2) = 2$ without proving it.

Exercize 2

Consider $(X_1, Y_1), \ldots, (X_n, Y_n)$ i.i.d. such that X_1 follows the uniform distribution on $[-1, 1]$ and $Y_1 = X_1 + Z_1$ where Z_1 follows the uniform distribution on $[-2, 2]$. Assume also that X_1 and Z_1 are independent.

1) For any $0 < K < +\infty$, show that the function $\theta \to \mathbb{E}((Y_1 - \theta X_1)^4)$ is continuous on $[-K, K]$ and has a unique global minimum at $\theta_0 = 1$.

Indication: You can use, without proof, that for W following the uniform distribution on $[-a, a]$, we have $\mathbb{E}(W^\ell) = 0$ if ℓ is odd and $\mathbb{E}(W^\ell) = a^\ell/(\ell + 1)$ if ℓ is even.

2) Show that there exist $0 < T < \infty$ and $\epsilon > 0$ such that, with probability going to one as $n \to \infty$,

$$\inf_{\substack{\theta \in \mathbb{R} \\ |\theta| \geq T}} \left(\frac{1}{n} \sum_{i=1}^{n} (Y_i - \theta X_i)^4 \right) - \frac{1}{n} \sum_{i=1}^{n} (Y_i - X_i)^4 \geq \epsilon.$$

3) Prove that for all $0 < T < \infty$

$$\sup_{\theta \in [-T,T]} \left| \left(\frac{1}{n} \sum_{i=1}^{n} (Y_i - \theta X_i)^4 \right) - \mathbb{E}((Y_1 - \theta X_1)^4) \right|$$

goes to 0 in probability.

4) Let

$$\hat{\theta} \in \text{argmin}_{\theta \in \mathbb{R}} \frac{1}{n} \sum_{i=1}^{n} (Y_i - \theta X_i)^4.$$

Show that $\hat{\theta}$ converges to 1 in probability.

Exercize 3

Consider a measure P on $\mathbb{R}$. Consider two sets $\mathcal{F}_1$ and $\mathcal{F}_2$ of functions from $\mathbb{R}$ to $\mathbb{R}$, that are summable with respect to P. Consider the set of functions from $\mathbb{R}$ to $\mathbb{R}$

$$\mathcal{F}_{1,2} = \{f_1 + f_2; f_1 \in \mathcal{F}_1, f_2 \in \mathcal{F}_2\}.$$

Show that for all $u > 0$,

$$N_{[]}(\mathcal{F}_{1,2}, L^1(P), 2u) \leq N_{[]}(\mathcal{F}_1, L^1(P), u) N_{[]}(\mathcal{F}_2, L^1(P), u).$$