Chap I: Introduction to collective behavior

I) Introduction

Collective motion or self-organisation is observed in various natural processes such as a fish schools, bird flocks, herds of bulls, cellular dynamics, pedestrian motion

Self organization does not occur by chance but rather due to the numerous, specific interactions among the agents (particles)
The underlying "forces" or phenomena leading to self-organization can be of various type

- physical mechanisms (gravity, electro-magnetic forces, nuclear forces)
- chemical mechanisms (pheromones, Van-der-Waals forces)
- instinctive survival mechanisms (fear, feeding, ...)

The latter is much more complicated to describe from a mathematical point of view.

Self organization systems obey evolution equations which are highly-non linear and non local

Such models take the form of ODE systems, or non-local transport PDEs

Mathematical study of "flocking" models began with Viscek and his collaborators

Ref T. Viscek et al. Novel type of phase transition in a system of self-driven particles, Phys. Rev Letters (1995)

It is a stochastic, time-discrete model

Later Cucker-Smale proposed a deterministic, time continuous model

Ref Cucker and Smale, Emergent behavior in Flocks IEEE Transactions on autonomous control (2007)

, On the mathematics of emergence, Japanese Journal of Mathematics (2007)

Many other models have been proposed later; we in particular mention the three-zone model, based on Reyr empirical rules

1. Flocking : the "desire" of agents to stay together, for safety, social reasons

(2) <u>Collision avoidance</u> : agents tend to repel (3) when coming too close

(3) <u>Velocity matching</u> : attempt to keep similar velocities and flying directions as its neighbours

II) Some examples of flocking models

We consider the system of N particles with positions and velocities $(x_i(t), v_i(t))_{1\leq i\leq N}$ and masses $m_i = 1$

We first give some definitions

<u>Definition</u> System $(x_i, v_i)_{1\leq i\leq N}$ is said to have an <u>asymptotic flocking pattern</u> if the following two conditions are satisfied

1) <u>aggregation</u> : the spatial diameter $D(t)$ of the particle cloud is uniformly bounded in time, meaning

$$\sup_{t\geq 0} D(t) < +\infty \quad \text{with } D(t) = \max_{1\leq i,j\leq N} \|x_i(t) - x_j(t)\|$$

2) <u>velocity alignment</u> the velocity diameter $A(t)$ of the particle cloud tends to zero as $t \to +\infty$

$$\lim_{t\to+\infty} A(t) = 0 \quad \text{with} \quad A(t) = \max_{1\leq i,j\leq N} \|(v_i - v_j)(t)\|$$

Another notion exists in the literature "swarming", which is less restrictive than flocking, requiring only cohesion,

$$\sup_{t \geq 0} \max_i \|x_i(t) - x_c(t)\| < +\infty$$

$$\sup_{t \geq 0} \max_i \|v_i(t) - v_c(t)\| < +\infty$$

where $x_c(t) = \frac{1}{N} \sum x_i(t)$

$v_c(t) = \frac{1}{N} \sum v_i(t)$

We now present two different models

1) The Cucker-Smale model

$$\begin{cases} x_i'(t) = v_i(t) \\ v_i'(t) = \frac{\lambda}{N} \sum_{j=1}^{N} \psi(\|x_i - x_j\|)(v_j - v_i) \end{cases}$$

ψ is the communication strenghts

$\psi_b(r) = \frac{\alpha}{(1 + r^2)^{\beta/2}}$ $\alpha > 0; \beta \geq 0$ bounded kernel

$\psi_s = \frac{\alpha}{r^{\beta}}$ $\alpha > 0$ $\beta \in \mathbb{R}^+$ singular kernel

$$\int_{r_0}^{+\infty} \psi(r)\,dr = +\infty \qquad \text{(long range conditions, heavy tail)}$$

$$\int_{0}^{r_0} \psi(r)\,dr = +\infty \qquad \text{(short range conditions)}$$

<u>Property</u>

$$v_c(t) = v_c(0)$$
$$x_c(t) = x_c(0) + v_c(0)\,t$$

Since ψ only depends on the relative positions

For simplicity, we assume that $x_c(0) = 0$
$v_c(0) = 0$

We have the following theorem

<u>Theorem</u> (Flocking in the bounded case)

Suppose that ψ is bounded and initial conditions are non-collisional ($x_i^0 \neq x_j^0 \ \forall i,j \ i\neq j$)

Then

(i) if $\beta \in [0,1]$ (long range), one has an <u>unconditional flocking</u> : $\exists\, d_m$ and d_M

$$0 \leq d_m \leq \sum \|x_i(t)\|^2 \leq d_M$$

and

$$\sum \|v_i(t)\|^2 \leq \sum \|v_i^0\|^2 \, e^{-2\lambda \psi_b(2d_M)t}$$

(ii) if $\beta \in]1,+\infty[$; one has <u>conditional flocking</u> for initial data

$$\|V_i^0\| < \int_{\|X_i^0\|}^{+\infty} \psi(r)\,dr$$

such that we get the previous estimate

In the strong singular case ; we have the following result

<u>Theorem</u> Let us consider $(x_i(t), v_i(t))_{1\leq i\leq N}$ a solution to the Cucker-Smale system

- If $\beta \in [0,1]$ and $x_i^0 \neq x_j^0$ $\forall i,j \in \{1,\dots,N\}$ then there exist x_m and x_M such that

$$x_m \leq \left(\sum_{i=1}^N \|x_i(t)\|^2\right)^{1/2} \leq x_M$$

$$\|v(t)\| = \left(\sum_{j=1}^N \|v_j(t)\|^2\right)^{1/2} \leq \|v(0)\| \, e^{-\lambda\psi(2x_M)t}$$

- If $\beta > 0$, $x_i^0 \neq x_j^0$ $\forall i,j \in \{1,\dots,N\}$ and

$$\|V(0)\| < \frac{\lambda\alpha}{2^\beta(\beta-1)} \|X(0)\|^{1-\beta}$$

then there exist x_m and x_M such that

$$x_m \leq \left(\sum_{i=1}^N \|x_i(t)\|^2\right)^{1/2} \leq x_M$$

$$\sum_{j=1}^N \|v_j(t)\|^2 \leq \sum_{j=0}^N \|v_j(0)\|^2 \, e^{-2\lambda\psi_s(2x_M)t}$$

<u>Reference</u> For bounded kernel ; we refer to S.Y Ha & J.G. Liu CMS (2009)

For singular kernel ; we refer to J.A. Carrillo , Y.P Choi , PM Mucha and J Peszek in Nonlinear Analysis (2017)

Remark We will prove the previous result in Lecture #2

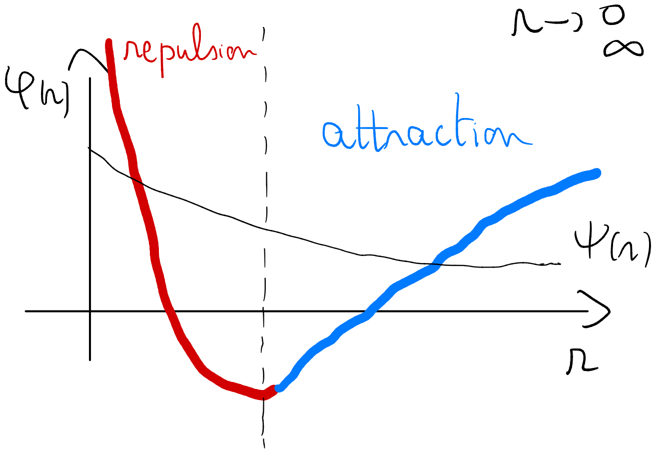
2) Three zone model

$$\begin{cases} x_i'(t) = v_i(t) \\ v_i'(t) = \frac{1}{N} \sum_j \psi(\|x_i - x_j\|)(v_j - v_i) \\ \qquad - \frac{1}{N} \sum_{j \neq i} \nabla_x \varphi(\|x_i - x_j\|) \end{cases}$$

where we suppose that the interacting potential φ is such that

$$\varphi \in \mathcal{C}^1(\mathbb{R}^+_*)$$

$$\varphi(r) > 0 \qquad \lim_{\substack{r \to 0 \\ \infty}} \varphi(r) = +\infty$$



- repulsion for $r \ll 1$
- alignment for $r \sim 1$
- attraction for $r \gg 1$

For this model; we have the following result

<u>Theorem</u> Suppose that ψ is bounded and φ satisfies the assumption above

Then

- For non-collisional initial data, there exists a unique global solution such that

$$0 < d_m \leq \|x_i(t) - x_j(t)\| \leq d_M \quad \forall t \geq 0$$

and $A(t) \xrightarrow[t \to +\infty]{} 0$

<u>Ref</u> Cao - Motsch - Reamy and Theisen, Math Bio Eng (2020)

To conclude this first part of the lectures, let us emphasize that there are different levels of description for this self-organization phenomena

III) Kinetic & fluid description

<u>Kinetic description</u> particles are replaced by a probability distribution function $f(t,x,v) \geq 0$ solution to a mean field model obtained as the limit of the particle model

Consider $f_N(t) = \sum_i \delta_0(x - x_i(t))\, \delta_0(v - v_i(t))$

The distribution f_N is named the empirical measure, it is bounded, compactly supported measure

<u>Remark</u> The Dirac distribution or measure δ_0 is a distribution such that

$$\langle \delta_0 , \theta \rangle_{\mathcal{D}',\mathcal{D}} = \theta(0) \quad \forall \theta \in \mathcal{E}_c^\infty(\mathbb{R})$$

Actually, it is enough to consider δ_0 in the dual space of $\mathcal{E}(\mathbb{R})$ (choose the test function only continuous)

Let us prove that f_N is solution to a PDE

<u>Proposition</u> Consider f_N the empirical measure

Then it is solution in the distribution sense to

$$\partial_t f_N + v \cdot \nabla_x f_N + \mathrm{div}_v \left(L[f_N] f_N + E_N f_N \right) = 0$$

where

$$L[f_N] = \varphi * J_N - (\varphi * \rho_N) v$$
$$E_N = - \nabla \varphi * \rho_N$$

with

$$J_N = \frac{1}{N} \sum_{j=1}^{N} v_j(t)\, \delta_0(x - x_j(t))$$

$$\rho_N = \frac{1}{N} \sum_{j=1}^{N} \delta(x - x_j(t))$$

Proof First we compute E_N and $L[f_N]$

$$E_N(t,x) = -(\nabla\varphi * \rho_N)(t,x)$$

$$= -\frac{1}{N}\sum_{k=1}^{N} \langle \delta_0(\cdot - x_k(t)), \nabla\varphi(x - \cdot)\rangle$$

$$= -\frac{1}{N}\sum_{k=1}^{N} \nabla\varphi(x - x_k(t))$$

We do the same for $L[f_N]$

$$L[f_N](t,x,v) = (\psi * J_N - \psi * \rho_N v)(t,x)$$

$$= \frac{1}{N}\sum_{k=1}^{N} v_k(t)\, \psi(\|x - x_k(t)\|) -$$

$$\frac{1}{N}\sum_{k=1}^{N} v\, \psi(\|x - x_k(t)\|)$$

$$= \frac{1}{N}\sum_{k=1}^{N} \psi(\|x - x_k(t)\|)\,(v_k(t) - v)$$

Observe that now $L[f_N]$ and E_N are smooth function of (t,x,v) (as a consequence of the regularity of ψ and φ)

Then; we show that f_N is a solution in the distribution sense of the transport equation

$\forall \Theta \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$

$$\langle \partial_t f_N + v \cdot \nabla_x f_N + \mathrm{div}_v \left((L[f_N] + E_N) f_N \right), \Theta \rangle_{\mathcal{D}', \mathcal{D}}$$

$$= - \langle f_N, \partial_t \Theta + v \cdot \nabla_x \Theta + (L[f_N] + E_N) \Theta \rangle_{\mathcal{D}', \mathcal{D}}$$

$$= - \sum_{j=1}^{N} \int_{\mathbb{R}} \left(\partial_t \Theta + v_j \nabla_x \Theta + (L[f_N] + E_N) \nabla_v \Theta \right)(t, x_j, v_j)\, dt$$

Using that (x_j, v_j) is solution to the Cucker-Smale system and from the computation of $L[f_N]$ and E_N

$$= - \sum_{j=1}^{N} \int_{\mathbb{R}} \left(\partial_t \Theta + \frac{dx_j}{dt} \nabla_x \Theta + \frac{dv_j}{dt} \nabla_v \Theta \right)(t, x_j, v_j)\, dt$$

$$= - \sum_{j=1}^{N} \int_{\mathbb{R}} \frac{d}{dt} \Theta(t, x_j(t), v_j(t))\, dt$$

$= 0$ since Θ is compactly supported $\square$

Here, we see that the dynamical system on (x_i, v_i) is equivalent to a measure solution of a kinetic transport equation.

More generally, when $N \to +\infty$, we may replace the empirical measure f_N by a distribution function f depending on time t, position x and velocity v.

Replacing f_N by any distribution function and sum and discrete convolutions by integrals and convolution it yields

$$\partial_t f + v \nabla_x f + \operatorname{div}\left(\left(L(f) - \nabla \varphi * \rho \right) f \right) = 0$$

$$L(f) = \psi * J - (\psi * \rho) v$$

$$J = \int f v \, dv \qquad \rho = \int f \, dv$$

<u>Fluid description</u> we study system of equations for macroscopic equations (ρ, J)

$$\begin{cases} \partial_t \rho + \mathrm{div}_x (J) \\ \partial_t J + \mathrm{div}_x \left(\int f v \otimes v \right) - (\Psi * J \rho - \Psi * \rho \, J) = 0 \end{cases}$$

we need a closure to eliminate f in the previous eq

For instance $f = \rho \, \delta(v - \frac{J}{\rho})$; this ansatz give

$$\begin{cases} \partial_t \rho + \mathrm{div}_x J = 0 \\ \partial_t J + \mathrm{div}_x \left(\frac{J \otimes J}{\rho} \right) = (\Psi * J) \rho - (\Psi * \rho) J \end{cases}$$

In the following we will study these kinetic and fluid models and investigate how there are related.

IV) Mathematical models sharing the same structure

1) Fitzhugh-Nagumo model

this model appears in neurosciences

it describe the time evolution of a potential membrane $V_i(t)$ of the neuron i and an adaptation variable $W_i(t)$

$$\begin{cases} dV_i = \left(N(V_i) - W_i + I_{ext}\right)dt + \sqrt{2}\, dB_i \\ dW_i = A(V_i, W_i) \end{cases}$$

- $N(V) = V - V^3$
$A(V,W) = aV - bW + c \qquad a, b, c \geq 0$

and I_{ext} describes the interactions between neurons

$$I_{ext}(t) = \frac{1}{N} \sum_j \Psi_{ij} (V_j - V_i)$$

Ψ_{ij} is related to a conductance and is given by the neural networks.

2) Kuramoto model

$$\dot{\Theta}_i(t) = \nu_i + \frac{1}{N} \sum_j a_{ij} \sin(\Theta_j - \Theta_i)$$

It can be written as a second order model (Θ_i, ω_i)

$$\begin{cases} \theta_i' = \omega_i \\ \omega_i' = \dfrac{1}{N} \displaystyle\sum_j \cos(\theta_j - \theta_i)\,(\omega_j - \omega_i) \end{cases}$$