Chap. II : Analysis of the microscopic model

This chapter is devoted to the proof of the results given in Chap I on the system of particles. The strategy is to deliver an abstract result which can be applied to various system of equations

I) Preliminary results

Let $(x_i, v_i)_{1\leq i\leq N} \in \mathbb{R}^d \times \mathbb{R}^d$ be the phase space coordinates of the i-th agent among N

We set $E = \mathbb{R}^{Nd}$ to be considered as a vector space with an inner product $\langle \cdot, \cdot \rangle$ and the associated norm

We set $x = (x_1, \dots, x_N) \in \mathbb{R}^{dN}$ and $v = (v_1, \dots, v_N) \in \mathbb{R}^{dN}$, we consider the following system

$$\begin{cases} \dfrac{dx}{dt} = v \\[2ex] \dfrac{dv}{dt} = -L(x)v \end{cases} \qquad (1)$$

with initial conditions $(x^0, v^0) \in E \times E$

and $L(x) : E \to E$ is a linear operator
$v \to L(x)v$

satisfying the coercivity condition : there is a non-negative and non increasing function $\phi(s)$ such that

$$\langle L(x)v, v\rangle \geq \phi(\|x\|)\,\|v\|^2 \quad \forall v \in E$$

Since ϕ is a non-negative and measurable function we can define Φ such that $\Phi' = \phi \geq 0$ then Φ is non-decreasing.

Therefore; we have $\pm \frac{d}{dt}\|x\|^2 = \pm 2\langle x, v\rangle$ and

$\frac{d}{dt}\|v\|^2 = -2\langle L(x)v, v\rangle \leq -2\phi(\|x\|)\,\|v\|^2$

which yields that

$$\begin{cases} \dfrac{d}{dt}\|v\| \leq -\phi(\|x\|)\,\|v\| \\[2ex] \left|\dfrac{d}{dt}\|x\|\right| \leq \|v\| \end{cases} \qquad (*)$$

A system satisfying $(*)$ is called a system of dissipative differential inequalities.

Let us prove that such a system admits Lyapunov functionals $\mathcal{E}_\pm(\|x\|, \|v\|)$ given by

$$\mathcal{E}_\pm(\|x\|, \|v\|) = \|v\| \pm \Phi(\|x\|)$$

Lemma Suppose that (x,v) is solution to (1). Then we have $\forall t \geq 0$

(i) $\mathcal{E}_\pm(\|x(t)\|, \|v(t)\|) \leq \mathcal{E}_\pm(\|x^0\|, \|v^0\|)$

(ii) $\|v(t)\| + \left| \int_{\|x^0\|}^{\|x(t)\|} \phi(s)\,ds \right| \leq \|v^0\|$

Proof We start by differentiating with respect to time the functionals $\mathcal{E}_\pm$; it yields that

$$\frac{d}{dt}\mathcal{E}_\pm(\|x(t)\|, \|v(t)\|) = \frac{d}{dt}\|v(t)\| \pm \phi(\|x(t)\|)\frac{d\|x(t)\|}{dt}$$
$$\leq \phi(\|x(t)\|)\underbrace{\left(-\|v(t)\| \pm \frac{d}{dt}\|x(t)\|\right)}_{\leq 0}$$

which gives the first result (i).

Now to prove (ii), we use the previous result and integrate between 0 and t:

$$\|v(t)\| - \|v(0)\| \leq \pm\left(\Phi(\|x(t)\|) - \Phi(\|x^0\|)\right)$$
$$\leq -\left|\Phi(\|x(t)\|) - \Phi(\|x^0\|)\right|$$
$$= -\left|\int_{\|x^0\|}^{\|x(t)\|}\phi(s)\,ds\right| \qquad \square$$

A direct consequence of this lemma is the following theorem

Theorem A Suppose that (x,v) is solution to (1) with ϕ non-negative and non-increasing. Then the following holds

(i) If $\|v^0\| < \int_0^{\|x^0\|} \phi(s)ds$, then there exists $x_m > 0$ such that

$$\|v^0\| = \int_{x_m}^{\|x^0\|} \phi(s)ds \quad \text{and} \quad \|x(t)\| \geq x_m$$

(ii) If $\|v^0\| < \int_{\|x^0\|}^{+\infty} \phi(s)ds$, then there exists $x_M \geq 0$ such that

$$\|v^0\| = \int_{\|x^0\|}^{x_M} \phi(s)ds \; ; \quad \|x(t)\| \leq x_M \text{ and } \|v(t)\| \leq \|v^0\| e^{-\phi(x_M)t}$$

<u>Proof</u> Let us start with (i) and suppose that the initial data is such that $\|v^0\| < \int_0^{\|x^0\|} \phi(s)ds$

On the one hand since the function $\phi \geq 0$ and measurable; we may define the function $f : \delta \longrightarrow \int_\delta^{\|x^0\|} \phi(s)ds$; which is non-increasing and continuous.

Hence, from the assumption on (x^0, v^0) ; we chose $x_m > 0$ as the smallest value such that $f(\|x^0\|) = 0 \leq \|v^0\| = \int_{x_m}^{\|x^0\|} \phi(s)ds < f(0)$

Now let us prove that $\forall t \geq 0 \quad \|x(t)\| \geq x_m$

We proceed by contradiction and suppose that there exists $t > 0$ such that $\|x(t)\| < x_m$.

Thus by continuity of f and since it is also non-increa-sing, we know that $0 = f(\|x^0\|) \leq f(x_m) < f(0)$

Moreover, by continuity of $s \longrightarrow \|x(s)\|$ and $\|x(s=0)\| = \|x^0\|$; we can choose $t^* \geq 0$ such that $\|x(t_*)\| < x_m$ and since x_m is the smallest value such that $f(x_m) = \|v^0\|$

and again f non-increasing; $f(x_m) < f(\|x(t_*)\|)$.
However, by application to the previous lemma; we know that for any solution (x,v) to (1); we have

$$\left| \int_{\|x^0\|}^{\|x(s)\|} \phi(s)ds \right| \leq \|v^0\|$$; which is a contradiction.

Now let us prove (ii) using the same arguments as in (i).
Suppose that $\|v^0\| < \displaystyle\int_{\|x^0\|}^{+\infty} \phi(s)ds$

then since $\phi \geq 0$ and measurable, the function $g : \delta \to \displaystyle\int_{\|x^0\|}^{\delta} \phi(s)ds$ is continuous and non-decreasing.
Since $\|x(0)\| = \|x^0\|$; we choose $x_M \geq 0$ as the largest value such that $\|v^0\| = \displaystyle\int_{\|x^0\|}^{x_M} \phi(s)ds$

To prove that $\|x(t)\| \leq x_M$; we again proceed by contradiction and the lemma allows to get the contradiction.

Finally using that $\|x(t)\| \leq x_M$ and since ϕ is non increasing; we have using $(*)$

$$\frac{d}{dt}\|v(t)\| \leq -\phi(\|x(t)\|)\,\|v(t)\|$$
$$\leq -\phi(x_M)\,\|v(t)\|$$

From the Gronwall lemma, we get that
$$\|v(t)\| \leq \|v(0)\|\, e^{-\phi(x_M)t}$$ $\square$

II) Application to the Cucker-Smale system

We consider a function ψ such that $\psi \geqslant 0$ and non-increasing, the Cucker-Smale system is given by $1 \leq i \leq N$

$$\begin{cases} \dfrac{dx_i}{dt} = v_i \\ \dfrac{dv_i}{dt} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \psi(\|x_i - x_j\|)\,(v_j - v_i) & t > 0 \\ x_i(0) = x_i^0 \text{ and } v_i(0) = v_i^0 & 1 \leq i \leq N \end{cases} \qquad (2)$$

We remind that we have conservation of the mean-velocity

$$v_c(t) = \frac{1}{N}\sum_{j=1}^{N} v_j(t) \qquad v_c'(t) = 0$$

$$\text{and } x_c(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t) = x_c(0) + v_c(0)\,t$$

Then to simplify the presentation; we suppose that $x_c(0) = 0$ and $v_c(0) = 0$.

Then; we have
$$\sum_{j=1}^{N} x_j(t) = 0$$
$$\sum_{j=1}^{N} v_j(t) = 0$$

Let us show that the Cucker-Smale system satisfies the $(*)$ condition

<u>Lemma</u> Let $(x_i, v_i)_{1 \leq i \leq N}$ be solution to (2) with ψ non-negative and non-increasing

We define
$$\|X(t)\| = \left(\sum_{j=1}^{N} \|x_j(t)\|^2\right)^{1/2}$$
$$\|V(t)\| = \left(\sum_{j=1}^{N} \|v_j(t)\|^2\right)^{1/2}$$

Then we have

$$\left|\frac{d}{dt}\|X(t)\|\right| \leq \|V(t)\| \;;\; \frac{d}{dt}\|V(t)\| \leq -2\lambda\,\psi(2\|X(t)\|)\,\|V(t)\|$$

Proof. We take the inner product

$$\pm \frac{d}{dt} \|x(t)\| = \pm 2 \left\langle \frac{dx}{dt}, v \right\rangle = \pm 2 \langle v, x \rangle \leq 2 \|x\| \, \|v\|$$

which gives the first inequality.

Then ; we use that $\displaystyle\max_{1 \leq i, j \leq N} \|x_i(t) - x_j(t)\| \leq 2 \|x(t)\|$

and since ψ is non-increasing

$$\frac{d}{dt} \|v(t)\|^2 = 2 \left\langle \frac{dv}{dt}, v \right\rangle$$

$$= -\frac{\lambda}{N} \sum_{j=1}^{N} \sum_{i=1}^{N} \psi(\|x_i(t) - x_j(t)\|) \, (v_j - v_i) \, 2 v_i$$

$$= -\frac{\lambda}{N} \sum_{1 \leq i, j \leq N} \psi(\|x_i(t) - x_j(t)\|) \, \|v_i - v_j\|^2$$

Now observing that

$$\sum_{1 \leq i, j \leq N} \|v_i - v_j\|^2 = 2N \sum_{i=1}^{N} \|v_i\|^2 - 2 \left\langle \sum_{i=1}^{N} v_i(t), \sum_{j=1}^{N} v_j(t) \right\rangle$$

$$= 2N \sum_{i=1}^{N} \|v_i\|^2$$

It yields that

$$\frac{d}{dt} \|v(t)\|^2 \leq -\frac{\lambda}{N} \sum_{1 \leq i, j \leq N} \psi(2 \|x(t)\|) \, \|v_i(t) - v_j(t)\|^2$$

$$= -2\lambda \sum_{i=1}^{N} \psi(2\|x(t)\|) \, \|v_i(t)\|^2$$

Hence we get the result $\square$

Now, we are in position to prove our first result on asymptotic flocking.

Let us first consider the case of a singular kernel

$$\psi(s) = \psi_S(s) = \frac{\alpha}{s^{\beta}} \qquad \alpha > 0 \text{ and } \beta \geq 0$$

We have the following result

Theorem 1 Consider the solution to (2)

- If $\beta \in [0,1]$ and $x_i^0 \neq x_j^0$ $\forall i,j \in \{1,\dots,N\}$ then there exists x_m and x_M such that

$$x_m \leq \|X(t)\| \leq x_M \qquad \forall t \geq 0$$
$$\|v(t)\| \leq \|v(0)\| e^{-2\lambda\psi(2x_M)t}$$

where x_m and x_M are given explicitely

- If $\beta > 1$ and $x_i^0 \neq x_j^0$ $\forall i,j \in \{1,\dots,N\}$ and

$$\|V^0\| < \frac{\lambda\alpha\, 2^{1-\beta}\|X^0\|^{1-\beta}}{(\beta-1)}$$

then there exists x_m and x_M such that

$$x_m \leq \|X(t)\| \leq x_M \qquad \forall t \geq 0$$
$$\|V(t)\| \leq \|V(0)\| e^{-2\lambda\psi(2x_M)t}$$

Proof $\boxed{\text{Part I : we consider } \beta \in [0,1]}$ and $x_i^0 = x_j^0$ $\forall i,j$

- If $\|V^0\| = 0$, then we have $\|V(t)\| = 0$ $\forall t \geq 0$ and then $\|X(t)\| \equiv \|X^0\|$ and the result holds

- If $\|V^0\| \neq 0$; we first prove the first estimate on the upper-bound. Since $\beta \in [0,1]$; we have

$$\int_{\|X^0\|}^{+\infty} \psi_s(2s)\,ds = \frac{1}{2}\int_{\|X^0\|}^{\infty} \frac{\alpha}{s^\beta}\,ds = +\infty$$ the condition is automatically satisfied.

It follows from Theorem A with $\phi(s) = \dfrac{2\lambda\alpha}{(2s)^\beta}$ such that

$$\begin{cases} \|V^0\| = \lambda\alpha \displaystyle\int_{2\|X^0\|}^{2x_M} \frac{ds}{s^\beta} \\ \|X(t)\| \leq x_M \\ \|V(t)\| \leq \|V(0)\| e^{-2\lambda\psi_s(2x_M)t} \end{cases}$$

Note that from the explicit expression of Ψ_s, we can have an explicit formula for x_M

Indeed, when $\beta \in [0,1[$, we have that

$$\|v^0\| = \frac{\lambda\alpha \, 2^{1-\beta}}{(1-\beta)} \left(x_M^{1-\beta} - \|x^0\|^{1-\beta} \right)$$

$$\Rightarrow \quad x_M = \left(\|x^0\|^{1-\beta} + \frac{(1-\beta)}{\lambda\alpha \, 2^{1-\beta}} \|v^0\| \right)^{1/(1-\beta)}$$

whereas when $\beta = 1$, we have

$$\|v^0\| = \lambda\alpha \int_{2\|x_0\|}^{2x_M} \frac{ds}{s} = \lambda\alpha \ln\left(\frac{x_M}{\|x^0\|}\right) \quad \text{i.e. } x_M = \|x^0\| \, e^{\frac{\|v^0\|}{\alpha\lambda}}$$

Now we proceed in the same way for the lower bound by applying Theorem A; we have

$$\int_0^{\|x^0\|} \phi(s)\,ds = \lambda\alpha \int_0^{2\|x^0\|} \frac{ds}{s^\beta} = \frac{\lambda\alpha \, 2^{1-\beta}}{(1-\beta)} \|x_0\|^{1-\beta}$$

and $$\int_{x_m}^{\|x^0\|} \phi(s)\,ds = \frac{\lambda\alpha \, 2^{1-\beta}}{(1-\beta)} \left(\|x^0\|^{1-\beta} - x_m^{1-\beta} \right)$$

- If $\beta \in [0,1[$, we suppose that $\|v^0\| < \frac{\lambda\alpha \, 2^{1-\beta}}{(1-\beta)} \|x^0\|^{1-\beta}$

  Then we define x_m as $\|v^0\| = \int_{x_m}^{\|x^0\|} \phi(s)\,ds$

  and when the condition is not satisfied; we choose $x_m = 0$
- If $\beta = 1$; the condition is automatically satisfied since

$$\int_0^{2\|x^0\|} \frac{ds}{s} = +\infty$$

We then apply Theorem A and we get the lower bound.

Part II: we consider $\beta > 1$

We will again apply Theorem A and proceed as in the previous situation

- If $\|v^0\| = 0$, the solution is trivial
- If $\|v^0\| \neq 0$ and observe that

$$\int_{\|x_0\|}^{+\infty} \phi(s)\,ds = \lambda\alpha \int_{2\|x^0\|}^{\infty} \frac{ds}{s^\beta} = \frac{\lambda\alpha\, 2^{1-\beta}}{\beta-1} \|x^0\|^{1-\beta} < +\infty$$

when $\|v^0\| < \frac{\lambda\alpha\, 2^{1-\beta}}{\beta-1} \|x^0\|^{1-\beta}$

We then choose x_M as in theorem A ; that is

$$\|v^0\| = \frac{\lambda\alpha\, 2^{1-\beta}}{(\beta-1)} \left(\|x^0\|^{1-\beta} - x_M^{1-\beta} \right)$$

and get the first estimate on the upper-bound

Finally since $\int_0^{\|x^0\|} \phi(s)\,ds = \lambda\alpha \int_0^{2\|x^0\|} \frac{ds}{s^\beta} = +\infty$

we can find the lower bound without any other condition on v^0 and choose x_m such that $\int_{x_m}^{\|x^0\|} \phi(s)\,ds = \|v^0\|$ $\square$

Using the same manner, we prove the theorem for bounded kernel

$$\psi(s) = \frac{\alpha}{(1+s^2)^{\beta/2}}$$

The proof is left as an exercise.