Chap. 4 : Flocking of kinetic equations

In this last chapter ; we want to prove flocking on the kinetic equation as we already performed at the micro-scopic level.

We now choose $\psi(s)$ such that $\exists \psi_m > 0$ such that $\psi(s) \geq \psi_m \quad \forall s \in \mathbb{R}^+$

We loose the integrability condition and have long range interactions.

We consider the (CS) model

$$\begin{cases} \partial_t f + v \nabla_x f + \operatorname{div}(L[f] f) + \beta \operatorname{div}((u - v) f) = 0 \\ L[f] = \psi * \rho u - (\psi * \rho) v \\ \psi \text{ is symmetric} \end{cases}$$

We define the following Lyapunov functionnals

$$\mathcal{E}_1(t) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f \, |v - u|^2 \, dx dv \qquad \text{relative energy}$$

and

$$\mathcal{E}_2(t) = \int_{\mathbb{R}^{2d}} \rho(t,x)\,\rho(t,y)\,|u(t,x) - u(t,y)|^2\,dx\,dy$$

$\mathcal{E}_1$ measures the local alignment whereas $\mathcal{E}_2$ measures alignment of the macroscopic velocities

We set $\boxed{\mathcal{E}(t) = \mathcal{E}_1(t) + \frac{1}{2}\,\mathcal{E}_2(t)}$

and prove the following theorem

<u>Theorem</u> $\mathcal{E}(t) \leq \mathcal{E}(0)\, e^{-Ct}$

where $C = 2\min(T, \varphi_m)$

and $\varphi_m = \min_{x,y} \varphi(x-y) > 0$

Let us give the main steps for the proof of this result

$$\frac{d\mathcal{E}_1}{dt} = 2\underbrace{\int f\,(u-v)\,\partial_t u}_{=0} + \int \partial_t f\,|u-v|^2\,dx\,dv$$

$$= 2\int \nabla_x u\,(u-v)\cdot v f\,dx\,dv \,\Big) = I_1$$

$$-2\int (u-v)\cdot L[f]\,f\,dx\,dv \,\Big) = I_2 \leq 0$$

$$\underbrace{-2\int |u-v|^2 f\,dx\,dv}_{2\mathcal{E}_1}$$

$$I_1 = 2\int \operatorname{div} P\cdot u\,dx \quad \text{where}$$

P is the stress tensor $P = \int (v-u)\otimes(v-u)\,f\,dv$

Hence we get that

$$\frac{d\mathcal{E}_1}{dt} \leq 2\int \operatorname{div}_x P\cdot u\,dx - 2\mathcal{E}_1$$

We next estimate

$$\frac{d\mathcal{E}_2}{dt} = 2\int \partial_t\rho(t,x)\,\rho(t,y)\,\overset{J_1}{\overset{\shortparallel}{|u(t,x)-u(t,y)|^2}}\,dx\,dy$$

$$+2\int \rho(t,x)\,\rho(t,y)\,\overset{J_2}{\overset{\shortparallel}{(u(t,x)-u(t,y))}}\,\partial_t(u(t,x)-u(t,y))\,dx\,dy$$

Then we use the equations satisfied by the macroscopic equations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0 \\ \rho \partial_t u + \rho u \cdot \nabla_x u + \mathrm{div}_x P = \int L(f) f \, dv \end{cases}$$

and get that

$$J_1 = 4 \int \rho u \cdot \nabla_x u \cdot u \, dx$$

$$- 4 \left(\int \rho u \cdot \nabla_x u \, dx \right) \cdot \int \rho u \, dx$$

$$J_2 = - J_1 - 4 \int \mathrm{div} P \cdot u \, dx$$

$$- 2 \int \psi(x-y) \rho(t,x) \rho(t,y) \, |u(t,x) - u(t,y)|^2 \, dx dy$$

Therefore we have

$$\frac{d \mathcal{E}_2}{dt} = - 4 \int \mathrm{div} P \cdot u - 2 \int \psi(x-y) \rho(t,x) \rho(t,y) \, |u(t,x) - u(t,y)|^2 \, dx dy$$

Gathering the two results; it yields

$$\frac{d\mathcal{E}}{dt} \leq -2\,\mathcal{E}_1 - \varphi_m\,\mathcal{E}_2$$
$$\leq -2\,\min(1, \varphi_m)\,\mathcal{E}$$

we conclude by a Gronwall lemma □