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Processus sur le groupe unitaire et probabilités libres

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Résumé

Cette thèse est consacrée à l'étude asymptotique d'objets liés au mouvement brownien sur le groupe unitaire en grande dimension, ainsi qu'à l'étude, dans le cadre des probabilités libres, des versions non-commutatives de ces objets. Elle se subdivise essentiellement en trois parties.

Dans le chapitre 2, nous résolvons le problème initial de cette thèse, à savoir la convergence de la transformation de Hall sur le groupe unitaire vers la transformation de Hall libre, lorsque la dimension tend vers l'infini. Pour résoudre ce problème, nous établissons des théorèmes d'existence de noyaux de transition pour la convolution libre. Enfin, nous utilisons ces résultats pour prouver que, pareillement au mouvement brownien sur le groupe unitaire, le mouvement brownien sur le groupe linéaire converge en distribution non-commutative vers sa version libre. Nous étudions les fluctuations autour de cette convergence dans le chapitre 3. Le chapitre 4 présente un morphisme entre les mesures infiniment divisibles pour la convolution libre additive d'une part et multiplicative de l'autre. Nous montrons que ce morphisme possède une version matricielle qui s'appuie sur un nouveau modèle de matrices aléatoires pour les processus de Lévy libres multiplicatifs.

Mots-clefs. Matrices aléatoires, Mouvement brownien, Transformation de Segal-Bargmann-Hall, Groupe unitaire, Groupe linéaire, Grande dimension, Probabilités libres, Convolution libre, Mesures infiniment divisibles.

Abstract

This thesis focuses on the asymptotic of objects related to the Brownian motion on the unitary group in large dimension, and on the study, in free probability, of the non-commutative versions of those objects. It subdivides into essentially three parts.

In Chapter 2, we solve the original problem of this thesis: the convergence of the Hall transform on the unitary group to the free Hall transform, as the dimension tends to infinity. To solve this problem, we establish theorems of existence of transition kernel for the free convolution. Finally, we use these results to prove that, exactly as the Brownian motion on the unitary group, the Brownian motion on the linear group converges in noncommutative distribution to its free version. Then we study the fluctuations around this convergence in Chapter 3. Chapter 4 presents a homomorphism between infinitely divisible measures for the free convolution, in respectively the additive case and the multiplicative case. We show that this homomorphism has a matricial version which is based on a new model of random matrices for the free multiplicative Lévy processes.

Keywords. Random matrices, Brownian motion, Segal-Bargmann-Hall transform, Unitary group, Linear group, Large dimension, Free probability, Inifinitely divisible measures.

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CHAPITRE 1

Introduction

Cette thèse est composée de trois textes principaux, les chapitres 2, 3 et 4, qui forment autant d'articles publiés ou destinés à l'être, et qui sont par conséquent rédigés en langue anglaise. L'objectif de cette introduction est de présenter de manière synthétique les différents résultats qui les composent. Je me suis efforcé de dégager deux axes de lecture : d'une part, le parallélisme entre la théorie des probabilités classiques et celle des probabilités libres ; d'autre part, l'analogie entre le cadre additif et le cadre multiplicatif.

Nous allons commencer par examiner un exemple simple, le mouvement brownien, exemple qui illustre parfaitement cette lecture à double entrée. Notons que cet objet sera au cœur des différents développements qui suivront.

1. Le mouvement brownien

1.1. Les mouvements browniens classiques. Pour tout réel $t > 0$, la *gaussienne* de variance t est la mesure $\mathcal{N}_t$ dont la densité par rapport à la mesure de Lebesgue est

$$\mathcal{N}_t(\mathrm{d}x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathrm{d}x.$$

DÉFINITION 1.1. Un *mouvement brownien réel standard* est un processus stochastique $(W_t)_{t\in\mathbb{R}_+}$, à valeur dans $\mathbb{R}$, tel que :

(1) $W_0 = 0$;
(2) pour tous temps $0 \leq s < t$, l'accroissement $W_t - W_s$ a pour loi la gaussienne $\mathcal{N}_{t-s}$;
(3) pour tous temps $0 \leq t_1 < \cdots < t_n$, les accroissements $W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ sont indépendants ;
(4) $(W_t)_{t\in\mathbb{R}_+}$ est presque sûrement continu.

Les différents mouvements browniens que nous allons rencontrer possèdent tous des accroissements indépendants et stationnaires, mais c'est la nature différente de ces accroissements qui fait leurs particularités. Le mouvement brownien réel standard que nous venons de définir possède des accroissements *additifs* qui sont indépendants pour l'*indépendance classique*. Nous pouvons généraliser cette définition dans deux directions. D'une part, remplacer les accroissements *additifs* par des accroissements *multiplicatifs*, d'autre part remplacer l'indépendance *classique* par la liberté, encore appelée indépendance *libre*.

Que se passe-t-il si nous imposons au processus des accroissements *multiplicatifs* indépendants et stationnaires ? Le morphisme $\mathbf{e} : x \mapsto e^{ix}$ de $\mathbb{R}$ vers le cercle unité $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, et qui transforme l'addition en multiplication, nous permet de répondre simplement à cette question. Notons $\mathbf{e}_*(\mathcal{N}_t)$ la mesure image d'une gaussienne $\mathcal{N}_t$ par $\mathbf{e}$, caractérisée par l'intégrale de toute fonction borélienne bornée f sur $\mathbb{U}$:

$$\int_{\mathbb{U}} f \mathrm{d}(\mathbf{e}_*(\mathcal{N}_t)) = \int_{\mathbb{R}} f(e^{ix}) \mathcal{N}_t(\mathrm{d}x).$$

Prenons un mouvement brownien réel standard $(W_t)_{t\in\mathbb{R}_+}$. Le processus $(e^{iW_t})_{t\in\mathbb{R}_+}$ est un mouvement brownien sur $\mathbb{U}$ au sens suivant.

DÉFINITION 1.2. Un *mouvement brownien sur* $\mathbb{U}$ est un processus stochastique $(V_t)_{t\in\mathbb{R}_+}$, à valeur dans $\mathbb{U}$, tel que :

(1) $V_0 = 1$;
(2) pour tous temps $0 \leq s < t$, l'accroissement $V_t V_s^{-1}$ a pour loi $\mathbf{e}_*(\mathcal{N}_{t-s})$;
(3) pour tous temps $0 \leq t_1 < \cdots < t_n$, les accroissements $V_{t_1}, V_{t_2}V_{t_1}^{-1}, \ldots, V_{t_n}V_{t_{n-1}}^{-1}$ sont indépendants ;
(4) $(V_t)_{t\in\mathbb{R}_+}$ est presque sûrement continu.

1.2. Espace de probabilité non-commutatif. Nous allons maintenant remplacer l'indépendance classique par l'indépendance libre. Pour cela, nous devons considérer la théorie des probabilités non-commutatives. Cette théorie a été créée par Dan-Virgil Voiculescu[76] dans les années 80 pour étudier certaines algèbres d'opérateurs en théorie des groupes libres, en utilisant le formalisme de la mécanique quantique. Présentons brièvement cette théorie, variante non-commutative des probabilités classiques.

Un *espace de probabilité non-commutatif* est la donnée d'une algèbre unitaire $\mathcal{A}$ munie d'une involution antilinéaire $A \mapsto A^*$ et d'une trace positive τ. En d'autres termes, τ est une forme linéaire telle que $\tau(1_{\mathcal{A}}) = 1$, pour tous $A, B \in \mathcal{A}$, $\tau(AB) = \tau(BA)$ (tracialité), et pour tout $A \in \mathcal{A}$, $\tau(A^*A) \geq 0$ (positivité). Les éléments de $\mathcal{A}$ sont appelés *variables aléatoires non-commutatives*. Remarquons qu'étant donné un espace de probabilité classique $(\Omega, \mathbb{P})$, l'algèbre $L^\infty(\Omega, \mathbb{P})$ munie de l'espérance $\mathbb{E}$ est un espace de probabilité non-commutatif.

Selon le degré de complexité de l'algèbre $\mathcal{A}$, il est possible d'y définir un calcul fonctionnel de plus en plus fin. Le calcul polynomial est le calcul fonctionnel le plus basique. Si $\mathcal{A}$ est une C^*-algèbre et τ est continue, on peut y définir un calcul fonctionnel continu pour tout élément *normal*, c'est-à-dire tel que $AA^* = A^*A$. On dit alors que $(\mathcal{A}, \tau)$ est un *C^*-espace de probabilité*. Un C^*-espace de probabilité est un *W^*-espace de probabilité* si A est une algèbre de von Neumann et τ est continue pour la topologie ultra-faible. Nous pouvons alors y définir un calcul borélien borné pour tout élément normal. Dans ce qui suit, les variables aléatoires non-commutatives seront considérées dans un même W^*-espace de probabilités non-commutatif $(\mathcal{A}, \tau)$.

La condition de positivité imposée à τ permet d'associer une distribution à ces variables aléatoires non-commutatives. Prenons une variable normale $A \in \mathcal{A}$. Il existe une unique mesure μ_A sur $\mathbb{C}$ telle que, pour tout polynôme P à deux variables,

$$\tau(P(A, A^*)) = \int_{\mathbb{C}} P(z, \bar{z}) \mathrm{d}\mu_A(z).$$

La mesure μ_A est appelée *distribution* de A. Bien entendu, on a alors plus généralement pour toute fonction borélienne f bornée sur le spectre de A,

$$\tau(f(A)) = \int_{\mathbb{C}} f(z) \mathrm{d}\mu_A(z).$$

Si A est auto-adjointe, c'est-à-dire que $A = A^*$, cette distribution est une mesure sur $\mathbb{R}$, et si A est unitaire, c'est-à-dire que $A^* = A^{-1}$, elle est concentrée sur $\mathbb{U}$. Dans le cadre de notre précédent exemple, la distribution d'une variable aléatoire de $L^\infty(\Omega, \mathbb{P})$ coïncide avec sa distribution au sens des probabilités classiques[1]. Cette notion de distribution est généralisée de la façon suivante.

[76] D.-V. VOICULESCU, Symmetries of some reduced free product C*-algebras (1985).

[1] Remarquons que si nous gardons dans une certaine mesure la notion de distribution en probabilités non-commutatives, nous perdons en revanche toute notion de réalisation.

Soit I un ensemble d'indices arbitraire et $\mathbb{C}\langle X_i : i \in I\rangle$ l'algèbre des polynômes à variables non-commutatives indexées par I. La *distribution*[2] d'une famille $\mathbf{A} = (A_i)_{i\in I}$ de variables aléatoires non-commutatives est la forme linéaire sur $\mathbb{C}\langle X_i : i \in I\rangle$ donnée par

$$\mu_{\mathbf{A}}(P) = \tau(P(\mathbf{A})).$$

Nous arrivons à la définition de l'indépendance libre, notion centrale dans cette thèse.

DÉFINITION 1.3. Soit I un ensemble d'indices. Pour tout $i \in I$, soit $\mathcal{B}_i$ une sous-algèbre de $\mathcal{A}$. Ces algèbres sont dites *librement indépendantes*, ou *libres*, si pour tout $n \in \mathbb{N}^*$, et tous indices $i_1 \neq \ldots \neq i_n$, les conditions $A_j \in B_{i_j}$ et $\tau(A_j) = 0$ pour tout $1 \leq j \leq n$ entraînent que $\tau(A_1 \cdots A_n) = 0$.

Cette notion d'indépendance est un analogue non-commutatif de l'indépendance classique, pour différentes raisons que nous n'expliquerons pas ici. Naturellement, une famille de variables aléatoires non-commutatives est dite libre si les algèbres respectivement engendrées par ces différentes variables aléatoires sont libres.

1.3. Les mouvements browniens libres. Nous sommes désormais presque en mesure d'énoncer la définition du mouvement brownien réel libre. Tout d'abord, il nous faut définir la mesure semi-circulaire, qui est l'analogue libre de la gaussienne[3]. Pour tout réel $t > 0$, la mesure *semi-circulaire* $\mathcal{S}_t$ de variance t est la mesure dont la densité par rapport à la mesure de Lebesgue est

$$\mathcal{S}_t(\mathrm{d}x) = \frac{1}{2\pi t}\sqrt{4a - x^2} \cdot 1_{[-2\sqrt{t},2\sqrt{t}]}(x)\mathrm{d}x.$$

Remarquons qu'une particularité remarquable qui distingue la mesure semi-circulaire de la mesure gaussienne est qu'elle est supportée par un compact. Nous pouvons transposer directement la définition du mouvement brownien réel standard dans ce cadre libre, à l'exception de la condition de continuité des trajectoires. En effet, cette notion n'a pas d'équivalent en probabilités libres.

DÉFINITION 1.4. Un *mouvement brownien réel libre* est une famille $(X_t)_{t\in\mathbb{R}_+}$ de variables aléatoires non-commutatives de $\mathcal{A}$, auto-adjointes, telle que :

(1) $X_0 = 0$;
(2) pour tous temps $0 \leq s < t$, l'accroissement $X_t - X_s$ a pour distribution une mesure semi-circulaire $\mathcal{S}_{t-s}$;
(3) pour tous temps $0 \leq t_1 < \cdots < t_n$, les accroissements $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ sont libres.

Reste donc à définir le mouvement brownien multiplicatif libre. Soit $(X_t)_{t\in\mathbb{R}_+}$ un mouvement brownien réel libre. Malheureusement, notre morphisme $\mathbf{e} : x \mapsto e^{ix}$ de $\mathbb{R}$ vers $\mathbb{U}$ ne nous permet pas de définir un processus à accroissements multiplicatifs stationnaires et librement indépendants. En effet, l'absence de commutativité nous empêche d'écrire un accroissement multiplicatif $e^{iX_t}e^{-iX_s}$ comme l'image $e^{i(X_t-X_s)}$ de l'accroissement $X_t - X_s$ par la fonction $\mathbf{e}$.

Reprenons notre exemple dans le cas classique. Soit $(W_t)_{t\in\mathbb{R}_+}$ un mouvement brownien réel standard. Le processus $(V_t)_{t\in\mathbb{R}_+} = (e^{iW_t})_{t\in\mathbb{R}_+}$ est un mouvement brownien sur $\mathbb{U}$. Une

[2] Pour une variable normale, nous identifions donc sa distribution avec la forme linéaire donnée par l'intégration des polynômes par rapport à cette distribution.

[3] La mesure semi-circulaire est la mesure limite qui apparait dans le théorème central limite lorsque l'indépendance libre remplace l'indépendance classique.

	Cas additif	**Cas multiplicatif**
Probabilités classiques	Brownien $(W_t)_{t\in\mathbb{R}_+}$	Brownien unitaire $(V_t)_{t\in\mathbb{R}_+}$
Probabilités libres	Brownien libre $(X_t)_{t\in\mathbb{R}_+}$	Brownien unitaire libre $(U_t)_{t\in\mathbb{R}_+}$

TABLE 1. Les quatre mouvements browniens

application de la formule d'Itô nous permet d'affirmer que $(V_t)_{t\in\mathbb{R}_+}$ est en fait la solution de l'équation différentielle stochastique linéaire suivante :

$$\left\{\begin{array}{lcl} V_0 & = & 1, \\ \mathrm{d}V_t & = & i\mathrm{d}W_t V_t - \frac{1}{2}V_t\mathrm{d}t. \end{array}\right.$$

Philippe Biane et Roland Speicher ont développé la notion de calcul stochastique libre[21], ce qui nous permet d'écrire l'équation correspondante et d'affirmer que l'équation différentielle stochastique libre

$$\left\{\begin{array}{lcl} U_0 & = & 1, \\ \mathrm{d}U_t & = & i\mathrm{d}X_t U_t - \frac{1}{2}U_t\mathrm{d}t \end{array}\right.$$

possède une unique solution dans le W^*-espace de probabilité $\mathcal{A}$. Pour tout temps t, U_t est unitaire, et sa distribution sera notée $\mathcal{B}_t$. Le processus $(U_t)_{t\in\mathbb{R}_+}$ est un mouvement brownien unitaire libre au sens suivant.

DÉFINITION 1.5. Le *mouvement brownien unitaire libre* est une famille $(U_t)_{t\in\mathbb{R}_+}$ de variables aléatoires non-commutatives de $\mathcal{A}$, unitaires, telle que :

(1) $U_0 = 1_{\mathcal{A}}$;
(2) pour tout temps $0 \leq s < t$, l'accroissement $U_t U_s^{-1}$ a pour loi $\mathcal{B}_t$;
(3) pour tout temps $0 \leq t_1 < \cdots < t_n$, les accroissements $U_{t_1}, U_{t_2}U_{t_1}^{-1}, \ldots, U_{t_n}U_{t_{n-1}}^{-1}$ sont indépendants.

Il n'existe malheureusement pas d'expression simple de la distribution $\mathcal{B}_t$ de U_t. Nous connaissons en revanche l'expression de ses moments : pour tout $t > 0$ et $n \in \mathbb{Z}$,

$$\int_{\mathbb{U}} z^n \mathrm{d}\mathcal{B}_t(z) = \tau(U_t^n) = e^{-\frac{t}{2}|n|} \sum_{k=0}^{|n|-1} \frac{(-t)^k}{k!} |n|^{k-1} \binom{|n|}{k+1}.$$

Nous pouvons résumer la situation dans le tableau 1. Nous avons déjà vu que cette double correspondance somme/produit d'une part et indépendance/liberté d'autre part n'est pas parfaite, et qu'il faut adapter notre point de vue sur les objets considérés pour obtenir, parfois très indirectement, les objets analogues dans chacune des situations. Précisons que l'objectif n'est pas de développer de façon parallèle quatre théories différentes. En réalité, il existe d'étroits lien entre ces quatre situations. Le lien principal qui va nous intéresser, et qui fait l'objet de la section suivante, s'exprime par le biais de l'étude asymptotique de matrices aléatoires.

1.4. Matrices aléatoires. Un résultat d'Eugene Wigner[79] stipule que la loi spectrale d'une matrice hermitienne dont les coefficients sont des gaussiennes de variances bien choisies converge vers la mesure semi-circulaire lorsque la dimension de la matrice tend vers l'infini. Dans les années 1990, Dan-Virgil Voiculescu[75] approfondit ce lien entre probabilité classique et probabilité libre lorsqu'il montre que deux de ces matrices, si elles sont indépendantes, sont asymptotiquement libres.

	Cas additif		Cas multiplicatif
Probabilités classiques	Brownien hermitien $(X_t^{(N)})_{t\in\mathbb{R}_+}$	calcul $\longrightarrow$ stochastique	Brownien unitaire $(U_t^{(N)})_{t\in\mathbb{R}_+}$
	$\downarrow N\to\infty$		$\downarrow N\to\infty$
Probabilités libres	Brownien libre $(X_t)_{t\in\mathbb{R}_+}$	calcul $\longrightarrow$ stochastique	Brownien unitaire libre $(U_t)_{t\in\mathbb{R}_+}$

Table 2. La limite des browniens matriciels

Présentons ce résultat. Pour tout $N\in\mathbb{N}^*$, soit $M_N(\mathbb{C})$ l'espace des matrices carrées de dimension N, et $\mathcal{H}_N\subset M_N(\mathbb{C})$ l'espace des matrices *hermitiennes*, c'est-à-dire des matrices H telles que $H=H^*$. Nous noterons $\mathrm{Tr}(\cdot)$ la trace usuelle et $\mathrm{tr}(\cdot)=\frac{1}{N}\mathrm{Tr}(\cdot)$ la trace normalisée telle que $\mathrm{tr}(I_N)=1$. Nous allons considérer un mouvement brownien sur $\mathcal{H}_N$, et pour cela nous fixons un produit scalaire $\langle\cdot,\cdot\rangle_{\mathcal{H}_N}$ sur cet espace :

$$\forall A,B\in\mathcal{H}_N, \langle A,B\rangle_{\mathcal{H}_N}=N\,\mathrm{Tr}(A^*B).$$

Considérons un *mouvement brownien sur l'espace des matrices hermiciennes* muni de ce produit scalaire. C'est le processus gaussien $(X_t^{(N)})_{t\in\mathbb{R}_+}$ à valeurs dans $\mathcal{H}_N$ de covariance suivante[4] :

$$\forall s,t\in\mathbb{R}_+,\forall A,B\in\mathcal{H}_N,\mathbb{E}\left[\langle A,X_s^{(N)}\rangle_{\mathcal{H}_N}\cdot\langle B,X_t^{(N)}\rangle_{\mathcal{H}_N}\right]=s\wedge t\cdot\langle A,B\rangle_{\mathcal{H}_N}.$$

Théorème[75]. *Soit $(X_t)_{t\in\mathbb{R}_+}$ un mouvement brownien réel libre. Le processus $(X_t^{(N)})_{t\in\mathbb{R}_+}$ converge vers $(X_t)_{t\in\mathbb{R}_+}$ en distribution non-commutative. En d'autres termes, pour tout entier $n\geq 1$, pour tout polynôme non-commutatif P en n variables, et tout choix de n temps positifs $t_1,\dots,t_n$, nous avons la convergence suivante :*

$$\lim_{N\to\infty}\mathbb{E}\left[\mathrm{tr}\left(P\left(X_{t_1}^{(N)},\dots,X_{t_n}^{(N)}\right)\right)\right]=\tau(P(X_{t_1},\dots,X_{t_n})).$$

Le pendant multiplicatif de ce résultat s'obtient en considérant un mouvement brownien sur le *groupe unitaire* $U(N)\subset M_N(\mathbb{C})$ composé des matrices unitaires, c'est-à-dire des matrices U telles que $U^*U=I_N$. Encore une fois, nous pouvons construire ce mouvement brownien multiplicatif à partir d'un mouvement brownien additif à l'aide du calcul stochastique. En effet, en reprenant le processus $(X_t^{(N)})_{t\in\mathbb{R}_+}$ à valeurs dans les matrices hermitiennes, l'équation différentielle stochastique multidimensionnelle

$$\begin{cases} U_0^{(N)} &= I_N,\\ \mathrm{d}U_t^{(N)} &= i\mathrm{d}X_t^{(N)}U_t^{(N)}-\frac{1}{2}U_t^{(N)}\mathrm{d}t\end{cases}$$

[4] De manière équivalente, soit $(B_{k,l},C_{k,l},D_k)_{k,l\geq 1}$ des mouvements browniens réels standards indépendants entre eux. Alors $X_t^{(N)}$ a la même distribution que la matrice aléatoire hermitienne dont les coefficients diagonaux sont les $\frac{1}{\sqrt{N}}D_k(t)$ et les coefficients au-dessus de la diagonale sont les $\frac{1}{\sqrt{2N}}(B_{k,l}(t)+iC_{k,l}(t))$.

[21] P. Biane et R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space (1998).

[75] D. Voiculescu, Limit laws for random matrices and free products (1991).

[79] E. P. Wigner, On the Distribution of the Roots of Certain Symmetric Matrices (1958).

a une unique solution $(U_t)_{t\in\mathbb{R}_+}$ à valeur dans $U(N)$. C'est en fait une des définitions du *mouvement brownien sur le groupe unitaire*. Le résultat suivant, dû à Philippe Biane, complète la situation décrite dans le tableau 2.

THÉORÈME[17]. *Soit $(U_t)_{t\in\mathbb{R}_+}$ un mouvement brownien unitaire libre. Alors, $(U_t^{(N)})_{t\in\mathbb{R}_+}$ converge vers $(U_t)_{t\in\mathbb{R}_+}$ en distribution non-commutative. En d'autres termes, pour tout entier $n \geq 1$, pour tout polynôme non-commutatif P en n variables, et tout choix de n temps positifs $t_1, \ldots, t_n$, nous avons la convergence suivante :*

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(P\left(U_{t_1}^{(N)}, \ldots, U_{t_n}^{(N)}\right)\right)\right] = \tau(P(U_{t_1}, \ldots, U_{t_n})).$$

L'analogie qui existe entre les probabilités classiques et les probabilités libres n'est donc pas uniquement formelle. Elle se concrétise au travers des matrices aléatoires. C'est dans cette démarche que s'inscrit la plupart des résultats de cette thèse : obtenir les structures définies dans le cadre non-commutatif comme limite en grande dimension de ces mêmes structures dans le cadre matriciel.

2. Transformation de Segal-Bargmann-Hall sur le groupe unitaire en grande dimension

La transformation de Segal–Bargmann est un outil mathématique important en mécanique quantique et en théorie des champs quantiques. Elle a été développée par Valentine Bargmann[8] et Irving Segal[66] dans les années 60. Nous allons en présenter ici les quatre versions correspondant aux quatre situations déjà décrites.

2.1. Transformation de Segal-Bargmann classique et libre. Fixons $t > 0$. La transformation de Segal-Bargmann opère sur l'espace $L^2(\mathbb{R}, \mathcal{N}_t)$ des fonctions de carré intégrable par rapport à une gaussienne $\mathcal{N}_t$ de variance t. Pour toute fonction $f \in L^2(\mathbb{R}, \mathcal{N}_t)$, la fonction

$$z \mapsto \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(z) e^{-\frac{(x-z)^2}{2t}} \mathrm{d}x,$$

définie sur $\mathbb{R}$ en convolant f avec $\mathcal{N}_t$, peut en fait s'étendre en une fonction entière $S_t(f)$ sur $\mathbb{C}$, donnée par la même formule. Cette fonction entière $S_t(f)$ est intégrable par rapport à une gaussienne complexe de variance t, c'est-à-dire la mesure $\mathcal{N}_t^{\mathbb{C}}$ dont la densité par rapport à la mesure de Lebesgue est

$$\mathcal{N}_t^{\mathbb{C}}(\mathrm{d}z) = \frac{1}{\pi t} e^{-\frac{|z|^2}{t}} \mathrm{d}z.$$

L'image de f par la transformation de Segal-Bargmann est par définition cette fonction $S_t(f)$. La propriété importante de la transformation de Segal-Bargmann est d'être un isomorphisme d'espaces de Hilbert entre $L^2(\mathbb{R}, \mathcal{N}_t)$ et le sous-espace fermé de $L^2(\mathbb{C}, \mathcal{N}_t^{\mathbb{C}})$ composé des fonctions entières[5].

Nous allons prendre ici un point de vue légèrement différent, qui n'est probablement pas le meilleur pour aborder la transformation de Segal-Bargmann, mais qui a l'avantage de mettre en valeur le parallélisme entre le monde classique et le monde libre. Prenons une variable aléatoire

[8] V. BARGMANN, On a Hilbert space of analytic functions and an associated integral transform. Part I (1961).

[17] P. BIANE, Free Brownian motion, free stochastic calculus and random matrices (1997).

[66] I. E. SEGAL, Mathematical characterization of the physical vacuum for a linear Bose-Einstein field (1962).

[5] La transformation de Segal-Bargmann a été initialement introduite pour entrelacer deux représentations des relations canoniques de commutation entre position et moment en mécanique quantique. La principale raison réside dans le fait que S_t échange les opérateurs ∂_x et $x - t\partial_x$ contre les opérateurs ∂_z et z.

W_t de distribution $\mathcal{N}_t$. L'espace $L^2(\mathbb{R}, \mathcal{N}_t)$ est alors isomorphe en tant qu'espace de Hilbert à l'espace de variables aléatoires

$$L^2(W_t) = \{f(W_t) : f \in L^2(\mathbb{R}, \mathcal{N}_t)\}.$$

Prenons une variable aléatoire $W_t^{\mathbb{C}}$ de distribution $\mathcal{N}_t^{\mathbb{C}}$. Définissons l'espace de Hilbert

$$L^2_{\mathrm{hol}}(W_t^{\mathbb{C}}) = \{f(W_t^{\mathbb{C}}) : f \text{ fonction entière de } L^2(\mathbb{C}, \mathcal{N}_t^{\mathbb{C}})\}.$$

Dans le théorème qui suit et son corollaire, nous présentons deux visions probabilistes possibles de la transformation de Segal-Bargmann.

Théorème[8]. *Soit $f \in L^2(\mathbb{R}, \mathcal{N}_t)$. La fonction $z \mapsto \mathbb{E}[f(z + W_t)]$ s'étend en une fonction entière F telle que $F \in L^2(\mathbb{C}, \mathcal{N}_t^{\mathbb{C}})$. De plus, la transformation linéaire*

$$S_t : f(W_t) \mapsto F(W_t^{\mathbb{C}})$$

est un isomorphisme d'espaces de Hilbert entre $L^2(W_t)$ et $L^2_{\mathrm{hol}}(W_t^{\mathbb{C}})$ que nous appellerons la transformation de Segal-Bargmann.

Corollaire 2.1. *Soit W_t une variable gaussienne réelle et $W_t^{\mathbb{C}}$ une variable gaussienne complexe de variance t qui sont indépendantes entre elles. La transformation linéaire*

$$P(W_t) \mapsto \mathbb{E}\left[P(W_t + W_t^{\mathbb{C}}) \,\middle|\, W_t^{\mathbb{C}}\right]$$

est une isométrie qui s'étend en un unique isomorphisme d'espaces de Hilbert entre $L^2(W_t)$ et $L^2_{\mathrm{hol}}(W_t^{\mathbb{C}})$. De plus, cet isomorphisme est la transformation de Segal-Bargmann.

Dans la suite de ce document, nous avons choisi de conserver ce point de vue probabiliste, où les fonctions boréliennes de carré intégrable par rapport à une mesure sont vues comme des variables aléatoires. Bien que cette formulation ne soit pas celle couramment utilisée, elle lui est équivalente et présente un avantage : les énoncés analogues en probabilités libres s'expriment plus facilement dans ce formalisme probabiliste. Il y a deux raisons pour cela. D'une part, la convolution libre d'une fonction avec une mesure n'a pas de sens, au contraire de la convolution classique d'une fonction avec une mesure[6]. D'autre part, la distribution d'une famille de variables aléatoires non-commutatives n'est en général pas une mesure, ce qui rend impossible l'expression analytique de la transformation de Segal-Bargmann libre à plusieurs variables.

Présentons maintenant la transformation de Segal-Bargmann libre sur $\mathbb{R}$. L'analogue libre de la gaussienne complexe est la variable circulaire. De la même façon qu'une gaussienne complexe est la somme de deux gaussiennes indépendantes, une variable aléatoire non-commutative $Z_t \in \mathcal{A}$ est dite *circulaire* de variance t si $(Z_t + Z_t^*)/\sqrt{2}$ et $(Z_t - Z_t^*)/(\sqrt{2}i)$ sont deux variables libres de distribution semi-circulaire de variance t. Plus généralement, un *mouvement brownien circulaire libre* est une famille $(Z_t)_{t \geq 0}$ de variables aléatoires non-commutatives de $\mathcal{A}$ telles que $(1/\sqrt{2})(Z_t + Z_t^*)_{t\geq 0}$ et $(1/\sqrt{2}i)(Z_t - Z_t^*)_{t \geq 0}$ sont deux mouvements browniens libres qui sont libres entre eux.

Soit X_t une variable aléatoire non-commutative semi-circulaire, et Z_t une variable aléatoire non-commutative circulaire. Les espaces $L^2(X_t, \tau)$ et $L^2_{\mathrm{hol}}(Z_t, \tau)$ désigneront respectivement les espaces de Hilbert complétés de la $*$-algèbre engendrée par X_t et de l'algèbre engendrée par Z_t pour la norme $\| \cdot \|_2 : A \mapsto \tau(A^* A)^{1/2}$. La transformation de Segal-Bargmann libre, définie par Philippe Biane[19], est un isomorphisme entre $L^2(X_t, \tau)$ et $L^2_{\mathrm{hol}}(Z_t, \tau)$. Pour la décrire,

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[6] Cela est dû au fait que la convolution libre des mesures n'est pas linéaire.

[19] P. Biane, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems (1997).

introduisons les polynômes de Tchebycheff $(T_n^t)_{n\in\mathbb{N}}$ de deuxième espèce, entièrement déterminés par leur fonction génératrice : pour tout $|z| < 1$ et $-2 < x < 2$,

$$\sum_{n=0}^{\infty} z^n T_n^t(x) = \frac{t}{t - xz + z^2}.$$

Soit $\mathcal{U}_t$ l'endomorphisme de $\mathbb{C}[X]$ défini pour tout $n \in \mathbb{N}$ par $\mathcal{U}_t(T_n^t) = X^n$.

Théorème[19]. *La transformation linéaire $P(X_t) \mapsto (\mathcal{U}_t(P))(Z_t)$ est une isométrie qui s'étend en un unique isomorphisme d'espaces de Hilbert entre $L^2(X_t, \tau)$ et $L^2_{\text{hol}}(Z_t, \tau)$ appelée la transformation de Segal-Bargmann libre.*

Nous proposons une description alternative de la transformation de Segal-Bargmann libre à l'aide de l'espérance conditionnelle. Si $\mathcal{B} \subset \mathcal{A}$ est une sous-algèbre de von Neumann, il existe une unique *espérance conditionnelle* de $\mathcal{A}$ vers $\mathcal{B}$ par rapport à τ que l'on dénotera $\tau(\cdot|\mathcal{B})$. C'est une contraction faiblement continue, complètement positive, qui préserve l'identité, et qui est caractérisée par la propriété suivante : pour tout $A \in \mathcal{A}$ et $B \in \mathcal{B}$, $\tau(AB) = \tau(\tau(A|\mathcal{B})B)$. Bien entendu, si $\mathbf{A} = (A_i)_{i\in I}$ est une famille de variables aléatoires non-commutatives, $\tau(\cdot|\mathbf{A})$ désignera l'espérance conditionnelle de $\mathcal{A}$ vers l'algèbre de von Neumann engendrée par les variables $(A_i)_{i\in I}$. La description de la transformation de Segal-Bargmann libre du théorème suivant est très proche de la description classique de la transformation de Segal-Bargmann telle que formulée dans le corollaire 2.1.

Théorème (cf. Th.10.1). *Soit X_t une variable semi-circulaire et Z_t une variable circulaire de variance t et qui sont libres entre elles. La transformation linéaire*

$$P(X_t) \mapsto \tau\left(P(X_t + Z_t)\,|Z_t\right)$$

est une isométrie qui s'étend en un unique isomorphisme d'espaces de Hilbert entre $L^2(X_t, \tau)$ et $L^2_{\text{hol}}(Z_t, \tau)$. De plus, cet isomorphisme est la transformation de Segal-Bargmann libre.

La procédure décrite précédemment pour construire la transformation de Segal-Bargmann classique sur $\mathbb{R}$ fonctionne en remplaçant $\mathbb{R}$ par un espace de Hilbert quelconque et $\mathbb{C}$ par son complexifié. Nous allons prendre comme cas particulier l'espace $\mathcal{H}_N$ des matrices hermitiennes. Le complexifié de $\mathcal{H}_N$ est l'espace $M_N(\mathbb{C})$ des matrices complexes. Le mouvement brownien que l'on considère sur $M_N(\mathbb{C})$ est le processus $(Z_t^{(N)})_{t\geq 0}$ tel que $(1/\sqrt{2})(Z_t^{(N)} + Z_t^{(N)^*})_{t\geq 0}$ et $(1/\sqrt{2}i)(Z_t^{(N)} - Z_t^{(N)^*})_{t\geq 0}$ sont deux mouvements browniens sur $\mathcal{H}_N$ indépendants entre eux[7]. Soit $(X_t^{(N)})_{t\in\mathbb{R}_+}$ un mouvement brownien sur $\mathcal{H}_N$. Posons

$$L^2(X_t^{(N)}) = \left\{ f(X_t^{(N)}) : f \text{ fonction borélienne sur } \mathcal{H}_N \text{ telle que } \mathbb{E}\left[\left|f(X_t^{(N)})\right|^2\right] < +\infty \right\}.$$

Soit $(Z_t^{(N)})_{t\in\mathbb{R}_+}$ un mouvement brownien sur $M_N(\mathbb{C})$. Posons

$$L^2_{\text{hol}}(Z_t^{(N)}) = \left\{ F(Z_t^{(N)}) : F \text{ fonction holomorphe telle que } \mathbb{E}\left[\left|F(Z_t^{(N)})\right|^2\right] < +\infty \right\}.$$

Théorème[8]. *Soit f une fonction borélienne sur $\mathcal{H}_N$ telle que $f(X_t^{(N)}) \in L^2(X_t^{(N)})$. La fonction $z \mapsto \mathbb{E}[f(z + X_t^{(N)})]$ s'étend en une fonction entière F telle que $F(Z_t^{(N)}) \in L^2_{\text{hol}}(Z_t^{(N)})$. De plus, la transformation linéaire*

$$S_t : f(X_t^{(N)}) \mapsto F(Z_t^{(N)})$$

[7] De manière équivalente, soit $(B_{k,l}, C_{k,l})_{k,l\geq 1}$ des mouvements browniens réels standards indépendants entre eux. Alors $Z_t^{(N)}$ a la même distribution que la matrice aléatoire dont les coefficients sont les $\frac{1}{\sqrt{2N}}(B_{k,l}(t) + iC_{k,l}(t))$.

est un isomorphisme d'espaces de Hilbert entre $L^2(X_t^{(N)})$ *et* $L^2_{\mathrm{hol}}(Z_t^{(N)})$ *que nous appellerons la transformation de Segal-Bargmann.*

Pour étudier la transformation de Segal-Bargmann S_t en grande dimension, Philippe Biane[19] a l'idée de considérer des variables aléatoires matricielles plutôt que des variables aléatoires scalaires. En 1997, il tensorise S_t en $S_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ dans la perspective d'étudier l'action de $S_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ sur les variables aléatoires matricielles obtenues par le calcul fonctionnel des matrices. Plus précisément, munissons $M_N(\mathbb{C})$ du produit scalaire $\langle X, Y\rangle_{M_N(\mathbb{C})} = \mathrm{Tr}(X^*Y)$. Pour tout $N > 0$, identifions l'espace de Hilbert

$$\left\{ f(X_t^{(N)}) : f \text{ fonction borélienne de } \mathcal{H}_N \text{ vers } M_N(\mathbb{C}) \text{ telle que } \mathbb{E}\left[\left\| f(X_t^{(N)})\right\|^2_{M_N(\mathbb{C})}\right] < +\infty \right\}$$

avec $L^2(X_t^{(N)}) \otimes M_N(\mathbb{C})$ et l'espace de Hilbert

$$\Big\{ F(Z_t^{(N)}) : F \text{ fonction holomorphe de } M_N(\mathbb{C}) \text{ vers } M_N(\mathbb{C}) \\ \text{telle que } \mathbb{E}\left[\left\| F(Z_t^{(N)})\right\|^2_{M_N(\mathbb{C})}\right] < +\infty \Big\}$$

avec $L^2_{\mathrm{hol}}(Z_t^{(N)}) \otimes M_N(\mathbb{C})$. Nous noterons $S_t^{(N)}$ l'isomorphisme $S_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ de $L^2(X_t^{(N)}) \otimes M_N(\mathbb{C})$ vers $L^2_{\mathrm{hol}}(Z_t^{(N)}) \otimes M_N(\mathbb{C})$. Biane parvient à montrer que cette transformation $S_t^{(N)}$ approche la transformation de Segal-Bargmann libre en grande dimension. Nous allons présenter une version édulcorée de son résultat où nous ne prenons en compte que les variables aléatoires obtenues par le calcul polynomial sur les matrices. Intuitivement, nous pouvons interpréter ce résultat de la façon suivante :

(1) le sous-espace $\left\{P(X_t^{(N)})\right\}_{P\in\mathbb{C}[X]} \subset L^2(X_t^{(N)}) \otimes M_N(\mathbb{C})$ converge vers $L^2(X_t, \tau)$;
(2) le sous-espace $\left\{P(Z_t^{(N)})\right\}_{P\in\mathbb{C}[X]} \subset L^2_{\mathrm{hol}}(Z_t^{(N)}) \otimes M_N(\mathbb{C})$ converge vers $L^2_{\mathrm{hol}}(Z_t, \tau)$;
(3) asymptotiquement, la transformation de Segal-Bargmann $S_t^{(N)}$ envoie isométriquement ces deux espaces l'un sur l'autre, et l'isométrie limite est la transformation de Segal-Bargmann libre de $L^2(X_t, \tau)$ vers $L^2_{\mathrm{hol}}(Z_t, \tau)$.

Alors que les deux premiers points sont une application directe du résultat de Voiculescu, le troisième point est original aussi bien dans sa forme même que dans sa preuve. Voici maintenant une version plus rigoureuse de ces différents points.

THÉORÈME[19]. *Soit* $t > 0$.

(1) Pour tout $n \in \mathbb{N}$, *et tout polynôme* $P \in \mathbb{C}[X]$, *nous avons*

$$\lim_{N\to\infty} \left\| P\left(X_t^{(N)}\right)\right\|^2_{L^2(X_t^{(N)})\otimes M_N(\mathbb{C})} = \left\| P\left(X_t\right)\right\|^2_{L^2(X_t,\tau)} ;$$

(2) pour tout $n \in \mathbb{N}$, *et tout polynôme* $P \in \mathbb{C}[X]$, *nous avons*

$$\lim_{N\to\infty} \left\| P\left(Z_t^{(N)}\right)\right\|^2_{L^2_{\mathrm{hol}}(Z_t^{(N)})\otimes M_N(\mathbb{C})} = \left\| P\left(Z_t\right)\right\|^2_{L^2_{\mathrm{hol}}(Z_t,\tau)} ;$$

[8] V. BARGMANN, On a Hilbert space of analytic functions and an associated integral transform. Part I (1961).

[19] P. BIANE, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems (1997).

(3) Pour tout polynôme $P \in \mathbb{C}[X]$, nous avons lorsque $N \to \infty$

$$\left\| S_t^{(N)} \Big(P \left(X_t^{(N)} \right) \Big) - \mathcal{U}_t(P) \Big(Z_t^{(N)} \Big) \right\|^2_{L^2_{\mathrm{hol}}(Z_t^{(N)}) \otimes M_N(\mathbb{C})} = O(1/N^2).$$

En réalité, Biane parvient par un argument d'approximation à généraliser ces résultats pour des classes plus générales de fonctions.

2.2. Transformation de Hall classique et libre. En 1994, Brian Hall prouve que la transformation de Segal-Bargmann existe toujours lorsque l'on remplace l'espace de Hilbert par un groupe de Lie et son complexifié par le complexifié du groupe de Lie. Pour nous, la transformation de Hall fera donc intervenir d'une part un mouvement brownien sur $U(N)$, d'autre part un mouvement brownien sur le groupe linéaire $GL_N(\mathbb{C})$ qui est l'ensemble des matrices inversibles de $M_N(\mathbb{C})$.

Définissons le mouvement brownien sur $GL_N(\mathbb{C})$. C'est le processus matriciel $(G_t^{(N)})_{t \geq 0}$ solution de l'équation différentielle stochastique multidimensionnelle suivante

$$\begin{cases} G_0^{(N)} & = & I_N, \\ \mathrm{d}G_t^{(N)} & = & \mathrm{d}Z_t^{(N)} G_t^{(N)}. \end{cases}$$

Le processus $(G_t^{(N)})_{t \geq 0}$ est inversible en tout temps $t \geq 0$. Pour $t \geq 0$, nous noterons $L^2(U_t^{(N)})$ l'espace de Hilbert

$$\left\{ f \left(U_t^{(N)} \right) : f \text{ est une fonction borélienne sur } U(N) \text{ telle que } \mathbb{E} \left[\left| f \left(U_t^{(N)} \right) \right|^2 \right] < +\infty \right\},$$

et nous noterons $L^2_{\mathrm{hol}}(G_t^{(N)})$ l'espace de Hilbert

$$\left\{ F \left(G_t^{(N)} \right) : F \text{ est une fonction holomorphe sur } GL_N(\mathbb{C}) \text{ telle que } \mathbb{E} \left[\left| F \left(G_t^{(N)} \right) \right|^2 \right] < +\infty \right\}.$$

Le fait que $L^2_{\mathrm{hol}}(G_t^{(N)})$ soit un espace de Hilbert est non-trivial. C'est une partie du résultat de Hall qui peut être exprimé de la façon suivante.

Théorème[44]. *Soit f une fonction borélienne sur $U(N)$ telle que $f(U_t^{(N)}) \in L^2(U_t^{(N)})$. La fonction $U \mapsto \mathbb{E}\left[f \left(U_t^{(N)} U \right) \right]$ possède une extension analytique F sur $GL_N(\mathbb{C})$. De plus, $F(G_t^{(N)}) \in L^2_{\mathrm{hol}}(G_t^{(N)})$ et l'application linéaire*

$$B_t : f \left(U_t^{(N)} \right) \mapsto F \left(G_t^{(N)} \right)$$

est un isomorphisme d'espaces de Hilbert entre $L^2(U_t^{(N)})$ et $L^2_{\mathrm{hol}}(G_t^{(N)})$.

La transformation B_t est connue sous le nom de transformation de Segal-Bargmann-Hall, ou plus simplement transformation de Hall. Gross et Malliavin ont montré qu'il est possible de construire la transformation de Hall comme restriction d'une version infini-dimensionnelle de la transformation de Segal-Bargmann, ce qui établit un lien entre le cas additif et le cas classique. En transposant le raisonnement de Gross et Malliavin dans le cadre des probabilités libres, Biane définit la transformation de Hall libre comme restriction d'une version infini-dimensionnelle de la transformation de Segal-Bargmann libre.

Ici encore nous devons présenter ce qui jouera le rôle du "complexifié" du mouvement brownien unitaire libre. Soit $(Z_t)_{t \geq 0}$ un mouvement brownien circulaire libre. Définissons le *mouvement*

[44] B. C. Hall, The Segal-Bargmann "Coherent State" Transform for Compact Lie Groups (1994).

brownien multiplicatif circulaire libre[17]. C'est le processus non-commutatif $(G_t)_{t\geq 0}$ dans $\mathcal{A}$ solution de l'équation différentielle stochastique suivante

$$\begin{cases} G_0 &= \mathrm{Id}, \\ \mathrm{d}G_t &= \mathrm{d}Z_t G_t. \end{cases}$$

Le processus $(G_t)_{t\geq 0}$ est inversible[17] en tout temps $t \geq 0$, en revanche, il n'est pas normal, au sens où $G_t^* G_t \neq G_t G_t^*$.

Soit $(U_t)_{t\geq 0}$ un mouvement brownien unitaire libre, et $(G_t)_{t\geq 0}$ un mouvement brownien multiplicatif circulaire libre. Fixons $t \geq 0$. Les espaces $L^2(U_t, \tau)$ et $L^2_{\mathrm{hol}}(G_t, \tau)$ désigneront respectivement les espaces de Hilbert complétés de la $*$-algèbre engendrée par U_t et U_t^{-1} et de l'algèbre engendrée par Z_t et Z_t^{-1} pour la norme $\|\cdot\|_2 : A \mapsto \tau(A^*A)^{1/2}$. La transformation de Hall libre, définie par Philippe Biane[19], est un isomorphisme entre $L^2(U_t, \tau)$ et $L^2_{\mathrm{hol}}(G_t, \tau)$. Un moyen rapide de la définir est de décrire son action sur les polynômes. Introduisons les polynômes $(P_n^t)_{n\in\mathbb{N}}$, définis par la série génératrice

$$\sum_{n=0}^{\infty} z^n P_n^t(x) = \frac{1}{1 - z e^{\frac{t}{2}\left(\frac{1+z}{1-z}\right)} x}.$$

Nous désignerons l'espace des polynômes de Laurent par $\mathbb{C}[X, X^{-1}]$. Pour tout $t \geq 0$, soit $\mathcal{G}_t$ l'opérateur linéaire sur $\mathbb{C}[X, X^{-1}]$ tel que, pour tout $n \in \mathbb{N}$, $\mathcal{G}_t\left(P_n^t(X)\right) = X^n$ et $\mathcal{G}_t\left(P_n^t(X^{-1})\right) = X^{-n}$.

THÉORÈME[19]. *Soit $t > 0$. L'application $\mathcal{F}_t : P(U_t) \mapsto (\mathcal{G}_t(P))(G_t)$ pour tout $P \in \mathbb{C}[X, X^{-1}]$ est une isométrie qui s'étend en un isomorphisme $\mathcal{F}_t$ d'espace de Hilbert entre $L^2(U_t, \tau)$ et $L^2_{\mathrm{hol}}(G_t, \tau)$ appelée transformation de Hall libre.*

Dans le théorème suivant, nous donnons une description alternative de la transformation de Hall libre, description intrinsèque à rapprocher du corollaire 2.1.

THÉORÈME (CF. TH.10.7). *Soit $t > 0$. On suppose les variables U_t et G_t libres entre elles. La transformation linéaire*

$$P(U_t) \mapsto \tau\left(P(U_t G_t)\,|G_t\right)$$

est une isométrie qui s'étend en un unique isomorphisme d'espaces de Hilbert entre $L^2(U_t, \tau)$ et $L^2_{\mathrm{hol}}(G_t, \tau)$. De plus, cet isomorphisme est la transformation de Hall libre.

2.3. Résultat principal. Nous allons maintenant présenter notre résultat principal, qui fait le lien entre la transformation de Hall classique sur $U(N)$ et la transformation de Hall libre. Pour tout $N > 0$, identifions

$$\left\{ f\left(U_t^{(N)}\right) : f \text{ fonction borélienne de } U(N) \text{ vers } M_N(\mathbb{C}) \text{ telle que } \mathbb{E}\left[\left\| f\left(U_t^{(N)}\right)\right\|^2_{M_N(\mathbb{C})}\right] < +\infty \right\}$$

avec $L^2(U_t^{(N)}) \otimes M_N(\mathbb{C})$ et l'espace

$$\left\{ F\left(G_t^{(N)}\right) : F \text{ fonction holomorphe de } GL_N(\mathbb{C}) \text{ vers } M_N(\mathbb{C}) \text{ telle que } \mathbb{E}\left[\left\| F\left(G_t^{(N)}\right)\right\|^2_{M_N(\mathbb{C})}\right] < +\infty \right\}$$

[17] P. BIANE, Free Brownian motion, free stochastic calculus and random matrices (1997).
[19] P. BIANE, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems (1997).

	Cas additif	**Cas multiplicatif**
Probabilités classiques	Transformation de Segal-Bargmann $S_t^{(N)}$	Transformation de Hall $B_t^{(N)}$
	$\downarrow N \to \infty$	$\downarrow N \to \infty$
Probabilités libres	Transformation de Segal-Bargmann libre $\mathcal{U}_t$	Transformation de Hall libre $\mathcal{G}_t$

TABLE 3. Transformations de Segal-Bargmann-Hall

avec $L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})$. Nous noterons $B_t^{(N)}$ l'isomorphisme $B_t\otimes \mathrm{Id}_{M_N(\mathbb{C})}$ de $L^2(U_t^{(N)})\otimes M_N(\mathbb{C})$ vers $L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})$. En 1997, dans l'article présentant la transformation de Hall libre[19], Biane suggère qu'elle est la limite de la transformation de Hall sur $U(N)$ de la même manière que la transformation de Segal-Bargmann libre est la limite de la transformation de Segal-Bargmann sur $\mathcal{H}_N$:

(1) le sous-espace $\left\{P(U_t^{(N)})\right\}_{P\in\mathbb{C}[X]}\subset L^2(U_t^{(N)})\otimes M_N(\mathbb{C})$ converge vers $L^2(U_t,\tau)$;
(2) le sous-espace $\left\{P(G_t^{(N)})\right\}_{P\in\mathbb{C}[X]}\subset L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})$ converge vers $L^2_{\mathrm{hol}}(G_t,\tau)$;
(3) asymptotiquement, la transformation de Hall $B_t^{(N)}$ envoie isométriquement ces deux espaces l'un sur l'autre, et l'isométrie limite est la transformation de Hall libre de $L^2(U_t,\tau)$ vers $L^2_{\mathrm{hol}}(G_t,\tau)$.

Le premier point est un corollaire de la convergence en distribution du mouvement brownien unitaire[17]. Une des motivations principales de la présente thèse est de prouver que les deux autres points sont vrais, dans le sens direct et quantitatif du théorème suivant.

THÉORÈME (CF. TH.7.4). *Soit $t>0$.*

(1) Pour tout $n\in\mathbb{N}$, et tout polynôme $P\in\mathbb{C}[X,X^{-1}]$, nous avons

$$\lim_{N\to\infty}\left\|P\left(U_t^{(N)}\right)\right\|^2_{L^2(U_t^{(N)})\otimes M_N(\mathbb{C})}=\|P\left(U_t\right)\|^2_{L^2(U_t,\tau)}\,;$$

(2) pour tout $n\in\mathbb{N}$, et tout polynôme $P\in\mathbb{C}[X,X^{-1}]$, nous avons

$$\lim_{N\to\infty}\left\|P\left(G_t^{(N)}\right)\right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})}=\|P\left(G_t\right)\|^2_{L^2_{\mathrm{hol}}(G_t,\tau)}\,;$$

(3) Pour tout polynôme $P\in\mathbb{C}[X,X^{-1}]$, nous avons lorsque $N\to\infty$

$$\left\|B_t^{(N)}\Big(P\left(U_t^{(N)}\right)\Big)-(\mathcal{G}_t(P))\Big(G_t^{(N)}\Big)\right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})}=O(1/N^2).$$

Parallèlement à l'élaboration de la démonstration de ce théorème, les trois auteurs Driver, Hall et Kemp ont également travaillé sur cette conjecture, et ont démontré le troisième point de ce théorème[38]. Bien que certaines parties de nos travaux soient conceptuellement proches, les deux preuves sont indépendantes et présentent chacune leur intérêt propre : alors que nous faisons une utilisation cruciale de résultats de probabilités libres, la démonstration de Driver

Hall et Kemp est nettement plus analytique. Les deux preuves[25,38] ont finalement été publiées simultanément dans le même numéro de *Journal of Functional Analysis.*

2.4. Laplacien, orientation et normalisation. Une autre manière de définir les différents mouvements browniens que nous venons de décrire consiste à se donner leurs générateurs. En tant que processus de Markov, ils possèdent tous un générateur qu'on appelle le laplacien, et que nous allons décrire dans le cas des mouvements browniens sur $U(N)$ et sur $GL_N(\mathbb{C})$.

L'algèbre de Lie $\mathfrak{u}(N)$ du groupe unitaire est composée des matrices anti-hermitiennes :

$$\mathfrak{u}(N) = \{X \in M_N(\mathbb{C}) : X^* + X = 0\}.$$

C'est cette algèbre qui paramètre les opérateurs différentiels invariants du premier ordre. Plus précisément, pour tout $X \in \mathfrak{u}(N)$, nous définissons le champ de vecteur invariant à gauche X^l qui agit sur les fonctions f dérivables sur $U(N)$ de la façon suivante : pour tout $g \in U(N)$, on a

$$X^l f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(ge^{tX}).$$

De la même façon, nous définissons le champ de vecteur invariant à droite X^r qui agit sur les fonctions f dérivables sur $U(N)$ de la façon suivante : pour tout $g \in U(N)$, on a

$$X^r f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(e^{tX}g).$$

On munit $\mathfrak{u}(N)$ du produit scalaire

$$(X,Y) \mapsto \langle X,Y\rangle_{\mathfrak{u}(N)} = N\,\mathrm{Tr}(X^*Y).$$

Ce produit scalaire est invariant par action de la conjugaison par un élément de $U(N)$. Ainsi, il induit une métrique riemannienne bi-invariante sur $U(N)$ et un laplacien bi-invariant $\Delta_{U(N)}$. Pour toute base orthonormale β de $\mathfrak{u}(N)$, le laplacien $\Delta_{U(N)}$ est l'opérateur différentiel du second ordre

$$\Delta_{U(N)} = \sum_{X\in\beta}(X^l)^2 = \sum_{X\in\beta}(X^r)^2.$$

Un mouvement brownien $(U_t^{(N)})_{t\geq 0}$ sur $U(N)$ est un processus de Markov partant de l'identité, et dont le générateur est $(1/2)\Delta_{U(N)}$.

La situation du groupe linéaire $GL_N(\mathbb{C})$ est quasiment identique. L'algèbre de Lie $\mathfrak{gl}_N(\mathbb{C})$ est composée de l'ensemble des matrices $M_N(\mathbb{C})$. Pour tout $X \in \mathfrak{gl}_N(\mathbb{C})$, nous définissons le champ de vecteur invariant à gauche X^l (respectivement à droite X^r) qui agit sur les fonctions f dérivables sur $GL_N(\mathbb{C})$ de la façon suivante : pour tout $g \in U(N)$, on a

$$X^l f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(ge^{tX}) \quad \left(\text{resp. } X^r f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(e^{tX}g)\right).$$

On munit $\mathfrak{gl}_N(\mathbb{C})$ du produit scalaire réel[8]

$$(X,Y) \mapsto \langle X,Y\rangle_{\mathfrak{gl}_N(\mathbb{C})} = N\Re\left(\mathrm{Tr}(X^*Y)\right).$$

[17] P. Biane, Free Brownian motion, free stochastic calculus and random matrices (1997).

[19] P. Biane, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems (1997).

[25] G. Cébron, Free convolution operators and free Hall transform (2013).

[38] B. K. Driver, B. C. Hall, et T. Kemp, The Large-N Limit of the Segal–Bargmann Transform on $\mathbb{U}_N$ (2013).

8 Bien que défini sur le même espace $\mathfrak{gl}_N(\mathbb{C}) = M_N(\mathbb{C})$, ce produit scalaire réel $\langle\cdot,\cdot\rangle_{\mathfrak{gl}_N(\mathbb{C})}$ ne doit pas être confondu avec le produit scalaire hermitien $\langle\cdot,\cdot\rangle_{M_N(\mathbb{C})}$ défini précédemment.

Ce produit scalaire n'est pas invariant par action de la conjugaison par un élément de $GL_N(\mathbb{C})$. Ainsi, il induit deux métriques riemanniennes sur $GL_N(\mathbb{C})$, une métrique invariante à gauche et une métrique invariante à droite. Soit $\Delta_{GL_N(\mathbb{C})}$ le laplacien correspondant à la métrique invariante à droite. Pour toute base orthonormale β de $\mathfrak{gl}_N(\mathbb{C})$, c'est l'opérateur différentiel du second ordre

$$\Delta_{GL_N(\mathbb{C})} = \sum_{X\in\beta} (X^r)^2.$$

Un mouvement brownien à *droite* $(G_t^{(N)})_{t\geq 0}$ sur $GL_N(\mathbb{C})$ est un processus de Markov partant de l'identité, et dont le générateur est $(1/4)\Delta_{GL_N(\mathbb{C})}$. Nous insistons sur cette définition du mouvement brownien qui inclut un facteur inhabituel de $1/4$, suivant la convention de Hall[44]. De cette manière, le mouvement brownien avance à une vitesse réduite de moitié[9] sur $GL_N(\mathbb{C})$.

Les mouvements browniens que l'on a considérés jusqu'à présent sont des mouvements browniens à droite, suivant la convention de Biane. Il semble que la littérature soit plus encline à travailler avec des mouvements browniens à gauche. Bien heureusement, il existe une dualité parfaite entre les deux situations, et nous laissons aux lecteurs consciencieux le soin de faire les modifications mineures nécessaires pour retrouver les résultats correspondant aux mouvements browniens à gauche.

Remarque 2.2. Dans les situations précédemment décrites, il y a essentiellement deux endroits où la normalisation joue un rôle crucial. D'une part, il faut que les normalisations des mouvements browniens sur $U(N)$ et sur $GL_N(\mathbb{C})$ à N fixé soient ajustées pour que la transformation de Hall soit une isométrie. D'autre part, il faut que les normalisations des mouvements browniens sur $U(N)$ soient ajustées lorsque la dimension N varie pour que la limite en grande dimension existe. Malgré cette absence de latitude, on retrouve dans la littérature différents produits scalaires, différentes vitesses et différents générateurs pour décrire les mêmes mouvements browniens que nous avons présentés. Cette apparente contradiction disparaît lorsqu'on comprend la propriété de changement d'échelle du mouvement brownien : si $c > 0$, Δ est le laplacien pour une métrique $\langle\cdot,\cdot\rangle$ et $(g_t)_{t\geq 0}$ un mouvement brownien dont le générateur est $(1/2)\Delta$, alors $(1/c)\Delta$ est le laplacien pour une métrique $c\langle\cdot,\cdot\rangle$ et $(g_{t/c})_{t\geq 0}$ est un mouvement brownien dont le générateur est $(1/2c)\Delta$. Avec cette liberté, nous pouvons décrire un même mouvement brownien avec autant de métriques, autant de générateurs et autant de vitesses différentes qu'il y a de réels positifs.

3. Opérateurs de convolution libre

Dans l'étude de la transformation de Segal-Bargmann libre, ou de la transformation de Hall libre, nous nous sommes intéressés au calcul de quantités de la forme $\tau(P(A+B)|B)$, ou encore $\tau(P(AB)|B)$. Nous pouvons nous inspirer du cas classique, mais nous allons voir que la situation s'avère très différente dans le cas non-commutatif.

3.1. Noyaux de transition pour les différentes convolutions. Soient A et B deux variables aléatoires complexes indépendantes de distributions μ_A et μ_B. La loi conditionnelle de $A+B$ sachant B est $k(y, \mathrm{d}x) = \mathrm{d}\mu_A(x-y)$; c'est le noyau de transition tel que, pour toute

9 De la même façon, le mouvement brownien standard complexe sur $\mathbb{C}$ est souvent défini comme étant le processus de Markov de générateur $(1/4)(\partial_x^2+\partial_y^2)$, de sorte qu'au temps 1, sa variance soit égale à 1. Bien entendu, il est toujours possible de voir le générateur du mouvement brownien sur $GL_N(\mathbb{C})$ comme $(1/2)\Delta$, avec $\Delta = \Delta_{GL_N(\mathbb{C})}/2$ étant le laplacien associé à la métrique $2\langle\cdot,\cdot\rangle_{\mathfrak{gl}_N(\mathbb{C})}$.

fonction f borélienne et bornée, on a[10]

$$\mathbb{E}[f(A+B)|B] = \int_{\mathbb{C}} f(x)k(B, \mathrm{d}x).$$

Nous pouvons le vérifier par le calcul suivant. Pour toutes fonctions f, g boréliennes et bornées,

$$\int_{\mathbb{C}^2} f(x+y)g(y)\mathrm{d}\mu_A(x)\mathrm{d}\mu_B(y) = \int_{\mathbb{C}} \left(\int_{\mathbb{C}} f(x)\mathrm{d}\mu_A(x-y) \right) g(y)\mathrm{d}\mu_B(y).$$

Bien entendu, ce noyau de transition ne dépend que de la distribution de A. Intuitivement, à chaque fois que l'on fixe une valeur y de B, la mesure de probabilité $k(y, \mathrm{d}x)$ nous donne le moyen de choisir de manière aléatoire un point x pour $A+B$.

En probabilités libres, nous ne pouvons pas fixer une réalisation d'une variable B, puis espérer comprendre le comportement de $A+B$ lorsque A est libre avec B. Dans un article de 1998, Biane[18] réussit pourtant à mettre au jour de telles lois conditionnelles. Le résultat suivant est en pratique accompagné d'une caractérisation du noyau de transition qui entre en jeu.

Théorème[18]. *Soient $A, B \in \mathcal{A}$ deux variables aléatoires non-commutatives qui sont auto-adjointes et libres entre elles. Il existe alors un noyau de transition $k(y, \mathrm{d}x)$ sur $\mathbb{R} \times \mathbb{R}$ tel que pour toute fonction f borélienne et bornée sur $\mathbb{R}$,*

$$\tau\left(f(A+B) \middle| B \right) = \int_{\mathbb{R}} f(x)k(B, \mathrm{d}x).$$

Revenons au cas classique. Si A et B sont indépendantes, la loi conditionnelle de AB sachant B est $k(y, \mathrm{d}x) = \mathrm{d}\mu_A(xy^{-1})$ si $y \neq 0$ et δ_0 si $y = 0$. Là encore, le noyau de transition ne dépend que de la distribution de A. La version multiplicative du théorème de Biane est la suivante.

Théorème[18]. *Soient $U, V \in \mathcal{A}$ deux variables aléatoires non-commutatives qui sont unitaires et libres entre elles. Il existe alors un noyau de transition $k(y, \mathrm{d}x)$ sur $\mathbb{U} \times \mathbb{U}$ tel que pour toute fonction f borélienne et bornée sur $\mathbb{U}$,*

$$\tau\left(f(UV) \middle| V \right) = \int_{\mathbb{U}} f(x)k(V, \mathrm{d}x).$$

Une caractéristique importante distingue le cas classique du cas libre. Les noyaux de transition dépendent des deux variables aléatoires à la fois. Pourtant, un exemple simple nous montre que le calcul de $\tau(P(A+B)|B)$ peut être fait sans tenir compte de B.

Exemple. Soit $t > 0$ et S_t une variable semi-circulaire de variance t. Pour toute variable B libre avec S_t, on a

$$\tau((S_t+B)^3|B) = B^3 + 2tB + t\tau(B).$$

En effet, pour toute variable A de l'algèbre engendrée par B,

$$\begin{aligned}
\tau\left((S_t+B)^3 A \right) =& \tau(S_t^3 A) + \tau(S_t^2 BA) + \tau(S_t B^2 A) + \tau(B^3 A) \\
&+ \tau(S_t B S_t A) + \tau(B S_t^2 A) + \tau(B S_t B A) \\
=& 0 + t\tau(BA) + 0 + \tau(B^3 A) + t\tau(B)\tau(A) + t\tau(BA) + 0 \\
=& \tau\left(\left(B^3 + 2tB + t\tau(B) \right) A \right)
\end{aligned}$$

ce qui nous permet d'identifier $B^3 + 2tB + t\tau(B)$ comme l'espérance conditionnelle de $(S_t+B)^3$ par rapport à B.

[10] Ici, $\int_{\mathbb{C}} f(x)k(B, \mathrm{d}x)$ doit être interprété comme la fonction $y \mapsto \int_{\mathbb{C}} f(x)k(y, \mathrm{d}x)$ appliquée à B.

[18] P. Biane, Processes with free increments (1998).

[44] B. C. Hall, The Segal-Bargmann "Coherent State" Transform for Compact Lie Groups (1994).

Dans l'exemple ci-dessus, bien que le polynôme $X^3 + 2tX + t\tau(B)$ dépende de B, le calcul a pu être effectué sans la donnée de B. Cela suggère qu'en changeant de paradigme, il est possible de s'affranchir de la donnée de B. La stratégie est la suivante. Nous allons définir des objets abstraits généraux sur lesquels effectuer les calculs, puis ces objets pourront être évalués en B. Dans le cadre de l'exemple, cela nous mène à un objet "$X^3 + 2tX + t\,\mathrm{tr}(X)$", puis l'évaluation[11] nous mène comme voulu à $B^3 + 2tB + t\tau(B)$.

La section suivante est consacrée à la définition de l'ensemble de ces objets abstraits, ce qui nous permettra ensuite de définir des opérateurs de transition.

3.2. L'espace $\mathbb{C}\{X_i : i \in I\}$. L'algèbre de polynômes non-commutatifs $\mathbb{C}\langle X_i : i \in I\rangle$ possède la propriété suivante : pour toute famille $(A_i)_{i\in I}$ d'une algèbre $\mathcal{A}$, il existe un unique morphisme d'algèbre $\varphi : \mathcal{A} \to \mathbb{C}\langle X_i : i \in I\rangle$ tel que $\varphi(X_i) = A_i$. Ce morphisme est donnée par la substitution des A_i aux indéterminées X_i d'un polynôme. Cette propriété définit l'algèbre $\mathbb{C}\langle X_i : i \in I\rangle$ à isomorphisme près : on dit que $\mathbb{C}\langle X_i : i \in I\rangle$ est la solution d'un problème universel. Intuitivement, peu importe la façon dont on a codé $\mathbb{C}\langle X_i : i \in I\rangle$, c'est sa propriété sus-citée qui nous intéresse.

De la même façon, nous allons définir l'espace $\mathbb{C}\{X_i : i \in I\}$ comme solution d'un problème universel similaire, mais auquel s'ajoute une structure supplémentaire. Soit $\mathcal{A}$ une algèbre unitaire. Le centre de $\mathcal{A}$ est l'algèbre unitaire $Z_{\mathcal{A}}$ constituée des éléments de $\mathcal{A}$ qui commutent avec tous les autres éléments de $\mathcal{A}$. La structure additionnelle est la suivante. On appelle espérance à valeurs centrales toute application linéaire τ de $\mathcal{A}$ à valeurs dans son centre $Z_{\mathcal{A}}$, et telle que :

(1) pour tout $A, B \in \mathcal{A}$, on a $\tau(\tau(A)B) = \tau(A)\tau(B)$;
(2) $\tau(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

Cette structure nous permet de définir le problème universel suivant.

Problème universel 8.1. *Soit I un ensemble quelconque d'indices. Soit $\mathcal{X}$ une algèbre munie d'une espérance à valeurs centrales* tr, *et de I éléments $(X_i)_{i\in I}$. Le triplet $\big(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I}\big)$ est solution du problème universel 8.1 pour les indices I si pour toute algèbre $\mathcal{A}$ munie d'une espérance à valeurs centrales τ, et de I éléments $(A_i)_{i\in I}$, il existe un unique morphisme d'algèbre f de $\mathcal{X}$ vers $\mathcal{A}$ tel que*

(1) pour tout $i \in I$, on a $f(X_i) = A_i$;
(2) pour tout $P \in \mathcal{X}$, on a $\tau(f(P)) = f(\mathrm{tr}(P))$.

On dira que f est le I-morphisme canonique de $\mathcal{X}$ vers $\mathcal{A}$. Le I-morphisme d'une solution du problème universel 8.1 vers une autre solution est bijectif : dans la pratique, on identifie volontiers deux solutions du problème universel, et on parle de *la* solution du problème universel 8.1. Dans l'appendice du chapitre 2, nous montrons qu'il existe une solution à ce problème universel, et c'est cette solution (unique à isomorphisme canonique près) que nous noterons $\big(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in I}\big)$ ou plus simplement $\mathbb{C}\{X_i : i \in I\}$.

La notation de X_i pour l'élément correspondant à $i \in I$ de $\mathbb{C}\{X_i : i \in I\}$ semble entrer en conflit avec la notation de X_i comme élément de $\mathbb{C}\langle X_i : i \in I\rangle$. Ce choix de notation est justifié par l'injectivité du morphisme d'algèbre $\varphi : \mathbb{C}\langle X_i : i \in I\rangle \to \mathbb{C}\{X_i : i \in I\}$ défini par $\varphi(X_i) = X_i$. L'algèbre $\mathbb{C}\langle X_i : i \in I\rangle$ peut donc être identifiée à une sous-algèbre de $\mathbb{C}\{X_i : i \in I\}$, et c'est exactement ce que nous ferons dans la suite de cette thèse :

$$\mathbb{C}\langle X_i : i \in I\rangle \subset \mathbb{C}\{X_i : i \in I\}.$$

On a par exemple $X_j^2 \in \mathbb{C}\{X_i : i \in I\}$ si $j \in I$. En utilisant la fonction tr, on a aussi $\mathrm{tr}(X_j^2) \in \mathbb{C}\{X_i : i \in I\}$, ou encore $X_j\,\mathrm{tr}(X_j^2) \in \mathbb{C}\{X_i : i \in I\}$. Plus généralement, nous prouvons que

[11] L'évaluation consiste à remplaçer formellement tr par τ et X par B.

l'ensemble

$$\{M_0 \operatorname{tr} M_1 \cdots \operatorname{tr} M_n : n \in \mathbb{N}, M_0, \ldots, M_n \text{ monômes de } \mathbb{C}\langle X_i : i \in I\rangle\}$$

forme une base de $\mathbb{C}\{X_i : i \in I\}$, appelée la base canonique.

Nous allons maintenant décrire le calcul fonctionnel associé à l'espace $\mathbb{C}\{X_i : i \in I\}$. Ce calcul fonctionnel est appelé $\mathbb{C}\{X_i : i \in I\}$-calcul et est défini à partir de la propriété universelle de $\mathbb{C}\{X_i : i \in I\}$. Soit $\mathcal{A}$ une algèbre munie d'une espérance à valeurs centrales τ, et de I éléments $\mathbf{A} = (A_i)_{i\in I}$. Soit $f : \mathbb{C}\{X_i : i \in I\} \to \mathcal{A}$ le I-morphisme canonique de $\mathbb{C}\{X_i : i \in I\}$ vers $\mathcal{A}$. Pour tout $P \in \mathbb{C}\{X_i : i \in I\}$, on dit que $f(P)$ est l'élément de $\mathcal{A}$ obtenu par substitution[12] de τ à tr, et des A_i aux indéterminées X_i, dans P, et on note cet élément $P(\mathbf{A})$. Concrètement, le $\mathbb{C}\{X_i : i \in I\}$-calcul est donné par son action sur la base canonique : pour tout $n \in \mathbb{N}$ et $M_0, \ldots, M_n$ monômes de $\mathbb{C}\langle X_i : i \in I\rangle$,

$$(M_0 \operatorname{tr} M_1 \cdots \operatorname{tr} M_n)(\mathbf{A}) = M_0(\mathbf{A})\tau(M_1(\mathbf{A})) \cdots \tau(M_n(\mathbf{A})).$$

Dans la section 8.2, on définit de la même façon les algèbres $\mathbb{C}\{X_i, X_i^{-1} : i \in I\}$, $\mathbb{C}\{X_i, X_i^* : i \in I\}$ et $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$, ainsi que les substitutions associées. Par exemple, pour tout algèbre A munie d'une involution $*$ et de I éléments inversibles, on peut définir un morphisme d'algèbre $P \mapsto P(\mathbf{A})$ de $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$ vers $\mathcal{A}$ par les seules conditions $X_i(\mathbf{A}) = A_i$, $X_i^*(\mathbf{A}) = A_i^*$, $X_i^{-1}(\mathbf{A}) = A_i^{-1}$, $X_i^{*-1}(\mathbf{A}) = A_i^{*-1}$ et $(\operatorname{tr} P)(\mathbf{A}) = \tau(P(\mathbf{A}))$.

Nous sommes maintenant en mesure d'énoncer notre théorème d'existence d'opérateurs de transition pour la somme de variables aléatoires libres. Nous le ferons dans la section 3.4. Toutefois, celui-ci mettant en scène les cumulants libres, coefficients dont l'importance en probabilités non-commutatives n'est plus à démontrer, nous leur consacrons d'abord la section qui suit.

3.3. Les cumulants et log-cumulants libres. Les cumulants libres ont été définis par Roland Speicher[71] pour étudier de manière combinatoire l'addition de variables aléatoires libres. Ils consistent en un ensemble de coefficients indexés par tous les n-uplets d'éléments de $\mathcal{A}$, où n parcourt $\mathbb{N}^*$. Autrement dit, c'est une fonction $\kappa : \bigsqcup_{m\in\mathbb{N}^*} \mathcal{A}^m \to \mathbb{C}$.

Le point de départ habituel pour définir les cumulants libres est l'ensemble des partitions non-croisées. Soit $n \in \mathbb{N}^*$. Une partition de $\{1, \ldots, n\}$ possède un croisement s'il existe quatre entiers i, j, k, l entre 1 et n tels que i et k appartiennent à un même bloc alors que j et l appartiennent à un autre bloc. L'ensemble des partitions de $\{1, \ldots, n\}$ qui ne possèdent pas de croisements, encore appelées *partitions non-croisées*, est noté $NC(n)$.

Definition 3.1. Les *cumulants libres* sont définis par la relation suivante. Pour tout $n \in \mathbb{N}^*$ et tout $A_1, \ldots, A_n \in \mathcal{A}$, on a

$$\tau(A_1 \cdots A_n) = \sum_{\pi\in NC(n)} \kappa[\pi](A_1, \ldots, A_n),$$

où $\kappa[\pi](A_1, \ldots, A_n) = \prod_{\{i_1<\cdots<i_k\}\in\pi} \kappa(A_{i_1}, \ldots, A_{i_k})$.

Cette relation peut en effet être inversée de manière implicite, et $\kappa(A_1, \ldots, A_n)$ exprimé comme un polynôme en les variables $\tau(A_{i_1}, \ldots, A_{i_k})$, avec $1 < i_1 < \ldots < i_k < n$. Nous donnerons plus de détails sur la théorie des cumulants libres dans la section 9.2.2 ; par ailleurs, nous renvoyons aux nombreux travaux de Nica et Speicher sur le sujet[60]. Pour le moment, contentons-nous de

[12] Bien entendu, lorsque $P \in \mathbb{C}\langle X_i : i \in I\rangle$, la substitution dans P vu comme un élément de $\mathbb{C}\{X_i : i \in I\}$ coïncide avec celle dans P vu comme un élément de $\mathbb{C}\langle X_i : i \in I\rangle$.

[60] A. Nica et R. Speicher, *Lectures on the Combinatorics of Free Probability* (2006).

[71] R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution (1994).

décrire la nouvelle caractérisation de la liberté à travers la proposition suivante. Son corollaire est immédiat : les cumulants libres linéarisent la somme de variables libres.

PROPOSITION 3.2. *Soit* $(\mathcal{B}_i)_{i\in I}$ *des sous-algèbres de* $\mathcal{A}$. *Ces algèbres sont libres si et seulement si leurs cumulants croisés sont nuls. Autrement dit : pour tout* $n \in \mathbb{N}^*$, *tout* $i_1, \dots, i_n \in I$ *et tout* $A_1, \dots, A_n \in \mathcal{A}$ *tels que chaque* A_j *appartient à* $\mathcal{B}_{i_j}$, *l'existence de* j *et* j' *tels que* $i_j \neq i_{j'}$ *entraîne que* $\kappa(A_1, \dots, A_n) = 0$.

COROLLAIRE 3.3. *Pour tout* $A_1, \dots, A_n \in \mathcal{A}$ *libres avec* $B_1, \dots, B_n \in \mathcal{A}$, *on a*

$$\kappa(A_1 + B_1, \dots, A_n + B_n) = \kappa(A_1, \dots, A_n) + \kappa(B_1, \dots, B_n).$$

EXEMPLE. Soit $t > 0$ et S_t une variable semi-circulaire de variance t. Les cumulants libres de S_t sont donnés par $\kappa_1(S_t) = 0$, $\kappa_2(S_t) = t$ et $\kappa_n(S_t) = 0$ pour $n > 2$. Ainsi, pour tout $n \leq 1$,

$$\tau(S_t^n) = \sum_{\pi \in NC(n)} \kappa[\pi](S_t, \dots, S_t) = \begin{cases} 0, & \text{si } n \text{ impair,} \\ t^{n/2} C_{n/2} & \text{si } n \text{ pair,} \end{cases}$$

où $C_{n/2}$, le $n/2$-ième nombre de Catalan, est le nombre d'appariements non-croisés de n éléments.

Bien que la notion de cumulants libres permette dans une certaine mesure de traiter le cas de produits de variables libres, nos calculs font plusieurs fois apparaître d'autres coefficients. Ces nouveaux coefficients, étudiés pour la première fois par Mastnak et Nica[58], jouent le rôle des cumulants libres pour le produit. La compréhension et l'étude de ces coefficients, que nous avons baptisés les log-cumulants libres, constituent une part non-négligeable de cette thèse. La section 9.4 reprend leur construction de manière plus détaillée – et illustrée !

Encore une fois, la définition des log-cumulants libres est combinatoire et fait intervenir l'ensemble $NC(n)$. Nous allons définir successivement dans les trois prochains paragraphes la relation de finesse $\preceq$, la complémentation de Kreweras et la notion de chaîne dans $NC(n)$. Nous pourrons ensuite définir les log-cumulants libres.

L'ensemble $NC(n)$ est un treillis pour la *relation de finesse* $\preceq$ définie comme ceci : pour σ et $\pi \in NC(n)$, σ est *plus fine* que π, ce qui se note $\sigma \preceq \pi$, si tout bloc de σ est inclus dans un bloc de π.

Dans le cas où $\sigma \preceq \pi$, il est alors possible de définir le *complémentaire de Kreweras*[52] $K_\pi(\sigma)$ de σ par rapport à π. C'est une bijection K_π de l'ensemble $\{\sigma \in NC(n) : \sigma \preceq \pi\}$ sur lui-même. Il en existe plusieurs définitions équivalentes. En voici une qui a le mérite d'être succinte. Munissons l'ensemble $\{1, 1', \dots, n, n'\}$ de l'ordre cyclique $(1, 1', \dots, n, n')$. Lorsque $\sigma' \in NC(n)$, nous pouvons former la partition $\sigma \cup \sigma'$ de $\{1, 1', \dots, n, n'\}$ en identifiant σ' à une partition de $\{1', \dots, n'\}$. Décomposons $\pi = \{V_1, \dots, V_l\} \in NC(n)$. Pour tout $1 \leq i \leq l$, notons $V_i' \subset \{1', \dots, n'\}$ l'image de V_i par l'isomorphisme $(1, \dots, n) \simeq (1', \dots, n')$. Enfin, notons $\tilde{\pi}$ la partition non-croisée $\tilde{\pi} = \{V_1 \cup V_1', \dots, V_n \cup V_n'\}$ de $\{1, 1', \dots, n, n'\}$. La partition $K_\pi(\sigma)$ est par définition le plus grand élément de $NC(n)$ tel que $\sigma \cup K_\pi(\sigma)$ soit une partition non-croisée de $\{1, 1', \dots, n, n'\}$ et soit plus fine que $\tilde{\pi}$.

Enfin, une *chaîne* dans le treillis $NC(n)$ est un $(l+1)$-uplet de la forme $\Gamma = (\pi_0, \dots, \pi_l)$ avec $\pi_0, \dots, \pi_l \in NC(n)$ et tel que $\pi_0 \prec \pi_1 \prec \cdots \prec \pi_l$. L'entier positif l est alors appelé longueur de la chaîne et on le notera $|\Gamma|$. Si, pour tout $1 \leq i \leq l$, $K_{\pi_i}(\pi_{i-1})$ a exactement un bloc qui possède plus de deux éléments, la chaîne Γ est dite *simple*.

[58] M. MASTNAK et A. NICA, Hopf algebras and the logarithm of the S-transform in free probability (2010).
[52] G. KREWERAS, Sur les partitions non croisées d'un cycle (1972).

	Cas additif	**Cas multiplicatif**
Probabilités libres	Cumulants libres	Log-cumulants libres

TABLE 4. Les cumulants libres

DEFINITION 3.4. Les *log-cumulants libres* sont les coefficients $L\kappa(A_1, \ldots, A_n)$, indexés par tous les n-uplets d'éléments de $\mathcal{A}$, et définis comme suit. Pour tout $A \in \mathcal{A}$ tels que $\tau(A) \neq 0$, on pose

$$L\kappa(A) = \mathrm{Log}(\tau(A)),$$

où Log désigne la branche principale du logarithme dont la partie imaginaire appartient à l'intervalle $(-\pi, \pi]$. Pour tout $n \geq 2$, et $A_1, \ldots, A_n \in \mathcal{A}$ tels que $\tau(A_1), \ldots, \tau(A_n)$ sont tous non-nuls, on pose

$$L\kappa(A_1, \ldots, A_n) = \frac{1}{\tau(A_1) \cdots \tau(A_n)} \sum_{\substack{\Gamma \text{ chaîne dans } NC(n) \\ \Gamma = (\pi_0, \ldots, \pi_{|\Gamma|}) \\ \pi_0 = 0_n, \pi_{|\Gamma|} = 1_n}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} \prod_{i=1}^{|\Gamma|} \kappa \left[K_{\pi_i}(\pi_{i-1}) \right] (A_1, \ldots, A_n),$$

où $\kappa[\pi](A_1, \ldots, A_n) = \prod_{\{i_1 < \cdots < i_k\} \in \pi} \kappa(A_{i_1}, \ldots, A_{i_k})$.

Le nom de log-cumulants est justifié par le résultat suivant, dû à Mastnak et Nica[58]. En le comparant au corollaire 3.3, on peut en effet interpréter les log-cumulants libres comme les analogues multiplicatifs des cumulants libres.

PROPOSITION (CF. PROP.9.11). *Soit $A, B \in \mathcal{A}$ libres et tels que $\tau(A)$ et $\tau(B)$ sont non-nuls. On a $L\kappa_1(AB) \equiv L\kappa_1(A) + L\kappa_1(B) \pmod{2i\pi}$, et pour tout $n \geq 2$:*

$$L\kappa_n(AB) = L\kappa_n(A) + L\kappa_n(B).$$

Mastnak et Nica ont aussi montré que les log-cumulants libres peuvent être utilisés pour caractériser la liberté, exactement comme dans la proposition 3.2.

PROPOSITION (CF. PROP.9.10). *Soit $(\mathcal{B}_i)_{i \in I}$ des sous-algèbres de $\mathcal{A}$. Ces algèbres sont libres si et seulement si leurs log-cumulants croisés sont nuls. Autrement dit : pour tout $n \in \mathbb{N}^*$, tout $i_1, \ldots, i_n \in I$ et tout $A_1, \ldots, A_n \in \mathcal{A}$ tels que $\tau(A_1) = \ldots = \tau(A_n) = 1$, et tels que chaque A_j appartient à $\mathcal{B}_{i_j}$, l'existence de j et j' tels que $i_j \neq i_{j'}$ entraîne que $L\kappa(A_1, \ldots, A_n) = 0$.*

La donnée des log-cumulants libres détermine entièrement la distribution des variables aléatoires. Dans la proposition suivante, nous avons même pu donner l'expression exacte de la distribution à l'aide des log-cumulants libres.

PROPOSITION (CF. COR.9.9). *Soit $n \geq 1$. Pour tous $A_1, \ldots, A_n \in \mathcal{A}$ tels que $\tau(A_1), \ldots, \tau(A_n)$ sont tous non-nuls, on a*

$$\tau(A_1 \cdots A_n) = e^{L\kappa(A_1)} \cdots e^{L\kappa(A_n)} \sum_{\substack{\Gamma \text{ chaîne simple dans } NC(k) \\ \Gamma = (\pi_0, \ldots, \pi_{|\Gamma|}), \pi_0 = 0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa \Big[K_{\pi_i}(\pi_{i-1}) \Big] (A_1, \ldots, A_n),$$

où, $\{i_1 < \cdots < i_k\}$ étant l'unique bloc de $K_{\pi_i}(\pi_{i-1})$ qui possède plus d'un élément,

$$L\kappa \Big[K_{\pi_i}(\pi_{i-1}) \Big] (A_1, \ldots, A_n) = L\kappa(A_{i_1}, \ldots, A_{i_k}).$$

Exemple. Soit $t > 0$ et U_t un mouvement brownien unitaire libre au temps t. Les log-cumulants libres de U_t sont donnés par $L\kappa_1(U_t) = -t/2$, $\kappa_2(S_t) = -t$ et $\kappa_n(S_t) = 0$ pour $n > 2$. On retrouve bien la distribution de U_t : pour tout $n \geq 1$,

$$\tau(U_t^n) = e^{nL\kappa(U_t)} \sum_{\substack{\Gamma \text{ chaîne simple dans } NC(k) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}),\pi_0=0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big]\,(U_t) = e^{-\frac{t}{2}n} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} n^{k-1} \binom{n}{k+1}.$$

En effet, le nombre de chaînes simples dans $NC(n)$ de longueur k est exactement $n^{k-1}\binom{n}{k+1}$.

3.4. Opérateurs de transition. Reprenons le fil de la section 3.2. Notre préoccupation reste l'existence d'opérateurs de transition sur l'espace $\mathbb{C}\{X_i : i \in I\}$ pour l'addition de variables aléatoires libres.

Théorème (cf. Th.9.4). *Soit I un ensemble quelconque d'indices. Soit $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$. Il existe un opérateur $\Delta_{\mathbf{A}}$ sur $\mathbb{C}\{X_i : i \in I\}$ tel que, pour tout $P \in \mathbb{C}\{X_i : i \in I\}$, et tout $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$ libre avec $(A_i)_{i\in I}$,*

$$\tau\left(P\left(\mathbf{A}+\mathbf{B}\right)\middle|\,\mathbf{B}\right) = \left(e^{\Delta_{\mathbf{A}}}P\right)(\mathbf{B}).$$

Avant de décrire $\Delta_{\mathbf{A}}$, comparons ce résultat avec le résultat de Biane de la section 3.1. Comme nous l'avions annoncé, l'opérateur de transition $e^{\Delta_{\mathbf{A}}}$ ne dépend plus des variables $\mathbf{B}$. La distribution de $\mathbf{B}$ n'intervient qu'a posteriori, lors de l'évaluation de l'élément $e^{\Delta_{\mathbf{A}}}P$ en $\mathbf{B}$. En revanche, nous perdons la généralité d'un calcul borélien pour un calcul seulement polynomial[13] : c'est le prix à payer pour pouvoir traiter plusieurs variables non-commutatives à la fois, et qui ne sont pas forcément normales. L'opérateur $\Delta_{\mathbf{A}}$ associé à $\mathbf{A}$ est de la forme suivante. Pour tout $n \in \mathbb{N}$ et $i(1), \ldots i(n) \in I$,

$$\begin{aligned} \Delta_{\mathbf{A}}\left(X_{i(1)}\cdots X_{i(n)}\right) = \sum_{\substack{1\leq m\leq n \\ 1\leq k_1<\ldots<k_m\leq n}} & \kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right) X_{i(1)}\cdots X_{i(k_1-1)} \\ & \cdot \operatorname{tr}\left(X_{i(k_1+1)}\cdots X_{i(k_2-1)}\right) \\ & \cdot \operatorname{tr}\left(X_{i(k_2+1)}\cdots X_{i(k_3-1)}\right) \\ & \cdots \\ & \cdot \operatorname{tr}\left(X_{i(k_{m-1}+1)}\cdots X_{i(k_m-1)}\right) \\ & \cdot X_{i(k_m+1)}\cdots X_{i(n)}. \end{aligned}$$

De plus, pour tout $M_0, \ldots, M_k$ monômes de $\mathbb{C}\langle X_i : i \in I\rangle$, on a

$$\Delta_{\mathbf{A}}\left(M_0 \operatorname{tr} M_1 \cdots \operatorname{tr} M_k\right) = \sum_{i=0}^{k} M_0 \operatorname{tr} M_1 \cdots \operatorname{tr}\left(\Delta_{\mathbf{A}}(M_i)\right) \cdots \operatorname{tr} M_k.$$

Exemple. Soit S_t une variable semi-circulaire de variance t. On calcule $\Delta_{S_t} X^3 = 2tX + t\operatorname{tr}(X)$ et $(\Delta_{S_t})^2 X^3 = \Delta_{S_t}(2tX + t\operatorname{tr}(X)) = 0$. Ainsi, $e^{\Delta_{S_t}}(X^3) = X^3 + \Delta_{S_t}X^3 + 0 = X^3 + 2tX + t\operatorname{tr}(X)$. Le théorème assure alors que, pour tout $B \in \mathcal{A}$ libre avec S_t,

$$\tau\left((S_t + B)^3 \,|B\right) = \left(e^{\Delta_{S_t}}(X^3)\right)(B) = B^3 + 2tB + t\tau(B).$$

Voici le théorème d'existence d'un opérateur de transition dans le cas multiplicatif. Pour tout $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$ et $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$, on notera $\mathbf{AB}$ la famille $(A_iB_i)_{i\in I} \in \mathcal{A}^I$.

[13] Rappelons que $\mathbb{C}\langle X_i : i \in I\rangle \subset \mathbb{C}\{X_i : i \in I\}$.

THÉORÈME (CF. TH.9.13). *Soit I un ensemble quelconque d'indices. Soit $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$ tel que, pour tout $i \in I$, $\tau(A_i) \neq 0$. Il existe un opérateur $\mathrm{D}_{\mathbf{A}}$ sur $\mathbb{C}\{X_i : i \in I\}$ tel que, pour tout $P \in \mathbb{C}\{X_i : i \in I\}$, et tout $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$ libre avec $(A_i)_{i\in I}$ et vérifiant $\tau(B_i) \neq 0$ pour $i \in I$, alors*

$$\tau\left(P(\mathbf{AB})|\,\mathbf{B}\right) = \left(e^{\mathrm{D}_{\mathbf{A}}}P\right)(\mathbf{B}).$$

L'hypothèse d'une trace non nulle pour les éléments entrant en jeu n'est pas étonnante. C'est une hypothèse technique assez courante lorsqu'il s'agit d'étudier le produit de variables aléatoires non-commutatives libres[14]. L'opérateur $\mathrm{D}_{\mathbf{A}}$ associé à $\mathbf{A}$ est de la forme suivante. Pour tout $n \in \mathbb{N}$ et $i(1), \ldots i(n) \in I$,

$$\begin{aligned}\mathrm{D}_{\mathbf{A}}\left(X_{i(1)}\cdots X_{i(n)}\right) \quad = \sum_{\substack{1\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}} & L\kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right)X_{i(1)}\cdots X_{i(k_1-1)}\\ & \cdot\operatorname{tr}\left(X_{i(k_1)}\cdots X_{i(k_2-1)}\right)\\ & \cdot\operatorname{tr}\left(X_{i(k_2)}\cdots X_{i(k_3-1)}\right)\\ & \cdots\\ & \cdot\operatorname{tr}\left(X_{i(k_{m-1})}\cdots X_{i(k_m-1)}\right)\\ & \cdot X_{i(k_m)}\cdots X_{i(n)}.\end{aligned}$$

De plus, pour tout $M_0, \ldots, M_k$ monômes de $\mathbb{C}\langle X_i : i \in I\rangle$, on a

$$\mathrm{D}_{\mathbf{A}}\left(M_0 \operatorname{tr} M_1 \cdots \operatorname{tr} M_k\right) = \sum_{i=0}^{k} M_0 \operatorname{tr} M_1 \cdots \operatorname{tr}\left(\mathrm{D}_{\mathbf{A}}(M_i)\right)\cdots \operatorname{tr} M_k.$$

EXEMPLE. Soit $t \geq 0$ et U_t un mouvement brownien unitaire libre au temps t. On calcule $\mathrm{D}_{U_t}(X^2) = -tX^2 - tX \operatorname{tr} X$, $\mathrm{D}_{U_t}(X \operatorname{tr} X) = -tX \operatorname{tr} X$. Ainsi, $e^{\mathrm{D}_{U_t}}X^2 = e^{-2t}\left(X^2 - tX \operatorname{tr} X\right)$. Le théorème nous dit alors que, pour tout $B \in \mathcal{A}$ libre avec U_t,

$$\tau\left((U_tB)^2\,|B\right) = e^{-2t}\left(B^2 - tB\tau(B)\right).$$

4. Mouvement brownien sur le groupe linéaire en grande dimension

L'espace $\mathbb{C}\{X\}$ ainsi que ses variantes nous offrent un cadre agréable pour calculer la distribution non-commutative de variables aléatoires définies par un calcul stochastique. Nous allons voir que ce formalisme nous permet d'étudier la limite en grande dimension du mouvement brownien sur le groupe linéaire lorsque la dimension tend vers l'infini.

4.1. Générateurs des mouvements browniens en grande dimension. Soit $(U_t)_{t\geq 0}$ un mouvement brownien unitaire libre et $(G_t)_{t\geq 0}$ un mouvement brownien circulaire multiplicatif libre. Rappelons que l'existence des opérateur D_{U_t} et D_{G_t} nous permet de calculer la distribution de U_t et de G_t à chaque temps t. En effet, pour tout polynôme $P \in \mathbb{C}[X]$ et tout temps $t \geq 0$,

$$\tau\left(P(U_t)\right) = \tau\left(P(U_t1_{\mathcal{A}})|\,1_{\mathcal{A}}\right) = \left(e^{\mathrm{D}_{U_t}}P\right)(1_{\mathcal{A}}).$$

En utilisant une preuve complètement différente grâce au calcul stochastique libre, il est possible de renforcer ce résultat pour obtenir la $*$-distribution jointe de (U_t, U_t^{-1}) et de (G_t, G_t^{-1}).

PROPOSITION (CF. PROP.10.4) ET PROP.10.6). *Il existe deux opérateurs Δ_U et Δ_{GL} sur $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ tels que, pour tout $t \geq 0$, on ait*

$$\tau\left(P(U_t)\right) = \left(e^{\frac{t}{2}\Delta_U}P\right)(1_{\mathcal{A}}) \quad et \quad \tau\left(P(G_t)\right) = \left(e^{\frac{t}{4}\Delta_{GL}}P\right)(1_{\mathcal{A}}).$$

[14] Ici, nous pourrions éventuellement nous en passer pour la famille $\mathbf{B}$ par un argument d'approximation.

Nous décrivons l'action de Δ_U et de Δ_{GL} sur $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ aux sections 10.4.1 et 10.5.1. Bien entendu, les opérateurs $(t/2)\Delta_U$ et $(t/4)\Delta_{GL}$ restreints au sous-espace $\mathbb{C}\{X\}$ ne sont autres que les opérateurs D_{U_t} et D_{G_t}.

Les preuves des théorèmes énoncés dans la section 2 reposent sur une observation élémentaire qui a son origine dans l'approche du laplacien donnée par Thierry Lévy[53]. Cette observation peut être résumée de la façon suivante : lorsque la dimension N tend vers l'infini, les opérateurs $\Delta_{U(N)}$ et $\Delta_{GL_N(\mathbb{C})}$ sont respectivement égaux aux opérateurs de transition Δ_U et Δ_G. Donnons une version plus précise de ce fait. On désignera par U la fonction identité sur $U(N)$. Puisque la trace normalisée agit sur l'algèbre $C^2(U(N), \mathrm{M}_N(\mathbb{C}))$ comme une trace à valeurs centrales[15], on peut considérer la fonction $P(U)$ obtenue par substitution de la trace normalisée à tr et de U à X pour tout $P \in \mathbb{C}\{X, X^{-1}\}$.

Lemme (cf. Lem.11.1). *Il existe un opérateur $\tilde{\Delta}_U$ sur $\mathbb{C}\{X, X^{-1}\}$ tel que, pour tout $P \in \mathbb{C}\{X, X^{-1}\}$,*

$$\Delta_{U(N)}\left(P(U)\right) = \left((\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)P\right)(U).$$

Ce lien entre le générateur $\Delta_{U(N)}/2$ du mouvement brownien $(U_t^{(N)})_{t\geq 0}$ sur $U(N)$ et le générateur $\Delta_U/2$ du mouvement brownien libre $(U_t)_{t\geq 0}$ rend limpide la convergence en distribution de $(U_t^{(N)})_{t\geq 0}$ vers $(U_t)_{t\geq 0}$. En effet, moyennant quelques justifications simples, nous pouvons écrire la convergence suivante pour tout polynôme $\mathbb{C}[X, X^{-1}]$ lorsque N tend vers l'infini :

$$\mathbb{E}\left[\mathrm{tr}\left(P(U_t^{(N)})\right)\right] = \left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}(\mathrm{tr}(P))\right)(I_N) \longrightarrow \left(e^{\frac{t}{2}\Delta_U}(\mathrm{tr}(P))\right)(1_{\mathcal{A}}) = \tau(P(U_t)). \tag{4.1}$$

De la même manière, nous désignerons par G la fonction identité de $GL_N(\mathbb{C})$. Pour tout $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$, on peut considérer la fonction $P(G) \in C^2(GL_N(\mathbb{C}), \mathrm{M}_N(\mathbb{C}))$ obtenue par substitution de la trace normalisée à tr et de G à X.

Lemme (cf. Lem.11.3). *Il existe un opérateur $\tilde{\Delta}_{GL}$ sur $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ tel que, pour tout $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$,*

$$\Delta_{GL_N(\mathbb{C})}\left(P(G)\right) = \left((\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})P\right)(G).$$

Un calcul analogue au calcul (4.1) nous conduit au résultat suivant.

Théorème (cf. Th.11.6). *Pour tout $N \in \mathbb{N}^*$, soit $(G_t^{(N)})_{t\geq 0}$ un mouvement brownien sur $GL_N(\mathbb{C})$ et soit $(G_t)_{t\geq 0}$ un mouvement brownien circulaire multiplicatif libre. Alors, pour tout $n \in \mathbb{N}$, tout $P_0, \ldots, P_n \in \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$ et tout $t \geq 0$, on a*

$$\mathbb{E}\left[\mathrm{tr}\left(P_0\left(G_t^{(N)}\right)\right)\cdots\mathrm{tr}\left(P_n\left(G_t^{(N)}\right)\right)\right] = \tau\left(P_0\left(G_t\right)\right)\cdots\tau\left(P_n\left(G_t\right)\right) + O(1/N^2)$$

lorsque N tend vers l'infini. En particulier, $(G_t^{(N)}, {G_t^{(N)}}^{-1})$ converge en $*$-distribution non-commutative vers (G_t, G_t^{-1}).

Ce théorème répond à une question posée par Philippe Biane[19] en 1997. Peu de temps après la prépublication de ce résultat[25] sur le site d'archivage arXiv, Todd Kemp proposa deux autres preuves[50,51] de la convergence en $*$-distribution non-commutative de $(G_t^{(N)}, {G_t^{(N)}}^{-1})$. Nous avons ensuite décidé d'étudier la question naturelle des fluctuations du mouvement brownien autour de sa moyenne $(G_t^{(N)})_{t\geq 0}$. La suite de cette section expose le théorème limite central qui résulte de cette collaboration.

[15] Il faut pour cela identifier $\mathbb{C}$ à la sous-algèbre $\mathbb{C} \cdot I_N$ de $\mathrm{M}_N(\mathbb{C})$.

4.2. Fluctuations browniennes (en collaboration avec Todd Kemp). Thierry Lévy et Mylène Maïda ont établi le théorème limite central suivant pour les fluctuations de $\mathrm{tr}(P(U_t^{(N)}))$ en grande dimension.

THÉORÈME[56]. *Soit* $(U_t^{(N)})_{t\geq 0}$ *un mouvement brownien sur* $U(N)$*. Soient* $P_1,\dots,P_n \in \mathbb{C}[X,X^{-1}]$ *des polynômes de Laurent, et* $T\geq 0$*. Lorsque* N *tend vers l'infini, le vecteur aléatoire*

$$N\left(\mathrm{tr}\left(P_i(U_T^{(N)})\right)-\mathbb{E}\left[\mathrm{tr}\left(P_i(U_T^{(N)})\right)\right]\right)_{1\leq i\leq n}$$

converge en loi vers un vecteur gaussien.

Il faut préciser que le théorème de Lévy et Maïda porte en réalité sur une classe de fonctions plus générale, celle des fonctions de $C^1(\mathbb{U})$ dont la dérivée est lipschitzienne. Remarquons l'ordre de grandeur inhabituel des fluctuations : il est de $1/N$ au lieu du classique $1/\sqrt{N}$. C'est l'ordre de grandeur standard pour les fluctuations de matrices aléatoires. Le calcul de la covariance est intéressant. Il fait apparaître trois mouvements browniens unitaires libres $(u_t)_{t\geq 0}$, $(v_t)_{t\geq 0}$ et $(w_t)_{t\geq 0}$ qui sont libres entre eux. Pour deux polynômes de Laurent $P,Q\in\mathbb{C}[X,X^{-1}]$, la covariance des variables aléatoires $\mathrm{Tr}\,P(U_T^{(N)})-\mathbb{E}[\mathrm{Tr}\,P(U_T^{(N)})]$ et $\mathrm{Tr}\,Q(U_T^{(N)})-\mathbb{E}[\mathrm{Tr}\,Q(U_T^{(N)})]$ est asymptotiquement égale[16] à

$$\int_0^T \tau\left(P'(v_{T-t}u_t)(Q'(w_{T-t}u_t))^*\right)\mathrm{d}t, \tag{4.2}$$

où les dérivées P' et Q' des polynômes P et Q sont les dérivées sur $\mathbb{U}$, données[17] pour tout entier k par $(X^k)'=ikX^k$ et $(X^{-k})'=-ikX^{-k}$.

La preuve de ce théorème repose en partie sur la compacité du groupe unitaire et n'est pas transposable directement au groupe linéaire. Très récemment, Antoine Dahlqvist a proposé une preuve combinatoire de ce théorème[31], ce qui lui permet de prouver un résultat similaire pour deux autres groupes compacts : le groupe orthogonal et le groupe symplectique. Dans le chapitre 3, nous utilisons une troisième approche, également combinatoire, dans l'optique de prouver le théorème limite central suivant pour les fluctuations de $\mathrm{tr}(P(G_t^{(N)}))$ en grande dimension.

THÉORÈME (CF. TH.15.1). *Soit* $(G_t^{(N)})_{t\geq 0}$ *mouvement brownien sur* $GL_N(\mathbb{C})$*. Soient* $P_1,\cdots,P_n\in\mathbb{C}\{X,X^{-1},X^*,X^{*-1}\}$*, et* $T\geq 0$*. Lorsque* N *tend vers l'infini, le vecteur aléatoire*

$$N\left(\mathrm{tr}\left(P_i(G_T^{(N)})\right)-\mathbb{E}\left[\mathrm{tr}\left(P_i(G_T^{(N)})\right)\right]\right)_{1\leq i\leq n}$$

converge en loi vers un vecteur gaussien.

De manière remarquable, la covariance est semblable à celle du théorème de Lévy et Maïda. En effet, fixons deux polynômes de Laurent P et $Q\in\mathbb{C}[X,X^{-1}]$. La covariance des variables

16 Le théorème de Lévy et Maïda porte sur des fonctions réelles sur $\mathbb{U}$. La linéarité d'un système gaussien nous permet d'étendre le résultat à des fonctions complexes, et en particulier à des polynômes de Laurent.

17 La dérivée d'une fonction f sur $\mathbb{U}$ est la fonction $f':z\mapsto \lim_{h\to 0}(f(ze^{ih})-f(z))/h$.

[19] P. BIANE, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems (1997).

[25] G. CÉBRON, Free convolution operators and free Hall transform (2013).

[31] A. DAHLQVIST, Dualité de Schur-Weyl, mouvement brownien sur les groupes de Lie compacts classiques et étude asymptotique de la mesure de Yang-Mills (2014).

[50] T. KEMP, Heat Kernel Empirical Laws on $\mathbb{U}_N$ and $\mathbb{GL}_N$ (2013).

[51] T. KEMP, The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$ (2013).

[53] T. LÉVY, Schur–Weyl duality and the heat kernel measure on the unitary group (2008).

[56] T. LÉVY et M. MAÏDA, Central limit theorem for the heat kernel measure on the unitary group (2010).

aléatoires $\operatorname{Tr} P(G_T^{(N)}) - \mathbb{E}[\operatorname{Tr} P(G_T^{(N)})]$ et $\operatorname{Tr} Q(G_T^{(N)}) - \mathbb{E}[\operatorname{Tr} Q(G_T^{(N)})]$ est asymptotiquement égale à (4.2), où $(u_t)_{t\geq 0}$, $(v_t)_{t\geq 0}$ et $(w_t)_{t\geq 0}$ désignent cette fois-ci trois mouvements browniens multiplicatifs circulaires qui sont libres entre eux. Lorsque P et Q ne sont plus des polynômes mais des éléments quelconques de $\mathbb{C}\{X, X^{-1}\}$, la covariance peut aussi s'exprimer à l'aide de trois mouvements browniens, mais son expression exacte n'a pas de formulation simple.

La preuve de ce théorème repose encore une fois sur la décomposition $\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL}$ de $\Delta_{GL_N(\mathbb{C})}$. En effet, l'égalité

$$e^{\frac{T}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})} = e^{\frac{T}{4}\Delta_{GL}} + \frac{1}{N^2}\int_0^T e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}\cdot \tilde{\Delta}_{GL}\cdot e^{\frac{T-t}{4}\Delta_{GL}}\mathrm{d}t$$

appliquée à $\mathbb{E}[(\operatorname{tr} P \operatorname{tr} Q^*)(G_T^{(N)})] = e^{\frac{T}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}(\operatorname{tr} P \operatorname{tr} Q^*)(I_N)$ nous permet d'une part d'identifier l'ordre de grandeur de $1/N$, d'autre part de rendre explicite le terme qui demeure après recentrage. Remarquons que cette explication rapide rend également compréhensible le terme de covariance (4.2). En effet, l'opérateur limite

$$\int_0^T e^{\frac{t}{4}\Delta_{GL}}\cdot \tilde{\Delta}_{GL}\cdot e^{\frac{T-t}{4}\Delta_{GL}}\mathrm{d}t$$

appliqué à $\operatorname{tr} P \operatorname{tr} Q^*$ se décompose heuristiquement en :

(1) $e^{\frac{T-t}{4}\Delta_{GL}}$, qui correspond à l'évolution de deux mouvements browniens libres dans respectivement $\operatorname{tr}(P)$ et $\operatorname{tr}(Q)^*$ jusqu'au temps $T-t$,
(2) $\tilde{\Delta}_{GL}$, qui fait apparaître $\operatorname{tr}(P'Q'^*)$ à la place de $\operatorname{tr} P \operatorname{tr} Q^*$,
(3) $e^{\frac{t}{4}\Delta_{GL}}$, qui correspond à l'évolution d'un troisième mouvement brownien pendant un temps t.

Précisons que l'analogie avec le théorème de Lévy et Maïda n'est pas fortuite. En réalité, la preuve de ce théorème central limite fonctionne pour une classe de mouvements browniens sur $GL_N(\mathbb{C})$ qui contient le mouvement brownien sur le sous-groupe $U(N) \subset GL_N(\mathbb{C})$. Mentionnons également que le théorème dans toute sa généralité permet de traiter une famille de mouvements browniens sur $GL_N(\mathbb{C})$ arrêtés en des temps différents.

5. Mesures infiniment divisibles

Une manière de généraliser la table 1 consiste à remplacer les différents mouvements browniens par des *processus de Lévy*, c'est-à-dire des processus à accroissements indépendants et stationnaires. Bien entendu, les accroissements seront additifs ou multiplicatifs, et l'indépendance sera classique ou libre selon la situation. Au niveau des distributions, l'addition ou la multiplication de variables aléatoires se traduit par des opérations de convolutions. Il y a alors une correspondance entre les processus de Lévy et les distributions infiniment divisibles pour ces convolutions. Le dernier chapitre de cette thèse est consacré aux liens entre les quatre situations, tantôt au niveau des distributions, tantôt au niveau des processus de Lévy.

5.1. Les convolutions classiques et libres. Soient μ et ν deux mesures de probabilités sur $\mathbb{R}$. La *convolution classique* des mesures μ et ν est définie comme l'unique mesure $\mu * \nu$ sur $\mathbb{R}$ telle que pour toute fonction mesurable bornée f,

$$\int_{\mathbb{R}} f \,\mathrm{d}(\mu * \nu) = \int_{\mathbb{R}^2} f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Observons que la convolution classique peut être obtenue concrètement comme la distribution d'une somme de deux variables aléatoires indépendantes. Plus précisément, soient A et B deux variables aléatoires indépendantes de distributions respectives μ et ν. La distribution de $A+B$

	Cas additif	**Cas multiplicatif**
Probabilités classiques	Convolution classique $*$ sur $\mathbb{R}$	Convolution classique $\circledast$ sur $\mathbb{U}$
Probabilités libres	Convolution libre $\boxplus$ sur $\mathbb{R}$	Convolution libre $\boxtimes$ sur $\mathbb{U}$

TABLE 5. Les quatre convolutions

est $\mu * \nu$, la convolution de μ et de ν. Pour définir la convolution libre, il suffit de remplacer l'indépendance par la liberté.

Soit μ et ν deux mesures de probabilités à support compact. Prenons A et $B \in \mathcal{A}$ deux variables aléatoires non-commutatives auto-adjointes libres de distributions respectives[18] μ et ν. Ici encore, la distribution μ_{A+B} de $A+B$ est déterminée de manière unique par μ et ν. Cette mesure de probabilité sur $\mathbb{R}$ est appelée la *convolution libre* de μ et ν, et elle est notée $\mu \boxplus \nu$. Définie comme ceci, la convolution libre $\boxplus$ est une opération sur les mesures de probabilités à support compact, mais cette définition peut être étendue à l'ensemble des mesures de probabilités sur $\mathbb{R}$ par continuité[14].

Bien entendu, nous définissons les convolutions multiplicatives classiques et libres de la même façon. Soient μ et ν deux mesures de probabilités sur $\mathbb{U}$. Soient A et B deux variables aléatoires indépendantes de distributions respectives μ et ν. La distribution de AB ne dépend que de μ et ν. C'est la *convolution* de μ et de ν, définie comme l'unique mesure $\mu \circledast \nu$ sur $\mathbb{U}$ telle que pour toute fonction mesurable bornée f,

$$\int_{\mathbb{U}} f \,\mathrm{d}(\mu \circledast \nu) = \int_{\mathbb{U}^2} f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y).$$

Prenons maintenant A et $B \in \mathcal{A}$ deux variables aléatoires non-commutatives unitaires libres de distributions respectives μ et ν. La distribution de AB est également déterminée de manière unique par μ et ν. On l'appelle la *convolution libre* de μ et ν et on la note $\mu \boxtimes \nu$.

5.2. Morphismes entre convolutions. Introduisons tout d'abord quelques définitions. Nous dirons qu'une suite de mesures finies $(\mu_n)_{n\in\mathbb{N}}$ sur $\mathbb{C}$ *converge faiblement* vers une mesure μ, et on notera[19]

$$\mu_n \xrightarrow[n\to+\infty]{(w)} \mu,$$

si, pour toute fonction complexe continue bornée f,

$$\lim_{n\to\infty} \int_{\mathbb{C}} f \mathrm{d}\mu_n = \int_{\mathbb{C}} f \mathrm{d}\mu.$$

Une mesure de probabilité sur $\mathbb{R}$ est dite $*$-*infiniment divisible* si, pour tout entier $n \in \mathbb{N}^*$, il existe une mesure de probabilités μ_n telle que

$$\underbrace{\mu_n * \cdots * \mu_n}_{n \text{ fois}} = \mu.$$

L'ensemble des mesures $*$-infiniment divisible est un semi-groupe pour l'opération $*$ que l'on notera $\mathcal{ID}(\mathbb{R}, *)$, et on définit de la même façon les ensembles $\mathcal{ID}(\mathbb{U}, \circledast)$, $\mathcal{ID}(\mathbb{R}, \boxplus)$ et $\mathcal{ID}(\mathbb{U}, \boxtimes)$.

[18] C'est justement pour pouvoir réaliser μ et ν comme distributions de variables aléatoires non-commutatives que celles-ci doivent être à support compact.

[14] H. BERCOVICI et D. VOICULESCU, Free convolution of measures with unbounded support (1993).

[19] La lettre w fait référence au terme anglais "weak convergence".

En 1999, Bercovici et Pata publient un article[13] avec un appendice de Biane sur le lien entre les mesures infiniment divisibles pour les convolutions $*$ et pour $\boxplus$. Ce lien peut être reformulé[20] dans les termes qui suivent.

Soit $\mu \in \mathcal{ID}(\mathbb{R}, *)$. Il existe alors un unique $\eta \in \mathbb{R}$, un unique $a \geq 0$ et une unique mesure ρ sur $\mathbb{R} \setminus \{0\}$ tels que, pour toute suite de mesures $(\mu_n)_{n\in\mathbb{N}}$, les deux assertions suivantes soient équivalentes :

(1) la mesure μ_n^{*n} converge faiblement vers μ,
(2) on a les convergences

$$\left(n\frac{x^2}{x^2+1}\mu_n(\mathrm{d}x)\right) \xrightarrow[n\to+\infty]{(w)} \left(\frac{x^2}{x^2+1}\rho(\mathrm{d}x) + a\delta_0\right) \quad \text{et} \quad \lim_{n\to\infty} n\int_{[-1,1]} x\mathrm{d}\mu_n(x) = \eta.$$

De plus, ceci reste vrai en remplaçant la convolution classique $*$ par la convolution libre $\boxtimes$. Le triplet (η, a, ρ) sera respectivement appelé le *triplet $*$-caractéristique*, ou *$\boxplus$-caractéristique*[21] de μ. En associant à chaque mesure $\mu \in \mathcal{ID}(\mathbb{R}, *)$ dont le triplet $*$-caractéristique est (η, a, ρ) la mesure $\Lambda(\mu) \in \mathcal{ID}(\mathbb{R}, \boxplus)$ dont le triplet $\boxplus$-caractéristique est (η, a, ρ), on définit alors un isomorphisme Λ entre $\mathcal{ID}(\mathbb{R}, *)$ et $\mathcal{ID}(\mathbb{R}, \boxplus)$ qui a le bon goût de préserver les théorèmes limites au sens suivant.

THÉORÈME[13]. *La bijection de Bercovici-Pata Λ a les propriétés suivantes :*

(1) pour toute suite $(\mu_n)_{n\in\mathbb{N}^*}$ de mesures de probabilités sur $\mathbb{R}$ et toute mesure $\mu \in \mathcal{ID}(\mathbb{R}, *)$, on a

$$\underbrace{\mu_n * \cdots * \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mu \iff \underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \Lambda(\mu) \ ;$$

(2) pour tout $\mu, \nu \in \mathcal{ID}(\mathbb{R}, *)$, $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$.

Le théorème de Bercovici et Pata est le premier d'une série de théorèmes infinitésimaux du type : « si la n-ième convolution de μ_n pour une certaine convolution converge faiblement lorsque n tend vers l'infini, alors la n-ième convolution de μ_n pour une autre convolution converge également faiblement.» C'est dans cette démarche que s'inscrit le travail de Chistyakov et Götze, qui s'intéressent aux convolutions multiplicatives. Malheureusement, la situation est plus complexe. On désignera par $\mathcal{M}_*$ l'ensemble des mesures de probabilités sur $\mathbb{U}$ telles que $m_1(\mu) \neq 0$. En 2008, Chistyakov et Götze ont montré que la convergence d'une suite $(\mu_n^{\boxtimes})_{n\in\mathbb{N}}$ vers une mesure $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$ entraîne la convergence de la suite $(\mu_n^{\circledast})_{n\in\mathbb{N}}$ vers une mesure de $\mu \in \mathcal{ID}(\mathbb{U}, \circledast)$ ne dépendant que de μ, et que l'on notera $\Gamma(\mu)$. Puisque $\mathcal{ID}(\mathbb{U}, \boxtimes) \setminus \mathcal{M}_*$ se réduit[15] à la mesure uniforme λ sur $\mathbb{U}$, nous pouvons étendre la définition de Γ à $\mathcal{ID}(\mathbb{U}, \boxtimes)$ par $\Gamma(\lambda) = \lambda$. Nous obtenons alors le résultat suivant.

THÉORÈME[28]. *La fonction Γ a les propriétés suivantes :*

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[20] La formulation équivalente originelle est faite à l'aide d'une *paire caractéristique* au lieu d'un triplet caractéristique.

[21] Le triplet caractéristique peut également se lire sur la transformée de Fourier (resp. la R-transformée) de μ, mais nous préférons cette présentation qui identifie (η, a, ρ) comme une sorte de générateur infinitésimal. Le réel η correspond à une dérive, le réel a à la partie brownienne (resp. semi-circulaire), et la mesure ρ représente la composante poissonnienne.

[13] H. BERCOVICI, V. PATA, et P. BIANE, Stable laws and domains of attraction in free probability theory (1999).

[15] H. BERCOVICI et D. VOICULESCU, Lévy-Hinčin type theorems for multiplicative and additive free convolution. (1992).

[28] G. P. CHISTYAKOV et F. GÖTZE, Limit theorems in free probability theory II (2008).

(1) pour toute suite $(\mu_n)_{n\in\mathbb{N}^*}$ *de mesures de* $\mathcal{M}_*$ *et toute mesure* $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)\cap\mathcal{M}_*$*, on a*

$$\underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mu \quad\Longrightarrow\quad \underbrace{\mu_n \circledast \cdots \circledast \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \Gamma(\mu)\ ;$$

(2) pour tout $\mu,\nu \in \mathcal{ID}(\mathbb{U},\boxtimes)$, $\Gamma(\mu\boxtimes\nu)=\Gamma(\mu)\circledast\Gamma(\nu)$.

La fonction Γ n'est pas injective, et par conséquent, la réciproque de ce théorème est fausse : la convergence faible de $\mu_n^{\circledast n}$ n'implique pas forcément la convergence de $\mu_n^{\boxtimes n}$.

Nous allons nous intéresser à l'existence de morphismes analogues reliant les convolutions additives aux convolutions multiplicatives. Dans le cas classique, le morphisme $\mathbf{e} : x \mapsto e^{ix}$ de $(\mathbb{R},+)$ vers $(\mathbb{U},\times)$ nous permet de transformer les théorèmes infinitésimaux relatifs à la convolution $*$ en des théorèmes relatifs à la convolution $\circledast$ à travers la notion de mesure image ; pour μ mesure de probabilité sur $\mathbb{R}$, la mesure image $\mathbf{e}_*(\mu)$ désigne la mesure de probabilité sur $\mathbb{U}$ caractérisée par l'intégrale de toute fonction borélienne bornée f :

$$\int_{\mathbb{U}} f\mathrm{d}(\mathbf{e}_*(\mu)) = \int_{\mathbb{R}} f(e^{ix})\mathrm{d}\mu(x).$$

Pour une variable aléatoire réelle A de distribution μ, la distribution de e^{iA} est la mesure image $\mathbf{e}_*(\mu)$. Ainsi, pour deux variables aléatoires A et B réelles et indépendantes, l'égalité $e^{iA}e^{iB}=e^{i(A+B)}$ implique que la fonction $\mathbf{e}_*$ est un morphisme. Le théorème suivant n'est que la conséquence de ce fait, et de la continuité de $\mathbf{e}_*$ pour la convergence faible.

THÉORÈME 5.1. *La fonction* $\mathbf{e}_*$*, dont la restriction à* $\mathcal{ID}(\mathbb{R},*)$ *est à valeurs dans* $\mathcal{ID}(\mathbb{U},\circledast)$*, a les propriétés suivantes :*

(1) pour toute suite $(\mu_n)_{n\in\mathbb{N}^*}$ *de mesures de probabilités sur* $\mathbb{R}$ *et toute mesure* $\mu\in\mathcal{ID}(\mathbb{R},*)$*, on a*

$$\underbrace{\mu_n * \cdots * \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mu \quad\Longrightarrow\quad \underbrace{\mathbf{e}_*(\mu_n) \circledast \cdots \circledast \mathbf{e}_*(\mu_n)}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_*(\mu)\ ;$$

(2) pour tout $\mu,\nu\in\mathcal{ID}(\mathbb{R},*)$, $\mathbf{e}_*(\mu*\nu)=\mathbf{e}_*(\mu)\circledast\mathbf{e}_*(\nu)$.

Bien sûr, pour deux variables aléatoires A et B auto-adjointes et libres, la non-commutativité entraîne que $e^{iA}e^{iB}\neq e^{i(A+B)}$. La fonction $\mathbf{e}_*$ n'est donc pas un morphisme : pour deux mesures de probabilités μ et ν sur $\mathbb{R}$, $\mathbf{e}_*(\mu\boxplus\nu)$ n'est pas forcément égale à $\mathbf{e}_*(\mu)\boxtimes\mathbf{e}_*(\nu)$. En nous restreignant aux mesures infiniment divisibles comme nous y incite le théorème de Bercovici et Pata, nous pouvons tout de même définir un morphisme entre $\boxplus$ et $\boxtimes$. Commençons par présenter ce qui joue le rôle de triplet caractéristique pour les mesures de $\mathcal{ID}(\mathbb{U},\boxtimes)$, en reprenant le résultat du corollaire 19.9.

Soit $\mu\in\mathcal{ID}(\mathbb{R},*)$. Il existe alors un unique $\omega\in\mathbb{U}$, un unique $b\geq 0$ et une unique mesure υ sur $\mathbb{U}\setminus\{1\}$ tels que, pour toute suite de mesures $(\mu_n)_{n\in\mathbb{N}}$, les deux assertions suivantes soient équivalentes :

(1) la mesure μ_n^{*n} converge faiblement vers μ,
(2) en notant ω_n l'angle $m_1(\mu_n)/|m_1(\mu_n)|$ on a les convergences

$$n(1-\Re(\zeta))\mathrm{d}\mu_n(\omega_n\zeta) \xrightarrow[n\to+\infty]{(w)} \left((1-\Re(\zeta))\upsilon(\mathrm{d}\zeta)+\frac{b}{2}\delta_1\right) \quad \text{et} \quad \lim_{n\to\infty}\omega_n^n=\omega.$$

Le triplet (ω,b,υ) sera appelé le triplet $\boxtimes$-caractéristique[22] de μ.

[22] Le triplet caractéristique peut également se lire sur la S-transformée, ou sur les log-cumulants. L'angle ω correspond à une dérive, le réel a à la partie brownienne libre, et la mesure υ représente la composante poissonnienne.

	Cas additif		**Cas multiplicatif**
Probabilités classiques	$\mathcal{ID}(\mathbb{R},*)$	$\xrightarrow{\mathbf{e}_*}$	$\mathcal{ID}(\mathbb{U},\circledast)$
	$\downarrow \Lambda$		$\uparrow \Gamma$
Probabilités libres	$\mathcal{ID}(\mathbb{R},\boxplus)$	$\xrightarrow{\mathbf{e}_\boxplus}$	$\mathcal{ID}(\mathbb{U},\boxtimes)$

Table 6. Les morphismes entre les quatre convolutions

Définition 5.2. Pour tout $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$ dont le triplet $\boxplus$-caractéristique est (η, a, ρ), on définit $\mathbf{e}_\boxplus(\mu)$ comme la mesure $\boxtimes$-infiniment divisible sur $\mathbb{U}$ dont le triplet $\boxtimes$-caractéristique est

$$(\omega, b, \upsilon) = \left(\exp\left(i\eta + i\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right).$$

La fonction $\mathbf{e}_\boxplus$ est lié aux morphismes Γ, Λ et $\mathbf{e}_*$ par la commutativité du diagramme de semigroupe présenté en table 6. Il a ceci de naturel qu'il préserve aussi les théorèmes infinitésimaux.

Théorème (cf. Th.17.1). *La fonction* $\mathbf{e}_\boxplus : \mathcal{ID}(\mathbb{R},\boxplus) \to \mathcal{ID}(\mathbb{R},\boxtimes)$ *a les propriétés suivantes :*

(1) pour toute suite $(\mu_n)_{n\in\mathbb{N}^*}$ de mesures de probabilités sur $\mathbb{R}$ et toute mesure $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$, on a

$$\underbrace{\mu_n \boxplus \cdots \boxplus \mu_n}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mu \implies \underbrace{\mathbf{e}_*(\mu_n) \boxtimes \cdots \boxtimes \mathbf{e}_*(\mu_n)}_{n \text{ fois}} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_\boxplus(\mu)\ ;$$

(2) pour tout $\mu, \nu \in \mathcal{ID}(\mathbb{R},\boxplus)$, $\mathbf{e}_\boxplus(\mu \boxplus \nu) = \mathbf{e}_\boxplus(\mu) \boxtimes \mathbf{e}_\boxplus(\nu)$.

Exemple 5.3. Nous avons déjà rencontré quatre exemples importants de mesures infiniment divisibles. Ce sont les distributions des différents mouvements browniens. C'est ainsi que, pour tout $t \geq 0$, la gaussienne $\mathcal{N}_t$ est $*$-infiniment divisible. De la même façon, $\mathcal{S}_t \in \mathcal{ID}(\mathbb{R},\boxplus)$, $\mathcal{B}_t \in \mathcal{ID}(\mathbb{U},\boxtimes)$ et $\mathbf{e}_*(\mathcal{N}_t) \in \mathcal{ID}(\mathbb{U},\circledast)$. Nous avons alors

$$\mathcal{N}_t \overset{\Lambda}{\longmapsto} \mathcal{S}_t \overset{\mathbf{e}_\boxplus}{\longmapsto} \mathcal{B}_t \overset{\Gamma}{\longmapsto} \mathbf{e}_*(\mathcal{N}_t).$$

Les différents théorèmes de cette section nous affirment donc que les mesures $\mathcal{S}_t$ et $\mathcal{B}_t$ vérifient des propriétés similaires à ceux d'une gaussienne classique. En particulier, fixons une suite $(\mu_n)_{n\in\mathbb{N}^*}$ de mesures de probabilités sur $\mathbb{R}$. Des trois assertions suivantes, les deux premières assertions sont équivalentes et impliquent la troisième :

(1) la mesure μ_n^{*n} converge faiblement vers $\mathcal{N}_t$;
(2) la mesure $\mu_n^{\boxplus n}$ converge faiblement vers $\mathcal{S}_t$;
(3) la mesure $(\mathbf{e}_*(\mu_n))^{\boxtimes n}$ converge faiblement vers $\mathcal{B}_t$.

5.3. Modèles matriciels. Nous allons maintenant retrouver le morphisme $\mathbf{e}_\boxplus$ comme limite en grande dimension d'un morphisme entre espaces de matrices.

On notera $*$ la convolution additive sur l'espace vectoriel des matrices hermiciennes $\mathcal{H}_N$: pour deux mesures de probabilités μ et ν sur $\mathcal{H}_N$, la mesure $\mu*\nu$ est définie comme l'unique mesure sur $\mathcal{H}_N$ telle que pour toute fonction mesurable bornée f, $\int_{\mathcal{H}_N} f\ \mathrm{d}(\mu*\nu) = \int_{\mathcal{H}_N^2} f(x+y)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$. De la même façon, la convolution multiplicative $\circledast$ de deux mesures μ et ν sur le groupe unitaire

$U(N)$ est l'unique mesure $\mu \circledast \nu$ sur $U(N)$ telle que pour toute fonction mesurable bornée f, $\int_{U(N)} f \ \mathrm{d}(\mu * \nu) = \int_{U(N)^2} f(xy)\mathrm{d}\mu(x)\mathrm{d}\nu(y)$. Nous noterons respectivement $\mathcal{ID}(\mathcal{H}_N, *)$ et $\mathcal{ID}(U(N), \circledast)$ l'ensemble des mesures infiniment divisible pour ces deux convolutions.

La fonction $\mathbf{e}$ a encore une signification dans ce contexte : pour tout $x \in \mathcal{H}_N$, on a $\mathbf{e}(x) = e^{ix} \in U(N)$. Comme précédemment, pour toute mesure μ sur $\mathcal{H}_N$, nous noterons $\mathbf{e}_*(\mu)$ la mesure image de μ par $\mathbf{e}$, définie sur $U(N)$. Nous n'avons pas en toute généralité $\mathbf{e}(x+y) = \mathbf{e}(x)\mathbf{e}(y)$, et par conséquent la fonction $\mathbf{e}_*$ n'est pas un morphisme : pour deux mesures de probabilités μ et ν sur $\mathcal{H}_N$, $\mathbf{e}_*(\mu * \nu)$ n'est pas forcément égale à $\mathbf{e}_*(\mu) \circledast \mathbf{e}_*(\nu)$. Nous pouvons nous en sortir grâce aux mesures infiniment divisibles. L'idée de base est classique[39] : multiplier les accroissements multiplicatifs obtenus par notre exponentielle $\mathbf{e}$ à partir d'accroissements additifs de plus en plus petits. Toute mesure $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$ s'inscrit dans un unique $*$-semigroupe $(\mu^{*t})_{t \geq 0}$ faiblement continu et tel que $\mu^{*0} = \delta_0$ et $\mu^{*1} = \mu$. Pour toute mesure $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$, la suite de mesures $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ converge faiblement vers une mesure sur $U(N)$ que nous noterons $\mathcal{E}_N$. Malheureusement, nous n'avons toujours pas l'égalité $\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$. Nous nous en sortons grâce à un ultime subterfuge, qui consiste à nous restreindre aux mesures unitairement invariantes. Une mesure sur $\mathcal{H}_N$ (ou $U(N)$) est unitairement invariante si pour tout $g \in U(N)$ et toute fonction mesurable bornée f,

$$\int f \ \mathrm{d}\mu = \int f(gxg^*)\mathrm{d}\mu(x).$$

Nous désignerons respectivement par $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$ et $\mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$ les mesures de $\mathcal{ID}(\mathcal{H}_N, *)$ et $\mathcal{ID}(U(N), \circledast)$ qui sont unitairement invariantes. Cette fois-ci, pour μ et $\nu \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$, nous avons l'égalité $\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$. Nous pouvons résumer la situation dans la proposition suivante.

Proposition-Définition (cf. Prop.22.2). *Pour tout $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$, la suite de mesures $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ converge faiblement vers une mesure $\mathcal{E}_N(\mu)$ sur $U(N)$. La fonction*

$$\mathcal{E}_N : \mathcal{ID}(\mathcal{H}_N, *) \to \mathcal{ID}(U(N), *)$$

est appelée l'exponentielle stochastique, elle préserve l'invariance unitaire, et pour tout μ, $\nu \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$, on a

$$\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu).$$

En 2005, Florent Benaych-Georges[12] et Thierry Cabanal-Duvillard[23] définissent simultanément un modèle pour les mesures $*$-infiniment divisibles. Plus précisément, pour tout $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$, ils construisent une mesure de $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$, que nous appellerons $\Pi_N(\mu)$, comme limite faible des mesures $(\nu_n)^{*n}$, où ν_n est la distribution de

$$g \begin{pmatrix} \lambda_{n,1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{n,N} \end{pmatrix} g^{-1}$$

avec $\lambda_{n,1}, \ldots, \lambda_{n,N}$ variables réelles distribuées selon $\mu^{\boxtimes 1/n}$ et indépendamment de la variable $g \in U(N)$ distribuée uniformément. Dans la section 23.1, nous donnons plus de détails[23] sur la

[12] F. Benaych-Georges, Classical and free infinitely divisible distributions and random matrices (2005).

[39] A. Estrade, Exponentielle stochastique et intégrale multiplicative discontinues. (1992).

[23] T. Cabanal-Duvillard, A Matrix Representation of the Bercovici-Pata Bijection (2005).

23 En particulier, nous y décrivons sa transformée de Fourier et son générateur infinitésimal. Mentionnons que le triplet caractéristique de (η, a, ρ) de μ nous donne respectivement une dérive qui dépend de η, un mouvement brownien hermitien qui dépend de a et une partie poissonnienne qui dépend de ρ.

	Cas additif		Cas multiplicatif
Probabilités classiques	$\mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$	$\xrightarrow{\mathcal{E}_N}$	$\mathcal{ID}_{\text{inv}}(U(N), \circledast)$
	$N\to\infty \downarrow\uparrow \Pi_N$		$N\to\infty \downarrow\uparrow \Gamma_N$
Probabilités libres	$\mathcal{ID}(\mathbb{R}, \boxplus)$	$\xrightarrow{\mathbf{e}_\boxplus}$	$\mathcal{ID}(\mathbb{U}, \boxtimes)$

TABLE 7. Modèles matriciels

définition de $\Pi_N(\mu)$. Outre la particularité d'être un morphisme, Π_N possède plusieurs caractéristiques intéressantes, dont la suivante.

THÉORÈME[12,23]. *Soit $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$. Pour tout $N \in \mathbb{N}^*$, soit H_N une matrice aléatoire distribuée selon $\Pi_N(\mu)$. Alors μ est la limite faible en espérance de la mesure spectrale de H_N. Autrement dit, pour toute fonction f continue bornée sur $\mathbb{R}$, on a*

$$\int_{\mathbb{R}} f \mathrm{d}\mu = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} \sum_{\substack{\lambda \text{ valeurs propres de } H_N \\ \text{(avec multiplicité)}}} f(\lambda) \right].$$

La situation décrite est schématisée dans la partie gauche de la table 7. Pour le cas particulier $N = 1$, nous avons $\Pi_1 = \Lambda^{-1}$ et $\mathcal{E}_1 = \mathbf{e}_*$. La table 7 n'est alors que la table 6. Nous allons maintenant compléter ce schéma, et en particulier définir une fonction $\Gamma_N : \mathcal{ID}(\mathbb{U}, \boxtimes) \to \mathcal{ID}_{\text{inv}}(U(N), \circledast)$ telle que, pour tout $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$, la mesure spectrale d'une matrice aléatoire distribuée selon $\Gamma_N(\mu)$ converge faiblement vers μ en espérance.

Soit $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$, et soit (ω, b, υ) son triplet $\boxtimes$-caractéristique. Nous allons définir la mesure $\Gamma_N(\mu)$ comme étant la distribution au temps 1 d'un processus de Lévy. Pour cela, introduisons successivement trois termes qui dépendent respectivement de ω, de b, et de υ.

- Soit $Y \in \mathfrak{u}(N)$ tel que $e^Y = \omega I_N$. L'opérateur différentiel du premier ordre Y^l va correspondre à une dérive déterministe qui aboutit au temps 1 en ωI_N.
- Soit D l'opérateur différentiel du second ordre
$$\frac{bN}{2(N+1)}\Delta_{U(N)} + \frac{b}{2(N+1)}((iI_N)^l)^2.$$
Cet opérateur va correspondre à la partie brownienne du processus. On pourrait plus simplement prendre $D = \frac{b}{2}\Delta_{U(N)}$ pour retrouver le mouvement brownien unitaire déjà rencontré dans les autres sections. Dans ce cas-là, on conserverait la bonne limite en grande dimension, mais on perdrait la relation $\Gamma_N \circ \mathbf{e}_\boxplus = \mathcal{E}_N \circ \Pi_N$.
- Soit υ_N la mesure image de $N\upsilon \otimes \text{Haar}$ par la fonction de $\mathbb{U} \times U(N)$ vers $U(N)$ définie par
$$(\zeta, g) \mapsto g \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} g^{-1}.$$

[12] F. BENAYCH-GEORGES, Classical and free infinitely divisible distributions and random matrices (2005).
[23] T. CABANAL-DUVILLARD, A Matrix Representation of the Bercovici-Pata Bijection (2005).

Ce dernier terme va correspondre à la partie saut du processus. Heuristiquement, υ_N désigne l'intensité des sauts effectués. Remarquons que lorsque N augmente, notre processus fera de plus en plus de sauts à cause du facteur N, mais que ceux-ci seront de plus en plus proches de l'identité, car il n'y a qu'une valeur propre qui est différente de 1.

DEFINITION 5.4. La mesure $\Gamma_N(\mu)$ est la distribution d'un processus de Lévy au temps 1 dont le générateur L est donné par

$$Lf(h) = Y^l f(h) + Df(h) + \int_{U(N)} f(hg) - f(h) - (i\Im(g))^l f(h)\ \upsilon_N(\mathrm{d}g).$$

La fonction Γ_N est un morphisme lié aux morphismes Π_N, $\mathcal{E}_N$ et $\mathbf{e}_\boxplus$ par la commutativité du diagramme de semigroupe présenté en table 7. Nous pouvons interpréter le théorème de matrices aléatoires suivant par le fait que $\mathbf{e}_\boxplus$ est la limite de $\mathcal{E}_N$ en grande dimension.

THÉORÈME (CF. COR.23.8). *Soit $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$. Pour tout $N \in \mathbb{N}^*$, soit U_N une matrice aléatoire distribuée selon $\Gamma_N(\mu)$. Alors μ est la limite faible en espérance de la mesure spectrale de U_N. Autrement dit, pour toute fonction f continue bornée sur $\mathbb{U}$, on a*

$$\int_{\mathbb{U}} f \mathrm{d}\mu = \lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N} \sum_{\substack{\lambda \text{ valeurs propres de } U_N \\ \text{(avec multiplicité)}}} f(\lambda)\right].$$

5.4. Processus de Lévy. Dans ce dernier paragraphe, nous allons utiliser notre résultat asymptotique sur $\Gamma_N(\mu)$ pour montrer l'existence d'un modèle de matrices aléatoires pour les processus de Lévy multiplicatifs libres. Notons que le résultat de Benaych-Georges et Cabanal-Duvillard permet de la même façon de construire un modèle de matrices aléatoires pour les processus de Lévy additifs libres.

Nous appellerons *processus de Lévy multiplicatif libre* une famille $(U_t)_{t\in\mathbb{R}_+}$ de variables non-commutatives unitaires de $\mathcal{A}$ telle que[24]

(1) $U_0 = 1_\mathcal{A}$;
(2) pour tous $0 \leq s \leq t$, la distribution de $U_t U_s^{-1}$ ne dépend que de $t-s$;
(3) pour tous $0 \leq t_1 < \ldots < t_n$, les variables $U_{t_1}, U_{t_2}U_{t_1}^{-1}, \ldots U_{t_n}U_{t_{n-1}}^{-1}$ sont libres ;
(4) la distribution de U_t converge faiblement vers δ_1 lorsque t tend vers 0.

Le mouvement brownien unitaire libre est un cas particulier de processus de Lévy multiplicatif libre. Le théorème suivant peut donc être vu comme une généralisation du théorème de Biane concernant la convergence du mouvement brownien unitaire vers le mouvement brownien unitaire libre.

THÉORÈME (CF. TH.17.3). *Soit $(U_t)_{t\in\mathbb{R}_+}$ un processus de Lévy multiplicatif libre de distributions marginales $(\mu_t)_{t\in\mathbb{R}_+}$ dans $\mathcal{M}_*$. Pour tout $N \geq 1$, il existe un processus de Lévy $(U_t^{(N)})_{t\in\mathbb{R}_+}$ sur le groupe unitaire de distributions marginales $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$; et dans ce cas-là, la distribution non-commutative de $(U_t^{(N)})_{t\in\mathbb{R}_+}$ converge en grande dimension vers celle de $(U_t)_{t\in\mathbb{R}_+}$, au sens où pour tout entier $n \geq 1$, pour tout polynôme non-commutatif P en n variables, et tout choix de n temps positifs $t_1, \ldots, t_n$, nous avons la convergence suivante :*

$$\lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(P\left(U_{t_1}^{(N)}, \ldots, U_{t_n}^{(N)}\right)\right)\right] = \tau(P(U_{t_1}, \ldots, U_{t_n})).$$

[24] Il n'est pas nécessaire ici de préciser une orientation. Puisque τ est une trace, les processus à accroissements multiplicatifs *à gauche* unitaires, libres et stationnaires sont les mêmes que les processus à accroissements multiplicatifs *à droite* unitaires, libres et stationnaires.

Pour établir ce théorème, on procède de la façon suivante. En usant de la qualité de morphisme de Γ_N, on remarque que $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$ est un semigroupe, et on en déduit l'existence d'un processus de Lévy $(U_t^{(N)})_{t\in\mathbb{R}_+}$ de marginales $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$. On observe alors que la convergence en distribution non-commutative de $U_t^{(N)}$ vers U_t n'est que la conséquence du théorème limite déjà établi, car pour tout polynôme $P\in\mathbb{C}[X]$, on a

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}(P(U_t^{(N)}))\right]=\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\sum_{\substack{\lambda \text{ valeurs propres de } U_t^{(N)}\\ \text{(avec multiplicité)}}}P(\lambda)\right]=\int_{\mathbb{U}}P\mathrm{d}\mu_t=\tau(P(U_t)).$$

On en déduit la convergence en distribution non-commutative de chaque accroissement. On conclut en remarquant que les accroissements de $(U_t^{(N)})_{t\in\mathbb{R}_+}$ sont asymptotiquement libres[25], car indépendants et unitairement invariants, d'après un résultat classique de Voiculescu.

La dernière partie de la preuve est plus délicate qu'elle ne le semble. En effet, tous les théorèmes du type «l'invariance par conjugaison unitaire associée à l'indépendance implique la liberté asymptotique» requièrent une condition de concentration, telle qu'une convergence presque sûre, ou un contrôle de la covariance. La démonstration donnée au chapitre 4 ne permet pas de prouver la convergence presque sûre. En revanche, elle met en lumière une propriété de factorisation qui nous permet de conclure[53,80]. Pour tous polynômes $P_1,\ldots,P_k\in\mathbb{C}[X]$, on a

$$\lim_{N\to\infty}\mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left(P_1(U_t^{(N)})\right)\cdots\frac{1}{N}\mathrm{Tr}\left(P_k(U_t^{(N)})\right)\right]=\int_{\mathbb{U}}P_1\mathrm{d}\mu_t\cdots\int_{\mathbb{U}}P_k\mathrm{d}\mu_t.$$

6. Conclusion et perspectives

En guise de conclusion, je vais reprendre les travaux les plus importants de cette thèse, et je vais y associer des développements possibles. En particulier, je vais mentionner quelques énoncés vraisemblablement exacts, mais dont les preuves restent pour le moment hors de portée.

(1) Le premier résultat est la convergence de la transformation de Hall sur le groupe unitaire vers la transformation de Hall libre lorsque la dimension du groupe tend vers l'infini. Ce problème a été la motivation principale du début de cette thèse, et j'y ai répondu positivement pour les fonctions polynomiales. Souvenons-nous cependant que Biane a réussi à étendre la convergence de la transformation de Segal-Bargmann pour une classe de fonctions plus générales grâce à un argument d'approximation. Cet argument échoue dans notre situation car le contrôle de la convergence dépend trop du degré du polynôme considéré. Pour être précis, il s'agit de trouver une norme sur $\mathbb{C}\{X\}$ pour laquelle les actions de Δ_U et de $\tilde{\Delta}_U$ soient bornées, et ainsi d'étendre nos résultats à un espace plus large que $\mathbb{C}\{X\}$. Les normes agréables seraient par exemple des normes de type Sobolev. Pour le moment, aucune norme répondant au problème n'est suffisamment indépendante du degré des polynômes pour être intéressante.

(2) Le second résultat va de pair avec le premier. Il concerne la convergence en distribution non-commutative du mouvement brownien sur le groupe linéaire en grande dimension. Cette thèse contient la démonstration de cette convergence, et même, en collaboration avec Todd Kemp, l'existence de fluctuations infinitésimales. Une question se pose naturellement. La mesure spectrale d'un mouvement brownien sur le groupe linéaire

[25] La liberté asymptotique ne signifie rien d'autre que la convergence en distribution vers des variables libres.

[53] T. Lévy, Schur–Weyl duality and the heat kernel measure on the unitary group (2008).

[80] F. Xu, A random matrix model from two-dimensional Yang-Mills theory (1997).

converge-t-elle en grande dimension ? Pour des matrices normales et bornées[26], la convergence en $*$-distribution non-commutative suffit à assurer la convergence de la mesure spectrale. Dans le cas de matrices non normales, la convergence de la mesure spectrale n'est pas immédiate, et les méthodes existantes reposent sur l'indépendance des coefficients de la matrice. Les techniques pour étudier le spectre du mouvement brownien sur le groupe linéaire restent donc à inventer. Tout au plus peut-on avancer un candidat pour la limite : la mesure de Brown du mouvement brownien circulaire multiplicatif libre. C'est la mesure qui joue le rôle de la mesure spectrale pour des variables aléatoires non-commutatives qui ne sont pas normales.

(3) Vient ensuite la construction de $\mathbb{C}\{X\}$ et de ses variantes, ainsi que les théorèmes d'existence d'opérateurs de transition. Ces objets me semblent particulièrement adaptés à l'étude des processus en probabilités libres. Yoann Dabrowski a déjà exposé dans une conférence[27] en 2014 une utilisation possible d'une version de $\mathbb{C}\{X\}$ dans l'étude du transport optimal libre, et j'ai pour ma part l'idée que cet espace pourrait être au cœur de futurs développements concernant le calcul de Malliavin libre. Mais attendons la suite...

(4) Le résultat du dernier chapitre concernant l'existence d'un morphisme entre les convolutions libres $\boxplus$ et $\boxtimes$ est dans la lignée du travail de Bercovici et Pata. Il serait intéressant de savoir dans quelle mesure ce morphisme s'étend à des mesures non nécessairement infiniment divisibles. Mentionnons que cette question est liée à la recherche d'une condition nécessaire et suffisante pour qu'une suite de coefficients soit une suite de cumulants libres, ou dans le cas multiplicatif, pour qu'une suite de coefficients soit une suite de log-cumulants libres.

(5) Enfin, la dernière contribution de cette thèse est la construction d'un modèle de matrices aléatoires pour les processus de Lévy multiplicatifs libres. La convergence de la mesure spectrale est alors à comprendre au sens de la convergence faible *en moyenne* de la mesure spectrale d'une matrice distribuée selon $\Gamma_N(\mu)$ lorsque N tend vers l'infini. En examinant la démonstration de cette convergence, nous pouvons contrôler la vitesse de convergence par $1/N$. Par une utilisation classique de l'inégalité de Markov, ce taux de convergence est suffisant pour en déduire la convergence faible *en probabilité* de la mesure spectrale. En revanche, pour la convergence faible *presque sûre* de la mesure spectrale, il faudrait un contrôle en $1/N^2$, afin d'appliquer le lemme de Borel-Cantelli. Mes tentatives en ce sens ont pour le moment échoué. Par ailleurs, cette vitesse en $1/N$ est peut-être le signe que, pour ce modèle de matrices aléatoires, les fluctuations sont d'ordre $1/\sqrt{N}$, et non d'ordre $1/N$ comme c'est habituellement le cas en matrices aléatoires.

[26] C'est le cas du mouvement brownien sur le groupe unitaire.
[27] *Free Probability and the Large N Limit (IV)*, organisée par Guionnet, Shlyakhtenko, et Voiculescu. Berkeley, Californie. Mars 2014.

CHAPTER 2

Free convolution operators

This self-contained chapter has been taken from the article [25].

Abstract. We define an extension of the polynomial calculus on a W^*-probability space by introducing an algebra $\mathbb{C}\{X_i : i \in I\}$ which contains polynomials. This extension allows us to define transition operators for additive and multiplicative free convolution. It also permits us to characterize the free Segal-Bargmann transform and the free Hall transform introduced by Biane, in a manner which is closer to classical definitions. Finally, we use this extension of polynomial calculus to prove two asymptotic results on random matrices: the convergence for each fixed time, as N tends to ∞, of the $*$-distribution of the Brownian motion on the linear group $GL_N(\mathbb{C})$ to the $*$-distribution of a free multiplicative circular Brownian motion, and the convergence of the classical Hall transform on $U(N)$ to the free Hall transform.

Contents

7. Introduction

Conditional expectations. Throughout this paper, $(\mathcal{A}, \tau)$ will denote a W^*-probability space, two random variables $A, B \in \mathcal{A}$ will be said to be free if the von Neumann algebras generated by A and B are free, and $\tau(\cdot|B)$ will denote the conditional expectation from $\mathcal{A}$ to the von Neumann algebra generated by B (see Section 9.1).

In [18], Biane shows how to compute some conditional expectations in free product of von Neumann algebras: let us recall Theorem 3.1 of [18], which is in practice accompanied by a characterization of the Feller Markov kernel involved.

Theorem (Biane [18]). *Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Let $A, B \in \mathcal{A}$ be two self-adjoint random variables which are free. Then there exists a Feller Markov kernel $K = k(x, du)$ on $\mathbb{R} \times \mathbb{R}$ such that for any Borel bounded function f on $\mathbb{R}$,*

$$\tau\left(f(A+B) \middle| B\right) = Kf(B)$$

(where $Kf(x) = \int_{\mathbb{R}} f(u)k(x, \mathrm{d}u)$).

In this theorem, the Feller Markov kernel K depends on both A and $B \in \mathcal{A}$. Informally, we might expect that for any Borel bounded function f on $\mathbb{R}$, there exists an object $R_A f$ which depends only on A such that, for all $B \in \mathcal{A}$ free from A,

$$\tau\left(f(A+B) \middle| B\right) = R_A f(B).$$

Let us summarize how we partly answer this question in Section 9.

We extend the notion of polynomial calculus to a more general calculus. Let $A \in \mathcal{A}$. We are more interested in random variables of the type $\tau(P_1(A)) \cdots \tau(P_n(A)) \cdot P_0(A)$, with $n \in \mathbb{N}$, and $P_0, \ldots, P_n \in \mathbb{C}[X]$, than in random variables of the type $P(A)$ with $P \in \mathbb{C}[X]$. One could object that the random variables are exactly the same. However, the interest resides in the fact that, for all $n \in \mathbb{N}, P_0, \ldots, P_n \in \mathbb{C}[X]$, the map

$$A \mapsto \tau(P_1(A)) \cdots \tau(P_n(A)) \cdot P_0(A)$$

cannot be represented by a polynomial calculus nor by a functional calculus, but these sorts of maps are essential in the following development.

Let us denote by $\mathbb{C}\{X\}$ the vector space generated by the formal vectors

$$\{P_0 \operatorname{tr}(P_1) \cdots \operatorname{tr}(P_n) : n \in \mathbb{N}, P_0, \ldots, P_n \in \mathbb{C}[X]\}.$$

For $P = P_0 \operatorname{tr}(P_1) \cdots \operatorname{tr}(P_n) \in \mathbb{C}\{X\}$ and $A \in \mathcal{A}$, let us denote by $P(A)$ the random variable $P(A) = \tau(P_1(A)) \cdots \tau(P_n(A)) \cdot P_0(A)$ and we extend this notation to all $\mathbb{C}\{X\}$ by linearity. We construct the vector space $\mathbb{C}\{X\}$ and the $\mathbb{C}\{X\}$-calculus more precisely in Section 8. In the same section, we construct similarly the space $\mathbb{C}\{X, X^{-1}\}$ (which naturally contains the space of Laurent polynomial $\mathbb{C}[X, X^{-1}]$) and the $\mathbb{C}\{X, X^{-1}\}$-calculus. More generally, for any index set I, we construct the space $\mathbb{C}\{X_i : i \in I\}$ and the $\mathbb{C}\{X_i : i \in I\}$-calculus. We are now able to formulate the following theorem, which is a version of Theorem 9.4 for one variable.

Theorem 7.1. *Let $A \in \mathcal{A}$. There exists a linear operator $\Delta_A : \mathbb{C}\{X\} \to \mathbb{C}\{X\}$ such that, for all polynomials $P \in \mathbb{C}[X]$, and all $B \in \mathcal{A}$ free from A, we have*

$$\tau(P(A+B)|B) = \left(e^{\Delta_A} P\right)(B).$$

Theorem 7.1 (or Theorem 9.4) has two advantages. Firstly, it deals with arbitrary non-commutative random variables, and not only with self-adjoint variables. Secondly, it introduces a transition kernel e^{Δ_A} which depends only on the variable A with which we are concerned with.

In [18], Biane established multiplicative versions of his theorem. There is also a multiplicative version of Theorem 7.1 in Theorem 9.13. Let us formulate a version for one variable.

Theorem 7.2. *Let $A \in \mathcal{A}$. Then there exists a linear operator $\mathrm{D}_A : \mathbb{C}\{X\} \to \mathbb{C}\{X\}$ such that, for all polynomials $P \in \mathbb{C}[X]$, and all $B \in \mathcal{A}$ free from A, we have*

$$\tau(P(AB)|B) = \left(e^{\mathrm{D}_A} P\right)(B).$$

Section 9 is devoted to the proofs of Theorems 9.4 and 9.13, which are multivariate versions of Theorems 7.1 and 7.2.

Free Hall transform. In Section 10, we use and extend Theorems 9.4 and 9.13 to give another description of the free Segal-Bargmann transform in Theorem 10.1 and of the free Hall transform in Theorem 10.7. Let us explain the result for the Hall transform.

Classical Hall transform. Let $N \in \mathbb{N}^*$. We endow the Lie algebra $\mathfrak{u}(N)$ of the unitary group $U(N)$ with the inner product $\langle X, Y\rangle_{\mathfrak{u}(N)} = N \operatorname{Tr}(X^*Y)$, and the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$ of the linear group $GL_N(\mathbb{C})$ with the real-valued inner product $\langle X, Y\rangle_{\mathfrak{gl}_N(\mathbb{C})} = N\Re \operatorname{Tr}(X^*Y)$ (see Section 11). These scalar products determine right-invariant Laplace operators $\Delta_{U(N)}$ and $\Delta_{GL_N(\mathbb{C})}$ respectively. The Brownian motion on $U(N)$ is a Markov process $(U_t^{(N)})_{t\geq 0}$ on $U(N)$, starting at the identity, and with generator $\frac{1}{2}\Delta_{U(N)}$. For all $t \geq 0$, we denote by $L^2(U_t^{(N)})$ the

Hilbert space

$$\left\{ f\left(U_t^{(N)}\right) : f \text{ is a complex Borel function on } U(N) \text{ such that } \mathbb{E}\left[\left|f\left(U_t^{(N)}\right)\right|^2\right] < +\infty \right\}.$$

The Brownian motion on $GL_N(\mathbb{C})$ is a Markov process $(G_t^{(N)})_{t\geq 0}$ on $GL_N(\mathbb{C})$, with generator $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}$, and starting at the identity. Observe that, for convenience, we have taken a definition of the Brownian motion on $GL_N(\mathbb{C})$ which includes an unusual factor of 2 in its generator. In such a way, the Brownian motion proceeds at "half speed" on $GL_N(\mathbb{C})$, just as a standard complex Brownian motion on $\mathbb{C}$ is sometimes defined to be a Markov process with generator $\frac{1}{4}(\partial_x^2+\partial_y^2)$. For all $t \geq 0$, we denote by $L^2_{\text{hol}}(G_t^{(N)})$ the Hilbert space

$$\left\{ F\left(G_t^{(N)}\right) : F \text{ is a holomorphic function on } GL_N(\mathbb{C}) \text{ such that } \mathbb{E}\left[\left|F\left(G_t^{(N)}\right)\right|^2\right] < +\infty \right\}.$$

The fact that $L^2_{\text{hol}}(G_t^{(N)})$ is a Hilbert space is not trivial. It is a part of Hall's theorem which may be stated as follows (see [36], [44] and Section 11.3).

Theorem (Hall [44]). *Let $t > 0$. Let f be a Borel function on $U(N)$ such that $f(U_t^{(N)}) \in L^2(U_t^{(N)})$. The function $e^{\frac{t}{2}\Delta_{U(N)}}f$ has an analytic continuation to a holomorphic function on $GL_N(\mathbb{C})$, also denoted by $e^{\frac{t}{2}\Delta_{U(N)}}f$. Moreover, $(e^{\frac{t}{2}\Delta_{U(N)}}f)(G_t^{(N)}) \in L^2_{\text{hol}}(G_t^{(N)})$ and the linear map*

$$B_t : f\left(U_t^{(N)}\right) \mapsto \left(e^{\frac{t}{2}\Delta_{U(N)}}f\right)\left(G_t^{(N)}\right)$$

is an isomorphism of Hilbert spaces between $L^2(U_t^{(N)})$ and $L^2_{\text{hol}}(G_t^{(N)})$. In particular, for all bounded Borel function f, we have $B_t(f(U_t^{(N)})) = F(G_t^{(N)})$, where F is the analytic continuation of $U \mapsto \mathbb{E}\left[f\left(U_t^{(N)}U\right)\right]$ to all $GL_N(\mathbb{C})$.

The transform B_t is referred to as the Segal-Bargmann transform, the Segal-Bargmann-Hall transform, or the Hall transform. We choose to use the third name. It should be remarked that this formulation of Hall's theorem is quite different (but equivalent) from the original one: the point of view from which it is considered is a probabilistic one. Indeed, we identify the space of square integrable Borel functions with respect to the law of $U_t^{(N)}$ with $L^2(U_t^{(N)})$, and the space of square integrable holomorphic functions with respect to the law of $G_t^{(N)}$ with $L^2_{\text{hol}}(G_t^{(N)})$. This identification between random variables and their functional representations is described in Section 11.3.2.

Free Hall transform. Let $(\mathcal{A}, \tau)$ be a W^*-probability space. Let $(U_t)_{t\geq 0}$ be a free unitary Brownian motion, and let $(G_t)_{t\geq 0}$ be a free circular multiplicative Brownian motion (see [19], or Section 10.3). Let $t \geq 0$. We denote by $L^2(U_t, \tau)$ the Hilbert completion of the *-algebra generated by U_t and U_t^{-1} for the norm $\|\cdot\|_2 : A \mapsto \tau\left(A^*A\right)^{1/2}$, and by $L^2_{\text{hol}}(G_t, \tau)$ the Hilbert completion of the algebra generated by G_t and G_t^{-1} for the same norm $\|\cdot\|_2$.

In [19], Biane defined the free Hall transform. This definition is based on a restriction of the free Segal-Bargmann transform in infinite dimensions, as Gross and Malliavin did in the classical case in [42]. In Theorem 7.3, a simplified version of Theorem 10.7, we give a description of the free Hall transform that is more direct and very close to the classical description thanks to the space $\mathbb{C}\{X, X^{-1}\}$ (see Section 10.6 for more details about those two theorems).

Theorem (Biane [19]). *Let $t > 0$. There exists a linear transform $\mathcal{G}_t$ of the space of Laurent polynomials $\mathbb{C}\left[X, X^{-1}\right]$ such that $\mathcal{F}_t : P\Big(U_t\Big) \mapsto \mathcal{G}_t(P)\Big(G_t\Big)$ is an isometric map which extends to a Hilbert space isomorphism $\mathcal{F}_t$ between $L^2(U_t, \tau)$ and $L^2_{\text{hol}}(G_t, \tau)$.*

THEOREM 7.3. *Let $t > 0$. For all $P \in \mathbb{C}[X]$, $\mathcal{F}_t(P(U_t)) = (e^{\mathrm{D}_{U_t}} P)(G_t)$, where D_{U_t} is given in Theorem 7.2. Moreover, if $(U_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ are free, for all $P \in \mathbb{C}\{X, X^{-1}\}$,*

$$\mathcal{F}_t\Big(P(U_t)\Big) = \tau\Big(P(U_t G_t)\Big|G_t\Big).$$

Random matrices. Section 11 contains an application of our formalism and of our previous results to random matrices. The two main results are summarized in Theorem 7.4: the large-N limit for each fixed time of the non-commutative distribution of the Brownian motion $(G_t^{(N)})_{t\geq 0}$ on $GL_N(\mathbb{C})$, and the large-N limit of the Hall transform for $U(N)$. It should be mentioned that, at the same time as the author, Kemp has studied the first question in [50] and [50], and Driver, Hall and Kemp have studied the second question in [38]. While the approaches are quite similar, they do not entirely overlap.

In [19], Biane suggests boosting B_t to $B_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ with the aim of studying the action of $B_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ on random variables given by the functional calculus for matrices. Let us explain how the space $\mathbb{C}\{X, X^{-1}\}$ allows us to better understand the action of $B_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ on variables given by the polynomial calculus. For all $N \in \mathbb{N}^*$, we endow $M_N(\mathbb{C})$ with the inner product $\langle X, Y\rangle_{M_N(\mathbb{C})} = \frac{1}{N}\operatorname{Tr}(X^*Y)$. Let us identify the space

$$\Big\{f\left(U_t^{(N)}\right) : f \text{ is a Borel function from } U(N) \text{ to } M_N(\mathbb{C}) \text{ such that } \mathbb{E}\left[\left\|f\left(U_t^{(N)}\right)\right\|^2_{M_N(\mathbb{C})}\right] < +\infty\Big\}$$

with $L^2(U_t^{(N)}) \otimes M_N(\mathbb{C})$ and the space

$$\Big\{F\left(G_t^{(N)}\right) : F \text{ is a holomorphic function from } GL_N(\mathbb{C}) \text{ to } M_N(\mathbb{C}) \text{ such that } \mathbb{E}\left[\left\|F\left(G_t^{(N)}\right)\right\|^2_{M_N(\mathbb{C})}\right] < +\infty\Big\}$$

with $L^2_{\mathrm{hol}}(G_t^{(N)}) \otimes M_N(\mathbb{C})$. For all $t > 0$ and $N \in \mathbb{N}^*$, we denote by $B_t^{(N)}$ the Hilbert space isomorphism $B_t \otimes \mathrm{Id}_{M_N(\mathbb{C})}$ from $L^2(U_t^{(N)}) \otimes M_N(\mathbb{C})$ into $L^2_{\mathrm{hol}}(G_t^{(N)}) \otimes M_N(\mathbb{C})$.

Unfortunately, the space of random variables $\{P(U_t^{(N)})\}_{P\in\mathbb{C}[X,X^{-1}]}$ is not transformed by $B_t^{(N)}$ into the space $\{P(G_t^{(N)})\}_{P\in\mathbb{C}[X,X^{-1}]}$. The $\mathbb{C}\{X, X^{-1}\}$-calculus offers us larger spaces which are stable under $B_t^{(N)}$. Indeed, the space of random variables $\{P(U_t^{(N)})\}_{P\in\mathbb{C}\{X,X^{-1}\}}$ is transformed by $B_t^{(N)}$ into the space of random variables $\{P(G_t^{(N)})\}_{P\in\mathbb{C}\{X,X^{-1}\}}$ (Proposition 11.5).

The use of the $\mathbb{C}\{X, X^{-1}\}$-calculus also allows us to study the limit in large dimension. It is already known that the free unitary Brownian motion is the limit in distribution of the Brownian motion on $U(N)$ (see [17], [53], [63] and [67]), which is the first item of the following theorem. For the two other items, see Theorem 11.6 and Theorem 11.7. As mentioned above, the concurrent papers [38] and [50, 51] address respectively the third item and the second item with complementary techniques and points of view.

THEOREM 7.4. *Let $t \geq 0$.*

(1) For all $n \in \mathbb{N}$, and all Laurent polynomials $P_0, \ldots, P_n \in \mathbb{C}[X, X^{-1}]$, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\operatorname{tr}\left(P_0\left(U_t^{(N)}\right)\right)\cdots\operatorname{tr}\left(P_n\left(U_t^{(N)}\right)\right)\right] = \tau\left(P_0\left(U_t\right)\right)\cdots\tau\left(P_n\left(U_t\right)\right).$$

(2) For all $n \in \mathbb{N}$, and all polynomials $P_0, \ldots, P_n \in \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$, we have

$$\lim_{N\to\infty} \mathbb{E}\left[\operatorname{tr}\left(P_0\left(G_t^{(N)}\right)\right)\cdots\operatorname{tr}\left(P_n\left(G_t^{(N)}\right)\right)\right] = \tau\left(P_0\left(G_t\right)\right)\cdots\tau\left(P_n\left(G_t\right)\right).$$

(3) For all Laurent polynomial $P \in \mathbb{C}[X, X^{-1}]$, and $t > 0$, as $N \to \infty$, we have

$$\left\| B_t^{(N)}\Big(P\left(U_t^{(N)}\right)\Big) - \mathcal{G}_t(P)\Big(G_t^{(N)}\Big) \right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)}) \otimes M_N(\mathbb{C})} = O(1/N^2).$$

In fact, the second item can be strengthened to the convergence of the full process $(G_t^{(N)})_{t\geq 0}$ to $(G_t)_{t\geq 0}$ (i.e. the convergence of all finite dimensional distributions). Indeed, since the increments of $(G_t)_{t\geq 0}$ are free, it remains to prove that the increments $(G_t^{(N)})_{t\geq 0}$ are asymptotically free, which is a consequence of the $\mathrm{Ad}(U(N))$-invariance of the law of each increment of $(G_t^{(N)})_{t\geq 0}$. A complete proof is presented in [50].

8. Functional calculus extension

In this section, we define the algebra $\mathbb{C}\{X_i : i \in I\}$. We denote by $\mathbb{C}\langle X_i : i \in I\rangle$ the space of polynomials in the non-commuting indeterminates $(X_i)_{i\in I}$. The algebra $\mathbb{C}\{X_i : i \in I\}$ is an extension of the algebra $\mathbb{C}\langle X_i : i \in I\rangle$. Intuitively, $\mathbb{C}\{X_i : i \in I\}$ is the free algebra generated by I indeterminates $(X_i)_{i\in I}$ and an indeterminate center-valued expectation tr.

We also define a $\mathbb{C}\{X_i : i \in I\}$-calculus which extends the polynomial calculus, but which depends also on the data of another center-valued trace τ. This algebra $\mathbb{C}\{X_i : i \in I\}$ and its functional calculus is the basis of all others sections.

8.1. The algebra $\mathbb{C}\{X_i : i \in I\}$. The algebra $\mathbb{C}\{X_i : i \in I\}$ will be defined by a universal property, which allows us to forget about its construction and to focus on its properties.

We present first the universal property and its immediate consequences before introducing the algebra $\mathbb{C}\{X_i : i \in I\}$ as its unique solution.

8.1.1. *Universal property.* Let $\mathcal{A}$ be a unital complex algebra. The center of $\mathcal{A}$ is the unital complex algebra $Z_{\mathcal{A}}$ formed by elements of $\mathcal{A}$ which commute with all the elements in $\mathcal{A}$. The algebra $\mathcal{A}$ is then a $Z_{\mathcal{A}}$-module. A center-valued expectation τ is a linear function from $\mathcal{A}$ to $Z_{\mathcal{A}}$ such that

(1) for all $A, B \in \mathcal{A}$, we have $\tau(\tau(A)B) = \tau(A)\tau(B)$;
(2) $\tau(1_{\mathcal{A}}) = 1_{\mathcal{A}}$.

Let us remark that the restriction for τ to be a morphism of $Z_{\mathcal{A}}$-modules is not required, since it is not needed in this paper.

Universal property 8.1. *Let I be an arbitrary index set. Let $\mathcal{X}$ be an algebra endowed with a center-valued expectation* tr*, and with I specified elements $(X_i)_{i\in I}$. The triplet $(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I})$ possesses Universal property 8.1 for index set I if for all algebras $\mathcal{A}$ endowed with a center-valued expectation τ, and with I elements $(A_i)_{i\in I}$, there exists a unique algebra homomorphism f from $\mathcal{X}$ to $\mathcal{A}$ such that*

(1) for all $i \in I$, we have $f(X_i) = A_i$;
(2) for all $X \in \mathcal{X}$, we have $\tau(f(X)) = f(\mathrm{tr}(X))$.

Such a homomorphism will be called an I-adapted homomorphism from $(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I})$ to $(\mathcal{A}, \tau, (A_i)_{i\in I})$, or more simply from $\mathcal{X}$ to $\mathcal{A}$. If an I-adapted homomorphism is bijective, we will call it an I-adapted isomorphism.

The nature of Universal property 8.1 induces some properties on its solutions summed up in the following proposition.

Proposition 8.2. *Let I be an arbitrary index set, and let $(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I})$ possess Universal property 8.1 for I.*

(1) The I-adapted isomorphism $\mathrm{id}_{\mathcal{X}}$ is the unique I-adapted algebra automorphism on $\mathcal{X}$.
(2) $(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I})$ is unique in the following sense: if $\left(\mathcal{Y}, \widetilde{\mathrm{tr}}, (Y_i)_{i\in I}\right)$ also possesses Universal property 8.1 for I, there exists a unique I-adapted isomorphism from $\mathcal{X}$ to $\mathcal{Y}$.
(3) There exists a unique algebra homomorphism f, which is injective, from $\mathbb{C}\langle Y_i : i \in I\rangle$ to $\mathcal{X}$ such that $f(Y_i) = X_i$ for all $i \in I$.

Thus, if I is an arbitrary index set, and $(\mathcal{X}, \mathrm{tr}, (X_i)_{i\in I})$ possesses Universal property 8.1 for I, we will always see $\mathbb{C}\langle X_i : i \in I\rangle$ as a subalgebra of $\mathcal{X}$.

Proof. The first assertion is immediate.

Let $\left(\mathcal{Y}, \widetilde{\mathrm{tr}}, (Y_i)_{i\in I}\right)$ also possess Universal property 8.1 for I. There exist a unique I-adapted homomorphism f from $\mathcal{X}$ to $\mathcal{Y}$ and an I-adapted homomorphism g from $\mathcal{Y}$ to $\mathcal{X}$. Then $g \circ f$ is an I-adapted homomorphism from $\mathcal{X}$ to $\mathcal{X}$, and consequently, $g \circ f = \mathrm{id}_{\mathcal{X}}$. Similarly, $f \circ g = \mathrm{id}_{\mathcal{Y}}$, and therefore, f is an I-adapted isomorphism from $\mathcal{X}$ to $\mathcal{Y}$.

For the third assertion, we endow $\mathbb{C}\langle Y_i : i \in I\rangle$ with the center-valued expectation τ such that $\tau(M) = 1$ for all monomials $M \in \mathbb{C}\langle Y_i : i \in I\rangle$. There exists a unique algebra homomorphism f from $\mathbb{C}\langle Y_i : i \in I\rangle$ to $\mathcal{X}$ such that $f(Y_i) = X_i$ for all $i \in I$. There exists an I-adapted homomorphism g from $\mathcal{X}$ to $\mathbb{C}\langle Y_i : i \in I\rangle$. Finally, $g \circ f$ is an algebra automorphism on $\mathbb{C}\langle Y_i : i \in I\rangle$ such that $g \circ f(Y_i) = Y_i$ for all $i \in I$, and thus it is equal to $\mathrm{id}_{\mathbb{C}\langle Y_i : i\in I\rangle}$. We deduce that f is injective. ☐

8.1.2. *Definition.* We now state an existence result in the following proposition.

Proposition-Definition 8.3. *Let I be an arbitrary index set. There exists an object satisfying Universal property 8.1 for I. This unique (up to an I-adapted isomorphism) object will be denoted by $(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in I})$. Furthermore,*

$$\{M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n : n \in \mathbb{N}, M_0, \dots, M_n \textit{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}$$

is a basis of $\mathbb{C}\{X_i : i \in I\}$, called the canonical basis.

Let us recall that $\mathbb{C}\langle X_i : i \in I\rangle$ is viewed as a subalgebra of $\mathbb{C}\{X_i : i \in I\}$. Thus, the set

$$\{M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n : n \in \mathbb{N}, M_0, \dots, M_n, \text{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}$$

is unambiguously defined in $\mathbb{C}\{X_i : i \in I\}$. Furthermore, an I-adapted isomorphism does not modify the definition of this set. Thus, Proposition-Definition 8.3 tells us that this set forms a basis in one particular realization and consequently, forms a basis in any realization of Universal property 8.1.

Proof. We will present a construction of $(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in I})$ in the appendix, which satisfies the characterization of the canonical basis, and a proof of its universal property. ☐

Proposition 8.4. *Let $J \subset I$ be two arbitrary index sets. There exists a unique J-adapted morphism, which is injective, from $(\mathbb{C}\{X_i : i \in J\}, \mathrm{tr}, (X_i)_{i\in J})$ to $(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in J})$.*

If $J \subset I$ are two arbitrary index sets, we will always see $\mathbb{C}\{X_i : i \in J\}$ as a subalgebra of $\mathbb{C}\{X_i : i \in I\}$.

Proof. Let $(Y_i)_{i\in I}$ be elements of $\mathbb{C}\{X_i : i \in J\}$ such that $Y_i = X_i$ for all $i \in J$ (one can set $Y_i = 1_{\mathbb{C}\{X_i : i\in J\}}$ for $i \notin J$).

There exists a unique J-adapted morphism f from $\mathbb{C}\{X_i : i \in J\}$ to $\mathbb{C}\{X_i : i \in I\}$. There exists an I-adapted homomorphism g from $(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in I})$ to $(\mathbb{C}\{X_i : i \in J\}, \mathrm{tr}, (Y_i)_{i\in I})$. Finally, $g \circ f$ is a J-adapted automorphism on $\mathbb{C}\{X_i : i \in J\}$, and thus $g \circ f$ is equal to $\mathrm{id}_{\mathbb{C}\{X_i : i\in J\}}$. We deduce that f is injective.

□

8.1.3. *Degrees.* Let us define a notion of degree on $\mathbb{C}\{X_i : i \in I\}$.

Let $n \in \mathbb{N}$, and $M_0, \ldots, M_n \in \mathbb{C}\langle X_i : i \in I\rangle$ be monomials whose degrees are respectively $k_0, \ldots, k_n \in \mathbb{N}$. The degree of $M_0 \operatorname{tr}(M_1) \cdots \operatorname{tr}(M_n)$ is defined to be $k_0 + \ldots + k_n$. For all $P \in \mathbb{C}\{X_i : i \in I\}$, the degree of P is defined to be the maximal degree of the elements of its decomposition in the canonical basis.

For $d \in \mathbb{N}$, we denote by $\mathbb{C}_d\{X_i : i \in I\}$ the subspace of $\mathbb{C}\{X_i : i \in I\}$ whose elements have degrees less than d. We have $\mathbb{C}\{X_i : i \in I\} = \cup_{d=0}^{\infty} \mathbb{C}_d\{X_i : i \in I\}$. If I is finite, each space $\mathbb{C}_d\{X_i : i \in I\}$ is a finite-dimensional space. In particular, the space $\mathbb{C}\{X_i : i \in I\}$ is the union of finite-dimensional spaces:

$$\mathbb{C}\{X_i : i \in I\} = \bigcup_{\substack{d \in \mathbb{N} \\ J \subset I,\ J \text{ finite}}} \mathbb{C}_d\{X_i : i \in J\}.$$

8.1.4. *Another product.* Let us define on $\mathbb{C}\{X_i : i \in I\}$ a second product which is bilinear and associative.

For all $P, Q \in \mathbb{C}\{X_i : i \in I\}$, the product $P \cdot_{\operatorname{tr}} Q$ is defined by $P \cdot_{\operatorname{tr}} Q = P \operatorname{tr} Q$. The bilinearity is due to the linearity of tr and the associativity is simply due to the fact that, for all P, Q and $R \in \mathbb{C}\langle X_i : i \in I\rangle$, we have

$$P \cdot_{\operatorname{tr}} (Q \cdot_{\operatorname{tr}} R) = P \operatorname{tr}(Q \operatorname{tr}(R)) = P \operatorname{tr} Q \operatorname{tr} R = (P \operatorname{tr} Q) \operatorname{tr} R = (P \cdot_{\operatorname{tr}} Q) \cdot_{\operatorname{tr}} R.$$

Thus, $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\operatorname{tr}})$ is a unital complex algebra. Moreover, this algebra is generated by the monomials of $\mathbb{C}\langle X_i : i \in I\rangle$. Indeed, for all $n \in \mathbb{N}$, $P_0, \ldots, P_n \in \mathbb{C}\{X_i : i \in I\}$, we have

$$P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n = P_0 \cdot_{\operatorname{tr}} P_1 \cdot_{\operatorname{tr}} \cdots \cdot_{\operatorname{tr}} P_n.$$

8.2. The $\mathbb{C}\{X_i : i \in I\}$-calculus. Let $\mathcal{A}$ be a unital complex algebra, and τ be a linear functional on $\mathcal{A}$ such that $\tau(1_{\mathcal{A}}) = 1$.

We define a $\mathbb{C}\{X_i : i \in I\}$-calculus on $(\mathcal{A}, \tau)$ in this way. Let $\mathbf{A} = (A_i)_{i \in I}$ be a family of elements in $\mathcal{A}$. Viewing $\mathbb{C}$ as contained in $\mathcal{A}$ as $\mathbb{C} \cdot 1_{\mathcal{A}}$, the algebra $\mathcal{A}$ is endowed with a center-valued expectation τ, and with I elements $(A_i)_{i \in I}$. Thus, there exists a unique algebra homomorphism f from $\mathbb{C}\{X_i : i \in I\}$ to $\mathcal{A}$ such that

(1) for all $i \in I$, we have $f(X_i) = A_i$;
(2) for all $X \in \mathbb{C}\{X_i : i \in I\}$, we have $\tau(f(X)) = f(\operatorname{tr}(X))$.

For $P \in \mathbb{C}\{X_i : i \in I\}$, we say that $f(P)$ is the element of $\mathcal{A}$ obtained by substitution of $(A_i)_{i \in I}$ for the indeterminates $(X_i)_{i \in I}$ and the substitution of τ for tr in P, and we denote this element by $P(A_i : i \in I) = f(P)$. Thus, the map $P \mapsto P(A_i : i \in I)$ is the unique algebra homomorphism from $(\mathbb{C}\{X_i : i \in I\}, \cdot)$ to $\mathcal{A}$ such that

(1) for all $j \in I$, we have $X_j(A_i : i \in I) = A_j$;
(2) for all $P \in \mathbb{C}\{X_i : i \in I\}$, we have $\tau(P(A_i : i \in I)) = (\operatorname{tr} P)(A_i : i \in I)$.

8.2.1. *The algebra $\mathbb{C}\{X_i, X_i^* : i \in I\}$.* From an arbitrary index set I, we construct the index set $\tilde{I} = I \cup (I \times \{*\})$. For all $i \in I$, the element $X_{(i,*)} \in \mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted by X_i^*, and the algebra $\mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted $\mathbb{C}\{X_i, X_i^* : i \in I\}$. We define a *-algebra structure on $\mathbb{C}\{X_i, X_i^* : i \in I\}$ with the involution * given naturally by taking for all $i \in I$, $(X_i)^* = X_i^*$, and for all $P \in \mathbb{C}\{X_i, X_i^* : i \in I\}$, $(\operatorname{tr} P)^* = \operatorname{tr}(P^*)$.

Let $\mathcal{A}$ be a unital complex *-algebra, and τ be a linear functional on $\mathcal{A}$ such that $\tau(1_{\mathcal{A}}) = 1$. Let us define a $\mathbb{C}\{X_i, X_i^* : i \in I\}$-calculus on $(\mathcal{A}, \tau)$ in this way. Let $(A_i)_{i \in I} \in \mathcal{A}^I$. For all $i \in I$, set $A_{(i,*)} = A_i^*$. For $P \in \mathbb{C}\{X_i, X_i^* : i \in I\}$, let us denote $P(A_i : i \in \tilde{I})$ by $P(A_i : i \in I)$. Thus, the map $P \mapsto P(A_i : i \in I)$ is a *-algebra homomorphism from $(\mathbb{C}\{X_i, X_i^* : i \in I\}, \cdot)$ to $\mathcal{A}$.

8.2.2. *The algebra* $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$. Similarly, from an arbitrary index set I, we construct the index set $\tilde{I} = I \cup (I \times \{*\}) \cup (I \times \{-1\}) \cup (I \times \{-*\})$. For all $i \in I$, the element $X_{(i,*)} \in \mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted by X_i^*, the element $X_{(i,-1)} \in \mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted by X_i^{-1}, and the element $X_{(i,-*)} \in \mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted by X_i^{*-1}. Finally, the algebra $\mathbb{C}\{X_i : i \in \tilde{I}\}$ will be denoted by $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$. We define a *-algebra structure on $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$ with the involution * given naturally by taking for all $i \in I$, $(X_i)^* = X_i^*$ and $(X_i^{-1})^* = X_i^{*-1}$, and for all $P \in \mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$, $(\mathrm{tr}\, P)^* = \mathrm{tr}(P^*)$.

Let $\mathcal{A}$ be a unital complex *-algebra, and τ be a linear functional on $\mathcal{A}$ such that $\tau(1_{\mathcal{A}}) = 1$. Let $P \in \mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$, and $(A_i)_{i\in I} \in \mathcal{A}^I$ be a family of invertible elements. For all $i \in I$, set $A_{(i,*)} = A_i^*$, $A_{(i,-1)} = A_i^{-1}$ and $A_{(i,-*)} = A_i^{*-1}$. For $P \in \mathbb{C}\{X_i, X_i^* : i \in I\}$, let us abuse notation slightly and denote $P(A_i : i \in \tilde{I})$ also by $P(A_i : i \in I)$, as this should cause no confusion. Thus, the map $P \mapsto P(A_i : i \in I)$ is a *-algebra homomorphism from $\mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i \in I\}$ to $\mathcal{A}$.

8.2.3. *Factorization by the distribution.* There exists a useful factorization of the $\mathbb{C}\{X_i : i \in I\}$-calculus by the $\mathbb{C}\langle X_i : i \in I\rangle$-calculus. Let us explain how it works.

Let $\mathcal{A}$ be a unital complex algebra, and τ be a linear functional on $\mathcal{A}$ such that $\tau(1_{\mathcal{A}}) = 1$ and $\tau \geq 0$ (i.e. $\tau(aa^*) \geq 0$ for all $a \in \mathcal{A}$). Such a space is called a non-commutative probability space. Elements of $\mathcal{A}$ are called (non-commutative) random variables. Let $\mathbf{A} = (A_i)_{i\in I}$ be a family of non-commutative variables of $\mathcal{A}$. The map

$$\begin{array}{rccc} \mu_{\mathbf{A}}: & \mathbb{C}\langle X_i : i \in I\rangle & \to & \mathbb{C} \\ & P & \mapsto & \tau(P(\mathbf{A})) \end{array}$$

will be called the distribution of $\mathbf{A}$. The algebra $\mathbb{C}\langle X_i : i \in I\rangle$ is then endowed with a center-valued expectation $\mu_{\mathbf{A}}$, and with I specified elements $(X_i)_{i\in I}$. Thus, there exists a unique algebra homomorphism f from $\mathbb{C}\{X_i : i \in I\}$ to $\mathbb{C}\langle X_i : i \in I\rangle$ such that

(1) for all $i \in I$, we have $f(X_i) = X_i$;
(2) for all $X \in \mathbb{C}\{X_i : i \in I\}$, we have $\mu_{\mathbf{A}}(f(X)) = f(\mathrm{tr}(X))$.

For $P \in \mathbb{C}\{X_i : i \in I\}$, we say that $f(P)$ is the element of $\mathbb{C}\langle X_i : i \in I\rangle$ obtained by substitution of $\mu_{\mathbf{A}}$ for tr in P, and we denote this element by $P|_{\mathbf{A}}$.

However, we can now use polynomial calculus, by substitution of $(A_i)_{i\in I}$ for $(X_i)_{i\in I}$ in $P|_{\mathbf{A}}$. Because the homomorphism from $\mathbb{C}\{X_i : i \in I\}$ to $\mathcal{A}$ given by $P \mapsto P|_{\mathbf{A}}(\mathbf{A})$ is an I-adapted homomorphism, we have that $P|_{\mathbf{A}}(\mathbf{A}) = P(\mathbf{A})$ using the universal property of $\mathbb{C}\{X_i : i \in I\}$.

9. Computation of some conditional expectations

In this section, we show the existence of operators on $\mathbb{C}\{X_i : i \in I\}$ which play the role of transition kernels in the context of free convolution. The first result, Theorem 9.4, deals with additive free convolution whereas the second one, Theorem 9.13, deals with multiplicative free convolution. Despite the analogy of the two theorems, the proofs are completely different. This is to be expected, since the non-commutativity means that the direct connection $e^{x+y} = e^x \cdot e^y$ between addition and multiplication is lost.

9.1. Generalities.

9.1.1. *W^*-probability spaces.* Let $(\mathcal{A}, \tau)$ be a non-commutative probability space such that $\mathcal{A}$ is a von Neumann algebra, and τ is a faithful normal tracial state. That is to say τ is a linear functional such that $\tau(1_{\mathcal{A}}) = 1$, and

(1) for all $A \in \mathcal{A}$, if $A \geq 0$, then $\tau(A) \geq 0$ (positivity),
(2) for all $A, B \in \mathcal{A}$, $\tau(AB) = \tau(BA)$ (traciality),

(3) τ is continuous for the ultraweak topology (normality),
(4) for all $A \in \mathcal{A}$, if $\tau(A^*A) = 0$, then $A = 0$ (faithfulness).

We call $(\mathcal{A}, \tau)$ a W^*-probability space. For all $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$, we denote by $W^*(\mathbf{A})$ the von Neumann subalgebra of $\mathcal{A}$ generated by $(A_i)_{i\in I}$.

9.1.2. *Freeness.* Let I be a set of indices. For all $i \in I$, let $\mathcal{B}_i$ be a von Neumann subalgebra of $\mathcal{A}$. These algebras are called free if, for all $n \in \mathbb{N}$, and all indices $i_1 \neq i_2 \neq \ldots \neq i_n$, whenever $A_j \in \mathcal{B}_{i_j}$ and $\tau(A_j) = 0$ for all $1 \leq j \leq n$, we have $\tau(A_1 \cdots A_n) = 0$.

For all $(A_i)_{i\in I} \in \mathcal{A}^I$, we say that the elements A_i are free for $i \in I$ if the algebras $W^*(A_i)$ are free for $i \in I$. For example, the operators $1_{\mathcal{A}}$ and $0_{\mathcal{A}}$ are free from all $A \in \mathcal{A}$.

9.1.3. *Conditional expectation.* If $\mathcal{B} \subset \mathcal{A}$ is a von Neumann subalgebra, there exists a unique conditional expectation from $\mathcal{A}$ to $\mathcal{B}$ with respect to τ, which we denote by $\tau(.|\mathcal{B})$. This map is a weakly continuous, completely positive, identity preserving, contraction, and it is characterized by the property that, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $\tau(AB) = \tau(\tau(A|\mathcal{B})B)$ (see e.g. [1] and [73]). For any index set I, and $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$, we denote by $\tau(\cdot|\mathbf{A})$ the conditional expectation $\tau(\cdot|W^*(\mathbf{A}))$.

9.2. Free cumulants. This section contains a succinct presentation of the theory of the free cumulants due to Speicher (see e.g. [60], [71], [70] and [69]).

9.2.1. *Non-crossing partitions.* Let S be a totally ordered set. A partition of the set S is said to have a crossing if there exist $i, j, k, l \in S$, with $i < j < k < l$, such that i and k belong to some block of the partition and j and l belong to another block. If a partition has no crossings, it is called non-crossing. The set of all non-crossing partitions of S is denoted by $NC(S)$. It is a lattice with respect to the fineness relation defined as follows: for all π_1 and $\pi_2 \in NC(S)$, $\pi_1 \preceq \pi_2$ if every block of π_1 is contained in a block of π_2.

Let $n \in \mathbb{N}$. When $S = \{1, \ldots, n\}$, with its natural order, we will use the notation $NC(n)$. This set has been first considered by Kreweras in [52]. We denote by 0_n and 1_n respectively the minimal element $\{\{1\}, \ldots, \{n\}\}$ of $NC(n)$, and the maximal element $\{\{1, \ldots, n\}\}$ of $NC(n)$.

9.2.2. *Free cumulants.* For all $n \in \mathbb{N}$, $S \subset \{1, \ldots, n\}$, $\pi \in NC(S)$, and $A_1, \ldots, A_n \in \mathcal{A}$, set

$$\tau[\pi](A_1, \ldots, A_n) = \prod_{V\in\pi} \tau(A_V)$$

where $A_V = A_{j_1} \cdots A_{j_k}$ if $V = \{j_1, ..., j_k\}$ is a block of the partition π, with $j_1 < j_2 < \ldots < j_k$. This way, $\tau[\pi]$ is an n-linear form on $\mathcal{A}$.

We use now the theory of Möbius inversion on lattices (see e.g. [72]). We denote by μ the Möbius function of the poset $(NC(n), \preceq)$, which is defined on

$$\{(\sigma, \pi) : \sigma \preceq \pi\} \subset NC(n) \times NC(n).$$

For all $n \in \mathbb{N}$, and $A_1, \ldots, A_n \in \mathcal{A}$, the free cumulant $\kappa(A_1, \ldots, A_n)$ is defined by

$$\kappa(A_1, \ldots, A_n) = \sum_{\pi\in NC(n)} \mu(\pi, 1_n)\tau[\pi](A_1, \ldots, A_n).$$

If $A_1 = \cdots = A_n = A$, we call $\kappa(A_1, \ldots, A_n)$ the free cumulant of order n of A, and we denote it by $\kappa_n(A)$. For all $n \in \mathbb{N}$, $S \subset \{1, \ldots, n\}$, $\pi \in NC(S)$, and $A_1, \ldots, A_n \in \mathcal{A}$, set

$$\kappa[\pi](A_1, \ldots, A_n) = \prod_{V\in\pi} \kappa(A_V)$$

where $A_V = (A_{j_1}, \ldots, A_{j_k})$ if $V = \{j_1, ..., j_k\}$ is a block of the partition π, with $j_1 < j_2 < \ldots < j_k$. Similarly to τ, $\kappa[\pi]$ is an n-linear form on $\mathcal{A}$. If $A_1 = \cdots = A_n = A$, we denote $\kappa[\pi](A_1, \ldots, A_n)$ by $\kappa[\pi](A)$.

For all $n \in \mathbb{N}$, $S \subset \{1, \ldots, n\}$, $\pi \in NC(S)$, and $A_1, \ldots, A_n \in \mathcal{A}$, we have (by the definition of the Möbius functions) the following relations

$$\tau[\pi](A_1, \ldots, A_n) = \sum_{\substack{\sigma \in NC(S) \\ \sigma \preceq \pi}} \kappa[\sigma](A_1, \ldots, A_n), \tag{9.1}$$

$$\kappa[\pi](A_1, \ldots, A_n) = \sum_{\substack{\sigma \in NC(S) \\ \sigma \preceq \pi}} \mu(\sigma, \pi) \tau[\sigma](A_1, \ldots, A_n). \tag{9.2}$$

The importance of the free cumulants is in large part due to the following characterization of freeness.

Proposition 9.1. *Let $(\mathcal{B}_i)_{i \in I}$ be subalgebras of $\mathcal{A}$. They are free if and only if their mixed cumulants vanish. That is to say: for all $n \in \mathbb{N}^*$, all $i_1, \ldots, i_n \in I$ and all $A_1, \ldots, A_n \in \mathcal{A}$ such that A_j belongs to some $\mathcal{B}_{i_j}$ for all $1 \leq j \leq n$, whenever there exist some j and j' with $i_j \neq i_{j'}$, we have $\kappa(A_1, \ldots, A_n) = 0$.*

By linearity, this property has the following consequence for the computation of cumulants.

Corollary 9.2. *For all $A_1, \ldots, A_n \in \mathcal{A}$ free from $B_1, \ldots, B_n \in \mathcal{A}$, we have*

$$\kappa(A_1 + B_1, \ldots, A_n + B_n) = \kappa(A_1, \ldots, A_n) + \kappa(B_1, \ldots, B_n).$$

9.2.3. *Semi-circular variables.* Let $t \geq 0$. A non-commutative random variable S_t is called semi-circular of variance t if S_t is self-adjoint and the free cumulants of S_t are $\kappa_1(S_t) = 0$, $\kappa_2(S_t) = t$ and $\kappa_n(S_t) = 0$ for all $n > 2$.

The distribution of S_t is given by (9.1): for all $n \in \mathbb{N}^*$, we have $\tau(S_t^n) = \sum_{\pi \in NC(n)} \kappa[\pi](S_t)$. The terms in the sum which are non-zero correspond to the non-crossing partitions composed of pairs of elements. For all $n \in \mathbb{N}^*$, we denote by $NC_2(n)$ the subset of $NC(n)$ of non-crossing partitions composed of pairs of elements. The cardinality of $NC_2(2n)$ is called the n-th Catalan number, and is denoted by C_n. Thus, for all $n \in \mathbb{N}^*$, we have $\tau(S_t^{2n}) = \sum_{\pi \in NC_2(2n)} t^n = t^n C_n$, and $\tau(S_t^{2n-1}) = 0$.

9.3. Additive transition operators. Let I be an arbitrary index set. Let $\mathbf{A} = (A_i)_{i \in I} \in \mathcal{A}^I$. Let us define a derivation $\Delta_{\mathbf{A}}$ associated to $\mathbf{A}$ on $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}})$ in the following way. For all $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$, we set

$$\begin{aligned} \Delta_{\mathbf{A}}\left(X_{i(1)} \cdots X_{i(n)}\right) = \sum_{\substack{1 \leq m \leq n \\ 1 \leq k_1 < \ldots < k_m \leq n}} & \kappa\left(A_{i(k_1)}, \ldots, A_{i(k_m)}\right) X_{i(1)} \cdots X_{i(k_1 - 1)} \\ & \cdot \operatorname{tr}\left(X_{i(k_1+1)} \cdots X_{i(k_2-1)}\right) \\ & \cdot \operatorname{tr}\left(X_{i(k_2+1)} \cdots X_{i(k_3-1)}\right) \\ & \cdots \\ & \cdot \operatorname{tr}\left(X_{i(k_{m-1}+1)} \cdots X_{i(k_m-1)}\right) \\ & \cdot X_{i(k_m+1)} \cdots X_{i(n)} \end{aligned}$$

and we extend $\Delta_{\mathbf{A}}$ to all $\mathbb{C}\{X_i : i \in I\}$ by linearity and by the relation of derivation

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ \Delta_{\mathbf{A}}(P \operatorname{tr} Q) = (\Delta_{\mathbf{A}} P) \operatorname{tr} Q + P \operatorname{tr}(\Delta_{\mathbf{A}} Q).$$

Proposition 9.3. *Let $\mathbf{A} = (A_i)_{i \in I}$ and $\mathbf{B} = (B_i)_{i \in I} \in \mathcal{A}^I$ be two families of elements of $\mathcal{A}$. Then $\Delta_{\mathbf{A}}$ and $\Delta_{\mathbf{B}}$ commute. Moreover, if $\mathbf{A}$ and $\mathbf{B}$ are free, we have $\Delta_{\mathbf{A}+\mathbf{B}} = \Delta_{\mathbf{A}} + \Delta_{\mathbf{B}}$.*

PROOF. We recall that, from Section 8.1.4, the monomials of $\mathbb{C}\langle X_i : i \in I\rangle$ generate the algebra $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}})$. Let M be a monomial of $\mathbb{C}\langle X_i : i \in I\rangle$. Let $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$ such that $M = X_{i(1)} \cdots X_{i(n)}$. For all $1 \leq k < l \leq n$, we will denote by $\mathrm{tr}_{k,l}$ the element $\mathrm{tr}(X_{i(k+1)} \cdots X_{i(l-1)})$. Since $\Delta_{\mathbf{B}}$ is a derivation for $\cdot_{\mathrm{tr}}$, we have

$$\begin{aligned}\Delta_{\mathbf{B}}\left(\Delta_{\mathbf{A}}\left(X_{i(1)}\cdots X_{i(n)}\right)\right) = & \sum_{\substack{1\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}} \kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right)\cdot \mathrm{tr}_{k_1,k_2}\cdots\mathrm{tr}_{k_{m-1},k_m}\\ &\qquad \cdot\Delta_{\mathbf{B}}\left(X_{i(1)}\cdots X_{i(k_1-1)}X_{i(k_m+1)}\cdots X_{i(n)}\right)\\ &+\sum_{\substack{1\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n\\ 1\leq p\leq m-1}} \kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right) X_{i(1)}\cdots X_{i(k_1-1)}\\ &\qquad \cdot\mathrm{tr}_{k_1,k_2}\cdots\Delta_{\mathbf{B}}(\mathrm{tr}_{k_p,k_{p+1}})\cdots\mathrm{tr}_{k_{m-1},k_m}\\ &\qquad \cdot X_{i(k_m+1)}\cdots X_{i(n)}.\end{aligned}$$

Let us fix $1 \leq m \leq n$ and $1 \leq k_1 < \ldots < k_m \leq n$ in the first sum above and compute the corresponding term of the sum:

$$\begin{aligned}\kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right)\cdot\Bigg(& \sum_{\substack{1\leq m'\leq n\\ 1\leq l_1<\ldots<l_{m'}<k_1}} \kappa\left(B_{i(l_1)},\ldots,B_{i(l_{m'})}\right) X_{i(1)}\cdots X_{i(l_1-1)}\\ &\qquad \cdot\mathrm{tr}_{l_1,l_2}\cdots\mathrm{tr}_{l_{m'-1},l_{m'}}\cdot X_{i(l_{m'}+1)}\cdots X_{i(k_1-1)}\\ &\qquad \cdot\mathrm{tr}_{k_1,k_2}\cdots\mathrm{tr}_{k_{m-1},k_m}\cdot X_{i(k_m+1)}\cdots X_{i(n)}\\ &+\sum_{\substack{1\leq m'\leq n\\ k_m<l_1<\ldots<l_{m'}\leq n}} \kappa\left(B_{i(l_1)},\ldots,B_{i(l_{m'})}\right) X_{i(1)}\cdots X_{i(k_1-1)}\\ &\qquad \cdot\mathrm{tr}_{k_1,k_2}\cdots\mathrm{tr}_{k_{m-1},k_m}\cdot X_{i(k_m+1)}\cdots X_{i(l_1-1)}\\ &\qquad \cdot\mathrm{tr}_{l_1,l_2}\cdots\mathrm{tr}_{l_{m'-1},l_{m'}}\cdot X_{i(l_{m'}+1)}\cdots X_{i(n)}\\ &+\sum_{\substack{1\leq m'\leq n\\ 1\leq q\leq m'-1\\ 1\leq l_1<\ldots l_q<k_1\\ k_m<l_{q+1}<\ldots<l_{m'}\leq n}} \kappa\left(B_{i(l_1)},\ldots,B_{i(l_{m'})}\right) X_{i(1)}\cdots X_{i(l_1-1)}\\ &\qquad \cdot\mathrm{tr}_{l_1,l_2}\cdots\mathrm{tr}_{l_{q-1},l_q}\cdot\mathrm{tr}_{k_1,k_2}\cdots\mathrm{tr}_{k_{m-1},k_m}\\ &\qquad \cdot\mathrm{tr}\left(X_{i(l_q+1)}\cdots X_{i(k_1-1)}X_{i(k_m+1)}\cdots X_{i(l_{q+1}-1)}\right)\\ &\qquad \cdot\mathrm{tr}_{l_{q+1},l_{q+2}}\cdots\mathrm{tr}_{l_{m'-1},l_{m'}}\cdot X_{i(l_{m'}+1)}\cdots X_{i(n)}\Bigg).\end{aligned}$$

When summing over the indices $1 \leq k_1 < \ldots < k_m \leq n$, the elements $\mathbf{A}$ and $\mathbf{B}$ play symmetric roles in the two first sums, while the last sum gives us the term

$$\begin{aligned}\sum_{\substack{1\leq m'\leq n\\ 1\leq l_1<\ldots<l_{m'}\leq n\\ 1\leq q\leq m'-1}} & \kappa\left(B_{i(k_1)},\ldots,B_{i(k_m)}\right) X_{i(1)}\cdots X_{i(l_1-1)}\\ &\cdot\mathrm{tr}_{l_1,l_2}\cdots\Delta_{\mathbf{A}}(\mathrm{tr}_{l_q,l_{q+1}})\cdots\mathrm{tr}_{l_{m'-1},l_{m'}}\\ &\cdot X_{i(l_{m'}+1)}\cdots X_{i(n)}\end{aligned}$$

which is exactly the term needed to achieve the symmetry of the roles of $\mathbf{A}$ and $\mathbf{B}$ in $\Delta_{\mathbf{B}}\Delta_{\mathbf{A}}(M)$. Thus, $\Delta_{\mathbf{A}}\Delta_{\mathbf{B}}(M) = \Delta_{\mathbf{B}}\Delta_{\mathbf{A}}(M)$ for all monomials of $\mathbb{C}\langle X_i : i \in I\rangle$, and the commutativity is extended by recurrence on all $\mathbb{C}\{X_i : i \in I\}$ because they are both derivations. Indeed, for all $P, Q \in \mathbb{C}\{X_i : i \in I\}$,

$$\Delta_{\mathbf{A}}\Delta_{\mathbf{B}}\left(P\,\mathrm{tr}\,Q\right) = (\Delta_{\mathbf{A}}\Delta_{\mathbf{B}}P)\,\mathrm{tr}\,(Q) + P\,\mathrm{tr}\,(\Delta_{\mathbf{A}}\Delta_{\mathbf{B}}Q) + (\Delta_{\mathbf{A}}P)\,\mathrm{tr}\,(\Delta_{\mathbf{B}}Q) + (\Delta_{\mathbf{B}}P)\,\mathrm{tr}\,(\Delta_{\mathbf{A}}Q).$$

If $\mathbf{A}$ and $\mathbf{B}$ are free, we have from Corollary 9.2 that $\Delta_{\mathbf{A}+\mathbf{B}}M = (\Delta_{\mathbf{A}} + \Delta_{\mathbf{B}})M$ for all monomials M of $\mathbb{C}\langle X_i : i \in I\rangle$, and since $\Delta_{\mathbf{A}+\mathbf{B}}$ and $\Delta_{\mathbf{A}} + \Delta_{\mathbf{B}}$ are both derivations, they are equal on $\mathbb{C}\{X_i : i \in I\}$. □

Let $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$. The operator $\Delta_{\mathbf{A}}$ makes the degree strictly decrease. Thus, the operator $e^{\Delta_{\mathbf{A}}}$ on $\mathbb{C}\{X_i : i \in I\}$ is defined by the formal series $\sum_{k=0}^{\infty}\frac{1}{k!}\Delta_{\mathbf{A}}^k$. For all $P \in \mathbb{C}_d\{X_i :$

$i \in I\}$, $e^{\Delta_{\mathbf{A}}} P$ is the finite sum $\sum_{k=0}^{d} \frac{1}{k!} \Delta_{\mathbf{A}}^{k} P$. The operator $\Delta_{\mathbf{A}}$ is a derivation, and we therefore have the Leibniz formula

$$\forall k \in \mathbb{N}, \forall P, Q \in \mathbb{C}\{X_i : i \in I\}, (\Delta_{\mathbf{A}})^k (P \operatorname{tr} Q) = \sum_{l=0}^{k} \binom{k}{l} \left(\Delta_{\mathbf{A}}^{l} P\right) \operatorname{tr} \left(\Delta_{\mathbf{A}}^{k-l} Q\right),$$

from which we deduce, using the standard power series argument, that the operator $e^{\Delta_{\mathbf{A}}}$ is multiplicative in the following sense:

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ e^{\Delta_{\mathbf{A}}} (P \operatorname{tr} Q) = \left(e^{\Delta_{\mathbf{A}}} P\right) \operatorname{tr} \left(e^{\Delta_{\mathbf{A}}} Q\right).$$

THEOREM 9.4. *Let I be an arbitrary index set. Let $\mathbf{A} = (A_i)_{i \in I} \in \mathcal{A}^I$. For all $P \in \mathbb{C}\{X_i : i \in I\}$, and all $\mathbf{B} = (B_i)_{i \in I} \in \mathcal{A}^I$ free from $(A_i)_{i \in I}$, we have*

$$\tau\left(P\left(\mathbf{A} + \mathbf{B}\right) \middle| \mathbf{B}\right) = \left(e^{\Delta_{\mathbf{A}}} P\right)(\mathbf{B}).$$

In particular, for all $P \in \mathbb{C}\{X_i : i \in I\}$, we have

$$\tau\left(P\left(\mathbf{A}\right)\right) = \left(e^{\Delta_{\mathbf{A}}} P\right)(0).$$

Example. Let $t \geq 0$. Let S_t be a semi-circular random variable of variance t. For all $B \in \mathcal{A}$ free from S_t, let us compute $\tau((S_t + B)^3 | B)$. We have $\Delta_{S_t} X^3 = 2tX + t \operatorname{tr}(X)$ and $(\Delta_{S_t})^2 X^3 = \Delta_{S_t}(2tX + t \operatorname{tr}(X)) = 0$. Thus, $e^{\Delta_{S_t}}(X^3) = X^3 + \Delta_{S_t} X^3 + 0 = X^3 + 2tX + t \operatorname{tr}(X)$. Using Theorem 9.13, we have, for all $B \in \mathcal{A}$ free from S_t,

$$\tau\left((S_t + B)^3 | B\right) = B^3 + 2tB + t\tau(B).$$

PROOF OF THEOREM 9.4. One could prove Theorem 9.4 directly but it would be very combinatorial and we prefer to present here a more dynamical proof where the combinatorics only appear infinitesimally. We shall prove first a dynamical lemma and a weaker version of Theorem 9.4 before proving the theorem in all generality.

Let us first prove Lemma 9.5, which will be useful in the rest of the paper.

LEMMA 9.5. *Let J be a finite index set and $d \in \mathbb{N}$. Let L be a linear operator on $\mathbb{C}_d\{X_i : i \in J\}$ and $(\phi_t)_{t \geq 0}$ be linear functionals on $\mathbb{C}_d\{X_i : i \in J\}$. Let us assume that, for all $P \in \mathbb{C}_d\{X_i : i \in J\}$, $t \mapsto \phi_t(P)$ is differentiable on $[0, \infty)$, and for all $t \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t(P) = \phi_t(LP).$$

If there exists $\mathbf{A} = (A_i)_{i \in J} \in \mathcal{A}^I$ such that $\phi_0(P) = P(\mathbf{A})$, then for all $t \geq 0$ and all $P \in \mathbb{C}_d\{X_i : i \in J\}$, we have

$$\phi_t(P) = e^{tL} P\left(\mathbf{A}\right).$$

PROOF. Let us fix a finite basis $\{P_b : b \in B\}$ of $\mathbb{C}_d\{X_i : i \in J\}$. For all $P \in \mathbb{C}_d\{X_i : i \in J\}$, let us denote by $(\alpha_b(P))_{b \in B}$ the coefficients of P in the basis $\{P_b : b \in B\}$. We claim that the functions $(t \mapsto \phi_t(P_b))_{b \in B}$ and $(t \mapsto e^{tL} P_b(\mathbf{A}))_{b \in B}$ are both the unique solution to the multidimensional differential equation

$$\begin{cases} y_b(0) &= P_b\left(\mathbf{A}\right), \\ y_b' &= \displaystyle\sum_{c \in B} \alpha_c\left(LP_b\right) \cdot y_c, \forall b \in B. \end{cases}$$

Indeed, let $b \in B$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tL} P_b(\mathbf{A}) = e^{tL} L P_b(\mathbf{A}) = \sum_{c \in B} \alpha_c(LP_b) \cdot e^{tL} P_c(\mathbf{A})$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \phi_t(P_b) = \phi_t(LP_b) = \sum_{c \in B} \alpha_c(LP_b) \cdot \phi_t(P_c).$$

Furthermore, $e^{0L} P_b(\mathbf{A}) = \phi_0(P_b) = P_b(\mathbf{A})$. Hence, we have $\phi_t(P_b) = e^{tL} P_b(\mathbf{A})$ for all $b \in B$. We extend the relation $\phi_t(P) = e^{tL} P(\mathbf{A})$ from $P \in \{P_b : b \in B\}$ to all $\mathbb{C}_d\{X_i : i \in J\}$ by linearity. ☐

The following lemma is a weak version of Theorem 9.4.

Lemma 9.6. *Let I be an arbitrary index set. Let $\mathbf{A} = (A_i)_{i \in I} \in \mathcal{A}^I$. For all $P \in \mathbb{C}\{X_i : i \in I\}$, we have*

$$\tau(P(\mathbf{A})) = (e^{\Delta_{\mathbf{A}}} P)(0).$$

Proof. For all $n \in \mathbb{N}$ and $\pi \in NC(n)$, let us denote by $|\pi|$ the number of blocks in π.

Let $t \geq 0$. For all $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$, we set (with the convention $0^0 = 1$)

$$\phi_t\left(X_{i(1)} \cdots X_{i(n)}\right) = \sum_{\pi \in NC(n)} t^{|\pi|} \kappa[\pi](A_{i(1)}, \ldots, A_{i(n)})$$

and we extend ϕ_t to all $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}})$ by linearity and by the multiplicative relation

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\}, \phi_t(P \operatorname{tr} Q) = \phi_t(P)\tau(\phi_t(Q)).$$

We remark that, using (9.1), we find $\tau(M(\mathbf{A})) = \phi_1(M)$ for all monomials M of $\mathbb{C}\langle X_i : i \in I\rangle$. Moreover, the map $P \mapsto \tau(P(\mathbf{A}))$ satisfies the same multiplicative relation as ϕ_1. Indeed, we have

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ \tau\Big(P \operatorname{tr} Q(\mathbf{A})\Big) = \tau\Big(P(\mathbf{A})\Big)\tau\Big(\tau\Big(Q(\mathbf{A})\Big)\Big).$$

It follows that $\phi_1(P) = \tau(P(\mathbf{A}))$ for all $P \in \mathbb{C}\{X_i : i \in I\}$. Thus, it remains to prove that, for all $\mathbb{C}\{X_i : i \in I\}$, we have $\phi_1(P) = (e^{\Delta_{\mathbf{A}}} P)(0)$. In order to use Lemma 9.5 in the third step, we will prove that, for all $P \in \mathbb{C}\{X_i : i \in I\}$, $\phi_0(P) = P(0)$ in the first step, and that, for all $t \geq 0$ and all $P \in \mathbb{C}\{X_i : i \in I\}$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(P) = \phi_t(\Delta_{\mathbf{A}} P)$ in the second step.

Step 1. For all $P \in \mathbb{C}\{X_i : i \in I\}$, $\phi_0(P) = P(0)$. Indeed, $\phi_0(1) = 1 = M(0)$ for $M = 1 \in \mathbb{C}\langle X_i : i \in I\rangle$ and $\phi_0(M) = 0 = M(0)$ for all monomials M of $\mathbb{C}\langle X_i : i \in I\rangle$ with non-zero degree. We infer the equality $\phi_0(P) = P(0)$ to all $\mathbb{C}\{X_i : i \in I\}$, since the evaluation satisfies the same multiplicative relation as ϕ_0. Indeed, we have

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ (P \operatorname{tr} Q)(0) = P(0)\tau(Q(0)).$$

Step 2. We prove now that, for all $t \geq 0$ and all $P \in \mathbb{C}\{X_i : i \in I\}$, we have $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(P) = \phi_t(\Delta_{\mathbf{A}} P)$.

Let $t \geq 0$ and M be a monomial of $\mathbb{C}\langle X_i : i \in I\rangle$. Let us fix $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$ such that $M = X_{i(1)} \cdots X_{i(n)}$. We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(X_{i(1)}\cdots X_{i(n)}\right) &= \sum_{\pi\in NC(n)} |\pi| t^{|\pi|-1}\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right)\\
&= \sum_{\pi\in NC(n)} \sum_{V\in\pi} t^{|\pi|-1}\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right)\\
&= \sum_{V\subset\{1,\ldots,n\}} \sum_{\substack{\pi\in NC(n)\\ V\in\pi}} t^{|\pi|-1}\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right).
\end{aligned}$$

Let us fix a subset $V = \{k_1, \ldots, k_m\}$ of $\{1, \ldots, n\}$ such that $1 \leq k_1 < \ldots < k_m \leq n$. Let us denote $W_1 = \{k_1+1, \ldots, k_2-1\}, W_2 = \{k_2+1, \ldots, k_3-1\}, \ldots, W_{m-1} = \{k_{m-1}+1, \ldots, k_m-1\}$ and $W_m = \{1, \ldots, k_1-1, k_m+1, \ldots, k_n\}$. To each partition $\pi \in NC(n)$ such that $V \in \pi$, we associate for all $1 \leq i \leq m$ the partition $\pi_i \in NC(W_i)$ induced by π on W_i. Conversely, to each m-tuple $\pi_1 \in NC(W_1), \ldots, \pi_m \in NC(W_m)$, we associate the non-crossing partition $\pi = \{V\} \cup \pi_1 \cup \cdots \cup \pi_m \in NC(n)$. Because of the non-crossing condition, this leads to a bijective correspondence

$$\{\pi \in NC(n) :\ V \in \pi\} \leftrightarrow \{(\pi_1, \ldots, \pi_m) \in NC(W_1) \times \cdots \times NC(W_m)\}$$

which allows us to sum separately each non-crossing partition on each subset $W_1, \ldots, W_m$.

We have to examine now how the terms of the sum are transformed. Let us fix $\pi \in NC(n)$ such that $V \in \pi$. Let us define $(\pi_1, \ldots, \pi_m) \in NC(W_1) \times \cdots \times NC(W_m)$ as before. We have

$$t^{|\pi|-1} = t^{|\{V\}\cup\pi_1\cup\cdots\cup\pi_m|-1} = t^{|\pi_1|}\cdots t^{|\pi_m|}$$

and

$$\kappa[\pi](A_{i(1)}, \ldots, A_{i(n)}) = \kappa(A_{i(k_1)}, \ldots, A_{i(k_m)}) \cdot \kappa[\pi_1](A_{i(1)}, \ldots, A_{i(n)}) \cdots \kappa[\pi_m](A_{i(1)}, \ldots, A_{i(n)}).$$

We infer

$$\begin{aligned}
&\sum_{\substack{\pi\in NC(n)\\ V\in\pi}} t^{|\pi|-1}\kappa[\pi](A_{i(1)},\ldots,A_{i(n)})\\
&= \kappa(A_{i(k_1)},\ldots,A_{i(k_m)})\\
&\qquad \cdot\left(\sum_{\pi_1\in NC(W_1)} t^{|\pi|}\kappa[\pi](A_{i(1)},\ldots,A_{i(n)})\right)\cdots\left(\sum_{\pi_m\in NC(W_m)} t^{|\pi|}\kappa[\pi](A_{i(1)},\ldots,A_{i(n)})\right)\\
&= \kappa(A_{i(k_1)},\ldots,A_{i(k_m)})\\
&\qquad \cdot\phi_t\left(X_{i(k_1+1)}\cdots X_{i(k_2-1)}\right)\\
&\qquad \cdots\\
&\qquad \cdot\phi_t\left(X_{i(k_{m-1}+1)}\cdots X_{i(k_m-1)}\right)\\
&\qquad \cdot\phi_t\left(X_{i(1)}\cdots X_{i(k_1-1)}X_{i(k_m+1)}\cdots X_{i(n)}\right).
\end{aligned}$$

Finally, we use the definition of $\Delta_{\mathbf{A}}$ and the multiplicative relation of ϕ_t to infer

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(X_{i(1)}\cdots X_{i(n)}\right)\\
&= \sum_{\substack{1\le m\le n\\ 1\le k_1<\ldots<k_m\le n}} \sum_{\substack{\pi\in NC(n)\\ \{k_1,\ldots,k_m\}\in\pi}} t^{|\pi|-1}\kappa[\pi](A_{i(1)},\ldots,A_{i(n)})\\
&= \sum_{\substack{1\le m\le n\\ 1\le k_1<\ldots<k_m\le n}} \kappa\left(A_{i(k_1)},\ldots,A_{i(k_m)}\right)\\
&\qquad \cdot\tau\left(\phi_t\left(X_{i(k_1+1)}\cdots X_{i(k_2-1)}\right)\right)\\
&\qquad \cdots\\
&\qquad \cdot\tau\left(\phi_t\left(X_{i(k_{m-1}+1)}\cdots X_{i(k_m-1)}\right)\right)\\
&\qquad \cdot\phi_t\left(X_{i(1)}\cdots X_{i(k_1-1)}X_{i(k_m+1)}\cdots X_{j(n)}\right)\\
&= \phi_t\left(\Delta_{\mathbf{A}}\left(X_{i(1)}\cdots X_{i(n)}\right)\right).
\end{aligned}$$

We extend the equality $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(P)=\phi_t\left(\Delta_{\mathbf{A}}P\right)$ from monomials of $\mathbb{C}\langle X_i : i\in I\rangle$ to all $P\in\mathbb{C}\{X_i : i\in I\}$ by linearity and by the following induction. If P and $Q\in\mathbb{C}\{X_i : i\in I\}$ verify $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(P)=\phi_t\left(\Delta_{\mathbf{A}}P\right)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(Q)=\phi_t\left(\Delta_{\mathbf{A}}Q\right)$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(P\operatorname{tr}Q\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_t\left(P\right)\tau\left(\phi_t(Q)\right)\Big)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_t\left(P\right)\right)\tau\left(\phi_t(Q)\right)+\phi_t\left(P\right)\tau\left(\frac{\mathrm{d}}{\mathrm{d}t}\phi_t(Q)\right)\\
&= \phi_t\left(\Delta_{\mathbf{A}}P\right)\tau\left(\phi_t(Q)\right)+\phi_t\left(P\right)\tau\left(\phi_t\left(\Delta_{\mathbf{A}}Q\right)\right)\\
&= \phi_t\Big(\Delta_{\mathbf{A}}P\cdot\tau\left(Q\right)+P\cdot\tau\left(\Delta_{\mathbf{A}}Q\right)\Big)\\
&= \phi_t\Big(\Delta_{\mathbf{A}}\left(P\operatorname{tr}Q\right)\Big).
\end{aligned}$$

Step 3. Let $P\in\mathbb{C}\{X_i : i\in I\}$. There exist a finite index set $J\subset I$ and $d\in\mathbb{N}$ such that $P\in\mathbb{C}_d\{X_i : i\in J\}$. We remark that $\Delta_{\mathbf{A}}$ is a linear operator on $\mathbb{C}_d\{X_i : i\in J\}$ and that $(\phi_t)_{t\ge 0}$ are linear functionals. We deduce from Lemma 9.5 and Steps 1 and 2 that, for all $t\ge 0$, we have $\phi_t(P)=\left(e^{t\Delta_{\mathbf{A}}}P\right)(0)$. In particular, $\left(e^{\Delta_{\mathbf{A}}}P\right)(0)=\phi_1(P)=\tau\left(P\left(\mathbf{A}\right)\right)$. ☐

Let us finish the proof of Theorem 9.4. Let $\mathbf{A}=(A_i)_{i\in I}\in\mathcal{A}^I$ and $\mathbf{B}=(B_i)_{i\in I}\in\mathcal{A}^I$ be free, and $P\in\mathbb{C}\{X_i : i\in I\}$. We remark that, thanks to Proposition 9.3, we know that $\Delta_{\mathbf{A}+\mathbf{B}}=\Delta_{\mathbf{A}}+\Delta_{\mathbf{B}}$, and that $\Delta_{\mathbf{A}}$ and $\Delta_{\mathbf{B}}$ commute. We deduce directly from Lemma 9.6 that

$$\tau\left(P(\mathbf{A}+\mathbf{B})\right)=\left(e^{\Delta_{\mathbf{A}+\mathbf{B}}}P\right)(0)=\left(e^{\Delta_{\mathbf{A}}+\Delta_{\mathbf{B}}}P\right)(0)=\left(e^{\Delta_{\mathbf{B}}}e^{\Delta_{\mathbf{A}}}P\right)(0)=\tau\Big(\left(e^{\Delta_{\mathbf{A}}}P\right)(\mathbf{B})\Big). \tag{9.3}$$

We will now use the following characterization of conditional expectation. The element $\tau(P(\mathbf{A}+\mathbf{B})|\mathbf{B})$ is the unique element of $W^*(\mathbf{B})$ such that, for all $B_{i_0}\in W^*(\mathcal{B})$,

$$\tau\left(P(\mathbf{A}+\mathbf{B})B_{i_0}\right)=\tau\left(\tau\left(P(\mathbf{A}+\mathbf{B})|\,\mathbf{B}\right)B_{i_0}\right).$$

Since $\left(e^{\Delta_{\mathbf{A}}}P\right)(\mathbf{B})\in W^*(\mathbf{B})$, it remains to prove that, for all $B_{i_0}\in W^*(\mathcal{B})$,

$$\tau\left(P(\mathbf{A}+\mathbf{B})B_{i_0}\right)=\tau\left(e^{\Delta_{\mathbf{A}}}P\left(\mathbf{B}\right)B_{i_0}\right).$$

In order to use (9.3), we prefer to work on $\mathbb{C}\{X_i : i\in I\cup\{i_0\}\}$. Let $R_{i_0} : P\mapsto PX_{i_0}$ be the operator of right multiplication by X_{i_0} on $\mathbb{C}\{X_i : i\in I\cup\{i_0\}\}$. Let $A_{i_0}=0$ and $B_{i_0}\in W^*(\mathcal{B})$. On one hand, we have $P(\mathbf{A}+\mathbf{B})B_{i_0}=(R_{i_0}P)(\mathbf{A}+\mathbf{B},A_{i_0}+B_{i_0})$, and using (9.3), we have $\tau\left(P(\mathbf{A}+\mathbf{B})B_{i_0}\right)=\tau\left(\left(e^{\Delta_{\mathbf{A},A_{i_0}}}R_{i_0}(P)\right)(\mathbf{B},B_{i_0})\right)$. On the other hand, $\tau\left(e^{\Delta_{\mathbf{A}}}P\left(\mathbf{B}\right)B_{i_0}\right)=\tau\left(\left(R_{i_0}e^{\Delta_{\mathbf{A},A_{i_0}}}(P)\right)(\mathbf{B},B_{i_0})\right)$. Thus, it remains to prove that the operators $\Delta_{\mathbf{A},A_{i_0}}$ and R_{i_0}

commute. Let us check this on monomials. For all $n \in \mathbb{N}$ and $i(1), \dots, i(n) \in I \cup \{i_0\}$, let us fix $i(n+1) = i_0$. For all $1 \le k < l \le n+1$, we will denote by $\mathrm{tr}_{k,l}$ the element $\mathrm{tr}(X_{i(k+1)} \cdots X_{i(l-1)})$. Because all free cumulants involving $A_{i_0} = 0$ are equal to zero, we have

$$\begin{aligned}
&\Delta_{\mathbf{A}} R_{i_0}\left(X_{i(1)} \cdots X_{i(n)}\right) \\
&= \sum_{\substack{1 \le m \le n \\ 1 \le k_1 < \dots < k_m \le n+1}} \kappa\left(A_{i(k_1)}, \dots, A_{i(k_m)}\right) X_{i(1)} \cdots X_{i(k_1-1)} \\
&\qquad\qquad \cdot \mathrm{tr}_{k_1,k_2} \cdots \mathrm{tr}_{k_{m-1},k_m} \cdot X_{i(k_m+1)} \cdots X_{i(n+1)} \\
&= \sum_{\substack{1 \le m \le n \\ 1 \le k_1 < \dots < k_m \le n}} \kappa\left(A_{i(k_1)}, \dots, A_{i(k_m)}\right) X_{i(1)} \cdots X_{i(k_1-1)} \\
&\qquad\qquad \cdot \mathrm{tr}_{k_1,k_2} \cdots \mathrm{tr}_{k_{m-1},k_m} \cdot X_{i(k_m+1)} \cdots X_{i(n)} X_{i_0} \\
&= R_{i_0} \Delta_{\mathbf{A}}\left(X_{i(1)} \cdots X_{i(n)}\right).
\end{aligned}$$

The commutativity is extended to all $\mathbb{C}\{X_i : i \in I \cup \{i_0\}\}$ by an immediate induction, because for all $P, Q \in \mathbb{C}\{X_i : i \in I \cup \{i_0\}\}$, we have

$$\Delta_{\mathbf{A}} R_{i_0}(P \,\mathrm{tr}\, Q) = \left(\Delta_{\mathbf{A}} R_{i_0}(P)\right) \mathrm{tr}\, Q + R_{i_0}\left(P \,\mathrm{tr}(\Delta_{\mathbf{A}}(Q))\right)$$

and

$$R_{i_0} \Delta_{\mathbf{A}}(P \,\mathrm{tr}\, Q) = \left(R_{i_0} \Delta_{\mathbf{A}}(P)\right) \mathrm{tr}\, Q + R_{i_0}\left(P \,\mathrm{tr}(\Delta_{\mathbf{A}}(Q))\right). \qquad \square$$

9.4. Free log-cumulants. Consider two free random variables $A, B \in \mathcal{A}$. The distribution of AB is determined by the distributions of A and B separately. The machinery for doing this computation, in the same spirit as Corollary 9.2, is the S-transform (see Section 3.6 in [74]).

Since we have to deal with more than two variables, it is necessary for our computations to introduce the free log-cumulants. There are some quantities introduced by Mastnak and Nica in [58], which have nice behavior with the product of free elements of $\mathcal{A}$. The name "log-cumulants", not present in [58], comes from analogous objects with that name in the context of classical multiplicative convolution [61]. In this section, we give a definition of such quantities, in a slightly different presentation from [58]. The link is made in Section 9.5, and we present some of the properties of the free log-cumulants in Section 9.6.

9.4.1. *The Kreweras complementation map.* There exist several equivalent definitions of the Kreweras complementation map. See the book [60] for a detailed presentation.

Let $n \in \mathbb{N}^*$. We endow the set $\{1, 1', \dots, n, n'\}$ with the cyclic order $(1, 1', \dots, n, n')$. For all $\pi_1, \pi_2 \in NC(n)$, we form a not necessarily non-crossing partition $\pi_1 \cup \pi_2$ of $\{1, 1', \dots, n, n'\}$ by identifying π_2 as a partition of $\{1', \dots, n'\}$, and merging π_1 and π_2. Let $\pi = \{V_1, \dots, V_l\} \in NC(n)$. For all $1 \le i \le l$, we denote by $V_i' \subset \{1', \dots, n'\}$ the image of V_i via the isomorphism $(1, \dots, n) \simeq (1', \dots, n')$. We denote by $\tilde{\pi}$ the non-crossing partition $\tilde{\pi} = \{V_1 \cup V_1', \dots, V_n \cup V_n'\}$ of $\{1, 1', \dots, n, n'\}$. It is in fact the "completion" of $\pi \cup \pi$ (i.e. it is the smallest non-crossing partition that coarsens both π as a partition of $\{1, \dots, n\}$ and π as a partition of $\{1', \dots, n'\}$).

Now, consider two partitions $\sigma \preceq \pi \in NC(n)$. The partition $K_\pi(\sigma)$ is by definition the largest element of $NC(n)$ such that $\sigma \cup K_\pi(\sigma)$ is a non-crossing partition of $\{1, 1', \dots, n, n'\}$ and $\sigma \cup K_\pi(\sigma) \preceq \tilde{\pi}$. The map K_π is a bijection from $\{\sigma \in NC(n) : \sigma \preceq \pi\}$ onto itself called the Kreweras complementation map with respect to π (see Figure 1 for an example). If $\pi = 1_n$, we set $K(\sigma) = K_{1_n}(\sigma)$.

9.4.2. *Chains of non-crossing partitions.* Let $n \in \mathbb{N}$. A multi-chain in the lattice $NC(n)$ is a tuple of the form $\Gamma = (\pi_0, \dots, \pi_l)$ with $\pi_0, \dots, \pi_l \in NC(n)$ such that $\pi_0 \preceq \pi_1 \preceq \dots \preceq \pi_l$ (notice that we do not impose $\pi_0 = 0_n$ or $\pi_l = 1_n$, unlike in [58]). The positive integer l appearing is called the length of the multi-chain, and is denoted by $|\Gamma|$. If $\pi_0 \ne \pi_1 \ne \dots \ne \pi_l$, we say that Γ

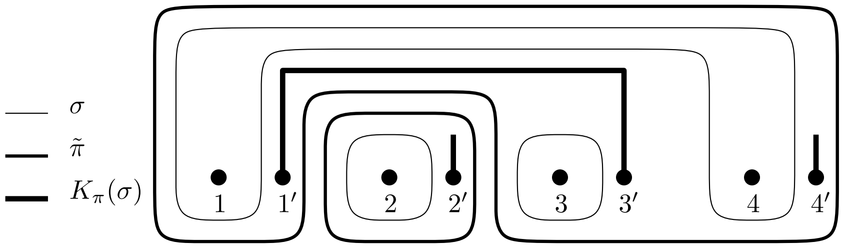

FIGURE 1. With $\pi = \{\{1,3,4\},\{2\}\}$ and $\sigma = \{\{1,4\},\{2\},\{3\}\}$, we have $K_\pi(\sigma) = \{\{1,3\},\{2\},\{4\}\}$.

is a chain in $NC(n)$. If, for all $1 \leq i \leq l$, $K_{\pi_i}(\pi_{i-1})$ has exactly one block which has more than two elements, we say that Γ is a simple chain in $NC(n)$.

Let us describe examples. In Figure 2, it is showed how a multi-chain can be represented graphically as nesting partitions. It could be helpful to imagine that, in a multi-chain $\Gamma = (\pi_0, \dots, \pi_l)$, each step from π_{i-1} to π_i is achieved by gluing some blocks of π_{i-1}.

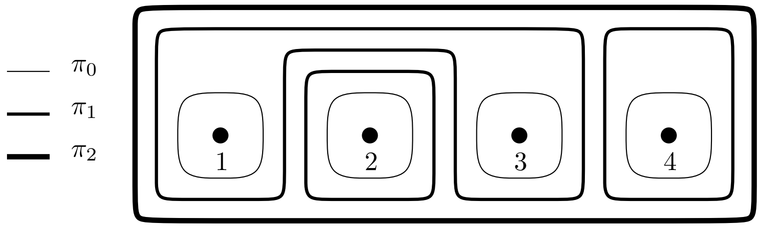

FIGURE 2. The chain (π_0, π_1, π_2) with $\pi_0 = 0_4$, $\pi_1 = \{\{1,3\},\{2\},\{4\}\}$ and $\pi_2 = 1_4$.

Notice that, for all $1 \leq i \leq l$, one can visualize $K_{\pi_i}(\pi_{i-1})$ by putting n dots labeled by $1', \dots, n'$ at the adequate places (which depend on i), just as in Figure 1. For example, in Figure 2, $K_{\pi_1}(\pi_0) = \{\{1,3\},\{2\},\{4\}\}$ and $K_{\pi_2}(\pi_1) = \{\{1,2\},\{3,4\}\}$. Thus, the multi-chain (π_0, π_1, π_2) is a chain but not a simple chain. A simple chain is obtained by gluing slowly the blocks of π_1. In Figure 3, $K_{\sigma_1}(\sigma_0) = \{\{1,3\},\{2\},\{4\}\}$ and $K_{\sigma_2}(\sigma_1) = \{\{1,2\},\{3\},\{4\}\}$. Thus, the multi-chain $(\sigma_0, \sigma_1, \sigma_2)$ is a simple chain.

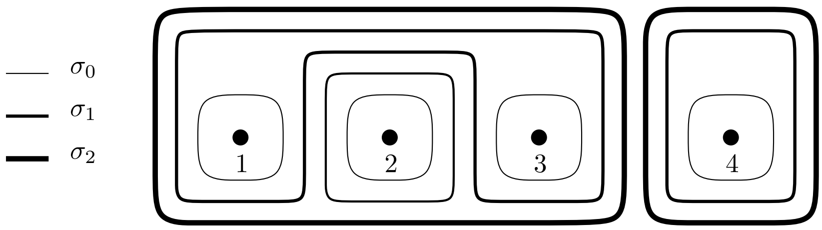

FIGURE 3. The simple chain $(\sigma_0, \sigma_1, \sigma_2)$ with $\sigma_0 = 0_4$, $\sigma_1 = \{\{1,3\},\{2\},\{4\}\}$ and $\sigma_2 = \{\{1,2,3\},\{4\}\}$.

9.4.3. *Free log-cumulants.* For all $n \geq 2$, and $A_1, \ldots, A_n \in \mathcal{A}$ such that $\tau(A_1) = \ldots = \tau(A_n) = 1$, set

$$L\kappa(A_1, \ldots, A_n) = \sum_{\substack{\Gamma \text{ chain in } NC(n) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}) \\ \pi_0=0_n, \pi_{|\Gamma|}=1_n}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} \prod_{i=1}^{|\Gamma|} \kappa\left[K_{\pi_i}(\pi_{i-1})\right](A_1, \ldots, A_n).$$

For all $n \geq 2$, and $A_1, \ldots, A_n \in \mathcal{A}$ such that $\tau(A_1), \ldots, \tau(A_n)$ are non-zero, set

$$L\kappa(A_1, \ldots, A_n) = L\kappa\left(\frac{A_1}{\tau(A_1)}, \ldots, \frac{A_n}{\tau(A_n)}\right).$$

If $n \geq 2$ and $A_1 = \cdots = A_n = A$, we call $L\kappa(A_1, \ldots, A_n)$ the free log-cumulant of order n of A, and we denote it by $L\kappa_n(A)$. Let us define also $L\kappa_1(A)$, or $L\kappa(A)$, the free log-cumulant of order 1 of A, by $\mathrm{Log}(\tau(A))$, using the principal value Log, the complex logarithm whose imaginary part lies in the interval $(-\pi, \pi]$.

9.5. The graded Hopf algebra $\mathcal{Y}^{(k)}$. We present here the formalism necessary for a detailed understand of the free log-cumulants and their relation with the free cumulants. We use the notation and results of Mastnak and Nica, and we refer to [58] for further detailed explanations.

Let $\{1, \ldots, k\} \subset \mathbb{N}$ be a fix index set, and $[k]^* = \cup_{n=0}^{\infty}\{1, \ldots, k\}^n$ be the set of all words of finite length over the alphabet $\{1, \ldots, k\}$. Let us denote by $\mathcal{Y}^{(k)}$ the commutative algebra of polynomials $\mathbb{C}\Big[Y_w : w \in [k]^*, 2 \leq |w|\Big]$. By convention, for a word $w \in [k]^*$ such that $|w| = 1$, set $Y_w = 1$. Moreover, for each word $w \in [k]^*$ such that $1 \leq |w| = n$, and all $\pi = \{S_1, \ldots, S_k\} \in NC(n)$, set $Y_{w;\pi} = Y_{w_1} \cdots Y_{w_k} \in \mathcal{Y}^{(k)}$ where, for all $1 \leq i \leq k$, w_i is the word w restricted to S_i.

9.5.1. *Definitions.* The comultiplication $\Delta : \mathcal{Y}^{(k)} \to \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(k)}$ is the unital algebra homomorphism uniquely determined by the requirement that, for every $w \in [k]^*$ with $2 \leq |w|$, we have

$$\Delta(Y_w) = \sum_{\pi \in NC(n)} Y_{w;\pi} \otimes Y_{w;K(\pi)}.$$

The counit $\varepsilon : \mathcal{Y}^{(k)} \mapsto \mathbb{C}$ is the unital algebra homomorphism uniquely determined by the requirement that, for every $w \in [k]^*$ with $2 \leq |w|$, we have $\varepsilon(Y_w) = 0$.

Proposition 3.6 of [58] reveals the bialgebra structure of $\mathcal{Y}^{(k)}$. The algebra $\mathcal{Y}^{(k)}$ is a bialgebra when we endow it with the comultiplication Δ and the counit ε, which means that

(1) $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta : \mathcal{Y}^{(k)} \to \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(k)}$ (coassociativity),
(2) $(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}$ (counity),
(3) Δ and ε are unital algebra homomorphisms.

Moreover, $\mathcal{Y}^{(k)}$ is a graded bialgebra. More precisely, for all $n \in \mathbb{N}$, the homogeneous elements of degree n are the elements of

$$\mathcal{Y}_n^{(k)} = \mathrm{span}\left\{ Y_{w_1} \cdots Y_{w_q} : \begin{array}{l} 1 \leq q, w_1, \ldots, w_q \in [k]^* \text{ with} \\ |w_1|, \ldots, |w_q| \geq 1 \text{ and } |w_1| + \ldots + |w_q| = n + q \end{array} \right\}.$$

This grading makes the bialgebra $\mathcal{Y}^{(k)}$ a graded connected bialgebra, which means that

(1) $\mathcal{Y}_0^{(k)} = \mathbb{C}$,
(2) $\mathcal{Y}^{(k)} = \oplus_{n=0}^{\infty} \mathcal{Y}_n^{(k)}$
(3) for all $m, n \in \mathbb{N}$, $\mathcal{Y}_n^{(k)} \cdot \mathcal{Y}_m^{(k)} \subset \mathcal{Y}_{m+n}^{(k)}$ and $\Delta(\mathcal{Y}_n^{(k)}) \subset \oplus_{i=0}^{n} \mathcal{Y}_i^{(k)} \otimes \mathcal{Y}_{n-i}^{(k)}$.

9.5.2. *The convolution.* Let $m : \mathcal{Y}^{(k)} \otimes \mathcal{Y}^{(k)} \to \mathcal{Y}^{(k)}$ be the linear map given by multiplication in $\mathcal{Y}^{(k)}$. Let ξ and η be two endomorphisms of $\mathcal{Y}^{(k)}$. The convolution product $\xi * \eta$ is defined by the formula

$$\xi * \eta = m \circ (\xi \otimes \eta) \circ \Delta.$$

It is an associative product on endomorphisms. Let us emphasize that a linear functional $\xi : \mathcal{Y}^{(k)} \to \mathbb{C}$ is considered as an endomorphism of $\mathcal{Y}^{(k)}$ since $\mathbb{C}$ is identified with the subalgebra $\mathbb{C} \cdot 1_{\mathcal{Y}^{(k)}}$ of $\mathcal{Y}^{(k)}$. Let ξ be an endomorphism of $\mathcal{Y}^{(k)}$ such that $\xi(1) = 0$. Because Δ respects the grading of $\mathcal{Y}^{(k)}$, the convolution powers $(\xi^{*l})_{l\in\mathbb{N}}$ of ξ are locally nilpotent. More precisely, for all $n \in \mathbb{N}$, ξ^{*l} vanishes on $\mathcal{Y}_n^{(k)}$ whenever $l > n$. Thus, for any sequence $(\alpha_l)_{l\in\mathbb{N}}$ in $\mathbb{C}$, we can unambiguously define $\sum_{l=0}^{\infty} \alpha_l \xi^{*l}$. For all $n \in \mathbb{N}$ and all $Y \in \mathcal{Y}^{(k)}$ of degree n, we have

$$\left(\sum_{l=0}^{\infty} \alpha_l \xi^{*l}\right) Y = \left(\sum_{l=0}^{n} \alpha_l \xi^{*l}\right) Y.$$

In particular, for any endomorphism ξ of $\mathcal{Y}^{(k)}$ such that $\xi(1) = 0$, let us define $\exp_*(\xi)$ by $\sum_{l=0}^{\infty} \frac{1}{l!}\xi^{*l}$, and for any endomorphism η of $\mathcal{Y}^{(k)}$ such that $\eta(1) = 1$, let us define $\log_*(\eta)$ by $-\sum_{l=0}^{\infty} \frac{1}{l}(\varepsilon - \eta)^{*l}$. The exponentiation $\exp_*$ maps the set of endomorphisms ξ of $\mathcal{Y}^{(k)}$ such that $\xi(1) = 0$ bijectively onto the set of endomorphisms η of $\mathcal{Y}^{(k)}$ such that $\eta(1) = 1$, and the inverse of this bijection is given by $\log_*$.

Let us indicate that $\mathcal{Y}^{(k)}$ is a Hopf algebra. Indeed, let us define the antipode S by the series $\varepsilon + \sum_{l=1}^{\infty} (\varepsilon - \mathrm{id})^{*l}$. Then the antipode S is such that $S * \mathrm{id} = \varepsilon = \mathrm{id} * S$.

9.5.3. *Free cumulants.* Let $\mathbf{A} = (A_1, \ldots, A_k) \in \mathcal{A}^k$. The character $\chi_{\mathbf{A}} : \mathcal{Y}^{(k)} \to \mathbb{C}$ associated to $\mathbf{A}$ is defined as follows (note that in [58], $\chi_{\mathbf{A}}$ would be denoted by $\chi_{\mu_{\mathbf{A}}}$ where $\mu_{\mathbf{A}}$ is the distribution of $\mathbf{A}$). The linear functional $\chi_{\mathbf{A}}$ is multiplicative, and for all $w = (i(1), \ldots, i(n)) \in [k]^*$ such that $2 \leq n$, we have $\chi_{\mathbf{A}}(Y_w) = \kappa(A_{i(1)}, \ldots, A_{i(n)})$.

Let us suppose that $\kappa(A_1) = \ldots = \kappa(A_k) = 1$. For all word $w = (i(1), \ldots, i(n)) \in [k]^*$ such that $1 \leq n$, and all $\pi \in NC(n)$, we have

$$\chi_{\mathbf{A}}(Y_{w;\pi}) = \kappa[\pi]\left(A_{i(1)}, \ldots, A_{i(n)}\right) \tag{9.4}$$

by the multiplicative properties of $\chi_{\mathbf{A}}$ and of the free cumulants. Proposition 4.5 of [58] links the free cumulants and the free log-cumulants as follows. For all $w = (i(1), \ldots, i(n)) \in [k]^*$ such that $2 \leq n$, we have

$$\log_* \chi_{\mathbf{A}}(Y_w) = L\kappa\left(A_{i(1)}, \ldots, A_{i(n)}\right). \tag{9.5}$$

Moreover, $\log_* \chi_{\mathbf{A}}$ is an infinitesimal character, which means that, for all $Y_1, Y_2 \in \mathcal{Y}^{(k)}$, we have

$$\log_* \chi_{\mathbf{A}}(Y_1 Y_2) = \log_* \chi_{\mathbf{A}}(Y_1) \cdot \varepsilon(Y_2) + \varepsilon(Y_1) \cdot \log_* \chi_{\mathbf{A}}(Y_2).$$

Let $w = (i(1), \ldots, i(n)) \in [k]^*$ be such that $2 \leq n$. Let $\pi \in NC(n)$ be such that π has exactly one block which has at least two elements. Let $\{j_1, \ldots, j_N\}$ be this block of π, with $j_1 < j_2 < \ldots < j_N$. Let us denote by $L\kappa[\pi](A_{i(1)}, \ldots, A_{i(n)})$ the free log-cumulant $L\kappa(A_{i(j_1)}, \ldots, A_{i(j_N)})$.

Lemma 9.7. *Let $(A_1, \ldots, A_k) \in \mathcal{A}^k$ be such that $\kappa(A_1) = \ldots = \kappa(A_k) = 1$. Let $w = (i(1), \ldots, i(n)) \in [k]^*$ be such that $2 \leq n$ and $\pi \in NC(n)$. We have*

$$\kappa[\pi]\left(A_{i(1)}, \ldots, A_{i(n)}\right) = \sum_{\substack{\Gamma \text{ simple chain in } NC(n) \\ \Gamma = (\pi_0, \ldots, \pi_{|\Gamma|}) \\ \pi_0 = 0_n, \pi_{|\Gamma|} = \pi}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big]\left(A_{i(1)}, \ldots, A_{i(n)}\right).$$

PROOF. We have $\kappa[\pi](A_{i(1)},\ldots,A_{i(n)}) = \chi_{\mathbf{A}}(Y_{w;\pi}) = \exp_*(\log_* \chi_{\mathbf{A}})(Y_{w;\pi})$. By definition of $\exp_*$, we have

$$\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right) = \sum_{l=0}^{\infty} \frac{1}{l!}(\log_* \chi_{\mathbf{A}})^{*l}(Y_{w;\pi}).$$

For all $1 \leq l$, let $\Delta^l : \mathcal{Y}^{(k)} \to (\mathcal{Y}^{(k)})^{\otimes l}$ denote the iterate of Δ. Following step-by-step the proof of Proposition 4.2 in [58] but with an arbitrary endpoint π_l, we have

$$\Delta^l(Y_{w;\pi}) = \sum_{\substack{\Gamma \text{ multi-chain in } NC(n)\\ \Gamma=(\pi_0,\ldots,\pi_l)\\ \pi_l=\pi}} Y_{w;K_{\pi_1}(\pi_0)} \otimes Y_{w;K_{\pi_2}(\pi_1)} \otimes \cdots \otimes Y_{w;K_{\pi_l}(\pi_{l-1})}.$$

Thus, we have

$$\begin{aligned}\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right) &= \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{\substack{\Gamma \text{ multi-chain in } NC(n)\\ \Gamma=(\pi_0,\ldots,\pi_l)\\ \pi_0=0_n,\pi_l=\pi}}\prod_{i=1}^{|\Gamma|}\log_* \chi_{\mathbf{A}}(Y_{w;K_{\pi_i}(\pi_{i-1})})\\ &= \sum_{\substack{\Gamma \text{ multi-chain in } NC(n)\\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|})\\ \pi_0=0_n,\pi_{|\Gamma|}=\pi}}\frac{1}{|\Gamma|!}\prod_{i=1}^{|\Gamma|}\log_* \chi_{\mathbf{A}}(Y_{w;K_{\pi_i}(\pi_{i-1})}).\end{aligned}$$

It remains to prove that, for all $\pi \in NC(n)$, $\log_* \chi_{\mathbf{A}}(Y_{w;\pi}) = L\kappa[\pi](A_{i(1)},\ldots,A_{i(n)})$ if π has exactly one block which has more than two elements, and 0 otherwise. Let $\pi \in NC(n)$ be such that π has exactly one block which has at least two elements. Let $\{j_1,\ldots,j_N\}$ be this block of π, with $j_1 < j_2 < \ldots < j_N$. Using (9.5), we have

$$\log_* \chi_{\mathbf{A}}(Y_{w;\pi}) = \log_* \chi_{\mathbf{A}}\left(Y_{(i(j_1),\ldots,i(j_N))}\right) = L\kappa\left(A_{i(j_1)},\ldots,A_{i(j_N)}\right) = L\kappa[\pi]\left(A_{i(1)},\ldots,A_{i(n)}\right).$$

If $\pi = 0_n \in NC(n)$, we have $\log_* \chi_{\mathbf{A}}(Y_{w;\pi}) = \log_* \chi_{\mathbf{A}}(1) = 0$. Finally, if $\pi \in NC(n)$ has two blocks which have at least two elements, there exist $Y_1, Y_2 \in \mathcal{Y}^{(k)} \setminus \mathcal{Y}_0^{(k)}$ such that $Y_{w;\pi} = Y_1Y_2$. We have $\varepsilon(Y_1) = \varepsilon(Y_2) = 0$. Thus,

$$\log_* \chi_{\mathbf{A}}(Y_{w;\pi}) = \log_* \chi_{\mathbf{A}}(Y_1)\varepsilon(Y_2) + \varepsilon(Y_1)\log_* \chi_{\mathbf{A}}(Y_2) = 0.$$ ☐

9.6. Properties of the free log-cumulants. This section presents some important properties of the free log-cumulants which arise from the Hopf algebra structure presented in the previous section. However, in their formulations, the different results do not involve the understanding of the previous section.

9.6.1. *Cumulants from log-cumulants.* Let $n \geq 2$. Let us consider $A_1,\ldots,A_n \in \mathcal{A}$ such that $\tau(A_1),\ldots,\tau(A_n)$ are non-zero. Let $\pi \in NC(n)$ be such that π has exactly one block which has at least two elements. Let $\{i_1,...,i_k\}$ be this block of π, with $i_1 < i_2 < \ldots < i_k$. Let us denote by $L\kappa[\pi](A_1,\ldots,A_n)$ the free log-cumulant $L\kappa(A_{i_1},\ldots,A_{i_k})$.

PROPOSITION 9.8. *Let $n \in \mathbb{N}^*$, and $A_1,\ldots,A_n \in \mathcal{A}$ be such that $\tau(A_1),\ldots,\tau(A_n)$ are non-zero. For all $\pi \in NC(n)$, we have*

$$\kappa[\pi](A_1,\ldots,A_n) = e^{L\kappa(A_1)}\cdots e^{L\kappa(A_n)}\sum_{\substack{\Gamma \text{ simple chain in } NC(n)\\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|})\\ \pi_0=0_n,\pi_{|\Gamma|}=\pi}}\frac{1}{|\Gamma|!}\prod_{i=1}^{|\Gamma|}L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big](A_1,\ldots,A_n). \tag{9.6}$$

PROOF. The degenerate case where $n = 1$ is verified because in this case, $\pi = \{\{1\}\}$,

$$\kappa[\pi](A_1) = \tau(A_1) = e^{L\kappa(A_1)} \text{ and } \sum_{\substack{\Gamma \text{ simple chain in } NC(1) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}) \\ \pi_0=0_n,\pi_{|\Gamma|}=\pi}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big]\,(A_1) = 1.$$

Let us suppose that $2 \le n$. Since the free log-cumulants of $(A_1, \ldots, A_n)$ involved are equal to those of $(A_1/\tau(A_1), \ldots, A_n/\tau(A_n))$, and

$$\kappa[\pi](A_1, \ldots, A_n) = e^{L\kappa(A_1)} \cdots e^{L\kappa(A_n)} \kappa[\pi](A_1/\tau(A_1), \ldots, A_n/\tau(A_n)),$$

we can assume that $\tau(A_1) = \ldots = \tau(A_n) = 1$, or equivalently that $\kappa(A_1) = \ldots = \kappa(A_n) = 1$. We end the proof using Lemma 9.7. □

From (9.1) and (9.6), we deduce the following corollary.

COROLLARY 9.9. *Let $n \in \mathbb{N}^*$, and $A_1, \ldots, A_n \in \mathcal{A}$ be such that $\tau(A_1), \ldots, \tau(A_n)$ are non-zero. We have*

$$\tau(A_1 \cdots A_n) = e^{L\kappa(A_1)} \cdots e^{L\kappa(A_n)} \sum_{\substack{\Gamma \text{ simple chain in } NC(k) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}),\pi_0=0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big]\,(A_1, \ldots, A_n)\,. \tag{9.7}$$

9.6.2. *Log-cumulants and freeness.* The LS-transform of $\mathbf{A}$ is the series in the non-commuting variables $z_1, \ldots, z_k$ defined by

$$\begin{aligned} LS_{\mathbf{A}}(z_1, \ldots, z_k) &= \sum_{\substack{2\le n \\ w=(i(1),\ldots,i(n))\in[k]^*}} \log_* \chi_{\mathbf{A}}(Y_w) \cdot z_{i(1)} \cdots z_{i(n)} \\ &= \sum_{\substack{2\le n \\ (i(1),\ldots,i(n))\in[k]^*}} L\kappa\left(A_{i(1)}, \ldots, A_{i(n)}\right) z_{i(1)} \cdots z_{i(n)}. \end{aligned}$$

The following two propositions are results analogous to Proposition 9.1 and Corollary 9.2. They are reformulations of Proposition 5.4 and Corollary 1.5 of [58].

PROPOSITION 9.10. *Let $(\mathcal{B}_i)_{i\in I}$ be subalgebras of $\mathcal{A}$. They are free if and only if their mixed free log-cumulants vanish. That is to say: for all $n \in \mathbb{N}^*$, all $i_1, \ldots, i_n \in I$ and all $A_1, \ldots, A_n \in \mathcal{A}$ such that $\tau(A_1) = \ldots = \tau(A_n) = 1$ and such that A_j belongs to some $\mathcal{B}_{i_j}$ for all $1 \le j \le n$, whenever there exist some j and j' with $i_j \ne i_{j'}$, we have $L\kappa(A_1, \ldots, A_n) = 0$.*

PROOF. Proposition 5.4 of [58] says that $A_1, \ldots, A_k$ are free if and only if one has

$$LS_{\mathbf{A}}(z_1, \ldots z_k) = LS_{A_1}(z_1) + \ldots + LS_{A_k}(z_k),$$

or equivalently, that $A_1, \ldots, A_k$ are free if and only if $L\kappa(A_{i(1)}, \ldots, A_{i(n)}) = 0$ each time there exist some j and j' such that $i_j \ne i_{j'}$. Proposition 9.10 follows immediately. □

PROPOSITION 9.11. *Let A and $B \in \mathcal{A}$ be free, and such that $\tau(A)$ and $\tau(B)$ are non-zero. We have $L\kappa_1(AB) \equiv L\kappa_1(A) + L\kappa_1(B) \pmod{2i\pi}$, and for all $n \ge 2$:*

$$L\kappa_n(AB) = L\kappa_n(A) + L\kappa_n(B).$$

PROOF. Let A and $B \in \mathcal{A}$ be free and such that $\tau(A)$ and $\tau(B)$ are non-zero. We have first $\tau(AB) = \tau(A)\tau(B)$, thus $L\kappa_1(AB) \equiv L\kappa_1(A) + L\kappa_1(B) \pmod{2i\pi}$. Set $\tilde{A} = A/\tau(A)$ and $\tilde{B} = B/\tau(B)$. Corollary 1.5 of [58] says that $LS_{\tilde{A}\tilde{B}} = LS_{\tilde{A}} + LS_{\tilde{B}}$. Thus, for all $2 \le n$, $L\kappa_n(AB) = L\kappa_n(\tilde{A}\tilde{B}) = L\kappa_n(\tilde{A}) + L\kappa_n(\tilde{B}) = L\kappa_n(A) + L\kappa_n(B)$. □

9.6.3. *S-transform.* In the 1-dimensional case, the free log-cumulants can be recovered from the S-transform. Indeed, let $A \in \mathcal{A}$ be such that $\tau(A) = 1$. Let us consider the R-transform of A, i.e. the formal series $R_A(z) = \sum_{n=1}^{\infty} \kappa_n(A) z^n$. Let S_A be the S-transform of A: it is the formal series S_A such that $zS_A(z)$ is the inverse under composition of $R_A(z)$. We remark that $\kappa_1(A) = \tau(A) = 1$, and by consequence, we have $S_A(0) = 1/\kappa_1(A) = 1$. Thus we can define the formal logarithm of S_A as the formal series $\log S_A(z) = -\sum_{n=1}^{\infty} \frac{1}{n}(1 - S_A(z))^n$. Corollary 6.12 of [58] establishes then $-z \log S_A(z) = \sum_{n=2}^{\infty} L\kappa_n(A) z^n = LS_A(z)$.

9.6.4. *Free unitary Brownian motion.* Let $t \geq 0$. A non-commutative random variable U_t is a free unitary Brownian motion in distribution at time t if U_t is unitary and the free log-cumulants of U_t are $L\kappa_1(U_t) = -t/2$, $L\kappa_2(U_t) = -t$ and $L\kappa_n(U_t) = 0$ for all $n > 2$. The distribution of U_t is given by (9.7): for all $n \in \mathbb{N}^*$, we have

$$\tau\left(U_t^n\right) = e^{-\frac{nt}{2}} \sum_{\substack{\Gamma \text{ simple chain in } NC(n) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}),\pi_0=0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big](U_t).$$

A simple chain $\Gamma = (0_n, \pi_1, \ldots, \pi_{|\Gamma|})$ in $NC(n)$ is called an increasing path if, for all $1 \leq i \leq l$, the block of $K_{\pi_i}(\pi_{i-1})$ which has more than two elements has exactly two elements. Proposition 6.6 of [53] tells us that, for all $k \geq 0$, the number of increasing paths of length k in $NC(n)$ is exactly $\binom{n}{k+1} n^{k-1}$ if $k \leq n-1$ and 0 if $k \geq n$. Thus,

$$\tau\left(U_t^n\right) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \left(\sum_{\substack{\Gamma \text{ increasing path in } NC(n) \\ |\Gamma|=k}} \frac{(-t)^k}{k!} \right) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-t)^k}{k!} \binom{n}{k+1} n^{k-1}.$$

In [17], Biane proved that it is indeed the distribution of a free unitary Brownian motion $(U_t)_{t \geq 0}$ at time t as defined in Section 9.6.4.

9.7. Multiplicative transition operators. Let $\mathbf{A} = (A_i)_{i \in I} \in \mathcal{A}^I$. Let us define a derivation $\mathrm{D}_{\mathbf{A}}$ associated to $\mathbf{A}$ on $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}})$ in the following way. For all $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$, we set

$$\begin{aligned} \mathrm{D}_{\mathbf{A}}\left(X_{i(1)} \cdots X_{i(n)}\right) = \sum_{\substack{1 \leq m \leq n \\ 1 \leq k_1 < \ldots < k_m \leq n}} & L\kappa\left(A_{i(k_1)}, \ldots, A_{i(k_m)}\right) X_{i(1)} \cdots X_{i(k_1-1)} \\ & \cdot \mathrm{tr}\left(X_{i(k_1)} \cdots X_{i(k_2-1)}\right) \\ & \cdot \mathrm{tr}\left(X_{i(k_2)} \cdots X_{i(k_3-1)}\right) \\ & \cdots \\ & \cdot \mathrm{tr}\left(X_{i(k_{m-1})} \cdots X_{i(k_m-1)}\right) \\ & \cdot X_{i(k_m)} \cdots X_{i(n)} \end{aligned}$$

and we extend $\mathrm{D}_{\mathbf{A}}$ to all $\mathbb{C}\{X_i : i \in I\}$ by linearity and by the relation of derivation

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ \mathrm{D}_{\mathbf{A}}(P \,\mathrm{tr}\, Q) = (\mathrm{D}_{\mathbf{A}} P) \,\mathrm{tr}\, Q + P \,\mathrm{tr}\,(\mathrm{D}_{\mathbf{A}} Q).$$

For any finite index set $J \subset I$ and $d \in \mathbb{N}$, the finite-dimensional space $\mathbb{C}_d\{X_i : i \in J\}$ is invariant for the operator $\mathrm{D}_{\mathbf{A}}$. Thus, we can define $e^{\mathrm{D}_{\mathbf{A}}}$ on each of those spaces. The operator $e^{\mathrm{D}_{\mathbf{A}}}$ on $\mathbb{C}\{X_i : i \in I\}$ is defined by the series $\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{D}_{\mathbf{A}}^k$. For all $P \in \mathbb{C}_d\{X_i : i \in J\}$, $e^{\mathrm{D}_{\mathbf{A}}} P$ is the convergent sum $\sum_{k=0}^{\infty} \frac{1}{k!} \mathrm{D}_{\mathbf{A}}^k P$. The operator $\mathrm{D}_{\mathbf{A}}$ is a derivation, and we have the Leibniz formula

$$\forall k \in \mathbb{N}, \forall P, Q \in \mathbb{C}\{X_i : i \in I\}, (\mathrm{D}_{\mathbf{A}})^k (P \,\mathrm{tr}\, Q) = \sum_{l=0}^{k} \binom{k}{l} \left(\mathrm{D}_{\mathbf{A}}^l P\right) \mathrm{tr}\left(\mathrm{D}_{\mathbf{A}}^{k-l} Q\right),$$

from which we deduce, using the standard power series argument, that the operator $e^{\mathrm{D}_{\mathbf{A}}}$ is multiplicative in the following sense:

$$\forall P, Q \in \mathbb{C}\{X_i : i \in I\},\ e^{\mathrm{D}_{\mathbf{A}}}\left(P \operatorname{tr} Q\right) = \left(e^{\mathrm{D}_{\mathbf{A}}} P\right) \operatorname{tr}\left(e^{\mathrm{D}_{\mathbf{A}}} Q\right).$$

Let us denote by $\tau(\mathbf{A})$ the family $(\tau(A_i))_{i\in I}$ and by $\mathbf{A}/\tau(\mathbf{A})$ the family $(A_i/\tau(A_i))_{i\in I}$.

Proposition 9.12. *Let $\mathbf{A} = (A_i)_{i\in I}$ and $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$ be such that $\tau(A_i) \neq 0$ and $\tau(B_i) \neq 0$ for all $i \in I$. The operators $\mathrm{D}_{\tau(\mathbf{A})}$, $\mathrm{D}_{\tau(\mathbf{B})}$ and $\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ commute, $\mathrm{D}_{\mathbf{A}} = \mathrm{D}_{\tau(\mathbf{A})} + \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$, and $e^{\mathrm{D}_{\mathbf{A}}} = e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}} e^{\mathrm{D}_{\tau(\mathbf{A})}}$.*

Proof. The free log-cumulants of $\tau(\mathbf{A})$ are zero except $L\kappa(\tau(A_i)) = L\kappa(A_i)$ for all $i \in I$ and the free log-cumulants of $\mathbf{A}/\tau(\mathbf{A})$ are those of $\mathbf{A}$, except $L\kappa(A_i/\tau(A_i)) = 0$ for all $i \in I$.

We recall that, from Section 8.1.4, the monomials of $\mathbb{C}\langle X_i : i \in I\rangle$ generate the algebra $(\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}})$. Let $n \in \mathbb{N}$ and $i(1), \ldots, i(n) \in I$ such that $M = X_{i(1)} \cdots X_{i(n)}$. For all $1 \leq k < l \leq n$, we denote by $\mathrm{tr}_{k,l}$ the element $\mathrm{tr}(X_{i(k)} \cdots X_{i(l-1)})$. We compute

$$\begin{aligned}
&\mathrm{D}_{\tau(\mathbf{B})}\, \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})} \left(X_{i(1)} \cdots X_{i(n)}\right) \\
&\quad = \sum_{1\leq l\leq n} \sum_{\substack{2\leq m\leq n \\ 1\leq k_1<\ldots<k_m\leq n}} L\kappa\left(B_{i(l)}\right) L\kappa\left(A_{i(k_1)}, \ldots, A_{i(k_m)}\right) \\
&\qquad\qquad \cdot X_{i(1)} \cdots X_{i(k_1-1)} \cdot \mathrm{tr}_{k_1,k_2} \cdots \mathrm{tr}_{k_{m-1},k_m} \cdot X_{i(k_m)} \cdots X_{i(n)} \\
&\quad = \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}\, \mathrm{D}_{\tau(\mathbf{B})} \left(X_{i(1)} \cdots X_{i(n)}\right).
\end{aligned}$$

Thus, $\mathrm{D}_{\tau(\mathbf{B})}\, \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}\, M = \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}\, \mathrm{D}_{\tau(\mathbf{B})}\, M$ for all monomials $M \in \mathbb{C}\langle X_i : i \in I\rangle$, and we extend the commutativity on all $\mathbb{C}\{X_i : i \in I\}$ by induction because $\mathrm{D}_{\tau(\mathbf{B})}$ and $\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ are derivations. Similarly, we verify the commutativity of $\mathrm{D}_{\tau(\mathbf{A})}$, $\mathrm{D}_{\tau(\mathbf{B})}$ and $\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ on monomials of $P \in \mathbb{C}\langle X_i : i \in I\rangle$, and we extend it on all $\mathbb{C}\{X_i : i \in I\}$ by induction.

Finally, the operators $\mathrm{D}_{\mathbf{A}}$ and $\mathrm{D}_{\tau(\mathbf{A})} + \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ are two derivations which coincide on monomials, so $\mathrm{D}_{\mathbf{A}} = \mathrm{D}_{\tau(\mathbf{A})} + \mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$, and $e^{\mathrm{D}_{\mathbf{A}}} = e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}} e^{\mathrm{D}_{\tau(\mathbf{A})}}$ is a direct consequence of the two first assertions. ☐

For all $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$ and $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$, let us denote $(A_iB_i)_{i\in I} \in \mathcal{A}^I$ by $\mathbf{AB}$.

Theorem 9.13. *Let I be an arbitrary index set. Let $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$ be such that $\tau(A_i) \neq 0$ for all $i \in I$. For all $P \in \mathbb{C}\{X_i : i \in I\}$, and all $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$ free from $(A_i)_{i\in I}$ and such that $\tau(B_i) \neq 0$ for all $i \in I$, we have*

$$\tau\left(P\left(\mathbf{AB}\right) \middle|\, \mathbf{B}\right) = \left(e^{\mathrm{D}_{\mathbf{A}}} P\right)(\mathbf{B}).$$

In particular, for all $P \in \mathbb{C}\{X_i : i \in I\}$, we have

$$\tau\left(P\left(\mathbf{A}\right)\right) = \left(e^{\mathrm{D}_{\mathbf{A}}} P\right)(1).$$

Note that, from Proposition 9.12 and Theorem 9.13, we have for all $P \in \mathbb{C}\{X_i : i \in I\}$

$$\tau\left(P\left(\mathbf{A}\right)\right) = \left(e^{\mathrm{D}_{\mathbf{A}}} P\right)(1) = \left(e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}} \left(e^{\mathrm{D}_{\tau(\mathbf{A})}} P\right)\right)(1).$$

In fact, the last expression $e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}} e^{\mathrm{D}_{\tau(\mathbf{A})}} P$ is easier to calculate because in practice $\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ is locally nilpotent, i.e. it is nilpotent on any finite dimensional space, and $\mathrm{D}_{\tau(\mathbf{A})}$ simply multiplies each element of the canonical basis of $P \in \mathbb{C}\{X_i : i \in I\}$ by a factor which depends on its degree (see the example in the next section).

Proof of Theorem 9.13. We start by proving a weak version of Theorem 9.13 in the following proposition.

PROPOSITION 9.14. *Let I be an arbitrary index set. Let $\mathbf{A} = (A_i)_{i\in I} \in \mathcal{A}^I$. Let $P \in \mathbb{C}\{X_i : i \in I\}$, and $\mathbf{B} = (B_i)_{i\in I} \in \mathcal{A}^I$ be free from $(A_i)_{i\in I}$. Let us assume that, for all $i \in I$, $\tau(A_i) = \tau(B_i) = 1$. We have*

$$\tau\left(P\left(\mathbf{AB}\right)\right) = \tau\left(\left(e^{\mathrm{D}_{\mathbf{A}}}P\right)\left(\mathbf{B}\right)\right).$$

PROOF. For any $P \in \mathbb{C}\{X_i : i \in I\}$, P depends on only finitely-many indices. Thus, we can suppose that I is finite. Let us say that $I = \{1, \ldots, k\} \subset \mathbb{N}$.

Let us assume first that P is a monomial. Let $w = (i(1), \ldots, i(n)) \in [k]^*$ be such that $P = X_{i(1)} \cdots X_{i(n)}$. Thanks to (9.1), we have

$$\tau\left(A_{i(1)}B_{i(1)} \cdots A_{i(n)}B_{i(n)}\right) = \sum_{\pi\in NC(2n)} \kappa\left[\pi\right]\left(A_{i(1)}, B_{i(1)}, \cdots, A_{i(n)}, B_{i(n)}\right).$$

The elements $\mathbf{A}$ and $\mathbf{B}$ are free. By Proposition 9.1, the only elements of $NC(2n)$ which contribute to the sum are the partitions of the form $\pi_1 \cup \pi_2$ where $\pi_1 \in NC(\{1, 3, \ldots, 2n-1\})$ and $\pi_2 \in NC(\{2, 4, \ldots, 2n\})$. We use now the definition of $K(\pi)$. We have

$$\begin{aligned}
&\tau\left(A_{i(1)}B_{i(1)} \cdots A_{i(n)}B_{i(n)}\right) \\
&\quad= \sum_{\pi_1\in NC(\{1,3,\ldots,2n-1\})} \sum_{\substack{\pi_2\in NC(\{2,4,\ldots,2n\}) \\ \pi_1\cup\pi_2\in NC(2n)}} \kappa\left[\pi_1 \cup \pi_2\right]\left(A_{i(1)}, B_{i(1)}, \ldots, A_{i(n)}, B_{i(n)}\right) \\
&\quad= \sum_{\pi_1\in NC(n)} \sum_{\substack{\pi_2\in NC(n) \\ \pi_2\preceq K(\pi_1)\in NC(n)}} \kappa\left[\pi_1\right]\left(A_{i(1)}, \ldots, A_{i(n)}\right) \cdot \kappa\left[\pi_2\right]\left(B_{i(1)}, \ldots, B_{i(n)}\right).
\end{aligned}$$

Applying now (9.1), we see that

$$\tau\left(A_{i(1)}B_{i(1)} \cdots A_{i(n)}B_{i(n)}\right) = \sum_{\pi\in NC(n)} \kappa\left[\pi\right]\left(A_{i(1)}, \ldots, A_{i(n)}\right) \cdot \tau\Big[K(\pi)\Big]\left(B_{i(1)}, \ldots, B_{i(n)}\right).$$

We recognize here the comultiplication Δ. More precisely, let us define $\eta_{\mathbf{B}}$ to be the character associated to $\mathbf{B}$ as below. The linear functional $\eta_{\mathbf{B}}$ is multiplicative, and for all $w = (i(1), \ldots, i(n)) \in [k]^*$, we have $\eta_{\mathbf{B}}(Y_w) = \tau(B_{i(1)}, \ldots, B_{i(n)})$. For all $w = (i(1), \ldots, i(n)) \in [k]^*$ we have

$$\begin{aligned}
\tau\left(A_{i(1)}B_{i(1)} \cdots A_{i(n)}B_{i(n)}\right) &= \sum_{\pi\in NC(n)} \kappa\left[\pi\right]\left(A_{i(1)}, \ldots, A_{i(n)}\right) \cdot \tau\Big[K(\pi)\Big]\left(B_{i(1)}, \ldots, B_{i(n)}\right) \\
&= \sum_{\pi\in NC(n)} \chi_{\mathbf{A}}(Y_{w;\pi})\eta_{\mathbf{B}}(Y_{w;K(\pi)}) \\
&= (\chi_{\mathbf{A}} * \eta_{\mathbf{B}})(Y_w).
\end{aligned}$$

We arrive at the Hopf algebra $\mathcal{Y}^{(k)}$, and consequently we introduce the algebra homomorphism $\rho : (\mathbb{C}\{X_i : i \in I\}, \cdot_{\mathrm{tr}}), \mapsto (\mathcal{Y}^{(k)}, \cdot)$. For all monomials $X_{i(1)} \cdots X_{i(n)} \in \mathbb{C}\{X_i : i \in I\}$, we set $\rho(X_{i(1)} \cdots X_{i(n)}) = Y_{(i(1),\ldots,i(n))}$, and we extend ρ by linearity and by products.

Since $P \mapsto \tau\left(P\left(\mathbf{AB}\right)\right)$ is multiplicative and $\chi_{\mathbf{A}} * \eta_{\mathbf{B}}$ is also multiplicative as a convolution of two characters, the relation $\tau\left(P\left(\mathbf{AB}\right)\right) = (\chi_{\mathbf{A}} * \eta_{\mathbf{B}})(\rho(P))$ extends from monomials to all $\mathbb{C}\{X_i : i \in I\}$. Similarly, the relation $\tau\left(P\left(\mathbf{B}\right)\right) = \eta_{\mathbf{B}}(\rho(P))$ extends from monomials to all $\mathbb{C}\{X_i : i \in I\}$.

Here we prove a general result which relates the composition exponentiation with the convolution exponentiation $\exp_*$.

LEMMA 9.15. *For all linear maps ξ from $\mathcal{Y}^{(k)}$ to $\mathbb{C}$, and for all endomorphisms η of $\mathcal{Y}^{(k)}$, we have $\xi * \eta = \eta \circ (\xi * \mathrm{id})$. Moreover, if $\xi(1) = 0$, we have $\exp_*(\xi) * \mathrm{id} = e^{\xi * \mathrm{id}}$.*

PROOF. For all $Y_1, Y_2 \in \mathcal{Y}^{(k)}$, we have

$$m(\xi(Y_1) \otimes \eta(Y)) = \xi(Y_1)\eta(Y_2) = \eta\Big(\xi(Y_1)Y_2\Big) = \eta \circ m(\xi(Y_1) \otimes Y_2).$$

Thus, $m \circ (\xi \otimes \eta) = \eta \circ m \circ (\xi \otimes \mathrm{id})$, and eventually,

$$\xi * \eta = m \circ (\xi \otimes \eta) \circ \Delta = \eta \circ m \circ (\xi \otimes \mathrm{id}) \circ \Delta = \eta \circ (\xi * \mathrm{id}).$$

For the second relation, let us suppose that $\xi(1) = 0$. It suffices to apply the previous result with $\eta = \xi * \mathrm{id}$. By an immediate induction, for all $l \in \mathbb{N}$, we have

$$\xi^{*l} * \mathrm{id} = \xi^{*(l-1)} * (\xi * \mathrm{id}) = (\xi * \mathrm{id}) \circ (\xi^{*(l-1)} * \mathrm{id}) = \ldots = (\xi * \mathrm{id})^{(l-1)} \circ (\xi * \mathrm{id}) = (\xi * \mathrm{id})^l,$$

and consequently $\exp_*(\xi) * \mathrm{id} = \sum_{l=0}^{\infty} \frac{1}{l!} \xi^{*l} * \mathrm{id} = \sum_{l=0}^{\infty} \frac{1}{l!} (\xi * \mathrm{id})^l = e^{\xi * \mathrm{id}}$. □

Because $\log_* \chi_{\mathbf{A}}$ maps $\mathcal{Y}^{(k)}$ to $\mathbb{C}$, using Lemma 9.15 twice, for all $P \in \mathbb{C}\{X_i : i \in I\}$ we have

$$\tau\left(P\left(\mathbf{AB}\right)\right) = \chi_{\mathbf{A}} * \eta_{\mathbf{B}}(\rho(P)) = \eta_{\mathbf{B}} \circ (\chi_{\mathbf{A}} * \mathrm{id}) \circ \rho(P) = \eta_{\mathbf{B}} \circ e^{\log_* \chi_{\mathbf{A}} * \mathrm{id}} \circ \rho(P).$$

For all $P \in \mathbb{C}\{X_i : i \in I\}$, we also have that $\tau\left(\left(e^{\mathrm{D}_{\mathbf{A}}} P\right)(\mathbf{B})\right) = \eta_{\mathbf{B}} \circ \rho \circ e^{\mathrm{D}_{\mathbf{A}}}(P)$. Therefore, it remains to prove that $(\log_* \chi_{\mathbf{A}} * \mathrm{id}) \circ \rho = \rho \circ \mathrm{D}_{\mathbf{A}}$. Let us check this first on monomials. For all $w = (i(1), \ldots, i(n)) \in [k]^*$, we have

$$\begin{aligned}(\log_* \chi_{\mathbf{A}} * \mathrm{id}) \circ \rho\left(X_{i(1)} \cdots X_{i(n)}\right) &= \log_* \chi_{\mathbf{A}} * \mathrm{id}(Y_w) \\ &= \sum_{\pi \in NC(n)} \log_* \chi_{\mathbf{A}}(Y_{w;\pi}) \cdot Y_{w;K(\pi)}.\end{aligned}$$

In the proof of Proposition 9.7, we showed that, for all $\pi \in NC(n)$,

$$\log_* \chi_{\mathbf{A}}(Y_{w;\pi}) = L\kappa[\pi](A_{i(1)}, \ldots, A_{i(n)})$$

if π has exactly one block which has more than two elements, and 0 otherwise. In the case where π has exactly one block which has more than two elements, let us denote by $\{k_1, \cdots, k_m\}$ this block of π, with $k_1 < k_2 < \ldots < k_m$. We have $K(\pi) = \{\{1, \ldots, k_1 - 1, k_m, \ldots, n\}, \{k_1, \ldots, k_2 - 1\}, \ldots, \{k_{m-1}, \ldots, k_m - 1\}\}$, which we denote by $\pi(k_1, \ldots, k_m)$. Thus, we have

$$\begin{aligned}(\log_* \chi_{\mathbf{A}} * \mathrm{id}) \circ \rho(X_{i(1)} \cdots X_{i(k)}) &= \sum_{\pi \in NC(n)} \log_* \chi_{\mathbf{A}}(Y_{w;\pi}) \cdot Y_{w;K(\pi)} \\ &= \sum_{\substack{2 \leq m \\ 1 \leq k_1 < \ldots < k_m \leq n}} L\kappa\left(A_{i(k_1)}, \ldots, A_{i(k_m)}\right) Y_{w;\pi(k_1,\ldots,k_m)} \\ &= \rho(\mathrm{D}_{\mathbf{A}}\, X_{i(1)} \cdots X_{i(k)}).\end{aligned}$$

Since $\mathrm{D}_{\mathbf{A}}$ is a derivation, $(\log_* \chi_{\mathbf{A}} * \mathrm{id}) \circ \rho = \rho \circ \mathrm{D}_{\mathbf{A}}$ will be a direct consequence of the fact that $\log_* \chi_{\mathbf{A}} * \mathrm{id}$ is a derivation. One can verify this directly, but we remark that $e^{\log_* \chi_{\mathbf{A}} * \mathrm{id}} = \chi_{\mathbf{A}} * \mathrm{id}$, and because $\chi_{\mathbf{A}} * \mathrm{id} - \mathrm{id}$ is locally nilpotent because $\chi_{\mathbf{A}} * \mathrm{id} - \mathrm{id}$ makes the degree strictly decrease, we also have $\log_* \chi_{\mathbf{A}} * \mathrm{id} = \log(\chi_{\mathbf{A}} * \mathrm{id}) = -\sum_{l=0}^{\infty} \frac{1}{l}(\mathrm{id} - \chi_{\mathbf{A}} * \mathrm{id})^l$. Finally, $\chi_{\mathbf{A}} * \mathrm{id}$ is multiplicative, so $\log_* \chi_{\mathbf{A}} * \mathrm{id} = \log(\chi_{\mathbf{A}} * \mathrm{id})$ is a derivation. This concludes the proof of Proposition 9.14. □

Let us finish the proof of Theorem 9.13. Let $\mathbf{A} = (A_i)_{i \in I} \in \mathcal{A}^I$ be such that $\tau(A_i) \neq 0$ for all $i \in I$ and let $\mathbf{B} = (B_i)_{i \in I} \in \mathcal{A}^I$ be free from $(A_i)_{i \in I}$ such that $\tau(B_i) \neq 0$ for all $i \in I$.

Step 1. Proposition 9.12 says us that $\mathrm{D}_{\tau(\mathbf{A})}$, $\mathrm{D}_{\tau(\mathbf{B})}$ and $\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}$ commute, and that $e^{\mathrm{D}_\mathbf{A}} = e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}}e^{\mathrm{D}_{\tau(\mathbf{A})}}$. Let $(i(1),\dots,i(n))\in I$, and $P = X_{i(1)}\cdots X_{i(n)}$. Using Proposition 9.14, we have

$$\begin{aligned}\tau\left(P\left(\mathbf{AB}\right)\right) &= \tau\left(\tau\left(A_{i(1)}\right)\tau\left(B_{i(1)}\right)\cdots\tau\left(A_{i(n)}\right)\tau\left(B_{i(n)}\right)P\left(\mathbf{A}/\tau(\mathbf{A})\cdot\mathbf{B}/\tau(\mathbf{B})\right)\right)\\ &= \tau\left(e^{\mathrm{D}_{\tau(\mathbf{B})}}e^{\mathrm{D}_{\tau(\mathbf{A})}}P\left(\mathbf{A}/\tau(\mathbf{A})\cdot\mathbf{B}/\tau(\mathbf{B})\right)\right)\\ &= \tau\left(e^{\mathrm{D}_{\mathbf{A}/\tau(\mathbf{A})}}e^{\mathrm{D}_{\tau(\mathbf{B})}}e^{\mathrm{D}_{\tau(\mathbf{A})}}P\left(\mathbf{B}/\tau(\mathbf{B})\right)\right)\\ &= \tau\left(e^{\mathrm{D}_{\tau(\mathbf{B})}}e^{\mathrm{D}_\mathbf{A}}P\left(\mathbf{B}/\tau(\mathbf{B})\right)\right).\end{aligned}$$

This relation extends to all $\mathbb{C}\{X_i : i\in I\}$ by induction.

In particular, and because $e^{\mathrm{D}_\mathbf{A}}P\in\mathbb{C}\{X_i : i\in I\}$, we have

$$\tau\left(e^{\mathrm{D}_\mathbf{A}}P\left(\mathbf{B}\right)\right) = \tau\left(e^{\mathrm{D}_\mathbf{A}}P\left(\tau(\mathbf{B})\mathbf{B}/\tau(\mathbf{B})\right)\right) = \tau\left(e^{\mathrm{D}_{\tau(\mathbf{B})}}e^{\mathrm{D}_\mathbf{A}}P\left(\mathbf{B}/\tau(\mathbf{B})\right)\right) = \tau\left(P\left(\mathbf{AB}\right)\right). \tag{9.8}$$

Step 2. We will use the following characterization of conditional expectation. The element $\tau\left(P(\mathbf{AB})\middle|\,\mathbf{B}\right)$ is the unique element of $W^*(\mathbf{B})$ such that, for all $B_{i_0}\in W^*(\mathcal{B})$,

$$\tau\left(P(\mathbf{AB})B_{i_0}\right) = \tau\left(\tau\left(P(\mathbf{AB})\middle|\,\mathbf{B}\right)B_{i_0}\right),$$

and since $e^{\mathrm{D}_\mathbf{A}}P\left(\mathbf{B}\right)\in W^*(\mathbf{B})$, it remains to prove that, for all $B_{i_0}\in W^*(\mathcal{B})$,

$$\tau\left(P(\mathbf{AB})B_{i_0}\right) = \tau\left(\left(e^{\mathrm{D}_\mathbf{A}}P\right)\left(\mathbf{B}\right)B_{i_0}\right).$$

In order to use Lemma 9.14, we work on $\mathbb{C}\{X_i : i\in I\cup\{i_0\}\}$. Let $R_{i_0} : P\mapsto PX_{i_0}$ be the operator of right multiplication by X_{i_0} on $\mathbb{C}\{X_i : i\in I\cup\{i_0\}\}$. Let $A_{i_0} = 1$ and $B_{i_0}\in W^*(\mathcal{B})$. On one hand, we have $P(\mathbf{AB})B_{i_0} = (R_{i_0}P)(\mathbf{AB}, A_{i_0}B_{i_0})$, and using (9.8), we have $\tau\left(P(\mathbf{AB})B_{i_0}\right) = \tau\left(\left(e^{\mathrm{D}_{\mathbf{A},A_{i_0}}}R_{i_0}P\right)\left(\mathbf{B},B_{i_0}\right)\right)$. On the other hand, $\tau\left(\left(e^{\mathrm{D}_\mathbf{A}}P\right)\left(\mathbf{B}\right)B_{i_0}\right) = \tau\left(\left(R_{i_0}e^{\mathrm{D}_{\mathbf{A},A_{i_0}}}(P)\right)\left(\mathbf{B},B_{i_0}\right)\right)$.

Thus, it suffices to prove that the operators $\mathrm{D}_{\mathbf{A},A_{i_0}}$ and R_{i_0} commute, which is essentially the same verification as the end of the proof of Theorem 9.4 in Section 9.3. □

9.7.1. *Example.* Let $t\geq 0$. Let U_t be a free unitary Brownian motion at time t. Let us compute $\mathrm{D}_{\tau(U_t)}X^2 = -tX^2$, $\mathrm{D}_{U_t/\tau(U_t)}X^2 = -tX\operatorname{tr}X$, and $\left(\mathrm{D}_{U_t/\tau(U_t)}\right)^2X^2 = 0$. Thus,

$$e^{\mathrm{D}_{U_t}}X^2 = e^{\mathrm{D}_{U_t/\tau(U_t)}}e^{\mathrm{D}_{\tau(U_t)}}X^2 = e^{\mathrm{D}_{U_t/\tau(U_t)}}\left(e^{-t}X^2\right) = e^{-t}\left(X^2 - tX\operatorname{tr}X + 0\right).$$

Using Theorem 9.13, we have for all $B\in\mathcal{A}$ free from U_t, $\tau\left((U_tB)^2\,|B\right) = e^{-t}\left(B^2 - tB\tau(B)\right)$. Let us derive a more general fact which will be useful in the proof of Theorem 10.7. For all $n\in\mathbb{N}$, set $W_n = \operatorname{span}\{Q\operatorname{tr}R : Q\in\mathbb{C}_{n-1}[X], R\in\mathbb{C}\{X\}\}\subset\mathbb{C}\{X\}$. For all $n\in\mathbb{N}^*$, we have

$$e^{\mathrm{D}_{U_t}}X^n = e^{\mathrm{D}_{U_t/\tau(U_t)}}e^{\mathrm{D}_{\tau(U_t)}}X^n = e^{\mathrm{D}_{U_t/\tau(U_t)}}\left(e^{-nt/2}X^n\right) = e^{-nt/2}X^n + P$$

with $P\in W_{n-1}$. Indeed, $\mathrm{D}_{U_t/\tau(U_t)}$ maps the space W_n into the space W_{n-1}, so the unique term which is not in W_{n-1} in the sum $e^{-nt/2}\sum_{k=0}^\infty\frac{1}{k!}(\mathrm{D}_{U_t/\tau(U_t)})^kX^n$ is $e^{-nt/2}X^n$.

10. Free Segal-Bargmann and free Hall transform

This section is devoted to provide a new construction of the free Segal-Bargmann transform and of the free Hall transform defined in [19] using free convolution or, more precisely, using the free transition operators of Theorems 9.4 and 9.13. The free Segal-Bargmann transform is completely characterized in Theorem 10.1 and the free Hall transform in Theorem 10.7.

For convenience, fix a (sufficiently large) W^*-probability space $(\mathcal{A}, \tau)$ to use throughout this section. We start with a very succinct review of some results of [59]. Let us suppose that $(\mathcal{A}, \tau)$ is represented on a Hilbert space $\mathcal{H}$, in the sense that $\mathcal{A}$ is a subspace of bounded operators on $\mathcal{H}$. The operator norm of $\mathcal{A}$ is denoted by $\|\cdot\|_\infty$. A (not necessarily bounded or everywhere defined) operator A on $\mathcal{H}$ is said to be affiliated with $\mathcal{A}$ if $AU = UA$ for every unitary operator U in the commutant of $\mathcal{A}$. The set of closed densely-defined operators affiliated with $\mathcal{A}$ is denoted by $\mathfrak{M}(\mathcal{A})$. For all $A \in \mathcal{A}$, let us denote by $\|A\|_2$ the norm $\|A\|_2^2 = \tau(A^*A)$. If $A \in \mathfrak{M}(\mathcal{A})$, we can still define the norm $\|\cdot\|_2$ of A (not necessarily finite), by extending the trace τ to general positive operators. The space $L^2(\mathcal{A}, \tau) = \{A \in \mathfrak{M}(\mathcal{A}) : \|A\|_2 < \infty\}$ is a Hilbert space for the norm $\|\cdot\|_2$ in which $\mathcal{A}$ is dense. When we consider a Hilbert space completion for the norm $\|\cdot\|_2$, it will always be identified with a subset of $L^2(\mathcal{A}, \tau)$, and so with a subset of $\mathfrak{M}(\mathcal{A})$.

10.1. Semi-circular system and circular system. Let H be a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle_H$.

10.1.1. *Semi-circular system.* A linear map $\mathbf{s} = (s(h))_{h \in H}$ from H to $\mathcal{A}$ is called a semi-circular system if

(1) for each $h \in H$, the element $s(h)$ is a semi-circular random variable of variance $\|h\|_H^2$,
(2) for each orthogonal family $h_1, \dots, h_n$ in H, the elements $s(h_1), \dots, s(h_n)$ are free.

Let $\mathbf{s}$ be a semi-circular system. We denote by $L^2(\mathbf{s}, \tau)$ the Hilbert completion of the algebra generated by $s(H)$ for the norm $\|\cdot\|_2 : A \mapsto \tau(A^*A)^{1/2}$. Let us compute the free cumulants of $\mathbf{s}$. Let $n \in \mathbb{N}$ and $h_1, \dots, h_n \in H$. By Proposition 9.1, by the definition of a semi-circular random variable in Section 9.2.3, and by the linearity of the free cumulants, we have

$$\kappa(s(h_1), s(h_2)) = \kappa(s(h_1)^*, s(h_2)^*) = \kappa(s(h_1)^*, s(h_2)) = \kappa(s(h_1), s(h_2)^*) = \langle h_1, h_2 \rangle_H \,,$$

and $\kappa(s(h_1)^{\varepsilon_1}, \dots, s(h_n)^{\varepsilon_n}) = 0$ for any $n \neq 2$ and $\varepsilon_1, \dots, \varepsilon_n \in \{1, *\}$.

10.1.2. *Circular system.* A linear map $\mathbf{c} = (c(h))_{h \in H}$ from H to $\mathcal{A}$ is called a circular system if

(1) the maps $\sqrt{2}\Re(c) : h \mapsto \frac{1}{\sqrt{2}}(c(h) + c(h)^*)$ and $\sqrt{2}\Im(c) : h \mapsto \frac{1}{\sqrt{2}i}(c(h) - c(h)^*)$ are semi-circular systems,
(2) $\Re(c)(H)$ and $\Im(c)(H)$ are free.

Let $\mathbf{c}$ be a circular system. We denote by $L^2_{\mathrm{hol}}(\mathbf{c}, \tau)$ the Hilbert completion of the algebra generated by $c(H)$ for the norm $\|\cdot\|_2 : A \mapsto \tau(A^*A)^{1/2}$. Let us compute the free cumulants of $\mathbf{c}$. Let $n \in \mathbb{N}$ and $h_1, \dots, h_n \in H$. By the linearity of the free cumulants, we have

$$\begin{aligned} \kappa(c(h_1), c(h_2)) &= \kappa\left((\Re(c) + i\Im(c))(h_1), (\Re(c) + i\Im(c))(h_2) \right) \\ &= \frac{1}{2}\kappa\Big(\sqrt{2}\Re(c)(h_1), \sqrt{2}\Re(c)(h_2)\Big) - \frac{1}{2}\kappa\Big(\sqrt{2}\Im(c)(h_1), \sqrt{2}\Im(c)(h_2)\Big) \\ &= 0, \end{aligned}$$

and similarly, $\kappa(c(h_1)^*, c(h_2)) = \kappa(c(h_1), c(h_2)^*) = \langle h_1, h_2 \rangle_H$, $\kappa(c(h_1)^*, c(h_2)^*) = 0$, and $\kappa(c(h_1)^{\varepsilon_1}, \dots, c(h_n)^{\varepsilon_n}) = 0$ for any $n \neq 2$ and $\varepsilon_1, \dots, \varepsilon_n \in \{1, *\}$.

10.2. Free Segal-Bargmann transform. Let $\mathbf{s}$ be a semi-circular system, and $\mathbf{c}$ be a circular system from H to $\mathcal{A}$. The free Segal-Bargmann transform is an isomorphism from $L^2(\mathbf{s}, \tau)$ to $L^2_{\mathrm{hol}}(\mathbf{c}, \tau)$ defined by Biane in [19].

10.2.1. *Definition.* Let us define the Tchebycheff type II polynomials $(T_n)_{n\in\mathbb{N}}$ by their generating function: for all $|z|<1$ and $-2<x<2$, we have

$$\sum_{n=0}^{\infty} z^n T_n(x) = \frac{1}{1-xz+z^2}.$$

We remark that, for all $n\in\mathbb{N}$, the degree of T_n is n and the leading coefficient is the coefficient of $x^n z^n$ in the development and is therefore equal to 1.

A quick way to define the free Segal-Bargmann transform is to define it on polynomials. Let $(h_j)_{j\in J}$ be an orthonormal basis of H. Let us define $\mathcal{G}$ by the unique linear operator on $\mathbb{C}\langle X_{h_j} : j\in J\rangle$ such that, for all $n\in\mathbb{N}$, $k_1,\ldots,k_n\in\mathbb{N}^*$ and $j(1)\neq j(2)\neq\cdots\neq j(n-1)\neq j(n)$ elements of J, one has $\mathcal{G}\Big(T_{k_1}(X_{h_{j(1)}})\cdots T_{k_n}(X_{h_{j(n)}})\Big) = X_{h_{j(1)}}\cdots X_{h_{j(n)}}$. The following theorem due to Biane corresponds to Definition 4 of [19].

Theorem (Biane [19]). *The map $P(\mathbf{s})\mapsto \mathcal{G}(P)(\mathbf{c})$ for all $P\in\mathbb{C}\langle X_{h_j} : j\in J\rangle$ is an isometric map which extends to a Hilbert isomorphism between $L^2(\mathbf{s},\tau)$ and $L^2_{\text{hol}}(\mathbf{c},\tau)$ called the free Segal-Bargmann transform.*

10.2.2. *Another construction.* Theorem 10.1 links the free Segal-Bargmann transform and the free convolution. The link with the free convolution is already presented in [19] in the 1-dimensional case.

Theorem 10.1. *Let $\mathbf{s}$ be a semi-circular system, and $\mathbf{c}$ be a circular system from H to $\mathcal{A}$. For all $P\in\mathbb{C}\{X_h : h\in H\}$, $\mathcal{F} : P(\mathbf{s})\mapsto (e^{\Delta_{\mathbf{s}}}P)(\mathbf{c})$ is an isometric map which extends to a Hilbert space isomorphism $\mathcal{F}$ between $L^2(\mathbf{s},\tau)$ and $L^2_{\text{hol}}(\mathbf{c},\tau)$. Moreover, this isomorphism is the free Segal-Bargmann transform.*

In particular, if $\mathbf{s}$ and $\mathbf{c}$ are free, for all $P\in\mathbb{C}\{X_h : h\in H\}$,

$$\mathcal{F}\Big(P(\mathbf{s})\Big) = \tau\Big(P(\mathbf{s}+\mathbf{c})\Big|\mathbf{c}\Big).$$

It should be remarked that, for the map $\mathcal{F}$ to be well-defined, it must be true that, for all $P,Q\in\mathbb{C}\{X_h : h\in H\}$, if $P(\mathbf{s})=Q(\mathbf{s})$, then $(e^{\Delta_{\mathbf{s}}}P)(\mathbf{c}) = (e^{\Delta_{\mathbf{s}}}Q)(\mathbf{c})$. This fact is contained in the following proof.

Proof. We will prove in a first step that, for all $P\in\mathbb{C}\{X_h : h\in H\}$, we have

$$\|P(\mathbf{s})\|^2_{L^2(\mathbf{s},\tau)} = \Big\|(e^{\Delta_{\mathbf{s}}}P)(\mathbf{c})\Big\|^2_{L^2_{\text{hol}}(\mathbf{c},\tau)}. \tag{10.1}$$

This proves that, for all $P\in\mathbb{C}\{X_h : h\in H\}$, if $P(\mathbf{s})=Q(\mathbf{s})$, then

$$\Big\|(e^{\Delta_{\mathbf{s}}}P)(\mathbf{c}) - (e^{\Delta_{\mathbf{s}}}Q)(\mathbf{c})\Big\|^2_{L^2_{\text{hol}}(\mathbf{c},\tau)} = \Big\|(e^{\Delta_{\mathbf{s}}}(P-Q))(\mathbf{c})\Big\|^2_{L^2_{\text{hol}}(\mathbf{c},\tau)} = \|(P-Q)(\mathbf{s})\|^2_{L^2(\mathbf{s},\tau)} = 0,$$

and consequently $(e^{\Delta_{\mathbf{s}}}P)(\mathbf{c}) = (e^{\Delta_{\mathbf{s}}}Q)(\mathbf{c})$. Thus, $\mathcal{F}$ is a well-defined isometric map, and so it extends to a Hilbert isomorphism $\mathcal{F}$ between $L^2(\mathbf{s},\tau)$ and $\mathcal{F}(L^2(\mathbf{s},\tau))\subset L^2_{\text{hol}}(\mathbf{c},\tau)$. The surjectivity is clear since, for all $P\in\mathbb{C}\langle X_h : h\in H\rangle$,

$$P(\mathbf{c}) = (e^{\Delta_{\mathbf{s}}}(e^{-\Delta_{\mathbf{s}}}P))(\mathbf{c}) = \mathcal{F}\Big((e^{-\Delta_{\mathbf{s}}}P)(\mathbf{c})\Big).$$

The equality with the free Segal-Bargmann transform will be made in a second step, and the last part of Theorem 10.1 follows from Theorem 9.4, which says in this case that, for all $P\in\mathbb{C}\{X_h : h\in H\}$, $(e^{\Delta_{\mathbf{s}}}P)(\mathbf{c}) = \tau(P(\mathbf{s}+\mathbf{c})|\mathbf{c})$.

Step 1. In the aim to prove (10.1), we work on the *-algebra $\mathbb{C}\{X_h : h \in H \cup (H \times \{*\})\} = \mathbb{C}\{X_h, X_h^* : h \in H\}$ (see Section 8.2.1).

Let $\mathbf{s}^+$ and $\mathbf{s}^-$ be two semi-circular systems from H to $\mathcal{A}$ such that $\mathbf{s}^+$, $\mathbf{s}^-$ and $\mathbf{c}$ are free. Let us define five auxiliary maps from $H \cup (H \times \{*\})$ to $\mathcal{A}$. The first four are the natural extensions of $\mathbf{s}$, $\mathbf{c}$, $\mathbf{s}^+$ and $\mathbf{s}^-$. For all $h \in H$, we set $\mathbf{s}(h) = s(h) = \mathbf{s}((h,*)) = s(h)^*$, $\mathbf{c}(h) = c(h)$, $\mathbf{c}((h,*)) = c(h)^*$, $\mathbf{s}^+(h) = s^+(h) = \mathbf{s}^+((h,*)) = s^+(h)^*$ and $\mathbf{s}^-(h) = s^-(h) = \mathbf{s}^-((h,*)) = s^-(h)^*$. The last map $\tilde{\mathbf{s}}$ is defined for all $h \in H$ by $\tilde{\mathbf{s}}(h) = c(h) + s^+(h)$, and $\tilde{\mathbf{s}}((h,*)) = c(h)^* + s^-(h)$. We remark that $\mathbf{s}$, $\mathbf{s}^+$ and $\mathbf{s}^-$ have the same distribution, and that $\tilde{\mathbf{s}}$ is equal to $\mathbf{c} + \mathbf{s}^+$ on H and is equal to $\mathbf{c} + \mathbf{s}^-$ on $H \times \{*\}$.

Let $P \in \mathbb{C}\{X_h : h \in H\}$. Theorem 9.13 gives us that

$$\|P(\mathbf{s})\|^2_{L^2(\mathbf{s},\tau)} = \tau\Big(P(\mathbf{s})^* P(\mathbf{s})\Big) = \tau\Big(P^* P(\mathbf{s})\Big) = \Big(e^{\Delta_{\mathbf{s}}}(P^*P)\Big)(0)$$

and

$$\begin{aligned}
\left\|e^{\Delta_{\mathbf{s}}} P(\mathbf{c})\right\|^2_{L^2_{\mathrm{hol}}(\mathbf{c},\tau)} &= \tau\left(\Big(e^{\Delta_{\mathbf{s}}} P(\mathbf{c})\Big)^*\Big(e^{\Delta_{\mathbf{s}}} P(\mathbf{c})\Big)\right)\\
&= \tau\left(\tau\Big(P(\mathbf{c}+\mathbf{s}^-)\,|\mathbf{c},\mathbf{s}^+\Big)^* \tau\Big(P(\mathbf{c}+\mathbf{s}^+)\,|\mathbf{c},\mathbf{s}^-\Big)\right)\\
&= \tau\Big(P(\mathbf{c}+\mathbf{s}^-)^* P(\mathbf{c}+\mathbf{s}^+)\Big)\\
&= \tau\Big(P^*(\mathbf{c}+\mathbf{s}^-)\, P(\mathbf{c}+\mathbf{s}^+)\Big)\\
&= \tau\Big(P^*(\tilde{\mathbf{s}})\, P(\tilde{\mathbf{s}})\Big)\\
&= \tau\Big(P^* P(\tilde{\mathbf{s}})\Big)\\
&= \Big(e^{\Delta_{\tilde{\mathbf{s}}}}(P^*P)\Big)(0)
\end{aligned}$$

It remains to prove that $\Delta_{\mathbf{s}} = \Delta_{\tilde{\mathbf{s}}}$ on $\mathbb{C}\{X_h, X_h^* : h \in H\}$. By definition, it suffices to prove that the free cumulants of $(\mathbf{s}(h))_{H \cup (H \times \{*\})}$ and $(\tilde{\mathbf{s}}(h))_{H \cup (H \times \{*\})}$ coincide.

Recall that $0_{\mathcal{A}}$ is free from all element of $\mathcal{A}$, so all free cumulants involving $0_{\mathcal{A}}$ are equal to 0. The data of the free cumulants of a semi-circular system (see Section 10.1.1), of a circular system (see Section 10.1.2) and the linearity of the free cumulants allows us to deduce the free cumulants of $(\mathbf{s}(h))_{H \cup (H \times \{*\})}$ and $(\tilde{\mathbf{s}}(h))_{H \cup (H \times \{*\})}$.

We observe first that all free cumulants of order different from 2 involved are equal to 0. Thus, for all $n \in \mathbb{N}$ such that $n \neq 2$ and $h_1, \ldots, h_n \in H \cup (H \times \{*\})$, we have

$$\kappa(\mathbf{s}(h_1), \ldots, \mathbf{s}(h_n)) = 0 = \kappa(\tilde{\mathbf{s}}(h_1), \ldots, \tilde{\mathbf{s}}(h_n)).$$

Only cumulants of order 2 are non-trivial. Let $h_1, h_2 \in H$. We have

$$\begin{aligned}
\kappa\Big(\mathbf{s}(h_1), \mathbf{s}(h_2)\Big) &= \kappa(\mathbf{s}(h_1), \mathbf{s}((h_2,*))) = \kappa(\mathbf{s}((h_1,*)), \mathbf{s}((h_2,*))) = \kappa(s(h_1), s(h_2)) &&= \langle h_1, h_2\rangle_H,\\
\kappa\Big(\tilde{\mathbf{s}}(h_1), \tilde{\mathbf{s}}(h_2)\Big) &= \kappa\Big(c(h_1) + s^+(h_1), c(h_2) + s^+(h_2)\Big) = \kappa\Big(s^+(h_1), s^+(h_2)\Big) &&= \langle h_1, h_2\rangle_H,\\
\kappa\Big(\tilde{\mathbf{s}}(h_1), \tilde{\mathbf{s}}((h_2,*))\Big) &= \kappa\Big(c(h_1) + s^+(h_1), c(h_2)^* + s^-(h_2)\Big) = \kappa\Big(c(h_1), c(h_2)^*\Big) &&= \langle h_1, h_2\rangle_H,
\end{aligned}$$

and

$$\kappa\Big(\tilde{\mathbf{s}}((h_1,*)), \tilde{\mathbf{s}}((h_2,*))\Big) = \kappa\Big(c(h_1)^* + s^-(h_1), c(h_2)^* + s^-(h_2)\Big) = \kappa\Big(s^-(h_1), s^-(h_2)\Big) = \langle h_1, h_2\rangle_H.$$

At the end, for all $h_1, h_2 \in H \cup (H \times \{*\})$, we have $\kappa\Big(\mathbf{s}(h_1), \mathbf{s}(h_2)\Big) = \kappa\Big(\tilde{\mathbf{s}}(h_1), \tilde{\mathbf{s}}(h_2)\Big)$.

Step 2. We prove now that the isomorphism $\mathcal{F}$ is indeed the free Segal-Bargmann transform. That can be done using the factorization of Section 8.2.3. More precisely, with the aim of working on polynomials, we will use the identity $P(\mathbf{c}) = P|_{\mathbf{c}}(\mathbf{s})$ for all $P \in \mathbb{C}\{X_h : h \in H\}$.

For all $P \in \mathbb{C}\langle X_{h_j} : j \in J\rangle$, we have $(e^{\Delta_{\mathbf{s}}}P)|_{\mathbf{c}} = \mathcal{G}(P)$. This fact is easily obtained by induction on the degree of P. Indeed, $P \mapsto (e^{\Delta_{\mathbf{s}}}P)|_{\mathbf{c}}$ and $P \mapsto \mathcal{G}(P)$ both respect the degree of P, preserve leading coefficients and are isometries between $(\mathbb{C}\langle X_{h_j} : j \in J\rangle, \|\cdot\|_{\mathbf{s}})$ and $(\mathbb{C}\langle X_{h_j} : j \in J\rangle, \|\cdot\|_{\mathbf{c}})$ for the norms $\|\cdot\|_{\mathbf{s}} : P \mapsto \|P(\mathbf{s})\|_2$ and $\|\cdot\|_{\mathbf{c}} : P \mapsto \|P(\mathbf{c})\|_2$. This proves that, for all $P \in \mathbb{C}\langle X_{h_j} : j \in J\rangle$, we have $\mathcal{F}(P(\mathbf{s})) = e^{\Delta_{\mathbf{s}}}P(\mathbf{c}) = (e^{\Delta_{\mathbf{s}}}P)|_{\mathbf{c}}(\mathbf{c}) = \mathcal{G}(P)(\mathbf{c})$, establishing the equality between $\mathcal{F}$ and the free Segal-Bargmann transform on the algebra generated by $(s(h_j))_{j\in J}$, and because they are isomorphisms, this equality extends to $L^2(\mathbf{s}, \tau)$. ☐

10.3. Free stochastic calculus.

10.3.1. *Semi-circular and circular Brownian motion.* A free standard Brownian motion, or semi-circular Brownian motion, is a family $(X_t)_{t\geq 0}$ of self-adjoint elements in $\mathcal{A}$, such that

(1) $X_0 = 0$;
(2) For all $0 \leq s < t$, the element $X_t - X_s$ is semi-circular of variance $t - s$;
(3) For all $0 \leq t_1 < \ldots < t_n$, the elements $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ form a free family.

Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$, free from $W^*(X_t, t \geq 0)$. We denote by $\mathcal{B}_t$ the von Neumann algebra generated by $W^*(X_s, 0 \leq s \leq t) \cup \mathcal{B}$.

A free circular standard Brownian motion is a family $(Z_t)_{t\geq 0}$ of non-commutative random variables in $\mathcal{A}$ such that $\frac{1}{\sqrt{2}}(Z_t + Z_t^*)_{t\geq 0}$ and $(\frac{1}{\sqrt{2}i}(Z_t - Z_t^*))_{t\geq 0}$ are two free standard Brownian motions which are free from each other.

10.3.2. *Free stochastic integration.* Let us recall briefly some basic definitions of free stochastic calculus. For simplicity, let us consider a very restricted situation. For further information, and more developments, see [19], [21] or [49]. We will treat in parallel the semi-circular and the circular case.

Let $t \geq 0$. Let $s \mapsto A_s$ and $s \mapsto B_s$ be maps from $[0, t]$ to $\mathcal{A}$, continuous and uniformly bounded for the operator norm $\|\cdot\|_\infty$. We suppose that, for all $s \in [0, t]$, A_s and $B_s \in \mathcal{B}_s$ (respectively $W^*(Z_u, u \leq s)$). Such mappings are called bounded adapted semi-circular (respectively circular) processes on $[0, t]$. Then we can define the free stochastic integral $\int_0^t A_s \mathrm{d}X_s B_s \in \mathcal{B}_t$ with respect to X (respectively the free stochastic integrals $\int_0^t A_s \mathrm{d}Z_s B_s \in W^*(Z_s, s \leq t)$ and $\int_0^t A_s \mathrm{d}Z_s^* B_s \in W^*(Z_s, s \leq t)$ with respect to Z and to Z^*). A bounded adapted process on $\mathbb{R}_+ = [0, \infty)$ is a bounded adapted process on all $[0, t] \subset \mathbb{R}_+$.

The following lemma includes properties corresponding to Lemma 10 and Proposition 7 of [19], and to Proposition 3.2.3 of [21].

LEMMA 10.2. *Let A and B be two bounded semi-circular (or respectively circular) adapted processes. We have for all $t \geq 0$, $\tau\left(\int_0^t A_s \mathrm{d}X_s B_s \middle| \mathcal{B}\right) = 0$ and $\left(\int_0^t A_s \mathrm{d}X_s B_s\right)^* = \int_0^t B_s^* \mathrm{d}X_s A_s^*$ in the semi-circular case.*

We have for all $t \geq 0$, $\tau\left(\int_0^t A_s \mathrm{d}Z_s B_s\right) = 0$, $\tau\left(\int_0^t A_s \mathrm{d}Z_s^* B_s\right) = 0$ and $\left(\int_0^t A_s \mathrm{d}Z_s B_s\right)^* = \int_0^t B_s^* \mathrm{d}Z_s^* A_s^*$ in the circular case.

We have Itô formulas for free stochastic integrals (Proposition 8 of [19], or Theorem 4.1.2 of [21]) which are summed up below, using formal rules. Let A, B, C and D be bounded adapted semi-circular processes. We have

$$A_t \mathrm{d}t \cdot C_t \mathrm{d}t = A_t \mathrm{d}t \cdot C_t \mathrm{d}X_t D_t = A_t \mathrm{d}X_t B_t \cdot C_t \mathrm{d}t = 0,$$

and

$$A_t \mathrm{d}X_t B_t \cdot C_t \mathrm{d}X_t D_t = \tau\left(B_t C_t\right) A_t D_t \mathrm{d}t.$$

Let A, B, C and D be bounded adapted circular processes. We have

$$A_t\mathrm{d}t \cdot C_t\mathrm{d}t = A_t\mathrm{d}t \cdot C_t\mathrm{d}Z_t D_t = A_t\mathrm{d}Z_t B_t \cdot C_t\mathrm{d}t = A_t\mathrm{d}t \cdot C_t\mathrm{d}Z_t^* D_t = A_t\mathrm{d}Z_t^* B_t \cdot C_t\mathrm{d}t = 0,$$

$$A_t\mathrm{d}Z_t B_t \cdot C_t\mathrm{d}Z_t D_t = A_t\mathrm{d}Z_t^* B_t \cdot C_t\mathrm{d}Z_t^* D_t = 0$$

and

$$A_t\mathrm{d}Z_t^* B_t \cdot C_t\mathrm{d}Z_t D_t = A_t\mathrm{d}Z_t B_t \cdot C_t\mathrm{d}Z_t^* D_t = \tau\left(B_t C_t\right) A_t D_t\mathrm{d}t.$$

As an example, let us write the Itô formula in terms of free stochastic integrals in the semi-circular case. For all $t \geq 0$, we have

$$\begin{aligned}\int_0^t A_s\mathrm{d}X_s B_s \int_0^t C_s\mathrm{d}X_s D_s &= \int_0^t A_s\mathrm{d}X_s\left[B_s\int_0^s C_u\mathrm{d}X_u D_u\right] + \int_0^t\left[\int_0^s A_u\mathrm{d}X_u B_u C_s\right]\mathrm{d}X_s D_s\\ &\quad + \int_0^t \tau\left(B_s C_s\right) A_s D_s\mathrm{d}s.\end{aligned}$$

10.4. Free unitary Brownian motion. The (right) free unitary Brownian motion $(U_t)_{t\geq 0}$ is defined to be the unique bounded adapted semi-circular process which is the solution of the following free stochastic differential equation

$$\left\{\begin{array}{rcl} U_0 &=& \mathrm{Id},\\ \mathrm{d}U_t &=& i\mathrm{d}X_t U_t - \frac{1}{2}U_t\mathrm{d}t.\end{array}\right. \tag{10.2}$$

It can be constructed using Picard iteration (see for example [49]). From Lemma 10.2, we know that $(U_t^*)_{t\geq 0}$ is a bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{\begin{array}{rcl} U_0^* &=& \mathrm{Id},\\ \mathrm{d}U_t^* &=& -iU_t^*\mathrm{d}X_t - \frac{1}{2}U_t^*\mathrm{d}t.\end{array}\right. \tag{10.3}$$

Thanks to the free Itô formula, one obtains $U_0U_0^* = \mathrm{Id}$ and for all $t \geq 0$,

$$\mathrm{d}\left(U_tU_t^*\right) = iU_t\mathrm{d}X_tU_t^* - iU_t\mathrm{d}X_tU_t^* + U_tU_t^*\mathrm{d}t - \frac{1}{2}U_tU_t^*\mathrm{d}t - \frac{1}{2}U_tU_t^*\mathrm{d}t = 0.$$

Thus, for all $t \geq 0$, $U_t^* = U_t^{-1}$. The distribution of $(U_t)_{t\geq 0}$ was computed in [17], and has been recalled in Section 9.6.4.

10.4.1. *Extension of* D_{U_t}. The purpose of this section is to establish Proposition 10.4 below, which is an extension of Theorem 9.13 for the unitary Brownian motion. We begin by extending D_{U_t} on $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$.

Let $k \in \mathbb{N}$, and $X_1, \ldots, X_k \in \{X, X^*, X^{-1}, X^{*-1}\}$. For all $1 \leq i \leq k$, set $l(i) = 0$, $r(i) = 1$ and $\epsilon(i) = 1$ if $X_i = X$ or $X_i = X^{*-1}$, and $l(i) = 1$, $r(i) = 0$ and $\epsilon(i) = -1$ if $X_i = X^*$ or $X_i = X^{-1}$. Informally, the numbers $l(i)$ and $r(i)$ indicate if X_i is a left multiplicative process, or a right one, and the sign $\epsilon(i)$ reflects the coefficient in the stochastic equation verified by X_i.

For all $1 \leq i < j \leq k$, the term $\overbracket{X_i \cdots X_j}$ refers to the product $X_i^{r(i)}X_{i+1}\cdots X_{j-1}X_j^{l(j)}$, that is to say the product $X_i \cdots X_j$ possibly excluding X_i and/or X_j. We set

$$\Delta_U X_1\cdots X_k = -kX_1\cdots X_k - 2\sum_{1\leq i<j\leq k}\epsilon(i)\epsilon(j)X_1\cdots\widehat{\overbracket{X_i\cdots X_j}}\cdots X_k \operatorname{tr}\left(\overbracket{X_i\cdots X_j}\right),$$

where the hat means that we have omitted the term $\overbracket{X_i\cdots X_j}$ in the product $X_1\cdots X_k$. We extend Δ_U to all $\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$ by linearity and by the relation of derivation

$$\forall P,Q\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}, \Delta_U\left(P\operatorname{tr}Q\right)=\left(\Delta_U P\right)\operatorname{tr}Q+P\operatorname{tr}\left(\Delta_U Q\right). \tag{10.4}$$

Let $t\geq 0$. We observe that, if we consider $\frac{t}{2}\Delta_U$ on $\mathbb{C}\{X\}$, the term $\overbracket{X_i\cdots X_j}$ is always the product $X_iX_{i+1}\cdots X_{j-1}$, and consequently $\frac{t}{2}\Delta_U$ coincides with D_{U_t} on $\mathbb{C}\{X\}$, as defined in Section 9.7 using the free log-cumulants of U_t which are stated in Section 9.6.4.

LEMMA 10.3. *Let B be an invertible variable of $\mathcal{B}$. For all $P\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(P\left(U_tB\right)|\mathcal{B}\right)=\tau\left(\frac{1}{2}\Delta_U P\left(U_tB\right)\middle|\mathcal{B}\right).$$

PROOF. Let us define $(U_t^B=U_tB)_{t\geq 0}$. We deduce from (10.2) that $(U_t^B)_{t\geq 0}$ is a bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{\begin{array}{rcl} U_0^B & = & B,\\ \mathrm{d}U_t^B & = & i\mathrm{d}X_tU_t^B-\frac{1}{2}U_t^B\mathrm{d}t.\end{array}\right. \tag{10.5}$$

The process $(U_t^B)_{t\geq 0}$ has an inverse $(U_t^B)^{-1}=B^{-1}U_t^{-1}$ at any time $t\geq 0$. From (10.3), we know that $((U_t^B)^{-1})_{t\geq 0}$ is the bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{\begin{array}{rcl} \left(U_0^B\right)^{-1} & = & B^{-1},\\ \mathrm{d}\left(U_t^B\right)^{-1} & = & -i\left(U_t^B\right)^{-1}\mathrm{d}X_t-\frac{1}{2}\left(U_t^B\right)^{-1}\mathrm{d}t.\end{array}\right. \tag{10.6}$$

Similarly, from Lemma 10.2 we know that $((U_t^B)^*)_{t\geq 0}$ is a bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{\begin{array}{rcl} \left(U_0^B\right)^* & = & B^*,\\ \mathrm{d}\left(U_t^B\right)^* & = & -i\left(U_t^B\right)^*\mathrm{d}X_t-\frac{1}{2}\left(U_t^B\right)^*\mathrm{d}t,\end{array}\right. \tag{10.7}$$

and that $((U_t^B)^{*-1})_{t\geq 0}$ is a bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{\begin{array}{rcl} \left(U_0^B\right)^{*-1} & = & B^{*-1},\\ \mathrm{d}\left(U_t^B\right)^{*-1} & = & i\mathrm{d}X_t\left(U_t^B\right)^{*-1}-\frac{1}{2}\left(U_t^B\right)^{*-1}\mathrm{d}t.\end{array}\right. \tag{10.8}$$

Let $k\in\mathbb{N}$, and $U_1,\ldots,U_k\in\{U_t^B,U_t^{B*},U_t^{B-1},U_t^{B*-1}\}$.
For all $1\leq i\leq k$, set $l(i)=0$, $r(i)=1$ and $\epsilon(i)=1$ if $U_i=U_t^B$ or $U_i=U_t^{B*-1}$, and $l(i)=1$, $r(i)=0$ and $\epsilon(i)=-1$ if $U_i=U_t^{B*}$ or $U_i=U_t^{B-1}$. For all $1\leq i<j\leq k$, the term $\overbracket{U_i\cdots U_j}$ refers to the product $U_i^{r(i)}U_{i+1}\cdots U_{j-1}U_j^{l(j)}$, that is to say the product $U_i\cdots U_j$ possibly excluding U_i and/or U_j. We claim

$$\mathrm{d}\left(U_1\cdots U_k\right)=\sum_{i=1}^k U_1\cdots\mathrm{d}U_i\cdots U_k-\sum_{1\leq i<j\leq k}\epsilon(i)\epsilon(j)U_1\cdots\widehat{\overbracket{U_i\cdots U_j}}\cdots U_k\cdot\tau\left(\overbracket{U_i\cdots U_j}\right)\mathrm{d}t. \tag{10.9}$$

Indeed, using Itô's formula (see Section 10.3.2), the equation (10.9) follows by induction on $k\in\mathbb{N}$. Let us show how it works. We suppose that the formula is true for $k\in\mathbb{N}$. Let $U_{k+1}=U_t^{B^{-1}}$. We use the Itô formula and evolution equations (10.5), (10.6), (10.7) and (10.8) to infer

$$
\begin{aligned}
&\mathrm{d}\left(U_1\cdots U_k U_{k+1}\right)\\
&\quad = \sum_{i=1}^{k} U_1\cdots \mathrm{d}U_i\cdots U_{k+1} - \sum_{1\le i<j\le k}\epsilon(i)\epsilon(j)U_1\cdots \widehat{\overline{U_i\cdots U_j}}\cdots U_{k+1}\cdot\tau\left(\overline{U_i\cdots U_j}\right)\mathrm{d}t\\
&\qquad\qquad +U_1\cdots U_k\mathrm{d}U_{k+1} + \sum_{1\le i\le k}\epsilon(i)U_1\cdots\widehat{\overline{U_i\cdots U_{k+1}}}\cdot\tau\left(\overline{U_i\cdots U_{k+1}}\right)\mathrm{d}t,
\end{aligned}
$$

which is the formula at step $k+1$. The other cases $U_{k+1}=U_t^B, U_t^{B^*}$ or $U_t^{B^{*-1}}$ are treated similarly, justifying (10.9).

Evaluating the conditional expectation $\tau\left(\cdot|\,\mathcal{B}\right)$ on both sides of (10.9), and using Lemma 10.2 (i.e. that the conditional expectation of a stochastic integral with respect to $(X_t)_{t\ge 0}$ vanishes), we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(U_1\cdots U_k|\mathcal{B}\right) &= -\frac{k}{2}\tau\left(U_1\cdots U_k|\,\mathcal{B}\right)\\
&\quad -2\sum_{1\le i<j\le k}\epsilon(i)\epsilon(j)\tau\left(U_1\cdots\widehat{\overline{U_i\cdots U_j}}\cdots U_{k+1}\middle|\,\mathcal{B}\right)\tau\left(\overline{U_i\cdots U_j}\right).
\end{aligned}
$$

Equivalently, for all $k\in\mathbb{N}$ and $X_1,\ldots,X_k\in\{X,X^*,X^{-1},X^{*-1}\}$, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(X_1\cdots X_k\left(U_t^B\right)|\mathcal{B}\right)=\tau\left(\frac{1}{2}\Delta_U X_1\cdots X_k\left(U_t^B\right)|\mathcal{B}\right).
$$

We extend this identity to all $P\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$ by linearity and by the following induction. If P and $Q\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$ verify the lemma, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tau\left((P\operatorname{tr}Q)\left(U_t^B\right)|\mathcal{B}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\tau\left(P(U_t^B)|\mathcal{B}\right)\tau\left(Q\left(U_t^B\right)\right)\right)\\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(P(U_t^B)|\mathcal{B}\right)\right)\tau\left(Q\left(U_t^B\right)\right)+\tau\left(P(U_t^B)|\mathcal{B}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(Q\left(U_t^B\right)\right)\right)\\
&= \tau\left(\Delta_U P(U_t^B)|\mathcal{B}\right)\tau\left(Q\left(U_t^B\right)\right)\\
&\qquad +\tau\left(P(U_t^B)|\mathcal{B}\right)\tau\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(Q\left(U_t^B\right)|\mathcal{B}\right)\right)\right)\\
&= \tau\left(\frac{1}{2}\Delta_U P(U_t^B)|\mathcal{B}\right)\tau\left(Q\left(U_t^B\right)\right)\\
&\qquad +\tau\left(P(U_t^B)|\mathcal{B}\right)\tau\left(\tau\left(\frac{1}{2}\Delta_U Q\left(U_t^B\right)|\mathcal{B}\right)\right)\\
&= \tau\left(\left(\left(\frac{1}{2}\Delta_U P\right)\operatorname{tr}Q+P\operatorname{tr}\left(\frac{1}{2}\Delta_U Q\right)\right)(U_t^B)|\mathcal{B}\right)\\
&= \tau\left(\frac{1}{2}\Delta_U\left(P\operatorname{tr}Q\right)\left(U_t^B\right)|\mathcal{B}\right). \quad \square
\end{aligned}
$$

Since, for all $m\in\mathbb{N}$, $\frac{t}{2}\Delta_U$ leaves $\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$ invariant, let us define the endomorphism $e^{\frac{t}{2}\Delta_U}$ on $\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}=\bigcup_{m\in\mathbb{N}}\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$ by the convergent series $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{2}\right)^n\Delta_U^n$ on each finite dimensional space $\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$.

PROPOSITION 10.4. *Let $t\ge 0$ and $P\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$. For all invertible $B\in\mathcal{B}$, we have*

$$
\tau\left(P\left(U_tB\right)|\,\mathcal{B}\right)=\left(e^{\frac{t}{2}\Delta_U}P\right)(B).
$$

Proof. Let B be an invertible element of $\mathcal{B}$. Let $P \in \mathbb{C}\{X_i : i \in I\}$. There exist a finite index set $J \subset I$ and $d \in \mathbb{N}$ such that $P \in \mathbb{C}_d\{X_i : i \in J\}$. We remark that $\frac{t}{2}\Delta_U$ is a linear operator on $\mathbb{C}_d\{X_i : i \in J\}$ and that $(P \mapsto \tau\left(P\left(U_t B\right)\middle| \mathcal{B}\right))_{t\geq 0}$ are linear maps from $\mathbb{C}_d\{X_i : i \in J\}$ to $\mathcal{A}$. Consequently, we deduce Lemma 10.4 from Lemma 9.5 and Lemma 10.3. □

10.5. Free circular multiplicative Brownian motion. Following [17], let us define the (right) free circular multiplicative Brownian motion. It is the unique bounded adapted circular process $(G_t)_{t\geq 0}$ defined by the free stochastic differential equation

$$\left\{ \begin{array}{rcl} G_0 & = & \mathrm{Id}, \\ \mathrm{d}G_t & = & \mathrm{d}Z_t G_t. \end{array} \right. \tag{10.10}$$

According to [17], the process $(G_t)_{t\geq 0}$ has an inverse at any time. It is the bounded adapted circular process $(G_t^{-1})_{t\geq 0}$ defined by the free stochastic differential equation

$$\left\{ \begin{array}{rcl} G_0^{-1} & = & \mathrm{Id}, \\ \mathrm{d}G_t^{-1} & = & -G_t^{-1}\mathrm{d}Z_t. \end{array} \right. \tag{10.11}$$

From Lemma 10.2, we know that $(G_t^*)_{t\geq 0}$ is a bounded adapted circular process defined by the free stochastic differential equation

$$\left\{ \begin{array}{rcl} G_t^* & = & \mathrm{Id}, \\ \mathrm{d}G_t^* & = & G_t^*\mathrm{d}Z_t^*, \end{array} \right. \tag{10.12}$$

and that $(G_t^{*-1})_{t\geq 0}$ is a bounded adapted semi-circular process defined by the free stochastic differential equation

$$\left\{ \begin{array}{rcl} G_t^{*-1} & = & \mathrm{Id}, \\ \mathrm{d}G_t^{*-1} & = & -\mathrm{d}Z_t^* G_t^{*-1}. \end{array} \right. \tag{10.13}$$

10.5.1. *Extension of* D_{G_t}. We establish in this section a version of Proposition 10.4 for $(G_t)_{t\geq 0}$.

Let us define a derivation Δ_{GL} on $(\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}, \cdot_{\mathrm{tr}})$ analogously to the definition of Δ_U in Section 10.4.1. Let $k \in \mathbb{N}$, and $X_1, \ldots, X_k \in \{X, X^*, X^{-1}, X^{*-1}\}$. For all $1 \leq i \leq k$, set $l(i) = 0$, $r(i) = 1$ if $X_i = X$ or $X_i = X^{*-1}$, and $l(i) = 1$, $r(i) = 0$ if $X_i = X^*$ or $X_i = X^{-1}$. For all $1 \leq i < j \leq k$, the term $\overline{X_i \cdots X_j}$ refers to the product $X_i^{r(i)} X_{i+1} \cdots X_{j-1} X_j^{l(j)}$, that is to say the product $X_i \cdots X_j$ possibly excluding X_i and/or X_j. For all $1 \leq i \leq k$, set $\epsilon(i) = 1$ if $X_i = X$ or $X_i = X^*$, and $\epsilon(i) = -1$ if $X_i = X^{*-1}$ or $X_i = X^{-1}$. For all $1 \leq i < j \leq k$, set $\delta(i,j) = 0$ if $X_i, X_j \in \{X, X^{-1}\}$ or if $X_i, X_j \in \{X^*, X^{*-1}\}$, and $\delta(i,j) = 1$ otherwise. We set

$$\Delta_{GL} X_1 \cdots X_k = 4 \sum_{1\leq i<j\leq k} \delta(i,j)\epsilon(i)\epsilon(j) \cdot X_1 \cdots \widehat{\overline{X_i \cdots X_j}} \cdots X_k \,\mathrm{tr}\left(\overline{X_i \cdots X_j}\right).$$

We extend Δ_{GL} to all $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ by linearity and by the relation

$$\forall P, Q \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}, \Delta_{GL}\left(P \,\mathrm{tr}\, Q\right) = \left(\Delta_{GL} P\right) \mathrm{tr}\, Q + P \,\mathrm{tr}\left(\Delta_{GL} Q\right). \tag{10.14}$$

Lemma 10.5. *For all* $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(P\left(G_t\right)\right) = \tau\left(\frac{1}{4}\Delta_{GL} P\left(G_t\right)\right).$$

PROOF. Recall that $(G_t)_{t\geq 0}$, $(G_t^{-1})_{t\geq 0}$, $(G_t^*)_{t\geq 0}$ and $(G_t^{*-1})_{t\geq 0}$ are defined respectively by (10.10), (10.11), (10.12) and (10.13).

Let $k\in\mathbb{N}$, and $G_1,\dots,G_k\in\{G_t,{G_t}^*,{G_t}^{-1},{G_t}^{*-1}\}$. For all $1\leq i\leq k$, set $l(i)=0$, $r(i)=1$ if $G_i=G_t$ or $G_i=G_t^{*-1}$, and $l(i)=1$, $r(i)=0$ if $G_i=G_t^*$ or $G_i=G_t^{-1}$.
For all $1\leq i\leq k$, set $\epsilon(i)=1$ if $G_i=G_t$ or $X_i=G_t^*$, and $\epsilon(i)=-1$ if $G_i=G_t^{*-1}$ or $G_i=G_t^{-1}$. For all $1\leq i<j\leq k$, set $\delta(i,j)=0$ if $G_i,G_j\in\{G_t,G_t^{-1}\}$ or if $G_i,G_j\in\{G_t^*,G_t^{*-1}\}$, and $\delta(i,j)=1$ otherwise. We claim

$$\begin{aligned}(10.15)\quad &\mathrm{d}\left(G_1\cdots G_k\right)\\ &=\sum_{i=1}^k G_1\cdots \mathrm{d}G_i\cdots G_k+\sum_{1\leq i<j\leq k}\delta(i,j)\epsilon(i)\epsilon(j)\cdot G_1\cdots\widehat{\overline{G_i\cdots G_j}}\cdots G_k\,\mathrm{tr}\left(\overline{G_i\cdots G_j}\right)\mathrm{d}t.\end{aligned}$$

Indeed, using Itô's formula 10.3.2, the equation (10.15) follows by induction on $k\in\mathbb{N}$. One can see the proof of Lemma 10.3 to understand how it works, analogously to the case of U_t.

Evaluating the conditional expectation τ on both sides of this equation, and using the fact that the trace of a stochastic integral with respect to $(Z_t)_{t\geq 0}$ vanishes (Lemma 10.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(G_1\cdots G_k\right)=\sum_{1\leq i<j\leq k}\delta(i,j)\epsilon(i)\epsilon(j)\cdot\tau\left(G_1\cdots\widehat{\overline{G_i\cdots G_j}}\cdots G_k\right)\tau\left(\overline{G_i\cdots G_j}\right).$$

Equivalently, for all $k\in\mathbb{N}$, and $X_1,\dots,X_k\in\{X,X^*,X^{-1},X^{*-1}\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau\left(X_1\cdots X_k\left(G_t\right)\right)=\tau\left(\frac{1}{4}\Delta_{GL}X_1\cdots X_k\left(G_t\right)\right).$$

We extend this identity to all $P\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$ by linearity and by the same induction as the end of the proof of Lemma 10.3. □

Let $t\geq 0$. Since, for all $m\in\mathbb{N}$, $\frac{t}{2}\Delta_{GL}$ leaves $\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$ invariant, the endomorphism $e^{\frac{t}{4}\Delta_{GL}}$ on $\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}=\bigcup_{m\in\mathbb{N}}\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$ is defined to be the convergent series $\sum_{n=0}^\infty\frac{1}{n!}\left(\frac{t}{4}\right)^n(\Delta_{GL})^n$ on each finite dimensional space $\mathbb{C}_m\{X,X^*,X^{-1},X^{*-1}\}$.

PROPOSITION 10.6. *Let $t\geq 0$ and $P\in\mathbb{C}\{X,X^*,X^{-1},X^{*-1}\}$. We have*

$$\tau\left(P\left(G_t\right)\right)=(e^{\frac{t}{4}\Delta_{GL}}P)(1).$$

This proposition allows us to compute the distribution of G_t. For example, let us remark that Δ_{GL} vanishes on $\mathbb{C}\{X\}$. This implies that $e^{\frac{t}{4}\Delta_{GL}}=\mathrm{id}$ on $\mathbb{C}\{X\}$, and that, for all $P\in\mathbb{C}\{X\}$, we have $\tau\left(P\left(G_t\right)\right)=(e^{\frac{t}{4}\Delta_{GL}}P)(1)=P(1)$.

PROOF. The demonstration follows the proof of Proposition 10.4: we deduce Proposition 10.6 from Lemma 9.5 and Lemma 10.5 from the linearity of the maps $\left(P\mapsto\tau\left(P\left(G_t\right)\right)\right)_{t\geq 0}$. □

10.6. Free Hall transform. Let $(U_t)_{t\geq 0}$ be a free unitary Brownian motion, and let $(G_t)_{t\geq 0}$ be a free circular multiplicative Brownian motion. Let $t\geq 0$. We denote by $L^2(U_t,\tau)$ the Hilbert completion of the $*$-algebra generated by U_t for the norm $\|\cdot\|_2:A\mapsto\tau\left(A^*A\right)^{1/2}$, and by $L^2_{\mathrm{hol}}(G_t,\tau)$ the Hilbert completion of the algebra generated by G_t and G_t^{-1} for the norm $\|\cdot\|_2:A\mapsto\tau\left(A^*A\right)^{1/2}$.

10.6.1. *Definition.* For all $t \geq 0$, let us define the polynomials $\left(P_n^t\right)_{n\in\mathbb{N}}$ by the generating series

$$\sum_{n=0}^{\infty} z^n P_n^t(x) = \frac{1}{1 - ze^{\frac{t}{2}\left(\frac{1+z}{1-z}\right)}x}.$$

We remark that, for all $t \geq 0$ and $n \in \mathbb{N}$, the degree of P_n^t is n and the leading coefficient is the coefficient of $x^n z^n$ in the development and is therefore equal to $e^{nt/2}$.

A quick way to define the free Hall transform is to define it on Laurent polynomials. Let us denote by $\mathbb{C}[X, X^{-1}]$ the space of Laurent polynomials. For all $t \geq 0$, let us define $\mathcal{G}_t$ by the unique linear operator on $\mathbb{C}[X, X^{-1}]$ such that, for all $n \in \mathbb{N}$, one has $\mathcal{G}_t\left(P_n^t(X)\right) = X^n$ and $\mathcal{G}_t\left(P_n^t(X^{-1})\right) = X^{-n}$. The following theorem, due to Biane, corresponds to Theorem 9 and Lemma 18 of [19].

Theorem (Biane [19]). *Let $t > 0$. The map $\mathcal{F}_t : P(U_t) \mapsto \mathcal{G}_t(P)(G_t)$ for all $P \in \mathbb{C}[X, X^{-1}]$ is an isometric map which extends to a Hilbert space isomorphism $\mathcal{F}_t$ between $L^2(U_t, \tau)$ and $L^2_{\mathrm{hol}}(G_t, \tau)$ called the free Hall transform.*

We notice that the polynomials $\left(P_n^t\right)_{n\in\mathbb{N}}$ are defined in [19] by another generating series

$$\sum_{n=0}^{\infty} z^n P_n^t(x) = \frac{1}{1 - \frac{z}{1+z}e^{\frac{t}{2}(1+2z)}x},$$

but Philippe Biane kindly pointed out (personal communication, 2012) that it is necessary to replace this generating series by the series

$$\frac{1}{1 - ze^{\frac{t}{2}\left(\frac{1+z}{1-z}\right)}x}$$

for the proofs of Lemmas 18 and 19 of [19] to be correct.

10.6.2. *Another construction.* Theorem 10.7 has to be read in parallel with the classical Hall transform definition, which will be stated in Section 11.3.

Theorem 10.7. *Let $(U_t)_{t\geq 0}$ be a free unitary Brownian motion, and let $(G_t)_{t\geq 0}$ be a free circular multiplicative Brownian motion. Let $t > 0$.*

For all $P \in \mathbb{C}\{X, X^{-1}\}$,

$$\mathcal{F}_t : P(U_t) \mapsto (e^{\frac{t}{2}\Delta_U} P)(G_t)$$

is an isometric map which extends to a Hilbert isomorphism $\mathcal{F}_t$ between $L^2(U_t, \tau)$ and $L^2_{\mathrm{hol}}(G_t, \tau)$. Moreover, this isomorphism is the free Hall transform.

In particular, if $(U_t)_{t\geq 0}$ and $(G_t)_{t\geq 0}$ are free, for all $P \in \mathbb{C}\{X, X^{-1}\}$,

$$\mathcal{F}_t\left(P(U_t)\right) = \tau\left(P(U_t G_t) \middle| G_t\right).$$

It should be remarked that, for the map $\mathcal{F}_t$ to be well-defined, it must be true that, for all $P, Q \in \mathbb{C}\{X, X^{-1}\}$, if $P\left(U_t\right) = Q\left(U_t\right)$, then $(e^{\frac{t}{2}\Delta_U} P)(G_t) = (e^{\frac{t}{2}\Delta_U} Q)(G_t)$. This fact is contained in the proof below.

Theorem 10.7 allows us to compute explicitly the free Hall transform on the *-algebra generated by U_t. For example, we have from Section 9.7.1 that $e^{\frac{t}{2}\Delta_U} X^2 = e^{\mathrm{D}_{U_t}} X^2 = e^{-t}\left(X^2 - tX \operatorname{tr} X\right)$ from which we deduce that $\mathcal{F}_t(U_t^2) = e^{-t}\left(G_t^2 - tG_t\tau(G_t)\right)$. We computed in the previous section that, for all $P \in \mathbb{C}[X]$, we have $\tau(P(G_t)) = P(1)$. Thus, we have $\mathcal{F}_t(U_t^2) = e^{-t}G_t^2 - te^{-t}G_t$.

Proof. We will prove in a first step that, for all $P \in \mathbb{C}\{X, X^{-1}\}$, we have

$$\|P(U_t)\|^2_{L^2(U_t,\tau)} = \left\|e^{\frac{t}{2}\Delta_U} P(G_t)\right\|^2_{L^2_{\mathrm{hol}}(G_t,\tau)} \tag{10.16}$$

This proves that, for all $P, Q \in \mathbb{C}\{X, X^{-1}\}$, if $P(U_t) = Q(U_t)$, then

$$\left\|(e^{\frac{t}{2}\Delta_U}P)(G_t) - (e^{\frac{t}{2}\Delta_U}Q)(G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)} = \left\|(e^{\frac{t}{2}\Delta_U}(P-Q))(G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)} = \|(P-Q)(U_t)\|^2_{L^2(U_t,\tau)} = 0,$$

and consequently $(e^{\frac{t}{2}\Delta_U}P)(G_t) = (e^{\frac{t}{2}\Delta_U}Q)(G_t)$. Thus, $\mathcal{F}_t$ is a well-defined isometric map, and so it extends to a Hilbert space isomorphism $\mathcal{F}$ between $L^2(U_t,\tau)$ and $\mathcal{F}(L^2(U_t,\tau)) \subset L^2_{\text{hol}}(G_t,\tau)$. The surjectivity is clear since, for all $P \in \mathbb{C}\{X, X^{-1}\}$,

$$P(G_t) = \left(e^{\frac{t}{2}\Delta_U}\left(e^{-\frac{t}{2}\Delta_U}P\right)\right)(G_t) = \mathcal{F}_t\left(\left(e^{-\frac{t}{2}\Delta_U}P\right)(U_t)\right).$$

The equality with the free Hall transform will be made in a second step, and the last part of Theorem 10.7 follows Proposition 10.4, which says that, for all $P \in \mathbb{C}\{X, X^{-1}\}$,

$$(e^{\frac{t}{2}\Delta_U}P)(G_t) = \tau\left(P(U_tG_t)|\, G_t\right).$$

Step 1. With the aim of proving (10.16), we work on the *-algebra $\mathbb{C}\{X, X^{-1}, X^*, X^{*-1}\}$. Let us define the subspace $\mathcal{C}$ of $\mathbb{C}\{X, X^{-1}, X^*, X^{*-1}\}$ which is generated by elements of

$$\left\{P_0 \operatorname{tr}(P_1)\cdots\operatorname{tr}(P_n) : n \in \mathbb{N}, P_0, \dots, P_n \text{ are monomials of } \mathbb{C}[X, X^{-1}]\right\}.$$

Replacing the pairs $XX^{-1} = X^{-1}X$ by 1, we observe that, for all $P \in \mathbb{C}\{X, X^{-1}, X^*, X^{*-1}\}$ there exists $\tilde{P} \in \mathcal{C}$ such that $\|P(U_t)\|^2_{L^2(U_t,\tau)} = \|\tilde{P}(U_t)\|^2_{L^2(U_t,\tau)}$. Moreover,

$$\left\|\left(e^{\frac{t}{2}\Delta_U}P\right)(G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)} = \|\tau(P(U_tG_t)|\,G_t)\|^2_{L^2_{\text{hol}}(G_t,\tau)} = \left\|\tau(\tilde{P}(U_tG_t)\big|\,G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)} = \left\|\left(e^{\frac{t}{2}\Delta_U}\tilde{P}\right)(G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)}.$$

Thus, it remains to prove that, for all $P \in \mathcal{C}$, we have $\|P(U_t)\|^2_{L^2(U_t,\tau)} = \|e^{\frac{t}{2}\Delta_U}P(G_t)\|^2_{L^2_{\text{hol}}(G_t,\tau)}$.

Let $P \in \mathcal{C}$. Proposition 10.4 gives us that

$$\begin{aligned}\|P(U_t)\|^2_{L^2(U_t,\tau)} &= \tau\Big(P(U_t)P(U_t)^*\Big)\\ &= \tau\Big((PP^*)(U_t)\Big)\\ &= \Big(e^{\frac{t}{2}\Delta_U}(PP^*)\Big)(1)\end{aligned}$$

and Proposition 10.6 gives us that

$$\begin{aligned}\left\|e^{\frac{t}{2}\Delta_U}P(G_t)\right\|^2_{L^2_{\text{hol}}(G_t,\tau)} &= \tau\left(\left(e^{\frac{t}{2}\Delta_U}P(G_t)\right)\left(e^{\frac{t}{2}\Delta_U}P(G_t)\right)^*\right)\\ &= \tau\left(\left(e^{\frac{t}{2}\Delta_U}P\left(e^{\frac{t}{2}\Delta_U}P\right)^*\right)(G_t)\right)\\ &= \left(e^{\frac{t}{4}\Delta_{GL}}\left(e^{\frac{t}{2}\Delta_U}P\left(e^{\frac{t}{2}\Delta_U}P\right)^*\right)\right)(1).\end{aligned}$$

It remains to prove that, for all $P \in \mathcal{C}$, $e^{\frac{t}{2}\Delta_U}(P^*P) = e^{\frac{t}{4}\Delta_{GL}}\left(e^{\frac{t}{2}\Delta_U}P\left(e^{\frac{t}{2}\Delta_U}P\right)^*\right)$.

Let us define the subspace $\mathcal{E}$ of $\mathbb{C}\{X, X^{-1}, X^*, X^{*-1}\}$ which is generated by elements of

$$\left\{P_0Q_0^* \operatorname{tr}(P_1Q_1^*)\cdots\operatorname{tr}(P_nQ_n^*) : n \in \mathbb{N}, P_0, Q_0, \dots, P_n, Q_n \text{ are monomials of } \mathbb{C}[X, X^{-1}]\right\}.$$

We remark that the space $\mathcal{E}$ is invariant by the product $\cdot_{\mathrm{tr}}$ and by the operators Δ_U and Δ_{GL}. Moreover, the algebra $(\mathcal{E},\cdot_{\mathrm{tr}})$ is generated as an algebra by the elements of

$$\left\{PQ^* : P,Q \text{ monomials of } \mathbb{C}[X,X^{-1}]\right\}.$$

Let us define the auxiliary derivations Δ_U^+ and Δ_U^- on $(\mathcal{E},\cdot_{\mathrm{tr}})$ in the following way. For all P,Q monomials of $\mathbb{C}[X,X^{-1}]$, we set $\Delta_U^+(PQ^*)=(\Delta_U P)\cdot Q^*$ and $\Delta_U^-(PQ^*)=P\cdot(\Delta_U Q)^*$, and we extend Δ_U^+ and Δ_U^- to all $\mathcal{E}$ by linearity and by the relations

$$\forall P,Q\in\mathcal{E}, \Delta_U^\pm\left(P\operatorname{tr}Q\right)=\left(\Delta_U^\pm P\right)\operatorname{tr}Q+P\operatorname{tr}\left(\Delta_U^\pm Q\right).$$

It should be remarked that the operators are well-defined because X_i and X_j^* are algebraically free: thus, if $P_1Q_1^*=P_2Q_2^*$, then there is a non-zero constant λ so that $P_1=\lambda P_1$ and $Q_2=\bar{\lambda}Q_1$.

The operators Δ_U^+ and Δ_U^- are such that, for all $P\in\mathcal{C}$, we have

$$\left(e^{\frac{t}{2}\Delta_U}P\right)\left(e^{\frac{t}{2}\Delta_U}P\right)^*=e^{\frac{t}{2}\Delta_U^+}\left(P(e^{\frac{t}{2}\Delta_U}P^*)\right)=e^{\frac{t}{2}\Delta_U^+}\left(e^{\frac{t}{2}\Delta_U^-}(PP^*)\right).$$

Let us observe that, for all $P\in\mathcal{C}$, we have by linearity that $PP^*\in\mathcal{E}$. We can now reformulate what we wanted to prove. It remains to verify that, for all $Q\in\mathcal{E}$, $e^{\frac{t}{2}\Delta_U}Q=e^{\frac{t}{4}\Delta_{GL}}e^{\frac{t}{2}\Delta_U^+}e^{\frac{t}{2}\Delta_U^-}Q$. More precisely, the end of the demonstration is devoted to proving that $\Delta_U=\Delta_{GL}/2+\Delta_U^++\Delta_U^-$, and that Δ_{GL}, Δ_U^+ and Δ_U^- commute on $\mathcal{E}$ (they do not commute on all $\mathbb{C}\{X,X^{-1},X^*,X^{*-1}\}$).

Because Δ_U,Δ_{GL}, Δ_U^+ and Δ_U^- are derivations on $\mathcal{E}$, it suffices to verify the properties on elements of $\left\{PQ^*:P,Q \text{ monomials of } \mathbb{C}[X,X^{-1}]\right\}$. For example, let $m,n\in\mathbb{N}$, and set $P=X^n$ and $Q=X^{-m}$. We have

$$\begin{aligned}
\Delta_U(PQ^*) &= (m+n)PQ^* -2\sum_{1\le i<j\le n}X^{n-i+j}\operatorname{tr}(X^{i-j})Q^*\\
&\quad -2\sum_{1\le i<j\le m}P(X^{-1*})^{m-i+j}\operatorname{tr}((X^{-1*})^{i-j})\\
&\quad +2\sum_{\substack{1\le i\le n\\ 1\le j\le m}}X^{i-1}(X^{-1*})^{m-j}\operatorname{tr}\left(X^{n+1-i}(X^{-1*})^j\right)\\
&= \left(nP -2\sum_{1\le i<j\le n}X^{n-i+j}\operatorname{tr}(X^{i-j})\right)Q^*\\
&\quad +P\left(mQ^* -2\sum_{1\le i<j\le m}(X^{-1*})^{m-i+j}\operatorname{tr}((X^{-1*})^{i-j})\right)\\
&\quad +\frac{1}{2}\Delta_{GL}(PQ^*)\\
&= \frac{1}{2}\Delta_{GL}(PQ^*) +\Delta_U^+(PQ^*)+\Delta_U^-(PQ^*).
\end{aligned}$$

The other three cases (depending on the signs of the exponents of P and Q) are treated similarly, and we have $\Delta_U=\Delta_{GL}/2+\Delta_U^++\Delta_U^-$.

The commutativity is less obvious. For all $m,n\in\mathbb{N}$, and $P=X^n$ and $Q=X^{-m}$, we have

$$\begin{aligned}
\Delta_U^+(PQ^*) &= nPQ^*-2\sum_{1\le i<j\le n}X^{n-i+j}\operatorname{tr}(X^{i-j})Q^*\\
&= nPQ^*-2\sum_{\substack{1\le i,j\le n\\ i+j=n}}iX^i\operatorname{tr}(X^j)Q^*.
\end{aligned}$$

Let us fix $m, n \in \mathbb{N}$, and $P = X^n$ and $Q = X^{-m}$. We have

$$\begin{aligned}
\Delta_U^+ \Delta_{GL}(PQ^*) &= \Delta_U^+ \left(-4 \sum_{\substack{1\le i\le n\\ 1\le l\le m}} X^{i-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^{n+1-i}(X^{-1*})^l\right)\right) \\
&= -4n \sum_{\substack{1\le i\le n\\ 1\le l\le m}} X^{i-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^{n+1-i}(X^{-1*})^l\right) \\
&\quad +8 \overbrace{\sum_{\substack{1\le i\le n\\ 1\le l\le m}} \sum_{\substack{1\le j,k\le i-1\\ j+k=i-1}} jX^j \operatorname{tr}(X^k)(X^{-1*})^{m-l} \operatorname{tr}\left(X^{n+1-i}(X^{-1*})^l\right)}^{=\Sigma_1} \\
&\quad +8 \overbrace{\sum_{\substack{1\le i\le n\\ 1\le l\le m}} \sum_{\substack{1\le j,k\le n+1-i\\ j+k=n+1-i}} jX^{i-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^j(X^{-1*})^l\right) \operatorname{tr}\left(X^k\right)}^{=\Sigma_2},
\end{aligned}$$

while

$$\begin{aligned}
\Delta_{GL}\Delta_U^+(PQ^*) &= \Delta_{GL}\left(nPQ^* - 2\sum_{\substack{1\le i,j\le n\\ i+j=n}} iX^i \operatorname{tr}(X^j) Q^* \right) \\
&= -4n \sum_{\substack{1\le i\le n\\ 1\le l\le m}} X^{i-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^{n+1-i}(X^{-1*})^l\right) \\
&\quad +8 \overbrace{\sum_{\substack{1\le i,j\le n\\ i+j=n}} \sum_{\substack{1\le k\le i\\ 1\le l\le m}} i \operatorname{tr}(X^j) X^{k-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^{i+1-k}(X^{-1*})^l\right)}^{=\Sigma_3}.
\end{aligned}$$

We compute

$$\begin{aligned}
\Sigma_3 &= \sum_{\substack{1\le i,j\le n\\ i+j=n}} \sum_{\substack{1\le k\le i\\ 1\le l\le m}} i \operatorname{tr}(X^j) X^{k-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^{i+1-k}(X^{-1*})^l\right) \\
&= \sum_{\substack{1\le i,j,k\le n\\ i+j+k=n+1}} \sum_{1\le l\le m} (i+k-1) \operatorname{tr}(X^j) \operatorname{tr}\left(X^i(X^{-1*})^l\right) X^{k-1}(X^{-1*})^{m-l} \qquad [i \leftarrow (i-k+1)] \\
&= \sum_{\substack{1\le i,j,k\le n\\ i+j+k=n+1}} \sum_{1\le l\le m} i \operatorname{tr}(X^j) \operatorname{tr}\left(X^i(X^{-1*})^l\right) X^{k-1}(X^{-1*})^{m-l} \\
&\qquad + \sum_{\substack{1\le i,j,k\le n\\ i+j+k=n+1}} \sum_{1\le l\le m} (k-1) \operatorname{tr}(X^j) \operatorname{tr}\left(X^i(X^{-1*})^l\right) X^{k-1}(X^{-1*})^{m-l}.
\end{aligned}$$

Finally, we conclude that $\Delta_{GL}\Delta_U^+(PQ^*) = \Delta_U^+\Delta_{GL}(PQ^*)$, using the following computation:

$$\begin{aligned}
\Sigma_3 &= \sum_{\substack{1\le i\le n\\ 1\le l\le m}} \sum_{\substack{1\le j,k\le n+1-i\\ j+k=n+1-i}} jX^{i-1}(X^{-1*})^{m-l} \operatorname{tr}\left(X^j(X^{-1*})^l\right) \operatorname{tr}\left(X^k\right) \qquad \begin{bmatrix} i \leftarrow k \\ j \leftarrow i \\ k \leftarrow i \end{bmatrix} \\
&\qquad + \sum_{\substack{1\le i,k\le n\\ 1\le l\le m}} \sum_{\substack{0\le j\le n-1\\ j+k=i-1}} jX^j \operatorname{tr}(X^k)(X^{-1*})^{m-l} \operatorname{tr}\left(X^{n+1-i}(X^{-1*})^l\right) \qquad \begin{bmatrix} i \leftarrow n+1-i \\ j \leftarrow k-1 \\ k \leftarrow j \end{bmatrix} \\
&= \Sigma_2 + \Sigma_1
\end{aligned}$$

where we remark that the case $j = 0$ does not contribute in the last sum.

The other cases of P, Q are similar, as well as the cases of the other couples of operators among the operators Δ_{GL}, Δ_U^+ and Δ_U^-.

Step 2. We prove now that the isomorphism $\mathcal{F}_t$ is indeed the free Hall transform. This can be done using the factorization of Section 8.2.3. More precisely, with the aim of working on polynomials, we will use the identity $P(G_t) = P|_{G_t}(G_t)$ for all $P \in \mathbb{C}\{X\}$.

Recall that $\frac{t}{2}\Delta_U$ coincides with D_{U_t} on $\mathbb{C}\{X\}$ (see Section 10.4.1). For all $P \in \mathbb{C}[X]$, we have $(e^{\mathrm{D}_{U_t}}P)|_{G_t} = \mathcal{G}_t(P)$. Indeed, $P \mapsto (e^{\mathrm{D}_{U_t}}P)|_{G_t}$ and $P \mapsto \mathcal{G}_t(P)$ both respect the degree of P, and are isometries between $(\mathbb{C}[X], \|\cdot\|_{U_t})$ and $(\mathbb{C}[X], \|\cdot\|_{G_t})$ for the norms $\|\cdot\|_{U_t} : P \mapsto \|P(U_t)\|_2$ and $\|\cdot\|_{G_t} : P \mapsto \|P(G_t)\|_2$. The equality is then obtained by induction on the degree of P, provided that the leading coefficients are multiplied by the same factor. This is, in fact, the case: the coefficient of the leading term is multiplied by $e^{-nt/2}$ if the degree of the polynomial is n (see the computation at the end of Section 9.7.1 for $P \mapsto (e^{\mathrm{D}_{U_t}}P)|_{G_t}$ and the definition of $\mathcal{G}_t$ in Section 11.3 for $P \mapsto \mathcal{G}_t(P)$).

This proves that, for all $P \in \mathbb{C}[X]$, we have $\mathcal{F}_t(P(U_t)) = e^{\frac{t}{2}\Delta_U}P(G_t) = e^{\mathrm{D}_{U_t}}P(G_t) = (e^{\mathrm{D}_{U_t}}P)|_{G_t}(G_t) = \mathcal{G}_t\Big(P\Big)(G_t)$, establishing the equality between $\mathcal{F}_t$ and the free Hall transform on the algebra generated by U_t. The same reasoning can be made on $\mathbb{C}\left[X^{-1}\right]$, and finally, $\mathcal{F}_t$ and the free Hall transform coincide on the algebra generated by U_t and U_t^{-1}. Because they are isomorphisms, this equality extends to $L^2(U_t, \tau)$. □

11. Random matrices

In this last part of the paper, we will give two applications of the previous results. The first one, Theorem 11.6, is the characterization of the distribution of a Brownian motion on $GL_N(\mathbb{C})$ at each fixed time in large-N limit. The second one, Theorem 11.7, establishes that the free Hall transform is the limit of the classical Hall transform for Laurent polynomial calculus.

Let us fix first some notation. Let N be an integer and let $M_N(\mathbb{C})$ be the space of matrices of dimension N, with its canonical basis $(E_{a,b})_{1\leq a,b\leq N}$. If $M \in M_N(\mathbb{C})$, we denote by M^* the adjoint of M. Let us denote by $\mathrm{Tr} : M_N(\mathbb{C}) \to \mathbb{R}$ the usual trace, and by $\mathrm{tr} : M_N(\mathbb{C}) \to \mathbb{R}$ the normalized trace $\frac{1}{N}\mathrm{Tr}$, which makes $(M_N(\mathbb{C}), \mathrm{tr})$ a non-commutative probability space.

11.1. Brownian motion on $U(N)$. Let $U(N)$ be the real Lie group of unitary matrices of dimension N:

$$U(N) = \{U \in M_N(\mathbb{C}) : U^*U = I_N\}.$$

Its Lie algebra $\mathfrak{u}(N)$ consists of skew-hermitian matrices:

$$\mathfrak{u}(N) = \{M \in M_N(\mathbb{C}) : M^* + M = 0\}.$$

We consider the following inner product on $\mathfrak{u}(N)$:

$$(X, Y) \mapsto \langle X, Y\rangle_{\mathfrak{u}(N)} = N\,\mathrm{Tr}(X^*Y).$$

This real-valued inner product is invariant under the adjoint action. Thus, it determines a bi-invariant Riemannian metric on $U(N)$ and a bi-invariant Laplace operator $\Delta_{U(N)}$. Let us fix $(X_1, \ldots, X_{N^2})$ an orthonormal basis of $\mathfrak{u}(N)$. Let us denote by $(\tilde{X}_1, \ldots, \tilde{X}_{N^2})$ the right-invariant vector fields on $U(N)$ which agree with $(X_1, \ldots, X_{N^2})$ at the identity. The Laplace operator $\Delta_{U(N)}$ is the second-order differential operator $\sum_{i=1}^{N^2} \tilde{X}_i^2$.

The (right) Brownian motion on $U(N)$ is a Markov process starting at the identity, and with generator $\frac{1}{2}\Delta_{U(N)}$.

11.1.1. *Computation of* $\Delta_{U(N)}$. In this section, we will compute $\Delta_{U(N)}$ with the help of the space $\mathbb{C}\{X, X^{-1}\}$. It should be remarked that the operator $\Delta_{U(N)}$ has been calculated in other frameworks (see the recent [38], and also [53], [63] and [67]) using similar computations. We have at our disposal an operator Δ_U on $\mathbb{C}\{X, X^{-1}\}$, defined in Section 11.1. We will see in Lemma 11.1 that the operator $\Delta_{U(N)}$ differs from Δ_U by an auxiliary operator $\frac{1}{N^2}\tilde{\Delta}_U$ when acting on the image of the $\mathbb{C}\{X, X^{-1}\}$-calculus.

Let us define this operator $\tilde{\Delta}_U$ on $\mathbb{C}\{X, X^{-1}\}$. Let $k, l \in \mathbb{N}$, and $X_1, \ldots, X_{k+l} \in \{X, X^{-1}\}$. For all $1 \leq i \leq k + l$, set $l(i) = 0$, $r(i) = 1$ and $\epsilon(i) = 1$ if $X_i = X$, and $l(i) = 1$, $r(i) = 0$ and $\epsilon(i) = -1$ if $X_i = X^{-1}$. For all $1 \leq i \leq k < j \leq k + l$, the term $\overline{X_i \cdots X_j}$ refers to the product $X_i^{r(i)} X_{i+1} \cdots X_{j-1} X_j^{l(i)}$, that is to say the product $X_i \cdots X_j$ possibly excluding X_i and/or X_j according to X_i and X_j, and the term $\overline{X_j \cdots X_i}$ refers to the product $X_j^{r(j)} X_{j+1} \cdots X_{k+l} X_1 \cdots X_{i-1} X_i^{l(i)}$, that is to say the product $X_j \cdots X_{k+l} X_1 \cdots X_i$ possibly excluding X_j and/or X_i. We set

$$\begin{aligned}
&\langle \nabla_U X_1 \cdots X_k, \nabla_U X_{k+1} \cdots X_{k+l} \rangle \\
&\qquad = - \sum_{1 \leq i \leq k < j \leq k+l} \epsilon(i)\epsilon(j) X_1 \cdots \widehat{\overline{X_i \cdots X_j}} \cdots X_{k+l} X_{k+1} \cdots \widehat{\overline{X_j \cdots X_i}} \cdots X_k,
\end{aligned}$$

where the hats mean that we have omitted the term $\overline{X_i \cdots X_j}$ in the product $X_1 \cdots X_{k+l}$, and the term $\overline{X_j \cdots X_i}$ in the product $X_{k+1} \cdots X_{k+l} X_1 \cdots X_k$. Notice that $\langle \nabla_U \cdot, \nabla_U \cdot \rangle$ is just a symbol for a bilinear map on $\mathbb{C}\langle X, X^{-1} \rangle$, but this notation is justified by Lemma 11.1 (see the remark that follows Lemma 11.1).

Let $n \in \mathbb{N}$, and $P_0, \ldots, P_n$ be monomials of $\mathbb{C}\langle X, X^{-1} \rangle$. We set

$$\begin{aligned}
&\tilde{\Delta}_U P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n \\
&\quad = 2 \sum_{m=1}^{n} \langle \nabla_U P_0, \nabla_U P_m \rangle \operatorname{tr} P_1 \cdots \widehat{\operatorname{tr} P_m} \cdots \operatorname{tr} P_n \\
&\qquad + \sum_{1 \leq m, m' \leq n} P_0 \operatorname{tr} P_1 \cdots \widehat{\operatorname{tr} P_m} \cdots \widehat{\operatorname{tr} P_{m'}} \cdots \operatorname{tr} P_n \operatorname{tr} \langle \nabla_U P_m, \nabla_U P_{m'} \rangle,
\end{aligned} \tag{11.1}$$

and we extend $\tilde{\Delta}_U$ to all $\mathbb{C}\{X, X^{-1}\}$ by linearity.

Let us denote by U the identity function of $U(N)$.

Lemma 11.1. *For all* $P \in \mathbb{C}\{X, X^{-1}\}$*, we have*

$$\Delta_{U(N)}\left(P(U)\right) = \Big((\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)P\Big)(U).$$

In particular, for P, Q two monomials of $\mathbb{C}\langle X, X^{-1} \rangle$, we have

$$\Delta_{U(N)}((PQ)(U)) = \Big((\Delta_U P) \cdot_{\operatorname{tr}} Q + \frac{2}{N^2} \langle \nabla_U P, \nabla_U Q \rangle + P \cdot_{\operatorname{tr}} (\Delta_U Q)\Big)(U).$$

Proof. Recall that $(X_1, \ldots, X_{N^2})$ is an orthonormal basis of $\mathfrak{u}(N)$, that $(\tilde{X}_1, \ldots, \tilde{X}_{N^2})$ are the right-invariant vector fields on $U(N)$ which agree with $(X_1, \ldots, X_{N^2})$ at the identity, and that the Laplace operator $\Delta_{U(N)}$ is the second-order differential operator $\sum_{a=1}^{N^2} \tilde{X}_a^2$.

It suffices to prove that, for all P and Q monomials of $\mathbb{C}\langle X, X^{-1} \rangle$, we have

$$\Delta_{U(N)} P(U) = (\Delta_U P)(U) \tag{11.2}$$

and

$$\sum_{a=1}^{N^2} \tilde{X}_a\left(P(U)\right) \operatorname{tr} \tilde{X}_a\left(Q(U)\right) = \frac{1}{N^2}\left\langle \nabla_U P, \nabla_U Q\right\rangle (U). \tag{11.3}$$

Indeed, let $P_0, \ldots, P_n$ be monomials of $\mathbb{C}\left\langle X, X^{-1}\right\rangle$. It follows from (11.2) and (11.3) that

$$\begin{aligned}
&\Delta_{U(N)}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(U)\\
&\quad = \left(\Delta_{U(N)} P_0(U)\right) \operatorname{tr} P_1(U) \cdots \operatorname{tr} P_n(U)\\
&\qquad + \sum_{m=1}^{n} P_0(U) \operatorname{tr} P_1(U) \cdots \operatorname{tr}\left(\Delta_{U(N)} P_m(U)\right) \cdots \operatorname{tr} P_n(U)\\
&\qquad + 2\sum_{m=1}^{n}\sum_{a=1}^{N^2} \left(\tilde{X}_a P_0(U)\right) \operatorname{tr} P_1(U) \cdots \operatorname{tr}\left(\tilde{X}_a P_m(U)\right) \cdots \operatorname{tr} P_n(U)\\
&\qquad + \sum_{1\le m,m'\le n}\sum_{a=1}^{N^2} P_0(U) \operatorname{tr} P_1(U) \cdots \operatorname{tr}\left(\tilde{X}_a P_m(U)\right) \cdots \operatorname{tr}\left(\tilde{X}_a P'_m(U)\right) \cdots \operatorname{tr} P_n(U).
\end{aligned}$$

Thanks to the structure equations (10.4) and (11.1) of Δ_U and $\tilde{\Delta}_U$, we deduce that

$$\begin{aligned}
&\Delta_{U(N)}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(U)\\
&\quad = \left(\Delta_U P_0\right)(U) \operatorname{tr} P_1(U) \cdots \operatorname{tr} P_n(U)\\
&\qquad + \sum_{m=1}^{n} P_0(U) \operatorname{tr} P_1(U) \cdots \operatorname{tr}\left(\Delta_U P_m\right)(U) \cdots \operatorname{tr} P_n(U)\\
&\qquad + \frac{2}{N^2}\sum_{m=1}^{n}\sum_{a=1}^{N^2} \left\langle \nabla_U P_0, \nabla_U P_m\right\rangle (U) \operatorname{tr} P_1(U) \cdots \operatorname{tr} \widehat{P_m(U)} \cdots \operatorname{tr} P_n(U)\\
&\qquad + \frac{1}{N^2}\sum_{1\le m,m'\le n}\sum_{a=1}^{N^2} P_0(U) \operatorname{tr} P_1(U) \cdots \operatorname{tr} \widehat{P_m(U)} \cdots \operatorname{tr} \widehat{P_{m'}(U)} \cdots \operatorname{tr} P_n(U)\\
&\qquad\qquad \cdot \operatorname{tr}\left\langle \nabla_U P_m, \nabla_U P_{m'}\right\rangle (U)\\
&\quad = \Delta_U\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(U) + \tfrac{1}{N^2}\tilde{\Delta}_U\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(U),
\end{aligned}$$

and Lemma 11.1 follows by linearity. Thus, we need only to prove (11.2) and (11.3).

The end of the proof is devoted to proving (11.2) and (11.3). For $1 \le a \le N^2$, we have $\tilde{X}_a(U) = X_a U$ and $\tilde{X}_a(U^{-1}) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(e^{tX_a}U)^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}(U^{-1}e^{-tX_a}) = -U^{-1}X_a$. Therefore, the expression of $\sum_{a=1}^{N^2} X_a \otimes X_a$ will be useful. Let us fix the following orthonormal basis of $\mathfrak{u}(N)$:

$$\begin{aligned}
&\{X_1, \ldots, X_{N^2}\}\\
&= \left\{\frac{i}{\sqrt{N}}E_{a,a}, 1 \le a \le N\right\} \cup \left\{\frac{E_{a,b} - E_{b,a}}{\sqrt{2N}}, 1 \le a < b \le N\right\} \cup \left\{i\frac{E_{a,b} + E_{b,a}}{\sqrt{2N}}, 1 \le a < b \le N\right\}.
\end{aligned}$$

We have

$$\begin{aligned}
N\sum_{a=1}^{N^2} X_a\otimes X_a &= -\sum_{a=1}^{N} E_{a,a}\otimes E_{a,a} + \frac{1}{2}\sum_{1\le a<b\le N}(E_{a,b}-E_{b,a})\otimes(E_{a,b}-E_{b,a})\\
&\quad -\frac{1}{2}\sum_{1\le a<b\le N}(E_{a,b}+E_{b,a})\otimes(E_{a,b}+E_{b,a})\\
&= -\sum_{a=1}^{N} E_{a,a}\otimes E_{a,a} - \sum_{1\le a<b\le N} E_{b,a}\otimes E_{a,b}C - \sum_{1\le a<b\le N} E_{a,b}\otimes E_{b,a}\\
&= -\sum_{1\le a,b\le N} E_{a,b}\otimes E_{b,a}.
\end{aligned}$$

From which we deduce that

$$\sum_{a=1}^{N^2}\left(\tilde{X}_a(U)\otimes\tilde{X}_a(U)\right) = -\frac{1}{N}\sum_{1\le a,b\le N} E_{a,b}U\otimes E_{b,a}U,$$

$$\sum_{a=1}^{N^2}\left(\tilde{X}_a(U)\otimes\tilde{X}_a(U^{-1})\right) = \frac{1}{N}\sum_{1\le a,b\le N} E_{a,b}U\otimes UE_{b,a},$$

and

$$\sum_{a=1}^{N^2}\left(\tilde{X}_a(U^{-1})\otimes\tilde{X}_a(U^{-1})\right) = -\frac{1}{N}\sum_{1\le a,b\le N} U^{-1}E_{a,b}\otimes U^{-1}E_{b,a}.$$

We can now compute (11.2) and (11.3). Let $k,l\in\mathbb{N}$, and $X_1,\ldots,X_{k+l}\in\{X,X^{-1}\}$. For all $1\le i\le k+l$, set $\epsilon(i)=1$ if $X_i=X$, and $\epsilon(i)=-1$ if $X_i=X^{-1}$. Thus, we have

$$\begin{aligned}
\Delta_{U(N)}\left(P(U)\right) &= \sum_{a=1}^{N^2}\tilde{X}_a^2\left(U^{\epsilon(1)}\cdots U^{\epsilon(k)}\right)\\
&= \sum_{a=1}^{N^2}\sum_{1\le i\le k} U^{\epsilon(1)}\cdots U^{\epsilon(i-1)}\tilde{X}_a^2\left(U^{\epsilon(i)}\right)U^{\epsilon(i+1)}\cdots U^{\epsilon(k)}\\
&\quad +\sum_{a=1}^{N^2} 2\sum_{1\le i<j\le k} U^{\epsilon(1)}\cdots\tilde{X}_a\left(U^{\epsilon(i)}\right)\cdots\tilde{X}_a\left(U^{\epsilon(j)}\right)\cdots U^{\epsilon(k)}\\
&= -k\frac{N}{N}U^{\epsilon(1)}\cdots U^{\epsilon(k)}\\
&\quad -2\sum_{1\le i<j\le k}\epsilon(i)\epsilon(j)\left(X_1\cdots\widehat{\overline{X_i\cdots X_j}}\cdots X_k\operatorname{tr}\left(\overline{X_i\cdots X_j}\right)\right)(U)\\
&= \left(\Delta_U P\right)(U)
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{a=1}^{N^2} \tilde{X}_a\left(P(U)\right) \operatorname{tr} \tilde{X}_a\left(Q(U)\right) \\
&= \sum_{a=1}^{N^2} \tilde{X}_a\left(U^{\epsilon(1)} \cdots U^{\epsilon(k)}\right) \operatorname{tr} \tilde{X}_a\left(U^{\epsilon(k+1)} \cdots U^{\epsilon(k+l)}\right) \\
&= \frac{1}{N} \sum_{1 \leq i \leq k < j \leq k+l} \sum_{a=1}^{N^2} \left(U^{\epsilon(1)} \cdots \tilde{X}_a\left(U^{\epsilon(i)}\right) \cdots U^{\epsilon(k)}\right) \operatorname{Tr}\left(U^{\epsilon(k+1)} \cdots \tilde{X}_a\left(U^{\epsilon(j)}\right) \cdots U^{\epsilon(k+l)}\right) \\
&= -\frac{1}{N^2} \sum_{1 \leq i \leq k < j \leq k+l} \epsilon(i)\epsilon(j) \left(X_1 \cdots \widehat{X_i \cdots X_j} \cdots X_{k+l} X_{k+1} \cdots \widehat{X_j \cdots X_i} \cdots X_k\right)(U) \\
&= \frac{1}{N^2} \left\langle \nabla_U P, \nabla_U Q \right\rangle (U)
\end{aligned}
$$

where we have used that, for all $A, B, C, D \in M_N(\mathbb{C})$, we have

$$
(11.4) \qquad \sum_{1 \leq a,b \leq N} A E_{a,b} B E_{b,a} C = \operatorname{Tr}(B) AC \text{ and } \sum_{1 \leq a,b \leq N} A E_{a,b} B \operatorname{Tr}(C E_{b,a} D) = ADCB. \qquad \square
$$

Corollary 11.2. *Let* $\left(U_t^{(N)}\right)_{t \geq 0}$ *be a Brownian motion on* $U(N)$.
Let $t \geq 0$ *and* $P \in \mathbb{C}\{X, X^{-1}\}$. *We have*

$$
\mathbb{E}\left[P\left(U_t^{(N)}\right)\right] = \left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)} P\right)(1).
$$

Proof. For all $t \geq 0$ and $P \in \mathbb{C}\{X, X^{-1}\}$, thanks to Lemma 11.1, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[P\left(U_t^{(N)}\right)\right] = \mathbb{E}\left[\left(\frac{1}{2}\Delta_{U(N)}\Big(P(U)\Big)\right)\left(U_t^{(N)}\right)\right] = \mathbb{E}\left[\left(\frac{1}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)P\right)\left(U_t^{(N)}\right)\right].
$$

We conclude with Lemma 9.5, since the maps $(t \mapsto \mathbb{E}\left[P\left(U_t^{(N)}\right)\right])_{t \geq 0}$ are linear. $\square$

11.2. Brownian motion on $GL_N(\mathbb{C})$. Let $GL_N(\mathbb{C})$ be the Lie group of invertible $N \times N$ matrices. We regard $GL_N(\mathbb{C})$ as a real Lie group, with its Lie algebra $\mathfrak{gl}_N(\mathbb{C}) = M_N(\mathbb{C})$. We endow $\mathfrak{gl}_N(\mathbb{C})$ with the following real-valued inner product:

$$
(X, Y) \mapsto \langle X, Y \rangle_{\mathfrak{gl}_N(\mathbb{C})} = N \Re \operatorname{Tr}(X^* Y).
$$

Unfortunately, this inner product is not invariant under the adjoint action: the left-invariant Riemannian metric which it determines does not coincide with the right-invariant Riemannian metric. Let us choose the right-invariant Riemannian metric. Let $\Delta_{GL_N(\mathbb{C})}$ be the Laplace operator corresponding to the right-invariant metric. Let $(Z_1, \ldots, Z_{2N^2})$ be an orthonormal basis of $M_N(\mathbb{C})$. Let us denote by $(\tilde{Z}_1, \ldots, \tilde{Z}_{2N^2})$ the right-invariant vector fields on $GL_N(\mathbb{C})$ which agree with $(Z_1, \ldots, Z_{2N^2})$ at the identity. The Laplace operator $\Delta_{GL_N(\mathbb{C})}$ is the second-order differential operator $\sum_{i=1}^{2N^2} \tilde{Z}_i^2$.

The (right) Brownian motion on $GL_N(\mathbb{C})$ is a Markov process starting at the identity, and with generator $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}$. As mentioned in the introduction, we have taken a definition of the Brownian motion on $GL_N(\mathbb{C})$ which differs by a factor of 2 from the usual definition for reasons of convenience.

11.2.1. *Computation of* $\Delta_{GL_N(\mathbb{C})}$. Similarly to $\Delta_{U(N)}$, we will see in Lemma 11.3 that the operator $\Delta_{GL_N(\mathbb{C})}$ differs from Δ_{GL} defined in Section 10.5.1 by an auxiliary operator $\frac{1}{N^2}\tilde{\Delta}_{GL}$ when acting on the image of the $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$-calculus.

Let us define this operator $\tilde{\Delta}_{GL}$ on $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$. Let $k, l \in \mathbb{N}$, and $X_1, \ldots, X_{k+l} \in \{X, X^*, X^{-1}, X^{*-1}\}$. For all $1 \leq i \leq k+l$, set $l(i) = 0$, $r(i) = 1$ if $X_i = X$ or $X_i = X^{*-1}$, and $l(i) = 1$, $r(i) = 0$ if $X_i = X^*$ or $X_i = X^{-1}$. For all $1 \leq i \leq k < j \leq k+l$, the term $\overbracket{X_i \cdots X_j}$ refers to the product $X_i^{r(i)} X_{i+1} \cdots X_{j-1} X_j^{l(i)}$, that is to say the product $X_i \cdots X_j$ possibly excluding X_i and/or X_j, and the term $\overbracket{X_j \cdots X_i}$ refers to the product $X_j^{r(j)} X_{j+1} \cdots X_{k+l} X_1 \cdots X_{i-1} X_i^{l(i)}$, that is to say the product $X_j \cdots X_{k+l} X_1 \cdots X_i$ possibly excluding X_j and/or X_i. For all $1 \leq i \leq k+l$, set $\epsilon(i) = 1$ if $X_i = X$ or $X_i = X^*$, and $\epsilon(i) = -1$ if $X_i = X^{*-1}$ or $X_i = X^{-1}$. For all $1 \leq i \leq k < j \leq k+l$, set $\delta(i,j) = 0$ if $X_i, X_j \in \{X, X^{-1}\}$ or if $X_i, X_j \in \{X^*, X^{*-1}\}$, and $\delta(i,j) = 1$ otherwise. We set

$$\begin{aligned}&\langle \nabla_{GL} X_1 \cdots X_k, \nabla_{GL} X_{k+1} \cdots X_{k+l} \rangle \\ &\quad = 2 \sum_{1 \leq i \leq k < j \leq k+l} \delta(i,j)\epsilon(i)\epsilon(j) X_1 \cdots \widehat{\overbracket{X_i \cdots X_j}} \cdots X_{k+l} X_{k+1} \cdots \widehat{\overbracket{X_j \cdots X_i}} \cdots X_k.\end{aligned}$$

Here again, it is important to note that $\langle \nabla_{GL} \cdot, \nabla_{GL} \cdot \rangle$ is just a notation for a bilinear map on $\mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$ which may be justified thanks to Lemma 11.3 (see the remark that follows Lemma 11.3).

Let $n \in \mathbb{N}$, and $P_0, \ldots, P_n$ be monomials of $\mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$. We set

$$\begin{aligned}&\tilde{\Delta}_{GL}(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n) \\ &\quad = 2 \sum_{m=1}^{n} \langle \nabla_{GL} P_0, \nabla_{GL} P_m \rangle \operatorname{tr} P_1 \cdots \widehat{\operatorname{tr} P_m} \cdots \operatorname{tr} P_n \\ &\qquad + \sum_{1 \leq m, m' \leq n} P_0 \operatorname{tr} P_1 \cdots \widehat{\operatorname{tr} P_m} \cdots \widehat{\operatorname{tr} P_{m'}} \cdots \operatorname{tr} P_n \operatorname{tr} \langle \nabla_{GL} P_m, \nabla_{GL} P_{m'} \rangle .\end{aligned} \tag{11.5}$$

We extend $\tilde{\Delta}_{GL}$ to all $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ by linearity.

Let us denote by G the identity function of $GL_N(\mathbb{C})$.

LEMMA 11.3. *For all* $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$, *we have*

$$\Delta_{GL_N(\mathbb{C})}\left(P(G)\right) = \left((\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})P\right)(G).$$

In particular, for P, Q two monomials of $\mathbb{C}\left\langle X, X^*, X^{-1}, X^{*-1}\right\rangle$, we have

$$\Delta_{GL_N(\mathbb{C})}((PQ)(U)) = \left((\Delta_{GL} P) \cdot_{\operatorname{tr}} Q + \frac{2}{N^2}\langle \nabla_{GL} P, \nabla_{GL} Q \rangle + P \cdot_{\operatorname{tr}} (\Delta_{GL} Q)\right)(U).$$

PROOF. The proof follows the demonstration of Lemma 11.1.

Recall that $(Z_1, \ldots, Z_{2N^2})$ is an orthonormal basis of $M_N(\mathbb{C})$, that $(\tilde{Z}_1, \ldots, \tilde{Z}_{2N^2})$ are the right-invariant vector fields on $U(N)$ which agree with $(Z_1, \ldots, Z_{2N^2})$ at the identity, and that the Laplace operator $\Delta_{GL_N(\mathbb{C})}$ is the second-order differential operator $\sum_{a=1}^{2N^2} \tilde{Z}_a^2$.

It suffices to prove that, for all P and Q monomials of $\mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$, we have

$$\Delta_{GL_N(\mathbb{C})} P(G) = (\Delta_{GL} P)(G) \tag{11.6}$$

and

$$\sum_{a=1}^{2N^2} \tilde{Z}_a\left(P(G)\right) \operatorname{tr} \tilde{Z}_a\left(Q(G)\right) = \frac{1}{N^2}\left\langle \nabla_{GL}P, \nabla_{GL}Q\right\rangle(G). \tag{11.7}$$

Indeed, let $P_0, \ldots, P_n$ be monomials of $\mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$. It follows from (11.6) and (11.7) that

$$\begin{aligned}
&\Delta_{GL_N(\mathbb{C})}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(G)\\
&\quad= \left(\Delta_{GL_N(\mathbb{C})}P_0(G)\right) \operatorname{tr} P_1(G) \cdots \operatorname{tr} P_n(G)\\
&\qquad + \sum_{m=1}^{n} P_0(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr}\left(\Delta_{GL_N(\mathbb{C})}P_m(G)\right) \cdots \operatorname{tr} P_n(G)\\
&\qquad + 2\sum_{m=1}^{n}\sum_{a=1}^{2N^2} \left(\tilde{Z}_a P_0(G)\right) \operatorname{tr} P_1(G) \cdots \operatorname{tr}\left(\tilde{Z}_a P_m(G)\right) \cdots \operatorname{tr} P_n(G)\\
&\qquad + \sum_{1\leq m,m'\leq n}\sum_{a=1}^{2N^2} P_0(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr}\left(\tilde{Z}_a P_m(G)\right) \cdots \operatorname{tr}\left(\tilde{Z}_a P'_m(G)\right) \cdots \operatorname{tr} P_n(G).
\end{aligned}$$

Thanks to the structure equations (10.14) and (11.5) of Δ_{GL} and $\tilde{\Delta}_{GL}$, we deduce that

$$\begin{aligned}
&\Delta_{GL_N(\mathbb{C})}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(G)\\
&\quad= \left(\Delta_{GL}P_0\right)(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr} P_n(G)\\
&\qquad + \sum_{m=1}^{n} P_0(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr}\left(\Delta_{GL}P_m\right)(G) \cdots \operatorname{tr} P_n(G)\\
&\qquad + \frac{2}{N^2}\sum_{m=1}^{n}\sum_{a=1}^{2N^2} \left\langle \nabla_{GL}P_0, \nabla_{GL}P_m\right\rangle(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr} \widehat{P_m(G)} \cdots \operatorname{tr} P_n(G)\\
&\qquad + \frac{1}{N^2}\sum_{1\leq m,m'\leq n}\sum_{a=1}^{2N^2} P_0(G) \operatorname{tr} P_1(G) \cdots \operatorname{tr} \widehat{P_m(G)} \cdots \operatorname{tr} \widehat{P_{m'}(G)} \cdots \operatorname{tr} P_n(G)\\
&\qquad\qquad \cdot \operatorname{tr}\left\langle \nabla_{GL}P_m, \nabla_{GL}P_{m'}\right\rangle(G)\\
&\quad= \Delta_{GL}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(G) + \frac{1}{N^2}\tilde{\Delta}_{GL}\left(P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(G),
\end{aligned}$$

and Lemma 11.3 follows by linearity. Thus, we need only prove (11.6) and (11.7). For $1 \leq a \leq 2N^2$, we have $\tilde{Z}_a(G) = Z_a G$,

$$\tilde{Z}_a\left(G^{-1}\right) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(e^{tZ_a}G\right)^{-1} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\left(G^{-1}e^{-tZ_a}\right) = -G^{-1}Z_a,$$

and similarly $\tilde{Z}_a(G^*) = G^* Z_a^*$, and $\tilde{Z}_a\left(G^{*-1}\right) = -Z_a^* G^{-1}$. Therefore, the expressions of $\sum_{a=1}^{2N^2} Z_a \otimes Z_a$, $\sum_{a=1}^{2N^2} Z_a \otimes Z_a^*$ and $\sum_{a=1}^{2N^2} Z_a^* \otimes Z_a^*$ will be useful. Let us fix the basis

$$\{Z_1, \ldots, Z_{2N^2}\} = \{\frac{1}{\sqrt{N}}E_{a,b}, \frac{i}{\sqrt{N}}Ea,b : 1 \leq a,b \leq N\}.$$

We have

$$\sum_{a=1}^{N^2} Z_a \otimes Z_a = \sum_{1\leq a,b\leq N} \frac{1}{N}E_{a,b} \otimes E_{a,b} + \sum_{1\leq a,b\leq N} \frac{-1}{N}E_{a,b} \otimes E_{a,b} = 0,$$

$$\sum_{a=1}^{N^2} Z_a \otimes Z_a^* = \sum_{1\leq a,b\leq N} \frac{1}{N}E_{a,b} \otimes E_{b,a} + \sum_{1\leq a,b\leq N} \frac{1}{N}E_{a,b} \otimes E_{b,a} = \frac{2}{N}\sum_{1\leq a,b\leq N} E_{a,b} \otimes E_{b,a},$$

and

$$\sum_{a=1}^{N^2} Z_a^* \otimes Z_a^* = \sum_{1\le a,b\le N} \frac{1}{N} E_{b,a} \otimes E_{b,a} + \sum_{1\le a,b\le N} \frac{-1}{N} E_{b,a} \otimes E_{b,a} = 0.$$

From this, we deduce that $\sum_{a=1}^{2N^2} \tilde{Z}_a^2(G^\varepsilon) = 0$ for all $\varepsilon \in \{1, -1, *, *-1\}$. Moreover,

$$\sum_{a=1}^{2N^2} \left(\tilde{Z}_a(G) \otimes \tilde{Z}_a(G^{-1}) \right) = \sum_{a=1}^{N^2} \left(\tilde{Z}_a(G^*) \otimes \tilde{Z}_a(G^{*-1}) \right) = 0,$$

$$\sum_{a=1}^{2N^2} \left(\tilde{Z}_a(G) \otimes \tilde{Z}_a(G^*) \right) = \frac{2}{N} \sum_{1\le a,b\le N} E_{a,b} G \otimes G^* E_{b,a},$$

$$\sum_{a=1}^{2N^2} \left(\tilde{Z}_a(G) \otimes \tilde{Z}_a(G^{*-1}) \right) = -\frac{2}{N} \sum_{1\le a,b\le N} E_{a,b} G \otimes E_{b,a} G^{*-1},$$

$$\sum_{a=1}^{2N^2} \left(\tilde{Z}_a(G^{-1}) \otimes \tilde{Z}_a(G^*) \right) = -\frac{2}{N} \sum_{1\le a,b\le N} G^{-1} E_{a,b} \otimes G^* E_{b,a},$$

and

$$\sum_{a=1}^{2N^2} \left(\tilde{Z}_a(G^{-1}) \otimes \tilde{Z}_a(G^{*-1}) \right) = \frac{2}{N} \sum_{1\le a,b\le N} G^{-1} E_{a,b} \otimes E_{b,a} G^{*-1}. \tag{11.8}$$

Let $k, l \in \mathbb{N}$, and $X_1, \dots, X_{k+l} \in \{X, X^*, X^{-1}, X^{*-1}\}$. For all $1 \le i \le k+l$, set $\epsilon(i) = 1$ if $X_i = X$ or $X_i = X^*$, and $\epsilon(i) = -1$ if $X_i = X^{*-1}$ or $X_i = X^{-1}$. For all $1 \le i \le k < j \le k+l$, set $\delta(i,j) = 0$ if $X_i, X_j \in \{X, X^{-1}\}$ or if $X_i, X_j \in \{X^*, X^{*-1}\}$, and $\delta(i,j) = 1$ otherwise. For all $1 \le i \le k+l$, set $G_i = X_i(G)$. We can now compute

$$\begin{aligned}
\Delta_{GL_N(\mathbb{C})}(P(G)) &= \sum_{a=1}^{2N^2} \tilde{Z}_a^2 (G_1 \cdots G_k) \\
&= \sum_{a=1}^{2N^2} 2 \sum_{1\le i<j\le k} G_1 \cdots \tilde{Z}_a(G_i) \cdots \tilde{Z}_a(G_j) \cdots G_k \\
&= -4 \sum_{1\le i<j\le k} \delta(i,j)\epsilon(i)\epsilon(j) \cdot \left(X_1 \cdots \widehat{\overline{X_i \cdots X_j}} \cdots X_k \operatorname{tr}\left(\overline{X_i \cdots X_j} \right) \right)(G) \\
&= (\Delta_{GL} P)(G)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{a=1}^{2N^2} \tilde{Z}_a(P(G)) \operatorname{tr} \tilde{Z}_a(Q(G)) \\
&= \sum_{a=1}^{2N^2} \tilde{Z}_a(G_1 \cdots G_k) \operatorname{tr} \tilde{Z}_a(G_{k+1} \cdots G_{k+l}) \\
&= \frac{1}{N} \sum_{1\le i\le k<j\le k+l} \sum_{a=1}^{N^2} \left(G_1 \cdots \tilde{Z}_a(G_i) \cdots G_k \right) \operatorname{Tr}\left(G_{k+1} \cdots \tilde{Z}_a(G_j) \cdots G_{k+l} \right) \\
&= \frac{2}{N^2} \sum_{1\le i\le k<j\le k+l} \delta(i,j)\epsilon(i)\epsilon(j) \left(X_1 \cdots \widehat{\overline{X_i \cdots X_j}} \cdots X_{k+l} X_{k+1} \cdots \widehat{\overline{X_j \cdots X_i}} \cdots X_k \right)(G) \\
&= \frac{1}{N^2} \langle \nabla_{GL} P, \nabla_{GL} Q \rangle (G),
\end{aligned}$$

where we have used (11.4). □

COROLLARY 11.4. *Let* $\left(G_t^{(N)}\right)_{t\geq 0}$ *be a Brownian motion on* $GL_N(\mathbb{C})$.
Let $t \geq 0$ *and* $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$. *We have*

$$\mathbb{E}\left[P\left(G_t^{(N)}\right)\right] = \left(e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}P\right)(1).$$

PROOF. For all $t \geq 0$ and $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$, thanks to Lemma 11.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[P\left(G_t^{(N)}\right)\right] = \mathbb{E}\left[\left(\frac{1}{4}\Delta_{GL_N(\mathbb{C})}\big(P(G)\big)\right)\left(G_t^{(N)}\right)\right] = \mathbb{E}\left[\left(\frac{1}{4}(\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})P\right)\left(G_t^{(N)}\right)\right].$$

We conclude with Lemma 9.5, since the maps $(t \mapsto \mathbb{E}\left[P\left(G_t^{(N)}\right)\right])_{t\geq 0}$ are linear. □

11.3. Hall transform. In this section, we introduce the Hall transform, and study it in the particular case of the unitary group $U(N)$. We refer to [36] or [44] for more details. The major difference is that Hall and Driver work with left-invariant metrics and Laplace operators. Because we chose to work with right Brownian motion, we will only consider right-invariant metrics and Laplace operators. The necessary modifications are minor.

11.3.1. *Definition.* Let K be a compact connected Lie group, with Lie algebra $\mathfrak{k}$. Let $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ denote a fixed $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$. It determines a bi-invariant Riemannian metric on K and a bi-invariant Laplace operator Δ_K. For all $t \geq 0$, let $\mathrm{d}\rho_t$ be the heat kernel measure on K at time t; that is to say, the probability measure on K which corresponds to the law at time t of a Markov process starting at the identity and with generator $\frac{1}{2}\Delta_K$. For all $t \geq 0$, the operator $e^{\frac{t}{2}\Delta_K}$ over $L^2(K, \mathrm{d}\rho_t)$ is given for all $f \in L^2(K, \mathrm{d}\rho_t)$ by $e^{\frac{t}{2}\Delta_K} f : x \mapsto \int_K f(yx)\mathrm{d}\rho_t(y)$.

Let G be the complexification of K, with Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Let us endow $\mathfrak{k}$ with the (real-valued) inner product given by $\langle K_1 + iK_2, K_3 + iK_4\rangle_{\mathfrak{g}} = \langle K_1, K_3\rangle_{\mathfrak{k}} + \langle K_2, K_4\rangle_{\mathfrak{k}}$ for all $K_1, K_2, K_3, K_4 \in \mathfrak{k}$: it determines a right-invariant Riemannian metric on G and a right-invariant Laplace operator Δ_G. For all $t \geq 0$, let $\mathrm{d}\mu_t$ be the "half heat kernel" measure on G at time t; that is to say, the probability measure on G which corresponds to the law at time t of a Markov process starting at the identity and with generator $\frac{1}{4}\Delta_G$.

For all $t > 0$, we denote by $L^2_{\mathrm{hol}}(G, \mathrm{d}\mu_t)$ the Hilbert space of holomorphic function in $L^2(G, \mathrm{d}\mu_t)$. The fact that $L^2_{\mathrm{hol}}(G, \mathrm{d}\mu_t)$ is a Hilbert space is not trivial. It is a part of Hall's theorem which may be stated as follows (see Theorem 1 of [44] or Theorem 1.16 of [36]).

THEOREM (Hall [44]). *Let* $t > 0$. *For all* $f \in L^2(K, \mathrm{d}\rho_t)$, *the function* $e^{\frac{t}{2}\Delta_K} f$ *has an analytic continuation to a holomorphic function on* G, *also denoted by* $e^{\frac{t}{2}\Delta_K} f$. *Moreover,* $e^{\frac{t}{2}\Delta_K} f \in L^2_{\mathrm{hol}}(G, \mathrm{d}\mu_t)$ *and the linear map* $B_t : f \mapsto e^{\frac{t}{2}\Delta_K} f$ *is an isomorphism of Hilbert spaces between* $L^2(K, \mathrm{d}\rho_t)$ *and* $L^2_{\mathrm{hol}}(G, \mathrm{d}\mu_t)$.

11.3.2. *The Hall transform on* $U(N)$. In this section, we take for K the Lie group $U(N)$, whose complexification is $GL_N(\mathbb{C})$. Let $(U_t^{(N)})_{t\geq 0}$ be a Brownian motion on $U(N)$ (with generator $\frac{1}{2}\Delta_{U(N)}$), and $(G_t^{(N)})_{t\geq 0}$ be a Brownian motion on $GL_N(\mathbb{C})$ (with generator $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}$). With the notation of the previous section, for all $t \geq 0$, $\mathrm{d}\rho_t$ is the law of $U_t^{(N)}$ and $\mathrm{d}\mu_t$ is the law of $G_t^{(N)}$. Hall's theorem gives us an isomorphism of Hilbert spaces between $L^2(U(N), \mathrm{d}\rho_t)$ and $L^2_{\mathrm{hol}}(GL_N(\mathbb{C}), \mathrm{d}\mu_t)$. We remind of the probabilistic formulation of Hall's theorem in the introduction: it expresses the Hall transform B_t as an isomorphism of Hilbert spaces between the spaces of random variables $L^2(U_t^{(N)})$ and $L^2_{\mathrm{hol}}(G_t^{(N)})$ as defined in the introduction. Let us explicit the identification made between the spaces of functions $L^2(U(N),\mathrm{d}\rho_t)$ and $L^2_{\mathrm{hol}}(GL_N(\mathbb{C}),\mathrm{d}\mu_t)$ and the spaces of random variables $L^2(U_t^{(N)})$ and $L^2_{\mathrm{hol}}(G_t^{(N)})$.

Let $t>0$. For all complex Borel function f, we have $\mathbb{E}[|f(U_t^{(N)})|^2]=\int_{U(N)}|f|^2\mathrm{d}\rho_t$ in $[0,+\infty]$. Thus, if f and g are two complex Borel functions such that $f(U_t^N)=g(U_t^N)$ *a.s.*, then $\int_{U(N)}|f-g|^2\mathrm{d}\rho_t$ =0 and therefore $f=g$ *a.s.* Moreover, the map $f\mapsto f(U_t^{(N)})$ is an isomorphism between $L^2(U(N),\mathrm{d}\rho_t)$ and $L^2(U_t^{(N)})$. Similarly, $F\mapsto F(G_t^N)$ is an isomorphism between $L^2_{\mathrm{hol}}(GL_N(\mathbb{C}),\mathrm{d}\mu_t)$ and $L^2_{\mathrm{hol}}(G_t^{(N)})$. Identifying the space $L^2(U(N),\mathrm{d}\rho_t)$ with $L^2(U_t^{(N)})$ and $L^2_{\mathrm{hol}}(GL_N(\mathbb{C}),\mathrm{d}\mu_t)$ with $L^2_{\mathrm{hol}}(G_t^{(N)})$, we deduce the version of Hall's theorem as formulated in the introduction: the Hall transform B_t is an isomorphism of Hilbert spaces between $L^2(U_t^{(N)})$ and $L^2_{\mathrm{hol}}(G_t^{(N)})$, and for all Borel function f on $U(N)$ such that $f(U_t^{(N)})\in L^2(U_t^{(N)})$, we have

$$B_t\Big(f(U_t^{(N)})\Big)=\Big(e^{\frac{t}{2}\Delta_{U(N)}}f\Big)\left(G_t^{(N)}\right).$$

11.3.3. *Boosting.* For all $N\in\mathbb{N}^*$, we endow $M_N(\mathbb{C})$ with the inner product $\langle X,Y\rangle_{M_N(\mathbb{C})}=\frac{1}{N}\operatorname{Tr}(X^*Y)$. For all $t>0$ and $N\in\mathbb{N}^*$, we denote by $B_t^{(N)}$ the Hilbert space isomorphism $B_t\otimes\mathrm{Id}_{M_N(\mathbb{C})}$ from $L^2(U_t^{(N)})\otimes M_N(\mathbb{C})$ into $L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})$.

Proposition 11.5. *Let $t>0$. For all $P\in\mathbb{C}\{X,X^{-1}\}$, we have*

$$B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right)=\left(e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P\right)\left(G_t^{(N)}\right).$$

Proof. Recall that we denote by U and G the identity functions of respectively $U(N)$ and $GL_N(\mathbb{C})$. We know that, for all $P\in\mathbb{C}\{X,X^{-1}\}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{t}{2}\Delta_{U(N)}}P(U)\right)=e^{\frac{t}{2}\Delta_{U(N)}}\left(\frac{1}{2}\Delta_{U(N)}P(U)\right).$$

Thus, thanks to Lemma 11.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{\frac{t}{2}\Delta_{U(N)}}(P(U))\right)=e^{\frac{t}{2}\Delta_{U(N)}}\left(\left(\frac{1}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)P\right)(U)\right).$$

Since the maps $(P\mapsto e^{\frac{t}{2}\Delta_{U(N)}}(P(U)))_{t\geq0}$ are linear, Lemma 9.5 tells us that, for all $P\in\mathbb{C}\{X,X^{-1}\}$ and $t\geq0$, we have $e^{\frac{t}{2}\Delta_{U(N)}}(P(U))=(e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P)(U)$.

Let $P\in\mathbb{C}\{X,X^{-1}\}$. Set $f=P(U)$. We have $e^{\frac{t}{2}\Delta_{U(N)}}f=(e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P)(U)$ which has an analytic continuation to a holomorphic function on $GL_N(\mathbb{C})$, denoted by F, and by definition

$$B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right)=F\left(G_t^{(N)}\right).$$

It remains to prove that $F=(e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P)(G)$, or equivalently to prove that the map $(e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P)(G)$ is holomorphic, which is evident since $e^{\frac{t}{2}(\Delta_U+\frac{1}{N^2}\tilde{\Delta}_U)}P\in\mathbb{C}\{X,X^{-1}\}$, and coordinates of G and G^{-1} are holomorphic. ∎

11.4. Large-N limit.

11.4.1. *Brownian motion on $GL_N(\mathbb{C})$.* The free circular multiplicative Brownian motion $(G_t)_{t\geq 0}$ is defined with a free stochastic differential equation driven by a free Brownian motion (see Section 10.5). It can be proved that the Brownian motion on $GL_N(\mathbb{C})$ verifies the same stochastic differential equation driven by a Brownian motion on $M_N(\mathbb{C})$ (see [19]). The following theorem is not surprising and reflects a well-known phenomenon: the free stochastic calculus is intuitively the limit of the stochastic calculus on $M_N(\mathbb{C})$ when $N \to \infty$. See also [50] and [51] for similar results with a larger class of heat kernel measures on $GL_N(\mathbb{C})$.

THEOREM 11.6. *For all $N \in \mathbb{N}^*$, let $(G_t^{(N)})_{t\geq 0}$ be a Brownian motion on $GL_N(\mathbb{C})$ and let $(G_t)_{t\geq 0}$ be a free circular multiplicative Brownian motion. For all $n \in \mathbb{N}$, and all $P_0, \dots, P_n \in \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$, for each $t \geq 0$, as $N \to \infty$, we have*

$$\mathbb{E}\left[\operatorname{tr}\left(P_0\left(G_t^{(N)}\right)\right)\cdots\operatorname{tr}\left(P_n\left(G_t^{(N)}\right)\right)\right] = \tau\left(P_0\left(G_t\right)\right)\cdots\tau\left(P_n\left(G_t\right)\right) + O(1/N^2).$$

PROOF. It is equivalent to prove that, for all $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$, we have

$$\mathbb{E}\left[\operatorname{tr}\left(P\left(G_t^{(N)}\right)\right)\right] = \tau\left(P\left(G_t\right)\right) + O(1/N^2).$$

Indeed, Theorem 11.6 is the particular case when $P = P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n$.

Let $P \in \mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ and $t \geq 0$. From Corollary 11.4, we have

$$\mathbb{E}\left[\operatorname{tr}\left(P\left(G_t^{(N)}\right)\right)\right] = \operatorname{tr}\left(\left(e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}P\right)(1)\right) = \left(e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}P\right)(1).$$

From Proposition 10.6, we have $\tau\left(P\left(G_t\right)\right) = \left(e^{\frac{t}{4}\Delta_{GL}}P\right)(1)$.

Let d be the degree of P. Because the exponential map is differentiable, we have

$$e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})} = e^{\frac{t}{4}\Delta_{GL}} + O(1/N^2)$$

in the finite dimensional space $\operatorname{End}(\mathbb{C}_d\{X, X^*, X^{-1}, X^{*-1}\})$. Since $A \mapsto (A(P))(1)$ is a linear map from $\operatorname{End}(\mathbb{C}_d\{X, X^*, X^{-1}, X^{*-1}\})$ to $\mathbb{C}$, it is therefore continuous and we have

$$\left(e^{\frac{t}{4}(\Delta_{GL}+\frac{1}{N^2}\tilde{\Delta}_{GL})}P\right)(1) = \left(e^{\frac{t}{4}\Delta_{GL}}P\right)(1) + O(1/N^2),$$

which completes the proof. ☐

11.4.2. *Hall transform.* We recall that, for all $t > 0$, $\mathcal{G}_t$ is the free Hall transform at the level of Laurent polynomials $\mathbb{C}[X, X^{-1}]$, and that $e^{\frac{t}{2}\Delta_U}$ is the free Hall transform at the level of the space $\mathbb{C}\{X, X^{-1}\}$ (see Section 10.6). The non-commutative random variables $U_t^{(N)}$ and $G_t^{(N)}$ give approximations to the free unitary Brownian motion U_t and the circular multiplicative Brownian motion G_t at time t. The following result states that the Hall transform also goes to the limit as N tends to ∞, on both the $\mathbb{C}[X, X^{-1}]$ and the $\mathbb{C}\{X, X^{-1}\}$-calculus. See also [38] for another proof of the first item.

THEOREM 11.7. *Let $t > 0$. For all $N \in \mathbb{N}^*$, let $(U_t^{(N)})_{t\geq 0}$ be a Brownian motion on $U(N)$, and $(G_t^{(N)})_{t\geq 0}$ be a Brownian motion on $GL_N(\mathbb{C})$. We have*

(1) for all Laurent polynomial $P \in \mathbb{C}[X, X^{-1}]$, as $N \to \infty$,

$$\left\|B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right) - \mathcal{G}_t(P)\left(G_t^{(N)}\right)\right\|^2_{L^2_{\text{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})} = O(1/N^2),$$

(2) for all $P \in \mathbb{C}\{X, X^{-1}\}$, as $N \to \infty$,

$$\left\|B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right) - \left(e^{\frac{t}{2}\Delta_U}P\right)\left(G_t^{(N)}\right)\right\|^2_{L^2_{\text{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})} = O(1/N^2).$$

PROOF. Let $t > 0$ and $P \in \mathbb{C}[X, X^{-1}]$. Let $N \in \mathbb{N}^*$. From Corollary 11.4 and Proposition 11.5, we have

$$\begin{aligned}
&\left\| B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right) - \mathcal{G}_t(P)\left(G_t^{(N)}\right)\right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})} \\
= & \left\| e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)} P\left(G_t^{(N)}\right) - \mathcal{G}_t(P)\left(G_t^{(N)}\right)\right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})} \\
= & \ e^{\frac{t}{4}(\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})}\left(\left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}P - \mathcal{G}_t(P)\right)^*\left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}P - \mathcal{G}_t(P)\right)\right)(1).
\end{aligned}$$

Let d be the degree of P. We will work in finite-dimensional spaces and consequently all the norms are equivalent and the linearity of a map implies its boundedness. The differentiability of the exponential map leads to

$$e^{\frac{t}{4}(\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})} = e^{\frac{t}{4}\Delta_{GL}} + O(1/N^2) \ \text{ and } e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)} = e^{\frac{t}{2}\Delta_U} + O(1/N^2)$$

in the finite dimensional space $\mathrm{End}(\mathbb{C}_d\{X, X^*, X^{-1}, X^{*-1}\})$. Since $A \mapsto A(P)$ is linear, we deduce that

$$\left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}P - \mathcal{G}_t(P)\right) = \left(e^{\frac{t}{2}\Delta_U}P - \mathcal{G}_t(P)\right) + O(1/N^2).$$

The linearity of $(A, P, Q) \mapsto (A(P^*Q))(1)$ from

$$\mathrm{End}(\mathbb{C}_d\{X, X^*, X^{-1}, X^{*-1}\}) \times \mathbb{C}_d\{X, X^*, X^{-1}, X^{*-1}\}^2$$

to $\mathbb{C}$ implies that

$$\begin{aligned}
& e^{\frac{t}{4}(\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})}\left(\left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}P - \mathcal{G}_t(P)\right)^*\left(e^{\frac{t}{2}(\Delta_U + \frac{1}{N^2}\tilde{\Delta}_U)}P - \mathcal{G}_t(P)\right)\right)(1) \\
= & \ e^{\frac{t}{4}\Delta_{GL}}\left(\left(e^{\frac{t}{2}\Delta_U}P - \mathcal{G}_t(P)\right)^*\left(e^{\frac{t}{2}\Delta_U}P - \mathcal{G}_t(P)\right)\right)(1) + O(1/N^2).
\end{aligned}$$

From Proposition 10.6, we have

$$\begin{aligned}
& e^{\frac{t}{4}\Delta_{GL}}\left(\left(e^{\frac{t}{2}\Delta_U}P - \mathcal{G}_t(P)\right)^*\left(e^{\frac{t}{2}\Delta_U}P - \mathcal{G}_t(P)\right)\right)(1) \\
= & \ \tau\left(\left((e^{\frac{t}{2}\Delta_U}P)(G_t) - \mathcal{G}_t(P)(G_t)\right)^*\left((e^{\frac{t}{2}\Delta_U}P)(G_t) - \mathcal{G}_t(P)(G_t)\right)\right),
\end{aligned}$$

and by Theorem 10.7, we have $e^{\frac{t}{2}\Delta_U}P(G_t) - \mathcal{G}_t\Big(P\Big)(G_t) = 0$. Finally, we have

$$\left\| B_t^{(N)}\left(P\left(U_t^{(N)}\right)\right) - \mathcal{G}_t(P)\left(G_t^{(N)}\right)\right\|^2_{L^2_{\mathrm{hol}}(G_t^{(N)})\otimes M_N(\mathbb{C})} = 0 + O(1/N^2).$$

The second assertion follows using the same argument substituting $e^{\frac{t}{2}\Delta_U}P$ for $\mathcal{G}_t(P)$. ☐

12. Appendix. The construction of $\mathbb{C}\{X_i : i \in I\}$

In this appendix, the algebra $\mathbb{C}\{X_i : i \in I\}$ is constructed as the direct limit of the inductive system formed by the family of finite tensor product $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ where $\odot$ is defined below. Then we verify that, when endowed with the appropriate product and center-valued expectation, $\mathbb{C}\{X_i : i \in I\}$ satisfies Universal property 8.1.

Construction by direct limit. Let $n \in \mathbb{N}$. Let E be the subspace of $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\otimes n}$ generated by the families

$$\{P_0 \otimes P_1 \otimes \cdots \otimes P_n - P_0 \otimes P_{\sigma(1)} \otimes \cdots \otimes P_{\sigma(n)} : P_0, \ldots, P_n \in \mathbb{C}\langle X_i : i \in I\rangle, \sigma \in \mathfrak{S}_n\}.$$

Let us denote by $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ the quotient space of the space $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\otimes n}$ by its subspace E. If $P_0, \ldots, P_n \in \mathbb{C}\langle X_i : i \in I\rangle$, we denote the equivalence class of $P_0 \otimes P_1 \otimes \cdots \otimes P_n$ in $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ by $P_0 \otimes P_1 \odot \cdots \odot P_n$.

For all $n \leq m \in \mathbb{N}$, we identify $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ as a subset of $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot m}$ thanks to the injective morphism

$$\begin{array}{rccc} i_{n,m} : & \mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n} & \longrightarrow & \mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot m} \\ & P_0 \otimes P_1 \odot \cdots \odot P_n & \longmapsto & P_0 \otimes P_1 \odot \cdots \odot P_n \odot 1 \odot \cdots \odot 1. \end{array}$$

The inclusion maps $(i_{n,m})_{n \leq m \in \mathbb{N}}$ make $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ an inductive system in the sense that, for all $l \leq n \leq m \in \mathbb{N}$, we have $i_{l,m} = i_{l,n} \circ i_{n,m}$. The vector space $\mathbb{C}\{X_i : i \in I\}$ is the direct limit $\mathbb{C}\{X_i : i \in I\} = \varinjlim \mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$. It is the minimal vector space which contains all the tensor products $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$ as subspaces identified via the inclusion maps $(i_{n,m})_{n \leq m \in \mathbb{N}}$. In particular, the space $\mathbb{C}\langle X_i : i \in I\rangle$ is a subspace of $\mathbb{C}\{X_i : i \in I\}$.

We have at our disposal a basis of $\mathbb{C}\{X_i : i \in I\}$. Indeed, for all $n \in \mathbb{N}$,

$$\{M_0 \otimes M_1 \odot \cdots \odot M_n : M_0, \ldots, M_n \text{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}$$

is a basis of $\mathbb{C}\langle X_i : i \in I\rangle \otimes \mathbb{C}\langle X_i : i \in I\rangle^{\odot n}$. Thus,

$$\{M_0 \otimes M_1 \odot \cdots \odot M_n : n \in \mathbb{N}, M_0, \ldots, M_n \text{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}$$

is a basis of $\mathbb{C}\{X_i : i \in I\}$, called the canonical basis.

Let us define now the product and the center-valued expectation of $\mathbb{C}\{X_i : i \in I\}$. For all $n \in \mathbb{N}$, $M_0, \ldots, M_n$ monomials of $\mathbb{C}\langle X_i : i \in I\rangle$, and $n' \in \mathbb{N}$, $N_0, \ldots, N_{n'}$ monomials of $\mathbb{C}\langle X_i : i \in I\rangle$, let us define the product $M \cdot N = MN$ of $M = M_0 \otimes M_1 \odot \cdots \odot M_n$ and $N = N_0 \otimes N_1 \odot \cdots \odot N_{n'}$ by

$$M \cdot N = MN = M_0 N_0 \otimes M_1 \odot \cdots M_n \odot N_1 \odot \cdots \odot N_{n'}.$$

We extend this product to all $\mathbb{C}\{X_i : i \in I\}$ by linearity. Endowed with this product, the space $(\mathbb{C}\{X_i : i \in I\}, \cdot)$ is a unital complex algebra.

Let $n \in \mathbb{N}$, and $M_0, \ldots, M_n \in \mathbb{C}\langle X_i : i \in I\rangle$. For all $M = M_0 \otimes M_1 \odot \cdots \odot M_n \in \mathbb{C}\{X_i : i \in I\}$, we set

$$\operatorname{tr} M = 1 \otimes M_0 \odot M_1 \odot \cdots \odot M_n,$$

and we extend the map tr to all $\mathbb{C}\{X_i : i \in I\}$ by linearity. Defined this way, tr is a center-valued expectation. Indeed, the center consists precisely of linear combinations of elements of the form $1 \otimes M_0 \odot M_1 \odot \cdots \odot M_n$.

We remark that, for all $n \in \mathbb{N}, P_0, \ldots, P_n \in \mathbb{C}\langle X_i : i \in I\rangle$, we have $P_0 \otimes P_1 \odot \cdots \odot P_n = P_0 \operatorname{tr} P_1 \cdots \operatorname{tr} P_n$. Thus, the canonical basis of $\mathbb{C}\{X_i : i \in I\}$ can be rewritten

$$\{M_0 \operatorname{tr} M_1 \cdots \operatorname{tr} M_n : n \in \mathbb{N}, M_0, \ldots, M_n \text{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}.$$

Universal property. The triplet $\left(\mathbb{C}\{X_i : i \in I\}, \mathrm{tr}, (X_i)_{i\in I}\right)$ is now constructed, and it remains to prove the universal property.

Let $\mathcal{A}$ be an algebra endowed with a center-valued expectation τ, and with a family of I elements $\mathbf{A} = (A_i)_{i\in I}$. Let $n \in \mathbb{N}$, and $M_0, \ldots, M_n$ be monomials of $\mathbb{C}\langle X_i : i \in I\rangle$. Let us define $f\left(M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n\right)$ by

$$f\left(M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n\right) = \tau\left(M_1\left(\mathbf{A}\right)\right) \cdots \tau\left(M_n\left(\mathbf{A}\right)\right) \cdot M_0\left(\mathbf{A}\right),$$

and we extend f to a map from $\mathbb{C}\{X_i : i \in I\}$ to $\mathcal{A}$ by linearity. The properties of polynomial calculus make f an algebra morphism. Moreover, by definition, for all $i \in I$, we have $f(X_i) = A_i$. Finally, we verify that, for all $P \in \mathbb{C}\{X_i : i \in I\}$, we have $\tau(f(P)) = f(\mathrm{tr}(P))$. Let $n \in \mathbb{N}$, and $P_0, \ldots, P_n \in \mathbb{C}\langle X_i : i \in I\rangle$. We have

$$\begin{aligned}
\tau\left(f\left(P_0 \,\mathrm{tr}\, P_1 \cdots \mathrm{tr}\, P_n\right)\right) &= \tau\left(\tau\left(P_1\left(\mathbf{A}\right)\right) \cdots \tau\left(P_n\left(\mathbf{A}\right)\right) \cdot P_0\left(\mathbf{A}\right)\right) \\
&= \tau\left(P_0\left(\mathbf{A}\right)\right) \tau\left(P_1\left(\mathbf{A}\right)\right) \cdots \tau\left(P_n\left(\mathbf{A}\right)\right) \\
&= f\left(1 \,\mathrm{tr}\, P_0 \cdots \mathrm{tr}\, P_n\right) \\
&= f\left(\mathrm{tr}\left(P_0 \,\mathrm{tr}\, P_1 \cdots \mathrm{tr}\, P_n\right)\right)
\end{aligned}$$

and by linearity, we deduce that, for all $P \in \mathbb{C}\{X_i : i \in I\}$, we have $\tau\left(f(P)\right) = f\left(\mathrm{tr}\left(P\right)\right)$.

Thus, there exists an algebra homomorphism f from $\mathbb{C}\langle X_i : i \in I\rangle$ to $\mathcal{A}$ such that

(1) for all $i \in I$, we have $f(X_i) = A_i$;
(2) for all $X \in \mathbb{C}\{X_i : i \in I\}$, we have $\tau\left(f(X)\right) = f\left(\mathrm{tr}\left(X\right)\right)$.

The uniqueness of such a morphism is clear since the action of f on $\mathbb{C}\langle X_i : i \in I\rangle$ is uniquely determined by the polynomial calculus, and by consequence f is uniquely determined on the basis

$$\{M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n : n \in \mathbb{N}, M_0, \ldots M_n \text{ are monomials of } \mathbb{C}\langle X_i : i \in I\rangle\}.$$

Indeed, let $n \in \mathbb{N}$, and $M_0, \ldots M_n$ be monomials of $\mathbb{C}\langle X_i : i \in I\rangle$. We have

$$\begin{aligned}
f\left(M_0 \,\mathrm{tr}\, M_1 \cdots \mathrm{tr}\, M_n\right) &= f(M_0) f\left(\mathrm{tr}\, M_1\right) \cdots f\left(\mathrm{tr}\, M_n\right) \\
&= f(M_0) \tau\left(f\left(M_1\right)\right) \cdots \tau\left(f\left(M_n\right)\right) \\
&= \tau\left(M_1\left(\mathbf{A}\right)\right) \cdots \tau\left(M_n\left(\mathbf{A}\right)\right) \cdot M_0\left(\mathbf{A}\right).
\end{aligned}$$

CHAPTER 3

Fluctuations of the brownian motion on the linear group in large dimension

This chapter is a work in collaboration with Todd Kemp [27].

ABSTRACT. The Brownian motion on the general linear group $GL_N(\mathbb{C})$ converges in noncommutative distribution to a free multiplicative Brownian motion when N tends to infinity. This paper discuss the fluctuations of this convergence: the collection of moments rescaled by the rate $1/N$ converges to a Gaussian system with covariance that can be described in terms of three freely independent free multiplicative Brownian motions.

Contents

13. Introduction

Let $M_N(\mathbb{C})$ denote the set of $N \times N$ complex matrices. Let $(B^N)_{N\geq 1}$ be a sequence of random matrices such that $B^N \in M_N(\mathbb{C})$. Generally, the first phenomenon studied is the convergence in non-commutative distribution of B^N, meaning that for each non-commutative polynomial P in two variables, we have the convergence of $\mathbb{E}[\mathrm{tr}(P(B^N, B^{N*}))]$, where tr is the normalized trace (so that $\mathrm{tr}(I_N) = 1$). In most cases, we have the stronger result of almost sure convergence of the random variable $\mathrm{tr}(P(B^N, B^{N*}))$ to its mean. It is therefore natural to ask what is the rate of convergence, or more precisely, wether the random variable $\mathrm{tr}(P(B^N, B^{N*})) - \mathbb{E}[\mathrm{tr}(P(B^N, B^{N*}))]$ converges in law after renormalization. The standard scaling for this kind of central limit theorem in random matrices is well-known to be $1/N$ instead of the classical $1/\sqrt{N}$ (see the seminal work of Johansson [47]). For example,

$$N\left(\mathrm{tr}(P(B^N, B^{N*})) - \mathbb{E}\left[\mathrm{tr}(P(B^N, B^{N*}))\right]\right)$$

is asymptotically Gaussian when

- B^N is a Wigner random matrix [24];
- B^N is a unitary random matrix whose distribution is the Haar measure [33];
- B^N is a unitary random matrix arising from a Brownian motion on the unitary group [56].

Our main result is a result of this type, when B^N is an invertible random matrix arising from a Brownian motion $(G_t^{(N)})_{t\geq 0}$ on the general linear group $GL_N(\mathbb{C})$ as defined in Chapter 2. Theorem 15.1 states that, for each non-commutative polynomial P and for each time $t > 0$,

$$N\left(\mathrm{tr}(P(G_t^{(N)}, G_t^{(N)*})) - \mathbb{E}[\mathrm{tr}(P(G_t^{(N)}, G_t^{(N)*}))]\right)$$

is asymptotically Gaussian when N tends to infinity.

The rest of this chapter is organized as follows. In Section 14, we recall some technical results from Chapter 2. Section 15 provides the statement of our main result Theorem 15.1, the description of the covariance involved, and the proof of Theorem 15.1. Finally, in Section 16, we give an alternative description of the covariance, using three non-commutative processes in the framework of free probability.

14. Background

14.1. Brownian motion on $GL_N(\mathbb{C})$. Let us recall that $GL_N(\mathbb{C})$ is the Lie group of invertible $N \times N$ matrices, and that its Lie algebra is $\mathfrak{gl}_N(\mathbb{C}) = M_N(\mathbb{C})$. For all $Z \in \mathfrak{gl}_N(\mathbb{C})$, let us denote by $\tilde{Z}$ the associated right-invariant vector field on $GL_N(\mathbb{C})$:

$$(\tilde{Z}f)(G) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(Ge^{tZ}), \qquad f \in C^\infty(GL_N(\mathbb{C})).$$

Define the real inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{gl}_N(\mathbb{C})$ by

$$\langle X, Y \rangle_{\mathfrak{gl}_N(\mathbb{C})} = N \Re \operatorname{Tr}(X^* Y).$$

Following Chapter 2, a (right) Brownian motion on $GL_N(\mathbb{C})$ is a diffusion process starting at the identity and with generator $(1/4)\Delta_{GL_N(\mathbb{C})}$, where $\Delta_{GL_N(\mathbb{C})}$ is the Laplace operator for the right-invariant metric induced by $\langle \cdot, \cdot \rangle_{\mathfrak{gl}_N(\mathbb{C})}$. To be more concretely, let us fix a orthonormal basis $(Z_1, \ldots, Z_{2N^2})$ of $M_N(\mathbb{C})$ for the inner-product $\langle \cdot, \cdot \rangle_{\mathfrak{gl}_N(\mathbb{C})}$. We have $\Delta_{GL_N(\mathbb{C})} = \sum_{i=1}^{2N^2} \tilde{Z}_a^2$.

Let us denote by G the identity function of $GL_N(\mathbb{C})$. We are interested in the functional on $GL_N(\mathbb{C})$ of the type $\operatorname{tr}(P_1(G)) \cdots \operatorname{tr}(P_n(G))$, with $P_1, \ldots, P_n \in \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$. In Chapter 2, we were able to compute the action of the Laplacian on those functionals thanks to the $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$-calculus. More precisely, let us define the subalgebra $\mathcal{P}$ of the algebra $\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}$ generated by the basis

$$\{\operatorname{tr} M_1 \cdots \operatorname{tr} M_n : n \in \mathbb{N}, M_1, \ldots, M_n \text{ are monomials of } \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle\}.$$

Equivalently, we have $\mathcal{P} = \operatorname{tr}\left(\mathbb{C}\{X, X^*, X^{-1}, X^{*-1}\}\right)$. Let us recall that, for all $P_1, \ldots, P_n \in \mathbb{C}\langle X, X^*, X^{-1}, X^{*-1}\rangle$, we have by definition

$$\left(\operatorname{tr} P_1 \cdots \operatorname{tr} P_n\right)(G) = \operatorname{tr}(P_1(G)) \cdots \operatorname{tr}(P_n(G)).$$

In respectively Section 10.5.1 and 11.2.1, we defined the operator Δ_{GL} and $\tilde{\Delta}_{GL}$. Thanks to Lemma 11.3, they are such that, for all $P \in \mathcal{P}$,

$$\Delta_{GL_N(\mathbb{C})}\left(P(G)\right) = \left((\Delta_{GL} + \frac{1}{N^2}\tilde{\Delta}_{GL})P\right)(G).$$

In order to simplify the notation, we denote by respectively D and L the operators $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}$ and $\frac{1}{4}\tilde{\Delta}_{GL_N(\mathbb{C})}$ restricted to $\mathcal{P}$. In the following theorem, we summarize the properties of D and L which follows from the definitions of $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}$ and $\frac{1}{4}\tilde{\Delta}_{GL_N(\mathbb{C})}$, and from Lemma 11.3.

THEOREM 14.1. *The operators* D *and* L *on* $\mathcal{P}$*, independent from* N*, are such that:*

(1) D *is a first-order operator, i.e. for all* $P, Q \in \mathcal{P}$*, we have* $\mathrm{D}(PQ) = \mathrm{D}(P)Q + P\,\mathrm{D}(Q)$*;*
(2) L *is a second-order operator, i.e. for all* $P, Q, R \in \mathcal{P}$*, we have*

$$\mathrm{L}(PQR) = \mathrm{L}(PQ)R + P\,\mathrm{L}(QR) + \mathrm{L}(PR)Q - \mathrm{L}(P)QR - P\,\mathrm{L}(Q)R - PQ\,\mathrm{L}(R);$$

(3) For all $P \in \mathcal{P}$*, we have* $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}\left(P(G)\right) = \left((\mathrm{D} + \frac{1}{N^2}\mathrm{L})P\right)(G)$*.*

We recall also the following result of Corollary 11.4.

COROLLARY 14.2. *Let* $\left(G_t^{(N)}\right)_{t\geq 0}$ *be a Brownian motion on* $GL_N(\mathbb{C})$*. For all* $t \geq 0$ *and* $P \in \mathcal{P}$*, we have*

$$\mathbb{E}\left[P\left(G_t^{(N)}\right)\right] = \left(e^{t(\mathrm{D} + \frac{1}{N^2}\mathrm{L})}P\right)(1).$$

14.2. The "carré du champ" operator. We consider the so-called "carré du champ" operator of $\Delta_{GL_N(\mathbb{C})}$, denoted by Γ_N, and given for all $f, g \in C^\infty(GL_N(\mathbb{C}))$ by

$$\Gamma_N(f,g) = \frac{1}{2}\left((\Delta_{GL_N(\mathbb{C})}(fg)) - (\Delta_{GL_N(\mathbb{C})}f)g - f(\Delta_{GL_N(\mathbb{C})}g)\right),$$

or equivalently by

$$\Gamma_N(f,g) = \sum_{a=1}^{2N^2}(\tilde{Z}_a f)(\tilde{Z}_a g).$$

As the operator $\Delta_{GL_N(\mathbb{C})}$ in Theorem 14.1, the operator Γ_N can be pushed forward on $\mathcal{P}$ as follows. Let us define the symmetric bilinear form on $\mathcal{P}\times\mathcal{P}$ by

$$\Gamma(P,Q) = \frac{1}{2}\left(\mathrm{L}(PQ) - \mathrm{L}(P)Q - P\,\mathrm{L}(Q)\right).$$

PROPOSITION 14.3. *For all* $P, Q \in \mathcal{P}$*, we have* $\frac{N^2}{4}\Gamma_N(P(G), Q(G)) = [\Gamma(P,Q)](G)$*.*

PROOF. Let us denote by D_N the operator $\mathrm{D} + \frac{1}{N^2}\mathrm{L}$. We have $\mathrm{D}(PQ) - \mathrm{D}(P)Q - P\,\mathrm{D}(Q) = 0$. As a consequence, $\Gamma(P,Q) = \frac{N^2}{2}\left(\mathrm{D}_N(PQ) - \mathrm{D}_N(P)Q - P\,\mathrm{D}_N(Q)\right)$. From the intertwining relation $\frac{1}{4}\Delta_{GL_N(\mathbb{C})}(P(G)) = ((\mathrm{D}_N)P)(G)$, we obtained that

$$[\Gamma(P,Q)](G) = \frac{1}{2}\left(\Delta_{GL_N(\mathbb{C})}((PQ)(G)) - (\Delta_{GL_N(\mathbb{C})}(P(G)))Q(G) - P(G)(\Delta_{GL_N(\mathbb{C})}(Q(G)))\right),$$

which is the "carré du champ" of $\Delta_{GL_N(\mathbb{C})}$, as wanted. □

The operator L is a second-order operator and it follows the following lemma.

LEMMA 14.4. *For all* $P, Q, R \in \mathcal{P}$*, we have*

$$\Gamma(PQ,R) = \Gamma(P,R)\cdot Q + P\cdot\Gamma(Q,R).$$

For all $P_1,\ldots,P_k \in \mathcal{P}$*, we have*

$$\mathrm{L}(P_1\cdots P_k) = \sum_{i=1}^{k} P_1\cdots\widehat{P_i}\cdots P_k\,\mathrm{L}(P_i) + 2\sum_{1\leq i<j\leq k} P_1\cdots\widehat{P_i}\cdots\widehat{P_j}\cdots P_n\Gamma(P_i,P_j),$$

where the hat means that we have omitted the term P_i *in the product*

PROOF. Using the second-order property of L, we compute

$$\begin{aligned}2\Gamma(PQ,R) =& \mathrm{L}(PQR) - \mathrm{L}(PQ)R - PQ\,\mathrm{L}(R)\\ =& \mathrm{L}(PR)Q - \mathrm{L}(P)QR - PQ\,\mathrm{L}(R)\\ &+ P\,\mathrm{L}(QR) + -P\,\mathrm{L}(Q)R - PQ\,\mathrm{L}(R)\\ =& 2\Gamma(P,R)\cdot Q + 2P\cdot\Gamma(Q,R).\end{aligned}$$

By a direct induction, we deduce that

$$\begin{aligned}\mathrm{L}(P_1\cdots P_k) &= \mathrm{L}(P_1\cdots P_{k-1})P_k + P_1\cdots P_{k-1}\,\mathrm{L}(P_k) + 2\Gamma(P_1\cdots P_{k-1},P_k)\\ &= \mathrm{L}(P_1\cdots P_{k-1})P_k + P_1\cdots P_{k-1}\,\mathrm{L}(P_k) + 2\sum_{1\le i\le k} P_1\cdots\widehat{P_i}\cdots P_{k-1}\Gamma(P_i,P_k)\\ &= \cdots\\ &= \sum_{i=1}^k P_1\cdots\widehat{P_i}\cdots P_k\,\mathrm{L}(P_i) + 2\sum_{1\le i<j\le k} P_1\cdots\widehat{P_i}\cdots\widehat{P_j}\cdots P_n\Gamma(P_i,P_j). \qquad \square\end{aligned}$$

15. Gaussian fluctuations

15.1. Main theorem. We are now ready to state our main result. Let $T \geq 0$. For all $P, Q \in \mathcal{P}$, define

$$Y_P^N \equiv N\left(P(G_T^{(N)}) - \mathbb{E}\left[P(G_T^{(N)})\right]\right)$$

and

$$\sigma_T(P,Q) \equiv 2\int_0^T \left[e^{t\,\mathrm{D}}\left(\Gamma(e^{(T-t)\,\mathrm{D}}P, e^{(T-t)\,\mathrm{D}}Q)\right)\right](1)\,\mathrm{d}t.$$

The following theorem says that the quantities of the form $\mathbb{E}(Y_{P_1}^N\cdots Y_{P_k}^N)$ satisfy a Wick formula in large dimension, with covariances given by σ_T. Let us denote by $\mathcal{P}_2(k)$ the set of pairings of $\{1,\dots,k\}$, that is to say the set of partitions in pairs.

Theorem 15.1. *For any $P_1,\dots,P_k \in \mathcal{P}$, we have, as N tends to infinity,*

$$\mathbb{E}(Y_{P_1}^N\cdots Y_{P_k}^N) = \sum_{\pi\in\mathcal{P}_2(k)}\prod_{\{i,j\}\in\pi}\sigma_T(P_i,P_j) + O\left(\frac{1}{N}\right).$$

We will now reformulate this result as a convergence towards a Gaussian field.

Lemma 15.2. *There exists a complex Gaussian Hilbert space K with some specified random variables $(\xi_P)_{P\in\mathcal{P}} \in K$ such that $P \mapsto \xi_P$ is linear, $\mathbb{E}(\xi_P\xi_Q) = \sigma_T(P,Q)$ and $\bar{\xi}_P = \xi_{P^*}$.*

Proof. Firstly, the map σ_T is symmetric, non-negative and bilinear on the subspace $\mathcal{P}_{sa}$ of self-adjoint elements of $\mathcal{P}$, and therefore, it exists a real Gaussian Hilbert space H and a linear map $P \mapsto \xi_P$ from $\mathcal{P}_{sa}$ to H such that $\mathbb{E}(\xi_P\xi_Q) = \sigma_T(P,Q)$. Let $K = H^{\mathbb{C}}$. For all $P \in \mathcal{P}$, we set $\xi_P = \xi_{(P+P^*)/2} + i\xi_{(P-P^*)/2i}$ which is obviously linear in P. By bilinearity of σ_T, $\mathbb{E}(\xi_P\xi_Q) = \sigma_T(P,Q)$. Finally,

$$\bar{\xi}_P = \xi_{(P+P^*)/2} - i\xi_{(P-P^*)/2i} = \xi_{(P^*+P^*)/2} + i\xi_{(P^*-P)/2i} = \xi_{P^*}. \quad \square$$

Corollary 15.3. *As N tends to infinity, $(Y_P^N)_{P\in\mathcal{P}}$ converges to $(\xi_P)_{P\in\mathcal{P}}$ in finite dimensional distribution: for all $P_1,\dots,P_k \in \mathcal{P}$,*

$$(Y_{P_1}^N,\dots,Y_{P_k}^N) \xrightarrow[N\to\infty]{(d)} (\xi_{P_1},\dots,\xi_{P_k}).$$

More generally, in the dual space $\mathcal{P}^*$ endowed with the topology of pointwise convergence, the random linear map $Y^N : P \mapsto X_P^N$ converge to the random linear map $\xi : P \mapsto \xi_P$ in distribution:

$$Y^N \xrightarrow[N\to\infty]{(d)} \xi.$$

Note that, for P and Q in $\mathcal{P}$, the asymptotic covariance of X_P^N and X_Q^N, or equivalently the covariance of ξ_P and ξ_Q, is $\mathbb{E}(\xi_P\overline{\xi_Q}) = \mathbb{E}(\xi_P\xi_{Q^*}) = \sigma_T(P,Q^*)$, which is different from $\sigma_T(P,Q)$.

PROOF. Let $k \in \mathbb{N}$ and $P_1, \ldots, P_k \in \mathcal{P}$. Because the vector $(\xi_{P_1}, \ldots, \xi_{P_k})$ is gaussian, it suffices to prove the convergence of the $*$-moments of $(Y^N_{P_1}, \ldots, Y^N_{P_k})$ to those of $(\xi_{P_1}, \ldots, \xi_{P_k})$. Let $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_m \leq k$. We want to prove that

$$\mathbb{E}(Y^N_{i_1} \cdots Y^N_{i_n} \overline{Y^N_{j_1}} \cdots \overline{Y^N_{j_n}}) \xrightarrow[N\to\infty]{} \mathbb{E}(\xi_{P_{i_1}} \cdots \xi_{P_{i_n}} \overline{\xi_{P_{j_1}}} \cdots \overline{\xi_{P_{j_n}}}).$$

We have

$$\mathbb{E}(Y^N_{P_{i_1}} \cdots Y^N_{P_{i_n}} \overline{Y^N_{P_{j_1}}} \cdots \overline{Y^N_{P_{j_m}}}) = \mathbb{E}(Y^N_{P_{i_1}} \cdots Y^N_{P_{i_n}} Y^N_{P^*_{j_1}} \cdots Y^N_{P^*_{j_m}})$$
$$\xrightarrow[N\to\infty]{} \mathbb{E}(\xi_{P_{i_1}} \cdots \xi_{P_{i_n}} \xi_{P^*_{j_1}} \cdots \xi_{P^*_{j_n}}) = \mathbb{E}(\xi_{P_{i_1}} \cdots \xi_{P_{i_n}} \overline{\xi_{P_{j_1}}} \cdots \overline{\xi_{P_{j_n}}}).$$

The general convergence in distribution follows because $\mathcal{P}$ is a countable-dimensional metric space. □

15.2. Demonstration of the main theorem. Let us say a few words about the demonstration of the Gaussian fluctuations. In the following computations, we shall use only the property of first-order operator of D, and the property of second-order operator of L. Consequently, this demonstration is valid for a larger class of Brownian motions, whenever their generators can be pushed forward on $\mathcal{P}$ as a sum of a first-order operator and $1/N^2$ times a second-order operator. For example, we recall from Lemma 11.1 that the generator of the unitary Brownian motion acts on $\mathbb{C}\{X, X^{-1}\}$ as $\Delta_U + (1/N^2)\tilde{\Delta}_U$. Moreover, Δ_U is a first-order operator and $\tilde{\Delta}_U$ is a second-order operator. We deduce that the distribution of a unitary Brownian motion have also Gaussian fluctuations, as already proved in [56]. We shall say more about this particular result in the next section.

PROOF OF THEOREM 15.1. Observing that the maps $(P_1, \ldots, P_k) \mapsto \mathbb{E}(Y^N_{P_1} \cdots Y^N_{P_k})$ and $(P_1, \ldots, P_k) \mapsto \sum_{\pi\in\mathcal{P}_2(k)} \prod_{(i,j)\in\pi} \sigma_T(P_i, P_j)$ are two symmetric multilinear forms on $\mathcal{P}$, it suffices by polarization to verify the asymptotic when $P_1 = \ldots = P_k = P$ (see Appendix D in [46]). In this case, set $Q_N = P - \mathbb{E}[P(G^{(N)}_T)]$. We want to prove that

$$N^k \mathbb{E}\left[Q^k_N(G^{(N)}_T)\right] = \sum_{\pi\in\mathcal{P}_2(k)} \prod_{(i,j)\in\pi} \sigma_T(P, P) + O\left(\frac{1}{N}\right).$$

And firstly, we remark that $\mathbb{E}[Q^k_N(G^{(N)}_T)] = \left[e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}(Q^k_N)\right](1)$ thanks to Corollary 14.2. The proof will consist in identifying the leading term in the development of $e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}$ with respect to N.

In order to control the insignificant term in the development, we will work on a finite dimensional space. Let $d \in \mathbb{N}$ be the degree of Q_N. The subalgebra $\mathcal{P}_{kd}$ of elements of $\mathcal{P}$ whose degrees are less than kd is finite dimensional and we endow it by a fixed unital algebra norm $\|\cdot\|_{(kd)}$. Let us denote by $|||\cdot|||_{(kd)}$ the operator norm on the finite dimensional algebra $\mathrm{End}(\mathcal{P}_{kd})$, and by $|||\cdot|||_{(d,d')}$ the usual norm of bilinear maps from $\mathcal{P}_d \times \mathcal{P}_{d'}$ to $\mathcal{P}_{d+d'}$ when $d + d' \leq kd$ (in the following development, we will often omit the indices (kd) or (d, d')). Throughout this proof, we will consider the operators D, L and Γ as bounded operators on the finite dimensional algebra $\mathcal{P}_{kd} \subset \mathcal{P}$. The differentiability of the exponential map leads to $e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})} = e^{T\,\mathrm{D}} + O(1/N^2)$. More precisely, we have the following result.

LEMMA 15.4. *For all $t \geq 0$, we have*

$$e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})} = e^{t\,\mathrm{D}} + \frac{1}{N^2}\int_0^t e^{t_1(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\,\mathrm{L}\,e^{(t-t_1)\,\mathrm{D}}\,\mathrm{d}t_1. \tag{15.1}$$

More generally, for all $k\in\mathbb{N}$, we have

$$\begin{aligned}e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})} =& e^{t\,\mathrm{D}}+\sum_{n=1}^{k}\frac{1}{N^{2n}}\int_{0\le t_n\le\cdots\le t_1\le t} e^{t_n\,\mathrm{D}}\,\mathrm{L}\,e^{(t_{n+1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(t-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\ &+\frac{1}{N^{2(k+1)}}\int_{0\le t_{k+1}\le\cdots\le t_1\le t} e^{t_{k+1}(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\,\mathrm{L}\,e^{(t_{k+1}-t_k)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(t-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}.\end{aligned}$$

PROOF. Let us define $S(t)=e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}e^{-t\,\mathrm{D}}$. The map S is differentiable, and we have

$$S'(t)=e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}(\mathrm{D}+\frac{1}{N^2}\mathrm{L}-\mathrm{D})e^{-t\,\mathrm{D}}=\frac{1}{N^2}S(t)e^{t\,\mathrm{D}}\,\mathrm{L}\,e^{-t\,\mathrm{D}}.$$

Thus, $S(t)=1+\frac{1}{N^2}\int_0^t S(t_1)e^{t_1\,\mathrm{D}}\,\mathrm{L}\,e^{-t_1\,\mathrm{D}}\mathrm{d}t_1$. Multiplying by $e^{t\,\mathrm{D}}$ on the right gives us the first formula. The second formula is obtained by induction over k, using at each step the first formula. □

For $n\in\mathbb{N}$, let us denote by $\Delta_n\subset\mathbb{R}^n$ the simplex

$$\Delta_n=\{(t_n,\dots,t_1)\in\mathbb{R}^n\colon 0\le t_n\le t_{n-1}\le\cdots\le t_1\le T\}.$$

Using the lemma at the step $[k/2]$, the study of the limit of $N^k\left[e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}(Q_N^k)\right](1)$ is decomposed into the study of the limits of:

1. $N^k\left[e^{T\,\mathrm{D}}(Q_N^k)\right](1)$,
2. $N^{k-2n}\left[\int_{\Delta_n} e^{t_n\,\mathrm{D}}\,\mathrm{L}\,e^{(t_{n+1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n(Q_N^k)\right](1)$ for $1\le n\le[k/2]$,
3. and $N^{k-2-[k/2]}\left[\int_{\Delta_{k+1}} e^{t_{k+1}(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\,\mathrm{L}\,e^{(t_{k+1}-t_k)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}(Q_N^k)\right](1)$,

which is the goal of respectively the three next steps. In the fourth step, we sum up the three convergences and conclude. We will see that the only term which does not vanish is the second term considered when $n=[k/2]$.

Step 1. Thanks to Corollary 14.2, we have $Q_N=P-\mathbb{E}[P(G_T^{(N)})]=P-\left[e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}(P)\right](1)$. Since the map $A\mapsto[A(P)](1)$ is linear, it is therefore bounded and we have $\left[e^{T(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}(P)\right](1)=[e^{T\,\mathrm{D}}(P)](1)+O(1/N^2)$. Consequently,

$$Q_N=P-[e^{T\,\mathrm{D}}(P)](1)+O(1/N^2)\tag{15.2}$$

and therefore $Q_N^k=(P-[e^{T\,\mathrm{D}}(P)](1))^k+O(1/N^{2k})$. Using a standard formal power series argument, we know that $e^{T\,\mathrm{D}}$ is multiplicative from the fact that D is a first-order operator. Thus

$$\begin{aligned}[e^{T\,\mathrm{D}}(Q_N^k)](1)&=\left[e^{T\,\mathrm{D}}(P-[e^{T\,\mathrm{D}}(P)](1))^k\right](1)+O(1/N^{2k})\\&=([e^{T\,\mathrm{D}}(P)](1)-[e^{T\,\mathrm{D}}(P)](1))^k+O(1/N^{2k})\\&=O(1/N^{2k}).\end{aligned}$$

Finally, $N^k\left[e^{T\,\mathrm{D}}(Q_N^k)\right](1)=O(1/N^k)$.

Step 2. We are assuming at this step that $2\le k$. For all $R\in\mathcal{P}$, $t\ge0$ and $n\ge2$, we have by Lemma 14.4

$$\begin{aligned}\mathrm{L}((e^{t\,\mathrm{D}}(Q_N))^n\cdot R)=&(e^{t\,\mathrm{D}}Q_N)^n\,\mathrm{L}(R)+2n(e^{t\,\mathrm{D}}Q_N)^{n-1}\Gamma(e^{t\,\mathrm{D}}Q_N,R)\\&+n(e^{t\,\mathrm{D}}Q_N)^{n-1}\,\mathrm{L}(e^{t\,\mathrm{D}}Q_N)R+n(n-1)(e^{t\,\mathrm{D}}Q_N)^{n-2}\Gamma(e^{t\,\mathrm{D}}Q_N,e^{t\,\mathrm{D}}Q_N)R.\end{aligned}$$

In others words, for all $d' \leq (k-1)d$, if we define the bilinear map $B_n : (S,R) \mapsto S \cdot \mathrm{L}(R) + 2n\Gamma(S,R) + n\,\mathrm{L}(S)\cdot R$ from $\mathcal{P}_d \times \mathcal{P}_{d'}$ to $\mathcal{P}_{d+d'}$, we have, for all $R \in \mathcal{P}_{d'}$,

$$\mathrm{L}((e^{t\,\mathrm{D}}(Q_N))^n \cdot R) = (e^{t\,\mathrm{D}}Q_N)^{n-1}B_n(e^{t\,\mathrm{D}}Q,R) + n(n-1)(e^{t\,\mathrm{D}}Q_N)^{n-2}\Gamma(e^{t\,\mathrm{D}}Q_N, e^{t\,\mathrm{D}}Q_N)R. \tag{15.3}$$

Let us denote by $\Gamma(t)$ the element $e^{t\,\mathrm{D}}\Gamma(e^{(T-t)\,\mathrm{D}}Q_N, e^{(T-t)\,\mathrm{D}}Q_N) \in \mathcal{P}_{2d}$. Using (15.3), we prove by induction on n the following lemma.

LEMMA 15.5. *For all n such that $1 \leq n \leq [k/2]$ and $0 \leq t_n \leq \cdots \leq t_0 = 1$, there exists $R_n \in \mathcal{P}_{(2n-1)d}$ bounded independently of $N, t_0, \ldots t_n$ such that*

$$\begin{aligned}\mathrm{L}\,e^{(t_{n-1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}(Q_N^k) \\ = \frac{k!}{(k-2n)!}(e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n}e^{-t_n\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_n)) + (e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+1}R_n.\end{aligned} \tag{15.4}$$

PROOF. Indeed, when $n=1$, setting $R_1 = k\,\mathrm{L}(e^{(T-t_1)\,\mathrm{D}}Q_N) \in \mathcal{P}_d$ which is bounded by $|||\mathrm{L}|||e^{|||\mathrm{D}|||}\|Q_N\|$, we have

$$\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}(Q_N^k) = k(k-1)(e^{(T-t_1)\,\mathrm{D}}Q_N)^{k-2n}\Gamma(e^{(T-t_1)\,\mathrm{D}}Q_N, e^{(T-t_1)\,\mathrm{D}}Q_N) + (e^{(T-t_1)\,\mathrm{D}}Q_N)^{k-1}R_1.$$

Because of (15.2), it is bounded independently of N, and so do R_1.

Assume now that $2 \leq n \leq [k/1]$ and that (15.4) has been verified up to level $n-1$. We compute

$$\begin{aligned}&\mathrm{L}\,e^{(t_{n-1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}(Q_N^k)\\
&=\mathrm{L}\,e^{(t_{n-1}-t_n)\,\mathrm{D}}\left(\frac{k!}{(k-2n+2)!}(e^{(T-t_{n-1})\,\mathrm{D}}Q_N)^{k-2n+2}e^{-t_{n-1}\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_{n-1}))\right.\\
&\qquad\qquad\left.+\,(e^{(T-t_{n-1})\,\mathrm{D}}Q_N)^{k-2n+3}R_{n-1}\right)\\
&=\frac{k!}{(k-2n+2)!}\,\mathrm{L}\left((e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+2}e^{-t_n\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_{n-1}))\right)\\
&\qquad\qquad+\mathrm{L}\left((e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+3}R_{n-1}\right).\end{aligned}$$

We use now (15.3) on each term. The first term leads to

$$\begin{aligned}&\frac{k!}{(k-2n)!}(e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n}e^{-t_n\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_n))\\
&\quad+\frac{k!}{(k-2n+2)!}(e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+1}B_{k-2n+2}\left(e^{(T-t_n)\,\mathrm{D}}Q_N, e^{-t_n\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_{n-1}))\right),\end{aligned}$$

and the second term to

$$\begin{aligned}&(e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+2}B_{k-2n+3}(e^{(T-t_n)\,\mathrm{D}}Q_N, R_{n-1})\\
&\quad+(k-2n+3)(k-2n+2)(e^{(T-t_n)\,\mathrm{D}}Q_N)^{k-2n+1}\Gamma(e^{(T-t_n)\,\mathrm{D}}Q_N, e^{(T-t_n)\,\mathrm{D}}Q_N)R_{n-1}.\end{aligned}$$

Thus, $R_n \in \mathcal{P}_{(2n-1)d}$ can be defined by

$$\begin{aligned}&\frac{k!}{(k-2n+2)!}B_{k-2n+2}\left(e^{(T-t_n)\,\mathrm{D}}Q_N, e^{-t_n\,\mathrm{D}}(\Gamma(t_1)\cdots\Gamma(t_{n-1}))\right)\\
&\quad+(e^{(T-t_n)\,\mathrm{D}}Q_N)B_{k-2n+3}(e^{(T-t_n)\,\mathrm{D}}Q_N, R_{n-1})\\
&\quad+(k-2n+3)(k-2n+2)\Gamma(e^{(T-t_n)\,\mathrm{D}}Q_N, e^{(T-t_n)\,\mathrm{D}}Q_N)R_{n-1}\end{aligned}$$

which verifies (15.4) and which is bounded by

$$\frac{k!}{(k-2n+2)!}|||B_{k-2n+2}|||_{(d,2(n-1)d)}e^{2|||\mathrm{D}|||}\|Q_N\|\|\Gamma(t_1)\|\cdots\|\Gamma(t_{n-1})\|$$
$$+e^{2|||\mathrm{D}|||}\|Q_N\|^2|||B_{k-2n+3}|||_{(d,(2n-1)d}\|R_{n-1}\|$$
$$+(k-2n+3)(k-2n+2)|||\Gamma|||_{(d,d)}e^{2|||\mathrm{D}|||}\|Q_N\|^2\|R_{n-1}\|.$$

Because of (15.2), it is bounded independently of N. We deduce also that

$$\Gamma(t_i)=e^{t_i\,\mathrm{D}}\Gamma(e^{(T-t_i)\,\mathrm{D}}Q_N,e^{(T-t_i)\,\mathrm{D}}Q_N)$$

is bounded by $|||\Gamma|||_{(d,d)}e^{2|||\mathrm{D}|||}\|Q_N\|^2$ and consequently is bounded independently of $N,t_1,\dots,t_n$. Finally, R_n is bounded independently of $N,t_1,\dots,t_n$. □

We recall that, because D is a first-order operator, $e^{t_n\,\mathrm{D}}$ is multiplicative. Applying $e^{t_n\,\mathrm{D}}$ on (15.4), we obtain that, for all $1\leq n\leq[k/2]$, for all $N\in\mathbb{N}$ and $(t_n,\dots,t_1)\in\Delta_n$, there exists $R_n\in\mathcal{P}_{(2n-1)d}$ bounded uniformly in $N,t_0,\dots t_n$ such that

$$e^{t_n\,\mathrm{D}}\,\mathrm{L}\,e^{(t_{n-1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}(Q_N^k)$$
$$=\frac{k!}{(k-2n)!}(e^{T\,\mathrm{D}}Q_N)^{k-2n}\Gamma(t_1)\cdots\Gamma(t_n)+(e^{T\,\mathrm{D}}Q_N)^{k-2n+1}R_n,$$

where $\Gamma(t)$ denotes the element $e^{t\,\mathrm{D}}\Gamma(e^{(T-t)\,\mathrm{D}}Q_N,e^{(T-t)\,\mathrm{D}}Q_N)\in\mathcal{P}_{2d}$.

From (15.2), we deduce that $\left[(e^{T\,\mathrm{D}}Q_N)^{k-2n}\right](1)=O(1/N^{2k-4n})$ and $\left[(e^{T\,\mathrm{D}}Q_N)^{k+1-2n}\right](1)=O(1/N^{2k+T-4n})$. We have already remarked in the proof of (15.4) that Q_N and

$$\Gamma(t_i)=e^{t_i\,\mathrm{D}}\Gamma(e^{(T-t_i)\,\mathrm{D}}Q_N,e^{(T-t_i)\,\mathrm{D}}Q_N)$$

are bounded independently of $N,t_1,\dots,t_n$. By consequence, we know that the two terms

$$N^{2k-4n}\frac{k!}{(k-2n)!}\left[(e^{T\,\mathrm{D}}Q_N)^{k-2n}\Gamma(t_1)\cdots\Gamma(t_n)\right](1)$$

and $N^{k+1-2n}\left[(e^{T\,\mathrm{D}}Q_N)^{k+1-2n}R_n\right](1)$ are bounded independently of $N,t_1,\dots,t_n$, and we deduce that

$$N^{k-2n}\left[\int_{\Delta_n}e^{t_n\,\mathrm{D}}\,\mathrm{L}\,e^{(t_{n+1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n(Q_N^k)\right](1)$$

is $O(1/N)$ if $k>2n$ and is $k!\int_{\Delta_n}(\Gamma(t_1)\cdots\Gamma(t_n))(1)\,\mathrm{d}t_1\cdots\mathrm{d}t_n+O(1/N)$ if $k=2n$.

In the case where $k=2n$, because the integrand is symmetric in $t_1,\dots,t_n$, the remaining term is equal to

$$\frac{k!}{n!}\int_{0\leq t_1,\dots,t_n\leq T}[\Gamma(t_1)\cdots\Gamma(t_n)](1)\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$
$$=\frac{k!}{n!}\left(\int_0^T[\Gamma(t)](1)\,\mathrm{d}t\right)^n=\frac{(2n)!}{2^nn!}\sigma_T(Q_N,Q_N)^n.$$

Note that L kills constants, and similarly $\Gamma(P+c,Q+d)=\Gamma(P,Q)$ for any $c,d\in\mathbb{C}$. As a consequence, $\sigma_T(Q_N,Q_N)=\sigma_T(P,P)$. To sum up,

$$N^{k-2n}\left[\int_{\Delta_n}e^{t_n\,\mathrm{D}}\,\mathrm{L}\,e^{(t_{n+1}-t_n)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_n(Q_N^k)\right](1)$$

is equal to $\frac{(2n)!}{2^nn!}\sigma_T(P,P)^n+O(1/N)$ if $k=2n$ and $O(1/N)$ if not.

Step 3. We have $Q_N^k = (P - [e^{T\,\mathrm{D}}(P)](1))^k + O(1/N^{2k})$ and

$$\left|\!\left|\!\left|\int_{\Delta_{k+1}} e^{t_{k+1}(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\,\mathrm{L}\,e^{(t_{k+1}-t_k)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}\right|\!\right|\!\right| \leq |||\mathrm{L}|||^n e^{|||\mathrm{L}|||+|||\mathrm{D}|||}.$$

Consequently

$$\left[\int_{\Delta_{k+1}} e^{t_{k+1}(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\,\mathrm{L}\,e^{(t_{k+1}-t_k)\,\mathrm{D}}\,\mathrm{L}\cdots\mathrm{L}\,e^{(T-t_1)\,\mathrm{D}}\,\mathrm{d}t_1\cdots\mathrm{d}t_{k+1}(Q_N^k)\right](1)$$

is bounded independently of N. On the other hand, $k-2([k/2]+1)\leq -1$ and $N^{k-2([k/2]+1)}$ is therefore $O(1/N)$. Thus, the term studied is $O(1/N)$.

Step 4. Finally, $N^k\mathbb{E}[Q_N^k] = \frac{k!}{2^{k/2}(k/2)!}\sigma_T(P,P)^{k/2} + O(1/N)$ if k is even and $O(1/N)$ if not. Because the cardinality of $\mathcal{P}_2(k)$ is $\frac{k!}{2^{k/2}(k/2)!}$ if k is even and 0 if not (see [46]), we have the wanted convergence

$$N^k\mathbb{E}\left[Q_N^k\right] = \sum_{\pi\in\mathcal{P}_2(k)}\prod_{(i,j)\in\pi}\sigma_T(Q,Q) + O\left(\frac{1}{N}\right). \qquad \square$$

16. Study of the covariance

In [56], Thierry Lévy and Mylène Maïda established a central limit theorem for random matrices arising from a unitary Brownian motion as defined in Section 11.1.

Theorem (Lévy and Maïda [56]). *Let $(U_t^{(N)})_{t\geq 0}$ be a unitary Brownian motion on $U(N)$. Let $P_1,\cdots,P_n\in\mathbb{C}[X,X^{-1}]$, and $T\geq 0$. When N tends to infinity, the random vector*

$$N\left(\mathrm{tr}\left(P_i(U_T^{(N)})\right) - \mathbb{E}\left[\mathrm{tr}\left(P_i(U_T^{(N)})\right)\right]\right)_{1\leq i\leq n}$$

converges in distribution to a Gaussian vector.

Interestingly, the limit covariance involves three free unitary brownian motion $(u_t)_{t\geq 0}$, $(v_t)_{t\geq 0}$ and $(w_t)_{t\geq 0}$ which are freely independent. Following [56], for all $P\in\mathbb{C}[X]$, we denote by $P'\in\mathbb{C}[X,X^{-1}]$ the derivative of P on the unit circle:

$$\forall z\in\mathbb{U}, P'(z) = \lim_{h\to 0}\frac{f(ze^{ih})-f(z)}{h}.$$

More concretely, for all $n\in\mathbb{N}$, if $P=X^n$, we have $P'=niX^n$, and if $P=X^{-n}$, we have $P'=-niX^{-n}$. For all $P,Q\in\mathbb{C}[X,X^{-1}]$, the covariance of the random variables $\mathrm{Tr}\,P(U_T^{(N)}) - \mathbb{E}[\mathrm{Tr}\,P(U_T^{(N)})]$ and $\mathrm{Tr}\,Q(U_T^{(N)}) - \mathbb{E}[\mathrm{Tr}\,Q(U_T^{(N)})]$ is asymptotically equal to

$$\int_0^T \tau\left(P'(v_{T-t}u_t)(Q'(w_{T-t}u_t))^*\right)\mathrm{d}t. \tag{16.1}$$

In this section, we relate our result to theirs by giving another expressions of the covariance, the last one being very similar to the covariance which they have found.

16.1. Two different expressions of the covariance.

Proposition 16.1. *Let $(G_t)_{t\geq 0},(H_t)_{t\geq 0},(K_t)_{t\geq 0}$ three independent Brownian motions on $GL_N(\mathbb{C})$ which are independent. For all $P,Q\in\mathcal{P}$, we have*

$$\sigma_T(P,Q) = \frac{N^2}{2}\int_0^T \mathbb{E}\left[\Gamma_N\left(P(H_{T-t}GG_T),Q(K_{T-t}GG_T)\right)(I_N)\right]\mathrm{d}t + O(1/N^2).$$

Proof. For all $P, Q \in \mathcal{P}$, we have

$$\sigma_T(P,Q) = 2\int_0^T \left[e^{t\,\mathrm{D}}\left(\Gamma(e^{(T-t)\,\mathrm{D}}P, e^{(T-t)\,\mathrm{D}}Q)\right)\right](1)\,\mathrm{d}t.$$

As in the proof of Theorem 15.1, we restrict our computations on a finite-dimensional space $\mathcal{P}_d$ (take d to be the sum of the degrees of P and Q). Because of Lemma 15.4, we have $N^2\left(e^{t\,\mathrm{D}} - e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\right)$ bounded independently of t and by consequence, it is straightforward to verify that

$$\sigma_T(P,Q) = 2\int_0^T \left[e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\left(\Gamma(e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}P, e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}Q)\right)\right](1)\,\mathrm{d}t + O(1/N^2).$$

The end of the proof consists in proving that, for all $0 \le t \le 1$, we have

$$\begin{aligned}(16.2)\quad &\left[e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\left(\Gamma(e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}P, e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}Q)\right)\right](1)\\ &\qquad = \frac{N^2}{4}\mathbb{E}\left[\Gamma_N\left(P(H_{T-t}GG_T), Q(K_{T-t}GG_T)\right)(I_N)\right].\end{aligned}$$

Let $0 \le t \le T$. We start from the left side to recover the right side. First of all, using Theorem 14.1 and its corollary, we have

$$\begin{aligned}&\left[e^{t(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}\left(\Gamma\left(e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}P, e^{(T-t)(\mathrm{D}+\frac{1}{N^2}\mathrm{L})}Q\right)\right)\right](1)\\ =&\frac{N^2}{4}\left[e^{\frac{t}{4}\Delta_{GL_N(\mathbb{C})}}\left(\Gamma_N\left(e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G)), e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)\right)\right](I_N)\\ =&\frac{N^2}{4}\mathbb{E}\left[\left(\Gamma_N\left(e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G)), e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)\right)(G_T)\right].\end{aligned}$$

For all $M \in GL_N(\mathbb{C})$, let us denote respectively by L_M and R_M the left and the right translation by M. We compute

$$\begin{aligned}&\left(\Gamma_N\left(e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G)), e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)\right)(G_T^{(N)})\\ =&\frac{1}{2}\sum_{a=1}^{2N^2}\left[\left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G))\right)\left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)\right](G_T)\\ =&\frac{1}{2}\sum_{a=1}^{2N^2}\left[R_{G_T}\circ\left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G))\right)(I_N)\right]\cdot\left[R_{G_T}\circ\left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)(I_N)\right].\end{aligned}$$

Here, in order to reverse the different operators, we introduce the left-invariant vector fields. For all $Z \in M_N(\mathbb{C})$, let us denote by $\hat{Z}$ the associated left-invariant vector field on $GL_N(\mathbb{C})$:

$$(\hat{Z}f)(g) = \left.\frac{d}{\mathrm{d}t}\right|_{t=0} f(ge^{tZ}), \qquad f \in C^\infty(GL_N(\mathbb{C})).$$

Note that, for all $Y, Z \in M_N(\mathbb{C})$, the vector fields $\tilde{Y}$ and $\hat{Z}$ commute. Moreover, for all $f \in C^\infty(GL_N(\mathbb{C}))$, we have $(\tilde{Z}f)(I_N) = (\hat{Z}f)(I_N)$. Using those two facts, we compute

$$\begin{aligned}\left(R_{G_T} \circ \left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G))\right)\right)(I_N) &= \left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(R_{G_T} \circ P(G))\right)(I_N)\\ &= \left(\hat{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(R_{G_T} \circ P(G))\right)(I_N)\\ &= \left(e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}\hat{Z}_a(R_{G_T} \circ P(G))\right)(I_N)\\ &= \mathbb{E}\left[\left(\hat{Z}_a(R_{G_T} \circ P(G))\right)(H_{T-t})\middle| G_T\right]\\ &= \mathbb{E}\left[L_{H_{T-t}} \circ \left(\hat{Z}_a(R_{G_T} \circ P(G))\right)(I_N)\middle| G_T\right]\\ &= \mathbb{E}\left[\left(\hat{Z}_a(L_{H_{T-t}} \circ R_{G_T} \circ P(G))\right)(I_N)\middle| G_T\right]\\ &= \mathbb{E}\left[\left(\hat{Z}_a(P(H_{T-t}GG_T))\right)(I_N)\middle| G_T\right]\end{aligned}$$

and similarly $\left(R_{G_T} \circ \left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)\right)(I_N) = \mathbb{E}\left[\left(\hat{Z}_a(P(K_{T-t}GG_T))\right)(I_N)\middle| G_T\right]$. It follows that

$$\begin{aligned}&\frac{1}{2}\sum_{a=1}^{2N^2}\left[R_{G_T} \circ \left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(P(G))\right)(I_N)\right] \cdot \left[R_{G_T} \circ \left(\tilde{Z}_a e^{\frac{(T-t)}{4}\Delta_{GL_N(\mathbb{C})}}(Q(G))\right)(I_N)\right]\\ =&\frac{1}{2}\sum_{a=1}^{2N^2}\mathbb{E}\left[\left(\hat{Z}_a(P(H_{T-t}GG_T))\right)(I_N)\left(\hat{Z}_a(P(K_{T-t}GG_T))\right)(I_N)\middle| G_T\right]\\ =&\mathbb{E}\left[\Gamma_N\left(P(H_{T-t}GG_T), Q(K_{T-t}GG_T)\right)(I_N)\middle| G_T\right].\end{aligned}$$

Taking the expectation leads to the right side of (16.2). □

In Proposition 16.1, we wrote the covariance with the help of three independent Brownian motions, and we want now to let the dimension tend to infinity in order to have a new expression of the covariance. We shall not review the definitions of free probability theory given in Chapter 2. Let us only recall that a circular multiplicative Brownian motion $(G_t)_{t\geq 0}$ is a non-commutative process which is the limit of the Brownian motion $(G_t^{(N)})_{t\geq 0}$ on the linear group in the sense of Theorem 7.4: for all $t \geq 0$, all $n \in \mathbb{N}$ and all $P \in \mathcal{P}$, we have

$$\lim_{N\to\infty}\mathbb{E}\left[P\left(G_t^{(N)}\right)\right] = P(G_t).$$

We precise also that independent Brownian motions $(G_t^{(N)})_{t\geq 0}$,$(H_t^{(N)})_{t\geq 0}$ and $(K_t^{(N)})_{t\geq 0}$ converge to free circular multiplicative Brownian motions $(G_t)_{t\geq 0}$, $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ which are mutually free: for all $t \geq 0$, all $n \in \mathbb{N}$, and all polynomials $P \in \mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i = 1, 2, 3\}$, we have

$$\lim_{N\to\infty}\mathbb{E}\left[\mathrm{tr}\left(P\left(G_t^{(N)}, H_t^{(N)}, K_t^{(N)}\right)\right)\right] = \tau(P(G_t, H_t, K_t)). \tag{16.3}$$

Indeed, it remains to prove that the independent Brownian motions are asymptotically free, which is a consequence of the $\mathrm{Ad}(U(N))$-invariance of the law of each Brownian motion. Let us recommend also [50] for a different and complete proof.

PROPOSITION 16.2. *For all $P, Q \in \mathcal{P}$, there exists*

$$\tilde{\Gamma}(P, Q) \in \mathbb{C}\{X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i = 1, 2, 3\}$$

such that, for all $N \in \mathbb{N}$, and all $B, H, K \in GL_N(\mathbb{C})$,

$$\frac{N^2}{2}\Gamma_N\left(P(HGB), Q(KGB)\right)(I_N) = \left[\tilde{\Gamma}(P,Q)\right](B,H,K) \tag{16.4}$$

and in this case, setting $(G_t)_{t\geq 0}$, $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be three free circular multiplicative Brownian motions which are free, we have

$$\sigma_T(P,Q) = \int_0^T \left[\tilde{\Gamma}(P,Q)\right](G_T, H_{T-t}, K_{T-t})\mathrm{d}t.$$

Proof. Let us suppose first that $P = \mathrm{tr}(X^{\varepsilon_1}\cdots X^{\varepsilon_n})$ and $Q = \mathrm{tr}(X^{\varepsilon_{n+1}}\cdots X^{\varepsilon_{n+m}})$, with $\varepsilon_1, \ldots, \varepsilon_{n+m} \in \{1, *, -1, *-1\}$. We can compute

$$\begin{aligned}\tilde{Z}_a\left(P(HGB)\right) &= \sum_{l=1}^n \mathrm{tr}((HB)^{\varepsilon_1}\cdots \tilde{Z}_a(HGB)^{\varepsilon_l}\cdots(HB)^{\varepsilon_n})\\ &= \sum_{l=1}^n \mathrm{tr}(\tilde{Z}_a(G^{\varepsilon_l})\cdot R(B,H,K)),\end{aligned}$$

where R is a monomial of $\mathbb{C}\langle X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i = 1,2,3\rangle$. Similarly,

$$\tilde{Z}_a\left(P(KGB)\right) = \sum_{l=n+1}^{m+n} \mathrm{tr}(\tilde{Z}_a(G^{\varepsilon_l})\cdot S(B,H,K)),$$

where S is a monomial of $\mathbb{C}\langle X_i, X_i^*, X_i^{-1}, X_i^{*-1} : i = 1,2,3\rangle$. Finally, using Equations (11.8), we have,

$$\frac{N^2}{2}\sum_{a=1}^{2N^2}\left(\tilde{Z}_a(P(HGB))\right)(I_N)\left(\tilde{Z}_a(Q(KGB))\right)(I_N) = \sum_{l=1}^n\sum_{k=n+1}^{m+n}\alpha_{k,l}\,\mathrm{tr}\left((SR)(B,H,K)\right),$$

where $\alpha = -1, 1$ or 0 depending on ε_l and ε_k. Thus, the element

$$\tilde{\Gamma}(P,Q) = \sum_{l=1}^n\sum_{k=n+1}^{m+n}\alpha_{k,l}\,\mathrm{tr}\left(SR\right)$$

satisfies (16.4).

We extend the definition of $\tilde{\Gamma}$ to all elements of $\mathcal{P}$ of the form $P_1\cdots P_k, Q_1\cdots Q_l \in \mathcal{P}$ by the relation

$$\tilde{\Gamma}(P_1\cdots P_k, Q_1\cdots Q_l) = \sum_{1\leq i\leq k}\sum_{1\leq j\leq l} P_1\cdots\widehat{P_i}\cdots P_kQ_1\cdots\widehat{Q_j}\cdots Q_l\tilde{\Gamma}(P_i,Q_j),$$

and finally, we extend $\tilde{\Gamma}$ to all elements of $\mathcal{P}$ by bilinearity. Because Γ_N fulfills the same relations as Γ, it follows (16.4). Thanks to Proposition 16.1, if $(G_t^{(N)})_{t\geq 0}$,$(H_t^{(N)})_{t\geq 0}$ and $(K_t^{(N)})_{t\geq 0}$ are three independent Brownian motion, we have

$$\begin{aligned}\sigma_T(P,Q) =& \frac{N^2}{2}\int_0^T \mathbb{E}\left[\Gamma_N\left(P(H_{T-t}^{(N)}GG_T^{(N)}), Q(K_{T-t}^{(N)}GG_T^{(N)})\right)(I_N)\right]\mathrm{d}t + O(1/N^2)\\ =& \int_0^T \mathbb{E}\left[\left[\tilde{\Gamma}(P,Q)\right](G_T^{(N)}, H_{T-t}^{(N)}, K_{T-t}^{(N)})\right]\mathrm{d}t + O(1/N^2)\\ =&\mathbb{E}\left[\left[\int_0^T \tilde{\Gamma}(P,Q)\mathrm{d}t\right](G_T^{(N)}, H_{T-t}^{(N)}, K_{T-t}^{(N)})\right] + O(1/N^2).\end{aligned}$$

We use (16.3), and it follows that

$$\sigma_T(P,Q) = \int_0^T \left[\tilde{\Gamma}(P,Q)\right](G_T, H_{T-t}, K_{T-t})\mathrm{d}t. \qquad \square$$

16.2. The simple case of polynomials. In the case of Laurent polynomials, it is possible to compute explicitly the term $\left[\tilde{\Gamma}(P,Q)\right]$ of Proposition 16.2, and thus recover the expression of covariance given by (16.1).

Proposition 16.3. *Let $P,Q \in \mathbb{C}[X,X^{-1}]$. Setting $(G_t)_{t\geq 0}$, $(H_t)_{t\geq 0}$ and $(K_t)_{t\geq 0}$ be three free circular multiplicative Brownian motions which are free, we have,*

$$\sigma_T(\operatorname{tr} P, \operatorname{tr} Q) = 0, \quad \sigma_T(\operatorname{tr} P^*, \operatorname{tr} Q^*) = 0$$

and

$$\sigma_T(\operatorname{tr} P, \operatorname{tr} Q^*) = \int_0^T \tau(P'(H_{T-t}G_T)(Q'(K_{T-t}G_T))^*)\mathrm{d}t.$$

In particular, this proposition shows that, for all $P \in \mathbb{C}[X,X^{-1}]$, the random variable $\left(\operatorname{Tr}(P(G_T^{(N)})) - \mathbb{E}\left[\operatorname{Tr}(P(G_T^{(N)}))\right]\right)$ is asymptotically a circularly-symmetric complex normal distribution of covariance $\int_0^T \tau(P'(H_{T-t}G_T)(P'(K_{T-t}G_T))^*)\mathrm{d}t$.

Proof. We will prove that, in Proposition 16.2, we can fix

$$\left(\tilde{\Gamma}(\operatorname{tr} P, \operatorname{tr} Q)\right) = 0, \quad \left(\tilde{\Gamma}(\operatorname{tr} P^*, \operatorname{tr} Q^*)\right) = 0$$

and

$$\left(\tilde{\Gamma}(\operatorname{tr} P, \operatorname{tr} Q^*)\right) = \operatorname{tr}(P'(X_2X_1)Q'(X_3X_1)^*).$$

Indeed, let $P = X^n$ and $Q = X^m$. Let $N \in \mathbb{N}$, and $B,H,K \in GL_N(\mathbb{C})$,Then

$$(\operatorname{tr} P)(HGB) = \operatorname{tr}((HGB)^n), \quad \text{and } (\operatorname{tr} Q^*)(KGB) = \operatorname{tr}((B^*G^*K^*)^m).$$

We compute, for all $1 \leq i \leq N^2$,

$$\tilde{Z}_i\left(\operatorname{tr}((HGB)^n)\right)(I_N) = n\operatorname{tr}(Z_i(HB)^n)$$

and

$$\tilde{Z}_i\left(\operatorname{tr}((B^*G^*K^*)^m)\right)(I_N) = m\operatorname{tr}(Z_i^*(B^*K^*)^m).$$

Finally, using Equations (11.8), we have

$$\begin{aligned}\frac{N^2}{2}\sum_{a=1}^{2N^2}\left(\tilde{Z}_a(P(HGB))\right)(I_N)\left(\tilde{Z}_a(Q(KGB))\right)(I_N) &= mn\operatorname{tr}((HB)^n((KB)^m)^*)\\ &= \left(\tilde{\Gamma}(\operatorname{tr} P, \operatorname{tr} Q^*)\right)(B,H,K)\end{aligned}$$

with $\left(\tilde{\Gamma}(\operatorname{tr} P, \operatorname{tr} Q^*)\right) = \operatorname{tr}(P'(X_2X_1)Q'(X_3X_1)^*)$. Similar computations lead to $\left(\tilde{\Gamma}(\operatorname{tr} P, \operatorname{tr} Q)\right) = 0$ and $\left(\tilde{\Gamma}(\operatorname{tr} P^*, \operatorname{tr} Q^*)\right) = 0$, and we extend the formulas to $P,Q \in \mathbb{C}[X,X^{-1}]$ by bilinearity. Finally, thanks to Proposition 16.2, we know that

$$\begin{aligned}\sigma_T(\operatorname{tr} P, \operatorname{tr} Q) &= \int_0^T \left[\tilde{\Gamma}(P,Q)\right](G_T, H_{T-t}, K_{T-t})\mathrm{d}t\\ &= \int_0^T \tau(P'(H_{T-t}G_T)Q'(K_{T-t}G_T)^*).\end{aligned}$$

The two others cases are treated similarly. $\square$

CHAPTER 4

Matricial model for the free multiplicative convolution

This self-contained chapter has been taken from the article [26].

ABSTRACT. This paper investigates homomorphisms à la Bercovici-Pata between additive and multiplicative convolutions. We also consider their matricial versions which are associated with measures on the space of Hermitian matrices and on the unitary group. The previous results combined with a matricial model of Benaych-Georges and Cabanal-Duvillard allow us to define and study the large N limit of a new matricial model on the unitary group for free multiplicative Lévy processes.

Contents

17. Introduction

The classical convolution $*$ on $\mathbb{R}$ and the classical multiplicative convolution $\circledast$ on the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, which correspond respectively to the addition and to the product of independent random variables, have analogues in free probability. Indeed, replacing the concept of classical independence by the concept of freeness, Voiculescu defined the free additive convolution $\boxplus$ on $\mathbb{R}$, and the free multiplicative convolution $\boxtimes$ on $\mathbb{U}$ (we refer the reader to [74] for an introduction to free convolutions). A probability measure μ on $\mathbb{R}$ is said to be $*$-*infinitely divisible* if, for all $n \in \mathbb{N}^*$, there exists a probability measure μ_n such that $\mu_n^{*n} = \mu$. The set of $*$-infinitely divisible probability measures endowed with the operation $*$ is a semigroup which we will denote by $\mathcal{ID}(\mathbb{R},*)$, and we consider analogously the sets $\mathcal{ID}(\mathbb{U},\circledast)$, $\mathcal{ID}(\mathbb{R},\boxplus)$ and $\mathcal{ID}(\mathbb{U},\boxtimes)$.

In [13], Bercovici and Pata identified an isomorphism of semigroups Λ between $\mathcal{ID}(\mathbb{R},*)$ and $\mathcal{ID}(\mathbb{R},\boxplus)$ which has a good behaviour with respect to limit theorems: for all $\mu \in \mathcal{ID}(\mathbb{R},*)$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$,

$$\mu_n^{*n} \xrightarrow[n\to+\infty]{(w)} \mu \iff \mu_n^{\boxplus n} \xrightarrow[n\to+\infty]{(w)} \Lambda(\mu)$$

where the convergence is the weak convergence of measures. Unfortunately, the situation is not as symmetric in the multiplicative case. Let $\mathcal{M}_*$ denote the set of probability measures μ on $\mathbb{U}$ such that $\int_{\mathbb{U}} \zeta \mathrm{d}\mu(\zeta) \neq 0$. In [28], Chistyakov and Götze proved that, given a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{U}$, the weak convergence of $\mu_n^{\boxtimes n}$ to any measure of $\mathcal{M}_*$ implies the weak convergence of $\mu_n^{\circledast n}$; but they also proved that the converse is false. It is thus only possible to define a homomorphism of semigroups Γ between $\mathcal{ID}(\mathbb{U},\boxtimes)$ and $\mathcal{ID}(\mathbb{U},\circledast)$ (see Definition 19.4) such that, for all $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes) \cap \mathcal{M}_*$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{U}$,

$$\mu_n^{\boxtimes n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mu_n^{\circledast n} \xrightarrow[n\to+\infty]{(w)} \Gamma(\mu).$$

Finally, the homomorphism $\mathbf{e} : x \mapsto e^{ix}$ from $(\mathbb{R},+)$ to $(\mathbb{U},\times)$ induces a homomorphism of semigroups $\mathbf{e}_*$ between $\mathcal{ID}(\mathbb{R},*)$ and $\mathcal{ID}(\mathbb{U},\circledast)$, given by the push-forward of measures, which enjoys a similar property: for all $\mu \in \mathcal{ID}(\mathbb{R},*)$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$,

$$\mu_n^{*n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mathbf{e}_*(\mu_n)^{\circledast n} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_*(\mu).$$

The first aim of this work is to complete the picture which we just sketched. In Definition 19.2, we shall introduce a new homomorphism of semigroups $\mathbf{e}_{\boxplus}$ between $\mathcal{ID}(\mathbb{R},\boxplus)$ and $\mathcal{ID}(\mathbb{U},\boxtimes)$, and which is linked to the previous homomorphisms in the following way.

THEOREM 17.1 (see Prop. 19.6 and Thm. 19.10). *The map $\mathbf{e}_{\boxplus} : \mathcal{ID}(\mathbb{R},\boxplus) \to \mathcal{ID}(\mathbb{U},\boxtimes)$ is such that:*

(1) For all $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$,

$$\mu_n^{\boxplus n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mathbf{e}_*(\mu_n)^{\boxtimes n} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_{\boxplus}(\mu);$$

(2) The following diagram commutes:

$$\begin{array}{ccc} \mathcal{ID}(\mathbb{R},*) & \xrightarrow{\Lambda} & \mathcal{ID}(\mathbb{R},\boxplus) \\ \mathbf{e}_* \downarrow & & \downarrow \mathbf{e}_{\boxplus} \\ \mathcal{ID}(\mathbb{U},\circledast) & \xleftarrow[\Gamma]{} & \mathcal{ID}(\mathbb{U},\boxtimes). \end{array} \tag{17.1}$$

In the highly non-commutative theory of Lie groups, there is a well-known process which connects additive infinitely divisible laws with multiplicative ones. It consists in passing to the limit the product of multiplicative little increments which are built from additive increments using the exponential map (see [39]). A natural question is whether there exists a matrix approximation of $\mathbf{e}_{\boxplus}$ which arises from this procedure.

Our starting point is a matricial model for $\mathcal{ID}(\mathbb{R},\boxplus)$ which has been constructed simultaneously by Benaych-Georges and Cabanal-Duvillard in [12] and [23]. For all $N \in \mathbb{N}$, let us consider the classical convolution $*$ on the set of Hermitian matrices $\mathcal{H}_N$, and denote by $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*)$ the set of infinitely divisible probability measures on $\mathcal{H}_N$ which are invariant under conjugation by unitary matrices. For all $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$, Benaych-Georges and Cabanal-Duvillard proved that there exists an element of $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*)$, which we shall denote by $\Pi_N(\mu)$ (see Section 23.1), such that:

(1) For all $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$, the spectral measure of a random matrix with distribution $\Pi_N(\mu)$ converges weakly to μ in probability as N tends to infinity;
(2) $\Pi_N : \mathcal{ID}(\mathbb{R},\boxplus) \to \mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*)$ is a homomorphism of semigroups.

On the other hand, the map $\mathbf{e} : H \mapsto e^{iH}$ from $\mathcal{H}_N$ to the unitary group $U(N)$ induces, with some care, a homomorphism of semigroups from $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*)$ to the set $\mathcal{ID}_{\text{inv}}(U(N),\circledast)$ of infinitely divisible measures on $U(N)$ which are invariant under conjugation. Indeed, for all $\mu \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*)$, the sequence $(\mathbf{e}_*(\mu^{*1/n})^{\circledast n})_{n\in\mathbb{N}^*}$ converges weakly to a measure $\mathcal{E}_N(\mu) \in$

$\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ (see Proposition-Definition 22.2). The situation can be summed up in the following diagram:

$$\begin{array}{ccc} \mathcal{ID}(\mathbb{R},\boxplus) & \xrightarrow{\Pi_N} & \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*) \\ {\scriptstyle \mathbf{e}_{\boxplus}}\downarrow & & \downarrow{\scriptstyle \mathcal{E}_N} \\ \mathcal{ID}(\mathbb{U},\boxtimes) & & \mathcal{ID}_{\mathrm{inv}}(U(N),\circledast). \end{array} \tag{17.2}$$

When $N=1$, we have $\Pi_1=\Lambda^{-1}$, $\mathcal{E}_1=\mathbf{e}_*$, and consequently the diagram (17.2) is exactly the top part of the diagram (17.1). The second main result of this work is the definition of a homomorphism of semigroups $\Gamma_N:\mathcal{ID}(\mathbb{U},\boxtimes)\to\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ which completes the picture as follows (see Section 23.2).

Theorem 17.2 (see Prop. 23.5 and Cor. 23.8). *The map Γ_N is such that:*

(1) for all $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$, the spectral measure of a random matrix $U^{(N)}$ with distribution $\Gamma_N(\mu)$ converges weakly to μ in expectation, in the sense that, for each continuous function f on $\mathbb{U}$, one has the convergence

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(f\left(U^{(N)}\right)\right)\right]=\int_{\mathbb{U}} f\,\mathrm{d}\mu;$$

(2) The following diagram commutes

$$\begin{array}{ccc} \mathcal{ID}(\mathbb{R},\boxplus) & \xrightarrow{\Pi_N} & \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*) \\ {\scriptstyle \mathbf{e}_{\boxplus}}\downarrow & & \downarrow{\scriptstyle \mathcal{E}_N} \\ \mathcal{ID}(\mathbb{U},\boxtimes) & \xrightarrow[\Gamma_N]{} & \mathcal{ID}_{\mathrm{inv}}(U(N),\circledast). \end{array} \tag{17.3}$$

This result can be expressed by saying that the map $\mathbf{e}_{\boxplus}$ is the limit of the map $\mathcal{E}_N$ as N tends to infinity. The first assertion of the theorem above is a generalisation of a result of Biane: in [17], he proved that the spectral measure of a Brownian motion on $U(N)$ with adequately chosen speed converges to the distribution of a free unitary Brownian motion at each fixed time. The distribution of a Brownian motion is indeed an infinitely divisible measure at each time, and this convergence can be viewed as a particular case of Theorem 17.2.

The proof itself of Theorem 17.2 is interesting at least for two reasons. It is the first time that the free log-cumulants, originated in [58], are used for proving an asymptotic result of random matrices. Secondly, the proof relies upon a key object, the symmetric group $\mathfrak{S}_n$, which is linked to both the combinatorics of free probability theory, and the computation of conjugate-invariant measures on $U(N)$. More precisely, in [53], Lévy established that the asymptotic distribution of a Brownian motion on the unitary group is closely related to the counting of paths in the Caley graph of $\mathfrak{S}_n$. Similarly, for all $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$, the asymptotic distribution of a random matrix with law $\Gamma_N(\mu)$ involves the counting of paths in $\mathfrak{S}_n$, each step of which is given by the following generator (see Lemma 23.7)

$$T(\sigma)=nL\kappa_1(\mu)\cdot\sigma+\sum_{\substack{2\leq m\leq n\\ c\ m\text{-cycle of }\mathfrak{S}_n\\ c\sigma\preceq\sigma}} L\kappa_m(\mu)\cdot c\sigma,$$

where $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ are the free log-cumulants of μ.

In fact, Biane proved in [17] a stronger result: the convergence of all finite dimensional distributions of the Brownian motion on $U(N)$ to the distribution of a free unitary Brownian motion. Similarly, a classical result of Voiculescu allows us to strengthen the previous asymptotic results of Theorem 17.2 as follows (see Section 23.4 for details).

THEOREM 17.3. *Let* $(U_t)_{t\in\mathbb{R}_+}$ *be a free unitary multiplicative Lévy process with marginal distributions* $(\mu_t)_{t\in\mathbb{R}_+}$ *in* $\mathcal{M}_*$. *For all* $N\in\mathbb{N}^*$, *let* $(U_t^{(N)})_{t\in\mathbb{R}_+}$ *be a Lévy process with marginal distributions* $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$. *Then,* $(U_t^{(N)})_{t\in\mathbb{R}_+}$ *converges to* $(U_t)_{t\in\mathbb{R}_+}$ *in non-commutative distribution. In other words, for each integer* $n\geq 1$, *for each non-commutative polynomial* P *in* n *variables, and each choice of* n *non-negative reals* $t_1,\dots,t_n$, *one has the convergence*

$$\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\mathrm{Tr}\left(P\left(U_{t_1}^{(N)},\dots,U_{t_n}^{(N)}\right)\right)\right]=\tau(P\left(U_{t_1},\dots,U_{t_n}\right)).$$

Moreover, independent copies of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ *converge to freely independent copies of* $(U_t)_{t\in\mathbb{R}_+}$.

The rest of the paper is organized as follows. In Section 18, we give an overview of the theory of infinitely divisible measures. In Section 19, we define $\mathbf{e}_{\boxplus}$ and Γ and we prove Theorem 17.1. Section 20 is devoted to the notion of free log-cumulants, which is an important tool for the proof of the asymptotic results of this paper. Section 21 presents a description of convolution semigroups on the unitary group, and studies more precisely those which are invariant by conjugation. Section 22 links together the measures on the Hermitian matrices with the measures on the unitary matrices through the stochastic exponentiation $\mathcal{E}_N$. Finally, Section 23 provides the definition of the random matrix models Π_N and Γ_N, and the proof of Theorem 17.2 and Theorem 17.3.

18. Infinite divisibility for unidimensional convolutions

In this section, we give the necessary background concerning $\mathcal{ID}(\mathbb{R},*)$, $\mathcal{ID}(\mathbb{U},\circledast)$, $\mathcal{ID}(\mathbb{R},\boxplus)$ and $\mathcal{ID}(\mathbb{U},\boxtimes)$. In particular, we give a description of the characteristic pair and the characteristic triplet of an infinitely divisible measure in each case.

We say that a sequence of finite measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$ *converges weakly* to a measure μ if for all continuous and bounded complex function f, $\lim_{n\to\infty}\int_{\mathbb{C}}f\mathrm{d}\mu_n=\int_{\mathbb{C}}f\mathrm{d}\mu$.

18.1. Classical infinite divisibility on $\mathbb{R}$. Let $\mu\in\mathcal{ID}(\mathbb{R},*)$. There exists a sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures such that, for all $n\in\mathbb{N}^*$, $\mu_n^{*n}=\mu$. The important fact is that the measures

$$\mathrm{d}\sigma_n(x)=n\frac{x^2}{x^2+1}\mu_n(\mathrm{d}x)$$

converge weakly to a measure σ and the reals

$$\gamma_n=n\int_{\mathbb{R}}\frac{x}{x^2+1}\mu_n(\mathrm{d}x)$$

converge to a constant $\gamma\in\mathbb{R}$. The pair (γ,σ) is known as the $*$-*characteristic pair* for μ and it is uniquely determined by μ. More generally, we have the following characterization.

THEOREM 18.1 ([13]). *Let* $\mu\in\mathcal{ID}(\mathbb{R},*)$ *with* $*$-*characteristic pair* (γ,σ). *Let* $k_1<k_2<\cdots$ *be natural numbers and* $(\mu_n)_{n\in\mathbb{N}^*}$ *be a sequence of probability measures on* $\mathbb{R}$. *The following assertions are equivalent:*

(1) the measures $\underbrace{\mu_n*\cdots*\mu_n}_{k_n\text{ times}}$ *converge weakly to* μ;

(2) the measures

$$\mathrm{d}\sigma_n(x) = k_n \frac{x^2}{x^2+1}\mu_n(\mathrm{d}x)$$

converge weakly to σ *and*

$$\lim_{n\to\infty} k_n \int_{\mathbb{R}} \frac{x}{x^2+1}\mu_n(\mathrm{d}x) = \gamma.$$

In addition to [13], we refer the reader to the very complete lecture notes [11]. We present now two additional properties of the $*$-characteristic pairs. Firstly, there is a one-to-one correspondence between $*$-infinitely divisible probability measures and pairs (γ, σ). Indeed, for all finite measure σ on $\mathbb{R}$, and all constant $\gamma \in \mathbb{R}$, there exists a unique $*$-infinitely divisible probability measure such that (γ, σ) is the $*$-characteristic pair for μ. Secondly, the $*$-characteristic pairs linearize the convolution: let μ_1 and μ_2 be two $*$-infinitely divisible measures with respective $*$-characteristic pairs (γ_1, σ_1) and (γ_2, σ_2). The measure $\mu_1 * \mu_2$ is a $*$-infinitely divisible measure with $*$-characteristic pair $(\gamma_1 + \gamma_2, \sigma_1 + \sigma_2)$.

Let us review another, perhaps more classical, characterization of infinitely divisible measures. Let μ be $*$-infinitely divisible and (γ, σ) be its $*$-characteristic pair. We set

(18.1)

$$a = \sigma(\{0\}),\ \rho(\mathrm{d}x) = \frac{1+x^2}{x^2}\cdot 1_{\mathbb{R}\setminus\{0\}}(x)\sigma(\mathrm{d}x),\ \text{and } \eta = \gamma + \int_{\mathbb{R}} x\left(1_{[-1,1]}(x) - \frac{1}{1+x^2}\right)\rho(\mathrm{d}x).$$

The triplet (η, a, ρ) is called the $*$-*characteristic triplet* for μ. Observe that ρ is such that the function $x \mapsto \min(1, x^2)$ is ρ-integrable and $\rho(\{0\}) = 0$. Such a measure is called a *Lévy measure* on $\mathbb{R}$. Conversely, for all (η, a, ρ) with $\eta \in \mathbb{R}$, $a \geq 0$ and ρ a Lévy measure on $\mathbb{R}$, there exists a unique $*$-infinitely divisible probability measure such that (η, a, ρ) is the $*$-characteristic triplet for μ.

EXAMPLE 18.2. Here are three important classes of $*$-infinitely divisible measures:

(1) For any constant η in $\mathbb{R}$, the Dirac distribution δ_η is in $\mathcal{ID}(\mathbb{R}, *)$, and its $*$-characteristic triplet is $(\eta, 0, 0)$;

(2) For any constant $a > 0$, the Gaussian distribution of variance a is

$$\mathcal{N}_a(\mathrm{d}x) = \frac{1}{\sqrt{2\pi a}} e^{-\frac{x^2}{2a}}\mathrm{d}x \in \mathcal{ID}(\mathbb{R}, *)$$

whose $*$-characteristic triplet is $(0, a, 0)$;

(3) For any constant $\lambda > 0$ and any probability measure $\rho \in \mathcal{P}(\mathbb{R})$, the compound Poisson distribution with rate λ and jump distribution ρ is

$$\mathrm{Poiss}^*_{\lambda,\rho} = e^{-\lambda}\sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!}\rho^{*n} \in \mathcal{ID}(\mathbb{R}, *)$$

whose $*$-characteristic triplet is $(\lambda \int_{[-1,1]} x\rho(\mathrm{d}x), 0, \lambda\rho_{|\mathbb{R}\setminus\{0\}})$. One important particular case is when $\rho = \delta_1$: the Poisson distribution Poiss^*_λ of mean λ is

$$\mathrm{Poiss}^*_\lambda(\mathrm{d}x) = \mathrm{Poiss}^*_{\lambda,\delta_1}(\mathrm{d}x) = e^{-\lambda}\sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!}\delta_n \in \mathcal{ID}(\mathbb{R}, *).$$

18.2. The Bercovici-Pata bijection. In [13], Bercovici and Pata proved that all results of the previous section stay true if one replaces the classical convolution $*$ by the free additive convolution $\boxplus$. This leads to the *Bercovici-Pata bijection* Λ from $\mathcal{ID}(\mathbb{R},*)$ to $\mathcal{ID}(\mathbb{R},\boxplus)$ which maps a $*$-infinitely divisible measure with $*$-characteristic pair (γ,σ) to the $\boxplus$-infinitely divisible measure with $\boxplus$-characteristic pair (γ,σ). Its importance is due to the following theorem.

THEOREM 18.3 ([13]). *The Bercovici-Pata bijection Λ has the following properties:*

(1) For all $\mu,\nu\in\mathcal{ID}(\mathbb{R},*)$, $\Lambda(\mu*\nu)=\Lambda(\mu)\boxplus\Lambda(\nu)$;
(2) For all natural numbers $k_1<k_2<\cdots$, all sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures on $\mathbb{R}$ and all $*$-infinitely divisible measure μ, the measures $\mu_n^{*k_n}$ converge weakly to μ if and only if the measures $\mu_n^{\boxplus k_n}$ converge weakly to $\Lambda(\mu)$.

EXAMPLE 18.4. Here are the free analogues of the measures presented in Example 18.2:

(1) For any constant η in $\mathbb{R}$, we have $\Lambda(\delta_\eta)=\delta_\eta\in\mathcal{ID}(\mathbb{R},\boxplus)$, and its $\boxplus$-characteristic triplet is $(\eta,0,0)$.
(2) For any constant $a>0$, the semi-circular distribution of variance a is
$$\mathcal{S}_a(\mathrm{d}x)=\frac{1}{2\pi a}\sqrt{4a-x^2}\cdot 1_{[-2\sqrt{a},2\sqrt{a}]}(x)\mathrm{d}x\in\mathcal{ID}(\mathbb{R},\boxplus)$$
whose characteristic triplet is $(0,a,0)$. We have $\Lambda(\mathcal{N}_a)=\mathcal{S}_a$.
(3) For any constant $\lambda>0$, the free Poisson distribution with mean λ, also called the Marçenko-Pastur distribution, is
$$\mathrm{Poiss}^{\boxplus}_{\lambda,\delta_1}(\mathrm{d}x)=\begin{cases}(1-\lambda)\delta_0+\dfrac{1}{2\pi x}\sqrt{(x-a)(b-x)}1_{a\le x\le b}\mathrm{d}x & \text{if } 0\le\lambda\le 1,\\ \dfrac{1}{2\pi x}\sqrt{(x-a)(b-x)}1_{a\le x\le b}\mathrm{d}x & \text{if } \lambda>1,\end{cases}$$
where $a=(1-\sqrt{\lambda})^2$ and $b=(1+\sqrt{\lambda})^2$. Its $\boxplus$-characteristic triplet is $(\lambda,0,\lambda\delta_1)$. More generally, for any constant $\lambda>0$ and probability measure $\rho\in\mathcal{P}(\mathbb{R})$, the free compound Poisson distribution with rate λ and jump distribution ρ is the measure $\mathrm{Poiss}^{\boxplus}_{\lambda,\rho}\in\mathcal{ID}(\mathbb{R},\boxplus)$ whose $\boxplus$-characteristic triplet is $(\lambda\int_{[-1,1]}x\rho(\mathrm{d}x),0,\lambda\rho)$. We have $\Lambda(\mathrm{Poiss}^{*}_{\lambda,\rho})=\mathrm{Poiss}^{\boxplus}_{\lambda,\rho}$.

We finish this section with a technical lemma, which is a straightforward reformulation of Theorem 18.1, using the relation given by (18.1).

LEMMA 18.5. *Let $\mu\in\mathcal{ID}(\mathbb{R},\boxplus)$ and (η,a,ρ) be its $\boxplus$-characteristic triplet. Let $k_1<k_2<\cdots$ be natural numbers and $(\mu_n)_{n\in\mathbb{N}^*}$ a sequence of probability measures on $\mathbb{R}$ such that the measures $\mu_n^{\boxplus k_n}$ converge weakly to μ. Then, for all $f:\mathbb{R}\to\mathbb{C}$ continuous, bounded, and such that $f(x)\sim_{x\to 0}f_0x^2$, we have*
$$\lim_{n\to\infty}k_n\int_{\mathbb{R}}f\mathrm{d}\mu_n=\int_{\mathbb{R}}f\mathrm{d}\rho+af_0,\ \textit{and}\ \lim_{n\to\infty}k_n\int x1_{[-1,1]}(x)\mathrm{d}\mu_n(x)=\eta.$$

18.3. Classical infinite divisibility on $\mathbb{U}$. As we will now see, the particularity of the set $\mathcal{ID}(\mathbb{U},\circledast)$ is the existence of idempotent measures, a infinite class which has no equivalent in $\mathcal{ID}(\mathbb{R},*)$, $\mathcal{ID}(\mathbb{R},\boxplus)$ or $\mathcal{ID}(\mathbb{U},\boxtimes)$. Our references in this section are [28, 62, 65].

A probability measure μ on $\mathbb{U}$ is said to be idempotent if $\mu\circledast\mu=\mu$. Each compact subgroup of $\mathbb{U}$ leads to an idempotent measure given by its Haar measure. More concretely, let $m\in\mathbb{N}$. The m-th roots of unity form a subgroup of $\mathbb{U}$, whose Haar measure is denoted by λ_m. We have $\lambda_m\circledast\lambda_m=\lambda_m$ and consequently $\lambda_m\in\mathcal{ID}(\mathbb{U},\circledast)$. We denote by λ, or λ_∞, the Haar measure on

$\mathbb{U}$, which is also $\circledast$-infinitely divisible. Fortunately, the measures $(\lambda_m)_{m\in\mathbb{N}\cup\{\infty\}}$ are the unique measures on $\mathbb{U}$ which are idempotent.

How to identify measures of $\mathcal{ID}(\mathbb{U},\circledast)$ which are not idempotent? Recall that $\mathcal{M}_*$ is the set of probability measures μ on $\mathbb{U}$ such that $\int_{\mathbb{U}}\zeta\mathrm{d}\mu(\zeta)\neq 0$. It is easy to see that measures in $\mathcal{M}_*$ are not idempotent, with the exception of δ_1. In fact, every measure in $\mathcal{ID}(\mathbb{U},\circledast)$ factorizes into the product of an idempotent measure with a measure in $\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$. For the study of $\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$, it is useful to introduce the *characteristic function*: for all probability measure μ on $\mathbb{U}$, it is the function $\widehat{\mu}:\mathbb{Z}\to\mathbb{C}$ defined for all $k\in\mathbb{Z}$ by

$$\widehat{\mu}(k)=\int_{\mathbb{U}}\zeta^k\mathrm{d}\mu(\zeta).$$

It is multiplicative for the convolution $\circledast$ in the sense that, for all μ,ν probability measures on $\mathbb{U}$, and all $k\in\mathbb{Z}$, we have

$$\widehat{\mu\circledast\nu}(k)=\widehat{\mu}(k)\cdot\widehat{\nu}(k). \tag{18.2}$$

For all $m\in\mathbb{N}^*$ and $k\in\mathbb{Z}$, we obviously have $\widehat{\lambda_m}(k)=1$ if k is divisible by m and 0 if not. Using the characteristic function, we can now characterize the measures in $\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$. Let $\mu\in\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$. There exists a finite measure ν on $\mathbb{U}$ and a real $\alpha\in\mathbb{R}$ such that, for all $k\in\mathbb{Z}$,

$$\widehat{\mu}(k)=e^{ik\alpha}\exp\Big(\int_{\mathbb{U}}\underbrace{\frac{\zeta^k-1-ik\Im(\zeta)}{1-\Re(\zeta)}}_{=-k^2\text{ if }\zeta=1}\mathrm{d}\nu(\zeta)\Big).$$

Unfortunately, the pair $(e^{i\alpha},\nu)$ is not unique, in contrast to what [65] can suggest at first reading (see the end of the current section). We say that $(e^{i\alpha},\nu)$ is *a $\circledast$-characteristic pair* for μ. Conversely, for all pair (ω,ν) such that $\omega\in\mathbb{U}$ and ν is a finite measure on $\mathbb{U}$, there exists a unique $\circledast$-infinitely divisible measure μ which admits (ω,ν) as a $\circledast$-characteristic pair.

Similarly to the additive case, we introduce now the characteristic triplet. Let $\mu\in\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$ and let (ω,ν) be a $\circledast$-characteristic pair for μ. We set

$$b=2\nu(\{1\})\quad\text{and}\quad\upsilon(\mathrm{d}\zeta)=\frac{1}{1-\Re\zeta}\cdot 1_{\mathbb{U}\setminus\{1\}}(\zeta)\nu(\mathrm{d}\zeta). \tag{18.3}$$

We have, for all $k\in\mathbb{Z}$,

$$\widehat{\mu}(k)=\omega^k\exp\Big(-\frac{1}{2}bk^2+\int_{\mathbb{U}}(\zeta^k-1-ik\Im(\zeta))\mathrm{d}\upsilon(\zeta)\Big).$$

We say that (ω,b,υ) is *a $\circledast$-characteristic triplet* for μ. Let us remark that $\upsilon(\{1\})=0$ and $\int_{\mathbb{U}}(1+\Re(\zeta))\mathrm{d}\upsilon(\zeta)<+\infty$. Such a measure is called a *Lévy measure* on $\mathbb{U}$. As expected, for all (ω,b,υ) with $\omega\in\mathbb{U}$, $b\geq 0$ and υ a Lévy measure on $\mathbb{U}$, there exists a unique $\circledast$-infinitely divisible probability measure such that (ω,b,υ) is a $\circledast$-characteristic triplet for μ. Moreover, for all μ_1 and μ_2 be two $\circledast$-infinitely divisible measures with $\circledast$-characteristic triplets $(\omega_1,b_1,\upsilon_1)$ and $(\omega_2,b_2,\upsilon_2)$, we see thanks to (18.2) that $\mu_1\circledast\mu_2\in\mathcal{ID}(\mathbb{U},\circledast)\cap\mathcal{M}_*$ with $\circledast$-characteristic triplet $(\omega_1\omega_2,b_1+b_2,\upsilon_1+\upsilon_2)$.

To sum up the previous discussion, for all $\mu\in\mathcal{ID}(\mathbb{U},\circledast)$, there exist $m\in\mathbb{N}\cup\{\infty\}$, $\omega\in\mathbb{U}$ and ν a finite measure on $\mathbb{U}$ such that, for all $k\in\mathbb{Z}$,

$$\widehat{\mu}(k)=\widehat{\lambda_m}(k)\cdot\omega^k\exp\Big(\int_{\mathbb{U}}\underbrace{\frac{\zeta^k-1-ik\Im(\zeta)}{1-\Re(\zeta)}}_{=-k^2\text{ if }\zeta=1}\mathrm{d}\nu(\zeta)\Big).$$

EXAMPLE 18.6. Here again, we can distinguish three classes of $\circledast$-infinitely divisible measures:

(1) For any constant $\omega \in \mathbb{U}$, $(\omega, 0, 0)$ is a $\circledast$-characteristic triplet of the Dirac distribution $\delta_\omega \in \mathcal{ID}(\mathbb{U}, \circledast)$;
(2) For any constant $b > 0$, the wrapped Gaussian distribution of parameter b is $\mathbf{e}_*(\mathcal{N}_b) \in \mathcal{ID}(\mathbb{U}, \circledast)$ whose one $*$-characteristic triplet is $(1, b, 0)$;
(3) For any constant $\lambda > 0$ and any probability measure υ on $\mathbb{U}$, the compound Poisson distribution with rate λ and jump distribution υ is
$$\mathrm{Poiss}^{\circledast}_{\lambda,\upsilon} = e^{-\lambda} \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} \upsilon^{\circledast n} \in \mathcal{ID}(\mathbb{U}, \circledast)$$
whose one $\circledast$-characteristic triplet is $(\exp(i\lambda \int_{\mathbb{U}} \Im \mathrm{d}\upsilon), 0, \lambda \upsilon_{|\mathbb{U}\setminus\{1\}})$.

We give now a case of $\circledast$-infinitely divisible measure which admits two $\circledast$-characteristic pairs which are different. Set

$$\mu = e^{-\pi} \left(\frac{\cosh(\pi) + 1}{2} \delta_1 + \frac{\cosh(\pi) - 1}{2} \delta_{-1} + \frac{\sinh(\pi)}{2} \delta_i + \frac{\sinh(\pi)}{2} \delta_{-i} \right).$$

For all $n \in \mathbb{Z}$, we have $\widehat{\mu}(4n) = 1$, $\widehat{\mu}(4n+1) = \widehat{\mu}(4n+3) = e^{-\pi}$ and $\widehat{\mu}(4n+2) = e^{-2\pi}$. It is immediate that, for $\upsilon = \pi\delta_i$ or $\upsilon = \pi\delta_{-i}$, we have

$$\widehat{\mu}(n) = \exp\Big(\int_{\mathbb{U}} (\zeta^n - 1 - in\Im(\zeta)) \mathrm{d}\upsilon(\zeta) \Big).$$

Thus, the measure μ admits $(1, 0, \pi\delta_i)$ and $(1, 0, \pi\delta_{-i})$ as $\circledast$-characteristic triplets. One can also see [28] for others examples.

18.4. The convolution $\boxtimes$ and the S-transform. The free multiplicative convolution $\boxtimes$ can be described succinctly in terms of the S-transform. Let us explain how it works.

Let μ be a finite measure on $\mathbb{U}$. For all $k \in \mathbb{N}$, we set $m_k(\mu) = \int_{\mathbb{C}} \zeta^k \mathrm{d}\mu(\zeta)$, which is finite, and we call $(m_k(\mu))_{k \in \mathbb{N}}$ the *moments* of μ. We consider the formal power series

$$M_\mu(z) = \sum_{k=0}^{\infty} m_k(\mu) z^k.$$

Let us assume that $\mu \in \mathcal{M}_*$. We define S_μ, the *S-transform* of μ, to be the formal power series such that $zS_\mu(z)/(1+z)$ is the inverse under composition of $M_\mu(z) - 1$. The S-transform is a $\boxtimes$-homomorphism (see [15]): for all μ and $\nu \in \mathcal{M}_*$,

$$S_{\mu \boxtimes \nu} = S_\mu \cdot S_\nu.$$

For all $\mu \in \mathcal{M}_*$, the series $S_\mu(z)$ is convergent in a neighbourhood of 0, and we can therefore identify S_μ with a function which is analytic in a neighborhood of zero. Sometimes it will be convenient to use the function

$$\Sigma_\mu(z) = S_\mu(z/(1-z))$$

which is also analytic in a neighborhood of 0.

18.5. Free infinite divisibility on $\mathbb{U}$. For the free multiplicative convolution, the existence of different proper subgroups of $\mathbb{U}$ does not imply the existence of different idempotent measures. Indeed, the Haar measure λ and δ_1 are the unique probability measures on $\mathbb{U}$ which are idempotent. Moreover, λ is an absorbing element for $\boxtimes$ and it is the unique $\boxtimes$-infinitely divisible measure in $\mathcal{ID}(\mathbb{U},\boxtimes)\setminus\mathcal{M}_*$ according to [15]. Consequently, we will focus our study on $\mathcal{ID}(\mathbb{U},\boxtimes)\cap\mathcal{M}_*$.

Let $\mu\in\mathcal{M}_*$ be a $\boxtimes$-infinitely divisible measure. From Theorem 6.7. of [15], there exists a unique finite measure $\nu\in\mathcal{M}_{\mathbb{U}}$ and a real $\alpha\in\mathbb{R}$ such that

$$\Sigma_\mu(z)=\exp\left(-i\alpha+\int_{\mathbb{U}}\frac{1+\zeta z}{1-\zeta z}\mathrm{d}\nu(\zeta)\right).$$

The pair $(e^{i\alpha},\nu)$ is called the $\boxtimes$-*characteristic pair* for μ, and, on the contrary to the classical case, it is uniquely determined by μ. We have

$$S_\mu(z)=e^{-i\alpha}\exp\left(\int_{\mathbb{U}}\frac{1+z+\zeta z}{1+z-\zeta z}\mathrm{d}\nu(\zeta)\right). \tag{18.4}$$

We observe that, for $\zeta\neq 1$, we have

$$\frac{1+z+\zeta z}{1+z-\zeta z}=\frac{1}{1-\Re\zeta}\left(i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\right),$$

which implies that, defining $\omega=e^{i\alpha}$, $b=2\nu(\{1\})$ and $\upsilon(\mathrm{d}\zeta)=\frac{1}{1-\Re\zeta}\cdot 1_{\mathbb{U}\setminus\{1\}}(\zeta)\nu(\mathrm{d}\zeta)$, we have

$$S_\mu(z)=\omega^{-1}\exp\left(\frac{b}{2}+bz+\int_{\mathbb{U}}i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\mathrm{d}\upsilon(\zeta)\right). \tag{18.5}$$

We will call (ω,b,υ) the $\boxtimes$-*characteristic triplet* for μ. Conversely, for all triplet (ω,b,υ) such that $\omega\in\mathbb{U}$, $b\in\mathbb{R}^+$ and υ is a Lévy measure on $\mathbb{U}$, there exists a unique $\boxtimes$-infinitely divisible measure μ whose $\boxtimes$-characteristic triplet is (ω,b,υ). Indeed, according to Theorem 6.7. of [15], if we define

$$v(z)=-\operatorname{Log}(\omega)+\frac{b}{2}+bz+\int_{\mathbb{U}}i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\mathrm{d}\upsilon(\zeta)$$

using the principal value Log, then the function $S(z)=\exp(v(z))$ is the S-transform of a unique $\boxtimes$-infinitely measure $\mu\in\mathcal{M}_*$.

Let $\mu_1,\mu_2\in\mathcal{M}_*$ be two $\boxtimes$-infinitely divisible measures with respective $\boxtimes$-characteristic triplets $(\omega_1,b_1,\upsilon_1)$ and $(\omega_2,b_2,\upsilon_2)$. The measure $\mu_1\boxtimes\mu_2\in\mathcal{M}_*$ is a $\boxtimes$-infinitely divisible measure with $\boxtimes$-characteristic triplet $(\omega_1\omega_2,b_1+b_2,\upsilon_1+\upsilon_2)$.

EXAMPLE 18.7. The three classes of $\boxtimes$-infinitely divisible measures are:

(1) For any constant $\omega\in\mathbb{U}$, $(\omega,0,0)$ is a $\boxtimes$-characteristic triplet of the Dirac distribution $\delta_\omega\in\mathcal{ID}(\mathbb{U},\boxtimes)$;
(2) For any constant $b>0$, the measure on $\mathbb{U}$ analogous to the Gaussian distribution law is the measure $\mathcal{B}_b\in\mathcal{ID}(\mathbb{U},\circledast)$ whose $*$-characteristic triplet is $(1,b,0)$; it is the law of a free unitary Brownian motion at time b;
(3) For any constant $\lambda>0$ and any probability measure υ on $\mathbb{U}$, the free compound Poisson distribution with rate λ and jump distribution υ is the measure $\mathrm{Poiss}^{\boxtimes}_{\lambda,\upsilon}\in\mathcal{ID}(\mathbb{U},\boxtimes)$ whose $\boxtimes$-characteristic triplet is $(\exp(i\lambda\int_{\mathbb{U}}\Im\mathrm{d}\upsilon),0,\lambda\upsilon_{|\mathbb{U}\setminus\{1\}})$.

19. Homomorphisms between $\mathcal{ID}(\mathbb{R},*)$, $\mathcal{ID}(\mathbb{U},\circledast)$, $\mathcal{ID}(\mathbb{R},\boxplus)$ and $\mathcal{ID}(\mathbb{U},\boxtimes)$.

In this section, we define $\mathbf{e}_\boxplus$ and Γ and prove Theorem 17.1. The definitions and the commutativity of (17.1) is a routine program. The very difficulty consists in proving the first item of Theorem 17.1, or equivalently Theorem 19.10. We shall do it in Section 19.2.

19.1. Definitions of $\mathbf{e}_\boxplus$ and Γ. In order to motivate the definition of $\mathbf{e}_\boxplus$, we start by indicating how a $*$-characteristic triplet is transformed by the homomorphism $\mathbf{e}_*$.

Let us recall that, for all measure μ on $\mathbb{R}$, $\mathbf{e}_*(\mu)$ denotes the push-forward measure of μ by the map $\mathbf{e}: x \mapsto e^{ix}$. Let us denote by $\mathbf{e}_*(\mu)_{|\mathbb{U}\setminus\{1\}}$ the measure induced by $\mathbf{e}_*(\mu)$ on $\mathbb{U}\setminus\{1\}$.

Proposition 19.1. *For all $\mu \in \mathcal{ID}(\mathbb{R},*)$ with $*$-characteristic triplet (η,a,ρ),*

$$(\omega,b,\upsilon) = \left(\exp\left(i\eta + i\int_{\mathbb{R}}(\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right)$$

is a $\circledast$-characteristic triplet of $\mathbf{e}_*(\mu)$.

Proof. First of all, the Fourier transform of a $*$-infinitely divisible measure is well-known (see [13, 64]): for all $\theta \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} e^{i\theta x}\mathrm{d}\mu(x) = \exp\left(i\eta\theta - \frac{1}{2}a\theta^2 + \int_{\mathbb{R}}(e^{i\theta x} - 1 - i\theta x 1_{[-1,1]}(x))\mathrm{d}\rho(x)\right).$$

Let $n \in \mathbb{N}$. We have

$$\begin{aligned}
\widehat{\mathbf{e}_*(\mu)}(n) &= \int_{\mathbb{U}} \zeta^n \mathrm{d}(\mathbf{e}_*(\mu))(\zeta) = \int_{\mathbb{R}} e^{inx}\mathrm{d}\mu(x)\\
&= \exp\left(i\eta n - \frac{1}{2}an^2 + \int_{\mathbb{R}}(e^{inx} - 1 - inx1_{[-1,1]}(x))\mathrm{d}\rho(x)\right)\\
&= \exp\left(i\eta n + in\int_{\mathbb{R}}(\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x) - \frac{1}{2}an^2 + \int_{\mathbb{R}}(e^{inx} - 1 - in\sin(x))\mathrm{d}\rho(x)\right)\\
&= \omega^n \exp\Big(-\frac{1}{2}bn^2 + \int_{\mathbb{U}}(\zeta^n - 1 - in\Im(\zeta))\mathrm{d}\upsilon(\zeta)\Big),
\end{aligned}$$

which proves that (ω,b,υ) is a $\circledast$-characteristic triplet of $\mathbf{e}_*(\mu)$. □

We define $\mathbf{e}_\boxplus : \mathcal{ID}(\mathbb{R},\boxplus) \to \mathcal{ID}(\mathbb{U},\boxtimes)$ by analogy with the previous proposition.

Definition 19.2. For all $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$ with $\boxplus$-characteristic triplet (η,a,ρ), we define $\mathbf{e}_\boxplus(\mu)$ to be the $\boxtimes$-infinitely divisible measure on $\mathbb{U}$ with $\boxtimes$-characteristic triplet

$$(\omega,b,\upsilon) = \left(\exp\left(i\eta + i\int_{\mathbb{R}}(\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right). \tag{19.1}$$

Proposition 19.3. *For all μ and $\nu \in \mathcal{ID}(\mathbb{R},\boxplus)$, we have $\mathbf{e}_\boxplus(\mu\boxplus\nu) = \mathbf{e}_\boxplus(\mu)\boxtimes\mathbf{e}_\boxplus(\nu)$.*

Proof. Let us denote by (η_1,a_1,ρ_1) and (η_2,a_2,ρ_2) the respective $\boxplus$-characteristic triplets of μ and ν. The $\boxtimes$-characteristic triplet of $\mu\boxplus\nu$ is $(\eta_1+\eta_2,a_1+a_2,\rho_1+\rho_2)$. As a consequence, denoting by $(\omega_1,b_1,\upsilon_1)$ and $(\omega_2,b_2,\upsilon_2)$ the $\boxtimes$-characteristic triplets of $\mathbf{e}_\boxplus(\mu)$ and $\mathbf{e}_\boxplus(\nu)$ defined by (19.1), $(\omega_1\omega_2, b_1+b_2, \upsilon_1+\upsilon_2)$ is the $\boxtimes$-characteristic triplet of both $\mathbf{e}_\boxplus(\mu\boxplus\nu)$ and $\mathbf{e}_\boxplus(\mu)\boxtimes\mathbf{e}_\boxplus(\nu)$. □

The definition of $\Gamma : \mathcal{ID}(\mathbb{U},\boxtimes) \to \mathcal{ID}(\mathbb{U},\circledast)$ is even simpler.

Definition 19.4. For all $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)\cap\mathcal{M}_*$ with characteristic triplet (ω,b,υ), we define $\Gamma(\mu)$ to be the $\circledast$-infinitely divisible measure on $\mathbb{U}$ with characteristic triplet (ω,b,υ). Moreover, for λ being the Haar measure of $\mathbb{U}$, we set $\Gamma(\lambda) = \lambda$.

Proposition 19.5. *For all μ and $\nu \in \mathcal{ID}(\mathbb{U},\boxtimes)$, we have $\Gamma(\mu\boxtimes\nu) = \Gamma(\mu)\circledast\Gamma(\nu)$.*

PROOF. Let μ and $\nu \in \mathcal{ID}(\mathbb{U},\boxtimes)$. If μ or ν is equal to λ, we have $\mu \boxtimes \nu = \lambda$. In this case, $\Gamma(\mu)$ or $\Gamma(\nu)$ is also equal to λ and consequently, $\lambda = \Gamma(\mu \boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$.

If $\mu, \nu \in \mathcal{ID}(\mathbb{U},\boxtimes) \cap \mathcal{M}_*$ with respective $\boxtimes$-characteristic triplets $(\omega_1, b_1, \upsilon_1)$ and $(\omega_2, b_2, \upsilon_2)$, the measure $\mu \boxtimes \nu$ is a $\boxtimes$-infinitely divisible measure with $\boxtimes$-characteristic triplet $(\omega_1\omega_2, b_1 + b_2, \upsilon_1 + \upsilon_2)$. By consequence, $(\omega_1\omega_2, b_1 + b_2, \upsilon_1 + \upsilon_2)$ is a $\circledast$-characteristic triplet of both $\Gamma(\mu \boxtimes \nu)$ and $\Gamma(\mu) \circledast \Gamma(\nu)$. □

We can now verify the commutativity of the diagram (17.1) in the following proposition.

PROPOSITION 19.6. *We have* $\Gamma \circ \mathbf{e}_{\boxplus} \circ \Lambda = \mathbf{e}_*$.

PROOF. For all $\mu \in \mathcal{ID}(\mathbb{R},*)$ with $*$-characteristic triplet (η, a, ρ),

$$\left(\exp\left(i\eta + i\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right)$$

is a $\circledast$-characteristic triplet of both $\mathbf{e}_*(\mu)$ and $\Gamma \circ \mathbf{e}_{\boxplus} \circ \Lambda(\mu)$. □

We summarize here the successive action of $\Lambda, \mathbf{e}_{\boxplus}, \Gamma$ and $\mathbf{e}_*$ on respectively a Dirac measure δ_η ($\eta \in \mathbb{R}$), a Gaussian measure $\mathcal{N}_b$ ($b > 0$), and a compound Poisson distribution with rate $\lambda > 0$ and jump distribution ρ (Example 18.2). As expected, their images are respectively their free analogues on $\mathbb{R}$ (Example 18.4), their free analogues on $\mathbb{U}$ (Example 18.7), and their multiplicative analogues on $\mathbb{U}$ (Example 18.6):

$$\begin{array}{ccccccc} & \Lambda & & \mathbf{e}_{\boxplus} & & \Gamma & \\ \delta_\eta & \longmapsto & \delta_\eta & \longmapsto & \delta_{e^{i\eta}} & \longmapsto & \delta_{e^{i\eta}} \\ \mathcal{N}_b & \longmapsto & \mathcal{S}_b & \longmapsto & \mathcal{B}_b & \longmapsto & \mathbf{e}_*(\mathcal{N}_b) \\ \mathrm{Poiss}^*_{\lambda,\rho} & \longmapsto & \mathrm{Poiss}^{\boxplus}_{\lambda,\rho} & \longmapsto & \mathrm{Poiss}^{\boxtimes}_{\lambda,\mathbf{e}_*(\rho)} & \longmapsto & \mathrm{Poiss}^{\circledast}_{\lambda,\mathbf{e}_*(\rho)}. \end{array}$$

19.2. A limit theorem. The definition of Γ is justified, if needed, by the following result of Chistyakov and Götze.

THEOREM 19.7 ([28]). *For all $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes) \cap \mathcal{M}_*$, all natural numbers $k_1 < k_2 < \cdots$ and all sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures in $\mathcal{M}_*$ such that the measures $\mu_n^{\boxtimes k_n}$ converge weakly to μ, the measures $\mu_n^{\circledast k_n}$ converge weakly to $\Gamma(\mu)$.*

The rest of this section is devoted to proving an analogous theorem for $\mathbf{e}_{\boxplus}$. This goal is achieved in Theorem 19.10. Let us start by a key result, interesting in its own, about the convergence towards a $\boxtimes$-infinitely divisible measure. The following proposition is the analogue of Theorem 18.1 for the convolution $\boxtimes$. We refer the reader to Theorem 4.3 of [16] and Theorem 2.3 of [28] for other similar criterions. The major difference between these results and ours is the shift of μ_n considered: in Proposition 19.8, we consider the angular part $\omega_n = m_1(\mu_n)/|m_1(\mu_n)|$ of the mean of μ_n.

For all measure μ_n on $\mathbb{U}$, all $\omega_n \in \mathbb{U}$ and all $k_n \in \mathbb{N}$, we denote by $k_n(1 - \Re(\zeta))\mathrm{d}\mu_n(\omega_n\zeta)$ the measure such that, for all bounded Borel functions f on $\mathbb{U}$,

$$\int_{\mathbb{U}} f(\zeta)k_n(1 - \Re(\zeta))\mathrm{d}\mu_n(\omega_n\zeta) = k_n \int_{\mathbb{U}} f(\omega_n^{-1}\zeta)(1 - \Re(\omega_n^{-1}\zeta))\mathrm{d}\mu_n(\zeta).$$

PROPOSITION 19.8. *Let $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)$ with $\boxtimes$-characteristic pair (ω, ν). Let $k_1 < k_2 < \cdots$ be a sequence of natural numbers. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}_*$ and $(\omega_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathbb{U}$ such that, for all $n \in \mathbb{N}$, $\omega_n = m_1(\mu_n)/|m_1(\mu_n)|$. The following assertions are equivalent:*

(1) the measures $\underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{k_n \text{ times}}$ *converge weakly to* μ*;*

(2) the measures

$$\mathrm{d}\nu_n(x) = k_n(1 - \Re(\zeta))\mathrm{d}\mu_n(\omega_n\zeta)$$

converge weakly to ν *and*

$$\lim_{n\to\infty} \omega_n^{k_n} = \omega.$$

In concrete cases, the second item is often easier to verify. For example, it allows us to infer that, for any constant $\lambda > 0$ and any probability measure υ on $\mathbb{U}$, the measure $\mathrm{Poiss}^{\boxtimes}_{\lambda,\upsilon}$ is the weak limit of $((1-\lambda/n)\delta_1 + (\lambda/n)\upsilon)^{\boxtimes n}$ as n tends to ∞.

We would point out the recent work [6] which proves that the convergence of Theorem 19.8 above implies local convergences of the probability densities.

Proof. The weak convergence of finite measures on $\mathbb{U}$ is equivalent to the convergence of the moments, or equivalently, for measures in $\mathcal{M}_*$, to the convergence of the S-transform. Thus, it suffices to prove that the following assertions are equivalent:

(1) $\lim_{n\to\infty} S_{\mu_n^{\boxtimes k_n}} = S_\mu$;

(2) $\lim_{n\to\infty} M_{\nu_n} = M_\nu$ and $\lim_{n\to\infty} \omega_n^{k_n} = \omega$

where the convergence of formal series is the convergence of each coefficient. Let us recall the useful information about the S-transform: it is a $\boxtimes$-homomorphism and $zS(z)/(1+z)$ is the inverse under composition of $M(z) - 1$ (see Section 18.4).

Let $n \in \mathbb{N}$. We set $r_n = |m_1(\mu_n)|$, so that $m_1(\mu_n) = r_n\omega_n$. We define also $\mu_n^\circ \in \mathcal{M}_*$ such that $\mathrm{d}\mu_n^\circ(\zeta) = \mathrm{d}\mu_n(\omega_n\zeta)$. The measure μ_n° will be the link between μ_n and ν_n. Observe that $M_{\mu_n^\circ}(z) = M_{\mu_n}(\omega_n^{-1}z)$, which implies that $S_{\mu_n^\circ}(z) = \omega_n S_{\mu_n}(z)$. The first step of the proof is to write M_{ν_n} with the help of $M_{\mu_n^\circ}$. For all $\zeta \in \mathbb{U}$ and $z \in \mathbb{C}$ sufficiently small, we have

$$2\frac{1-\Re\zeta}{1-\zeta z} = (z-1)\left[(1-z)\frac{\zeta}{1-\zeta z} - 1\right] + 1 - \bar{\zeta}.$$

Integrating with respect to μ_n°, and remarking that $\int_{\mathbb{U}} \bar{\zeta}\mathrm{d}\mu_n^\circ(\zeta) = \overline{\int_{\mathbb{U}} \zeta \mathrm{d}\mu_n(\zeta)/\omega_n} = \bar{r}_n = r_n$, we deduce that

$$\frac{2}{k_n}M_{\nu_n} = (z-1)\left[\frac{1-z}{z}(M_{\mu_n^\circ} - 1) - 1\right] + (1 - r_n). \tag{19.2}$$

Let us suppose that $\lim_{n\to\infty} S_{\mu_n^{\boxtimes k_n}} = S_\mu$. For all $n \in \mathbb{N}$, $S_{\mu_n^{\boxtimes k_n}} = (S_{\mu_n})^{k_n}$. Therefore, we have $\lim_{n\to\infty} m_1(\mu_n)^{-k_n} = \lim_{n\to\infty} S_{\mu_n}^{k_n}(0) = S_\mu(0)$. Thanks to (18.4), we know that $S_\mu(0) = \omega^{-1}e^{\nu(\mathbb{U})}$, which implies that $\lim_{n\to\infty} \omega_n^{k_n} = \omega$ and $\lim_{n\to\infty} r_n^{k_n} = e^{-\nu(\mathbb{U})}$. It remains now to prove $\lim_{n\to\infty} M_{\nu_n} = M_\nu$. At this stage of the proof, we need to inverse formal series, at least asymptotically, and instead of doing it term by term, we prefer to work in a quotient algebra where the negligible terms will be forgotten.

More precisely, let ℓ^∞ be the algebra of bounded complex sequences. We consider the ideal $\mathcal{I} \subset \ell^\infty$ composed of sequences x_n such that $\lim_{n\to\infty} k_n x_n = 0$; in other words, sequences which are $o(1/k_n)$. We find it convenient to work in the quotient algebra $\mathcal{B} = \ell^\infty/\mathcal{I}$. For example, $\lim_{n\to\infty} k_n \log(r_n) = -\nu(\mathbb{U})$ can be rewritten $\log(r_n) \cong -\frac{1}{k_n}\nu(\mathbb{U})$ in $\mathcal{B}$, which implies that $r_n \cong e^{-\frac{1}{k_n}\nu(\mathbb{U})} \cong 1 - \frac{1}{k_n}\nu(\mathbb{U})$. We will view sequences of formal series as elements of $\ell^\infty[[z]]$, and we will naturally identify $\ell^\infty[[z]]/\mathcal{I}[[z]]$ with $\mathcal{B}[[z]]$. For simplicity, equality in $\mathcal{B}$ or $\mathcal{B}[[z]]$ will be denoted by the symbol $\cong$.

Let us denote by $u(z)$ the series

$$u(z)=\int_{\mathbb{U}}\frac{1+z+\zeta z}{1+z-\zeta z}\mathrm{d}\nu(\zeta)-\nu(\mathbb{U}).$$

Thanks to (18.4), we have $m_1(\mu)S_\mu(z)=S_\mu(z)/S_\mu(0)=\exp(u(z))$, from which we deduce that $u(z)$ is equal to $\log(m_1(\mu)S_\mu)$, that is to say the series given by $-\sum_{k=1}^{\infty}\frac{1}{k}\left(1-m_1(\mu)S_\mu(z)\right)^k$. The formal series $k_n\log(m_1(\mu_n)S_{\mu_n})=\log((m_1(\mu_n)S_{\mu_n})^{k_n})$ tends to $\log(m_1(\mu)S_\mu)=u(z)$ as n tends to infinity. Consequently, $\log(m_1(\mu_n)S_{\mu_n})\cong\frac{1}{k_n}u(z)$. Thus, we have

$$\begin{aligned}S_{\mu_n^\circ}(z)=r_n^{-1}\cdot m_1(\mu_n)S_{\mu_n}(z)&\cong e^{\frac{1}{k_n}\nu(\mathbb{U})}\cdot\exp\left(\frac{1}{k_n}u(z)\right)\\&\cong\left(1+\frac{1}{k_n}\nu(\mathbb{U})\right)\left(1+\frac{1}{k_n}u(z)\right)\\&\cong 1+\frac{1}{k_n}\nu(\mathbb{U})+\frac{1}{k_n}u(z).\end{aligned}$$

With this new expression of $S_{\mu_n^\circ}$, it is easy to check that the inverse under composition in $\mathcal{B}[[z]]$ of $zS_{\mu_n^\circ}/(z+1)\cong z(1+\frac{1}{k_n}\nu(\mathbb{U})+\frac{1}{k_n}u(z))/(z+1)$ is exactly

$$\frac{z}{(1-z)}\left[1-\frac{1}{k_n}\frac{1}{1-z}\left(\nu(\mathbb{U})+u\left(\frac{z}{1-z}\right)\right)\right],$$

which is then the expression of $M_{\mu_n^\circ}(z)-1$ in $\mathcal{B}[[z]]$. Replacing $M_{\mu_n^\circ}(z)-1$ by this expression in (19.2) yields

$$\frac{2}{k_n}M_{\nu_n}\cong\frac{1}{k_n}\left(u\left(\frac{z}{1-z}\right)+2\nu(\mathbb{U})\right).$$

But we have $u\left(z/(1-z)\right)+2\nu(\mathbb{U})=\int_{\mathbb{U}}\frac{2}{1-\zeta z}\mathrm{d}\nu(\zeta)=2M_\nu$, and finally, $\frac{1}{k_n}M_{\nu_n}\cong\frac{1}{k_n}M_\nu$, or equivalently, $\lim_{n\to\infty}M_{\nu_n}=M_\nu$.

Conversely, if we suppose that $\lim_{n\to\infty}M_{\nu_n}=M_\nu$ and $\lim_{n\to\infty}\omega_n^{k_n}=\omega$, we can basically retrace our steps in order to arrive at $\lim_{n\to\infty}S_{\mu_n^{\boxtimes k_n}}=S_\mu$. ☐

COROLLARY 19.9. *Let $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$ with $\boxtimes$-characteristic triplet (ω,b,υ). Let $k_1<k_2<\cdots$ be a sequence of natural numbers. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}_*$ and $(\omega_n)_{n\in\mathbb{N}}$ be such that, for all $n\in\mathbb{N}$, $\omega_n=m_1(\mu_n)/|m_1(\mu_n)|$. The following assertions are equivalent:*

(1) the measures $\underbrace{\mu_n\boxtimes\cdots\boxtimes\mu_n}_{k_n\ \text{times}}$ converge weakly to μ;

(2) $\lim_{n\to\infty}\omega_n^{k_n}=\omega$ and the measures $\mathrm{d}\nu_n(x)=k_n(1-\Re(\zeta))\mathrm{d}\mu_n(\omega_n\zeta)$ converge weakly to $(1-\Re(\zeta))\mathrm{d}\upsilon(\zeta)+\frac{b}{2}\delta_1$.

We are now ready to prove the first main theorem of this paper.

THEOREM 19.10. *For all $\mu\in\mathcal{ID}(\mathbb{R},\boxplus)$, all natural numbers $k_1<k_2<\cdots$ and all sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures on $\mathbb{R}$ such that the measures $\mu_n^{\boxplus k_n}$ converge weakly to μ, the measures $\mathbf{e}_*(\mu_n)^{\boxtimes k_n}$ converge weakly to $\mathbf{e}_\boxplus(\mu)$.*

Let us derive right now some consequences of this theorem. It allows us to transfer limit theorems about $\boxplus$ into limit theorem about $\boxtimes$. For example, for all $b>0$, the semi-circular measure is such that $\mathcal{S}_{b/n}^{\boxplus n}=\mathcal{S}_b$. We deduce that $\mathcal{B}_b=\mathbf{e}_\boxplus(\mathcal{S}_b)$, which is the law of a free unitary Brownian motion at time b, is the weak limit of the measures $\mathbf{e}_*(\mathcal{S}_{b/n})^{\boxtimes n}$. Using Theorem 18.3, we know also that the measures $\mathcal{N}_{b/n}^{\boxtimes n}$ converge weakly to $\mathcal{S}_b$. By consequence, $\mathcal{B}_b$ is also the weak limit of $\mathbf{e}_*(\mathcal{N}_{b/n})^{\boxtimes n}$ as n tends to ∞.

Proof. Let (η, a, ρ) be the $\boxplus$-characteristic triplet of μ, and (ω, b, υ) be the $\boxtimes$-characteristic triplet of $\mathbf{e}_{\boxplus}(\mu)$ given by (19.1). In order to use Corollary 19.9, we first prove that $\mathbf{e}_*(\mu_n) \in \mathcal{M}_*$ for n sufficiently large.

Because $e^{ix} - 1 = ix1_{[-1,1]}(x) + (e^{ix} - 1 - ix1_{[-1,1]}(x))$, we have

$$\left(\int_{\mathbb{R}} e^{ix}\mathrm{d}\mu_n(x) - 1\right) = i\int_{\mathbb{R}} x1_{[-1,1]}(x)\mathrm{d}\mu_n(x) + \int_{\mathbb{R}} (e^{ix} - 1 - ix1_{[-1,1]}(x))\mathrm{d}\mu_n(x).$$

We use Lemma 18.5, and the fact that $e^{ix} - 1 - ix1_{[-1,1]}(x) \sim_{x\to 0} -\frac{1}{2}x^2$, to deduce that

$$\lim_{n\to\infty} k_n \left(\int_{\mathbb{R}} e^{ix}\mathrm{d}\mu_n(x) - 1\right) = i\eta - \frac{a}{2} + \int (e^{ix} - 1 - 1_{[-1,1]}(x)ix)\rho(\mathrm{d}x). \tag{19.3}$$

Consequently, $m_1(\mathbf{e}_*(\mu_n)) = \int_{\mathbb{R}} e^{ix}\mathrm{d}\mu_n(x)$ tends to 1 as n tends to ∞, and $\mathbf{e}_*(\mu_n) \in \mathcal{M}_*$ for n sufficiently large. Without loss of generality, we assume that $\mathbf{e}_*(\mu_n)$ is in $\mathcal{M}_*$ for all $n \in \mathbb{N}$. We set $(r_n, \omega_n)_{n\in\mathbb{N}}$ the sequence of $\mathbb{R}_+ \times [0, 1]$ such that, for all $n \in \mathbb{N}$, we have $m_1(\mathbf{e}_*(\mu_n)) = r_n\omega_n$. Thanks to Corollary 19.9, it suffices to prove that $\lim_{n\to\infty} \omega_n^{k_n} = \omega$ and to prove that the measure $k_n(1 - \Re(\zeta))\mathrm{d}(\mathbf{e}_*(\mu_n))(\omega_n\zeta)$ converge weakly to $(1 - \Re(\zeta))\mathrm{d}\upsilon + \frac{b}{2}\delta_1$ to conclude.

From (19.3), we deduce that

$$\lim_{n\to\infty} r_n^{k_n}\omega_n^{k_n} = \lim_{n\to\infty}\left(\int_{\mathbb{R}} e^{ix}\mathrm{d}\mu_n(x)\right)^{k_n} = \exp\left(i\eta - \frac{a}{2} + \int (e^{ix} - 1 - 1_{[-1,1]}(x)ix)\rho(\mathrm{d}x)\right),$$

and this result can be split into

$$\lim_{n\to\infty} r_n^{k_n} = \exp\left(-\frac{a}{2} + \int (\cos(x) - 1)\rho(\mathrm{d}x)\right)$$

and

$$\lim_{n\to\infty} \omega_n^{k_n} = \exp\left(i\eta + i\int (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x)\right) = \omega.$$

Using the real logarithm, we deduce that, as n tends to ∞,

$$r_n^{-1} = 1 + \frac{1}{k_n}\left(\frac{a}{2} - \int (\cos(x) - 1)\rho(\mathrm{d}x)\right) + o\left(\frac{1}{k_n}\right). \tag{19.4}$$

Using $\omega_n = r_n^{-1}\int_{\mathbb{R}} e^{ix}\mathrm{d}\mu_n(x)$, (19.3) and (19.4), it follows that, as n tends to ∞,

$$\omega_n = 1 + \frac{i}{k_n}\left(\eta + \int (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x)\right) + o\left(\frac{1}{k_n}\right) . \tag{19.5}$$

In order to prove that the measures $k_n(1-\Re(\zeta))\mathrm{d}\mathbf{e}_*\mu_n(\omega_n\zeta)$ converge weakly to $(1-\Re(\zeta))\mathrm{d}\upsilon+\frac{b}{2}\delta_1$, we shall use the method of moments and prove that, for all $m \in \mathbb{N}$,

$$\lim_{n\to\infty} k_n \int_{\mathbb{U}} \zeta^m(1 - \Re(\zeta))\mathrm{d}(\mathbf{e}_*(\mu_n))(\omega_n\zeta) = \int_{\mathbb{U}} \zeta^m(1 - \Re(\zeta))\mathrm{d}\upsilon(\zeta) + \frac{b}{2}.$$

Let $n \in \mathbb{N}$. We have

$$\begin{aligned} k_n \int_{\mathbb{U}} \zeta^m(1 - \Re(\zeta))\mathrm{d}(\mathbf{e}_*(\mu_n))(\omega_n\zeta) &= k_n \int_{\mathbb{U}} \omega_n^{-m}\zeta^m(1 - \Re(\omega_n^{-1}\zeta))\mathrm{d}(\mathbf{e}_*(\mu_n))(\zeta) \\ &= k_n\omega_n^{-m}\int_{\mathbb{R}} e^{imx}(1 - \Re(\omega_n^{-1}e^{ix}))\mathrm{d}\mu_n(x) \\ &= k_n\omega_n^{-m}\int_{\mathbb{R}} e^{imx}(1 - \Re(\omega_n)\cos(x) - \Im(\omega_n)\sin(x))\mathrm{d}\mu_n(x). \end{aligned}$$

Let us decompose the integral under study into four terms:

$$\begin{aligned} k_n \int_{\mathbb{U}} \zeta^m (1-\Re(\zeta))\mathrm{d}(\mathbf{e}_*(\mu_n))(\omega_n\zeta) =& k_n\omega_n^{-m}\Re(\omega_n)\int_{\mathbb{R}} e^{imx}(1-\cos(x))\mathrm{d}\mu_n(x) \\ &+ k_n(1-\Re(\omega_n))\omega_n^{-m}\int_{\mathbb{R}} e^{imx}\mathrm{d}\mu_n(x) \\ &+ k_n\Im(\omega_n)\omega_n^{-m}\int_{\mathbb{R}}(x1_{[-1,1]}(x)-e^{imx}\sin(x))\mathrm{d}\mu_n(x) \\ &- k_n\Im(\omega_n)\omega_n^{-m}\int_{\mathbb{R}} x1_{[-1,1]}(x)\mathrm{d}\mu_n(x). \end{aligned}$$

Thanks to Lemma 18.5, and because $\lim_{n\to\infty}\omega_n = 1$, we know the limit of the first term:

$$\begin{aligned} \lim_{n\to\infty} k_n\omega_n^{-m}\Re(\omega_n)\int_{\mathbb{R}} e^{imx}(1-\cos(x))\mathrm{d}\mu_n(x) &= \int_{\mathbb{R}} e^{imx}(1-\cos(x))\mathrm{d}\rho(x) + \frac{b}{2} \\ &= \int_{\mathbb{U}} \zeta^n(1-\Re(\zeta))\mathrm{d}\upsilon(\zeta) + \frac{b}{2}. \end{aligned}$$

The three others terms tend to 0. Indeed, (19.5) implies that $k_n(1-\Re(\omega_n)) = o(1/k_n)$ and $\Im(\omega_n) = O(1/k_n)$ when n tends to ∞. We know that $\omega_n^{-m} = O(1)$ and $\int_{\mathbb{R}} e^{imx}\mathrm{d}\mu_n(x) = O(1)$ when n tends to ∞. Finally, Lemma 18.5 tells us that $\int_{\mathbb{R}}(x1_{[-1,1]}(x) - e^{imx}\sin(x))\mathrm{d}\mu_n(x) = O(1/k_n)$ and $\int_{\mathbb{R}} x1_{[-1,1]}(x)\mathrm{d}\mu_n(x) = O(1/k_n)$ as n tends to ∞. Thus,

$$k_n(1-\Re(\omega_n))\omega_n^{-m}\int_{\mathbb{R}} e^{imx}\mathrm{d}\mu_n(x),$$

$$k_n\Im(\omega_n)\omega_n^{-m}\int_{\mathbb{R}}(x1_{[-1,1]}(x)-e^{imx}\sin(x))\mathrm{d}\mu_n(x)$$

and

$$-k_n\Im(\omega_n)\omega_n^{-m}\int_{\mathbb{R}} x1_{[-1,1]}(x)\mathrm{d}\mu_n(x)$$

are $o(1)$ as n tends to ∞, and the result follows. □

20. Free log-cumulants

We are at the beginning of the second part of the paper, the aim of which is to prove Theorem 17.2 and Theorem 17.3. This goal is achieved in Section 23. While Section 21 and Section 22 investigates the distributions of certain classes of random matrices, the current section is devoted to establish Proposition 20.1 which is the result of free probability needed for the asymptotic theorems proved in the last section of the paper. As a consequence, Section 20 can be read independently of Section 21 and Section 22.

Mastnak and Nica explain in [58] that, in order to treat the multidimensional free multiplicative convolution, it is preferable to work with a logarithmic version of the S-transform. This leads to a sequence of coefficients which in [25] are called the free log-cumulants. In this section, we use the theory of free log-cumulants to establish Proposition 20.1 which links in an explicit formula the moments of a ⊠-infinitely divisible measure to its ⊠-characteristic triplet. We start by stating Proposition 20.1, after which we introduce the free log-cumulants, which will be used only in the proof of Proposition 20.1.

20.1. Moments of a $\boxtimes$-infinitely divisible measure. Proposition 20.1 involves combinatorics on the symmetric group $\mathfrak{S}_n$. We first present the poset structure of $\mathfrak{S}_n$.

Let $n \in \mathbb{N}^*$. Let $\mathfrak{S}_n$ be the group of permutations of $\{1,\dots,n\}$. For all permutation $\sigma \in \mathfrak{S}_n$, we denote by $\ell(\sigma)$ the numbers of cycles of σ and we set $|\sigma| = n - \ell(\sigma)$. The minimal number of transpositions required to write σ is $|\sigma|$ and we have $|\sigma| = 0$ if and only if σ is the identity $1_{\mathfrak{S}_n}$. We define a distance on $\mathfrak{S}_n$ by $d(\sigma_1,\sigma_2) = |\sigma_1^{-1}\sigma_2|$. The set $\mathfrak{S}_n$ can be endowed with a partial order by the relation $\sigma_1 \preceq \sigma_2$ if $d(1_{\mathfrak{S}_n},\sigma_1) + d(\sigma_1,\sigma_2) = d(1_{\mathfrak{S}_n},\sigma_2)$, or similarly if σ_1 is on a geodesic between $1_{\mathfrak{S}_n}$ and σ_2. The minimal element of $\mathfrak{S}_n$ is thus $1_{\mathfrak{S}_n}$.

For all $\sigma \in \mathfrak{S}_n$, we denote by $[1_{\mathfrak{S}_n},\sigma]$ the segment between $1_{\mathfrak{S}_n}$ and the σ, that is, the set $\{\pi \in \mathfrak{S}_n : \pi \preceq \sigma\}$. It is a lattice with respect to the partial order. A $(l+1)$-tuple $\Gamma = (\sigma_0,\dots,\sigma_l)$ of $[1_{\mathfrak{S}_n},\sigma]$ such that

$$\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_l \preceq \sigma$$

is called a *simple chain* if and only if, for all $1 \le i \le l$, $\sigma_{i-1}^{-1}\sigma_i$ is a non-trivial cycle. The length k of a k-cycle c will be denoted by $\sharp c$. We are now ready to state the main result of this section.

PROPOSITION 20.1. *Let $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)$ with $\boxtimes$-characteristic triplet (ω,b,υ). For all $n \in \mathbb{N}^*$ and all $\sigma \in \mathfrak{S}_n$, we have*

$$\prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},\sigma] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu),$$

where

(1) $L\kappa_1(\mu) = \operatorname{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\,\mathrm{d}\upsilon(\zeta)$,
(2) $L\kappa_2(\mu) = -b + \int_{\mathbb{U}}(\zeta-1)^2 \mathrm{d}\upsilon(\zeta)$
(3) and $L\kappa_n(\mu) = \int_{\mathbb{U}}(\zeta-1)^n \mathrm{d}\upsilon(\zeta)$ *for all* $n \ge 3$.

The proof of Proposition 20.1 requires the notion of free log-cumulants and we postpone it until Section 20.4. In the mean time, we review the properties of the free log-cumulants that we shall use.

20.2. The non-crossing partitions. The definition of the free log-cumulants involves combinatorial formulae which are related to non-crossing partitions. We describe here the poset structure of the set of non-crossing partitions $NC(n)$, and we shall see that it is intimately linked to the poset structure of $\mathfrak{S}_n$.

A partition of the set $\{1,\dots,n\}$ is said to have a crossing if there exist $1 \le i < j < k < l \le n$, such that i and k belong to some block of the partition and j and l belong to another block. If a partition has no crossings, it is called non-crossing. The set of all non-crossing partitions of $\{1,\dots,n\}$ is denoted by $NC(n)$. It is a lattice with respect to the relation of fineness defined as follows: for all π_1 and $\pi_2 \in NC(n)$, we declare that $\pi_1 \preceq \pi_2$ if every block of π_1 is contained in a block of π_2. We denote respectively by 0_n and 1_n the minimal element $\{\{1\},\dots,\{n\}\}$ of $NC(n)$, and the maximal element $\{\{1,\dots,n\}\}$ of $NC(n)$.

In [20], Biane describes an isomorphism between the posets $NC(n)$ and $[1_{\mathfrak{S}_n},(1\cdots n)] \subset \mathfrak{S}_n$. It consists simply in defining, from every partition $\pi \in NC(n)$, the permutation σ_π which is the product, over all blocks $\{i_1 < \cdots < i_k\}$ of π, of the k-cycle $(i_1 \cdots i_k)$. In other words, take the cycles of σ_π to be the blocks of π with the cyclic order induced by the natural order of $\{1,\dots,n\}$. Note that $\sigma_{0_n} = 1_{\mathfrak{S}_n}$ and $\sigma_{1_n} = (1\cdots n)$.

LEMMA 20.2. *The map $\pi \mapsto \sigma_\pi$ is a poset isomorphism between $NC(n)$ and $[1_{\mathfrak{S}_n},(1\cdots n)]$.*

Let $\pi \in NC(n)$. It is immediate that the map $\sigma \mapsto \sigma^{-1}\sigma_\pi$ is an order-reversing bijection of $[1_{\mathfrak{S}_n}, \sigma_\pi]$. The corresponding decreasing bijection K_π of $\{\pi' \in NC(n) : \pi' \preceq \pi\}$ is called the Kreweras complementation map with respect to π. If $\pi = 1_n$, we set $K(\sigma) = K_{1_n}(\sigma)$.

Let $n \in \mathbb{N}$. A chain in the lattice $NC(n)$ is a $(l+1)$-tuple of the form $\Gamma = (\pi_0, \ldots, \pi_l)$ with $\pi_0, \ldots, \pi_l \in NC(n)$ such that $\pi_0 \prec \pi_1 \prec \cdots \prec \pi_l$ (notice that we do not impose $\pi_0 = 0_n$ nor $\pi_l = 1_n$, unlike in [58]). The positive integer l appearing is called the length of the chain, and is denoted by $|\Gamma|$. If, for all $1 \leq i \leq l$, $K_{\pi_i}(\pi_{i-1})$ has exactly one block which has more than two elements, we say that Γ is a simple chain in $NC(n)$. This way, we have an one-to-one correspondence between simple chains in $NC(n)$ and simple chains in $[1_{\mathfrak{S}_n}, (1 \cdots n)]$ via the isomorphism of Lemma 20.2.

20.3. Free log-cumulants. Let $\mu \in \mathcal{M}_*$. We denote by $W_\mu(z)$ the inverse under composition of $zM_\mu(z)$, and we denote by $C_\mu(z)$ the formal power series $M_\mu(W_\mu(z))$. The coefficients $(\kappa_k(\mu))_{k\in\mathbb{N}^*}$ of

$$C_\mu(z) = 1 + \sum_{k=1}^{\infty} \kappa_k(\mu) z^k$$

are known as the *free cumulants* of μ. Let $\pi \in NC(n)$. We set

$$\kappa\left[\pi\right](\mu) = \prod_{B \text{ block of } \pi} \kappa_{|B|}(\mu).$$

For all $n \geq 2$, we set

$$L\kappa_n(\mu) = m_1(\mu)^{-n} \sum_{\substack{\Gamma \text{ chain in } NC(n) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}) \\ \pi_0=0_n, \pi_{|\Gamma|}=1_n}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} \prod_{i=1}^{|\Gamma|} \kappa\left[K_{\pi_i}(\pi_{i-1})\right](\mu).$$

We shall call the coefficients $(L\kappa_k(\mu))_{n\leq 2}$ the *free log-cumulants* of μ. We define also the *LS-transform* of μ by

$$LS_\mu(z) = \sum_{n=2}^{\infty} L\kappa_n(\mu) z^n.$$

Let us define also $L\kappa_1(\mu)$, or $L\kappa(\mu)$, the free log-cumulant of order 1 of μ, by $\mathrm{Log}(m_1(\mu))$, where Log is the principal logarithm.

Remark 20.3. From Proposition 4.5 of [58], we see that this definition of LS_μ extends the definition of the LS-transform of μ given by Definition 1.4 of [58] in the case $m_1(\mu) \neq 1$. The definition of the free log-cumulants $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ follows [25], but we observe that $L\kappa_n(\mu)$ would be denoted by $L\kappa_n(A)$ in [25], where A would be a random variable whose law is μ.

As the free cumulants linearise $\boxplus$, the free log-cumulants linearise $\boxtimes$.

Proposition 20.4 (Proposition 9.11, or Corollary 1.5 of [58]). *For all $\mu, \nu \in \mathcal{M}_*$, we have $L\kappa_1(\mu \boxtimes \nu) \equiv L\kappa_1(\mu) + L\kappa_1(\nu) \pmod{2i\pi}$ and, for all $n \geq 2$,*

$$L\kappa_n(\mu \boxtimes \nu) = L\kappa_n(\mu) + L\kappa_n(\nu).$$

For concrete calculations, one would prefer to have an analytical description of the free log-cumulants. We have $S_\mu(0) = 1/m_1(\mu)$ and by consequence, we can define the formal logarithm of $m_1(\mu) \cdot S_\mu$ as the formal series $\log(m_1(\mu) \cdot S_\mu) = -\sum_{n=1}^{\infty} \frac{1}{n}(1 - m_1(\mu)S_\mu(z))^n$.

Proposition 20.5 (Corollary 6.12 of [58]). *Let $\mu \in \mathcal{M}_*$. We have*

$$LS_\mu(z) = -z \log(m_1(\mu) \cdot S_\mu(z)).$$

REMARK 20.6. Technically, Corollary 6.12 of [58] only deals with measures, or more precisely linear functionals on $\mathbb{C}[X]$, such that $m_1(\mu) = 1$. One can adapt the proof presented in [58]. Alternatively, argue as follows. From a measure $\mu \in \mathcal{M}_*$, we can define $\varphi_\mu : \mathbb{C}[X] \to \mathbb{C}$ such that $\varphi_\mu(X^k) = m_1(\mu)^{-k} m_k(\mu)$. Then, we observe that $S_{\varphi_\mu} = m_1(\mu) \cdot S_\mu(z)$ and $LS_{\varphi_\mu} = LS_\mu$. As a consequence, $LS_\mu(z) = LS_{\varphi_\mu} = -z \log(S_{\varphi_\mu}) = -z \log(m_1(\mu) \cdot S_\mu(z))$.

Let $\pi \in NC(n)$ be such that π has exactly one block which has at least two elements. Let $\{j_1, \ldots, j_N\}$ be this block of π, with $j_1 < \ldots < j_N$. Let us denote by $L\kappa\,[\pi]\,(\mu)$ the free log-cumulant $L\kappa_N(\mu)$.

PROPOSITION 20.7 (Corollary 9.9). *Let $\mu \in \mathcal{M}_*$ and $n \in \mathbb{N}^*$. We have*

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } NC(n) \\ \Gamma=(\pi_0,\ldots,\pi_{|\Gamma|}),\pi_0=0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\Big[K_{\pi_i}(\pi_{i-1})\Big]\,(\mu)\,. \tag{20.1}$$

20.4. Proof of Proposition 20.1. Let us formulate a more general formula than (20.1) with the help of the symmetric group.

LEMMA 20.8. *Let $\mu \in \mathcal{M}_*$ and $n \in \mathbb{N}^*$. For all $\sigma \in \mathfrak{S}_n$, we have*

$$\prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},\sigma] \\ \Gamma=(\sigma_0,\ldots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}\,(\mu)\,. \tag{20.2}$$

PROOF. The analogue formula of (20.1) for simple chains in $[1_{\mathfrak{S}_n}, (1\cdots n)]$ is obtained via the isomorphism of Lemma 20.2, remarking that, for a l-cycle $\sigma_1^{-1}\sigma_2$ of $[1_{\mathfrak{S}_n}, (1\cdots n)]$, we have $l = n - \ell(\sigma_1^{-1}\sigma_2) + 1 = d(\sigma_1, \sigma_2) + 1$. By consequence, we have

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},(1\cdots n)] \\ \Gamma=(\sigma_0,\ldots,\sigma_{|\Gamma|}),\sigma_0=1}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}\,(\mu)\,.$$

Applying the Kreweras complementation $\sigma \mapsto \sigma^{-1}(1\cdots n)$ which is an isomorphism and preserves simple chains, we obtain

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},(1\cdots n)] \\ \Gamma=(\sigma_0,\ldots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=(1\cdots n)}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}\,(\mu)\,.$$

We now use the fact that for a cycle c of length $\sharp c$, the segment $[1_{\mathfrak{S}_n}, c] \subset \mathfrak{S}_n$ is isomorphic as a lattice to $[1_{\mathfrak{S}_{\sharp c}}, (1\cdots \sharp c)] \subset \mathfrak{S}_{\sharp c}$, and by consequence, (20.2) is true if σ is a cycle.

For an arbitrary permutation σ, we decompose it into cycles $c_1, \ldots, c_{\ell(\sigma)}$. Constructing a simple chain of length k ending at σ is equivalent to constructing $\ell(\sigma)$ simple chains ending respectively at $c_1, \ldots, c_{\ell(\sigma)}$, whose lengths $l_1, \ldots, l_{\ell(\sigma)}$ add up to k, and shuffling the steps of these paths, that is choosing a sequence $(C_1, \ldots, C_{\ell(\sigma)})$ of subsets of $\{1, \ldots, k\}$ which partition $\{1, \ldots, k\}$ and whose cardinals are $l_1, \ldots, l_{\ell(\sigma)}$ respectively. Using the formula (20.2) for cycles, this remark leads to (20.2) for an arbitrary $\sigma \in \mathfrak{S}_n$. ☐

In order to conclude the proof of Proposition 20.1, it suffices to compute explicitly the free log-cumulants of a ⊠-infinitely divisible measure.

PROPOSITION 20.9. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ with ⊠-characteristic triplet (ω, b, υ). We have*

(1) $L\kappa_1(\mu) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$,
(2) $L\kappa_2(\mu) = -b + \int_{\mathbb{U}} (\zeta - 1)^2 \mathrm{d}\upsilon(\zeta)$
(3) and $L\kappa_n(\mu) = \int_{\mathbb{U}} (\zeta - 1)^n \mathrm{d}\upsilon(\zeta)$ *for all* $n \geq 3$.

Proof. The data of $S_\mu(z)$ is given by (18.5). We first remark that

$$m_1(\mu) = S_\mu(0)^{-1} = \omega e^{-b/2 - \int_{\mathbb{U}} (i\Im(\zeta) + 1 - \zeta)\mathrm{d}\upsilon(\zeta)},$$

from which we deduce that $L\kappa_1(\mu) = \mathrm{Log}(m_1(\mu)) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$. We also have $m_1(\mu) S_\mu(z) = S_\mu(z)/S_\mu(0) = \exp\left(bz + \int_{\mathbb{U}} \frac{1-\zeta}{1+z(1-\zeta)} - (1 - \zeta)\mathrm{d}\upsilon(\zeta)\right)$. Therefore,

$$LS_\mu(z) = -z \log(m_1(\mu) \cdot S_\mu(z)) = -bz^2 + \int_{\mathbb{U}} \frac{z^2(\zeta - 1)^2}{1 - z(\zeta - 1)} \mathrm{d}\upsilon(\zeta).$$

We identify $(L\kappa_n(\mu))_{n\geq 2}$ as the coefficients of $LS_\mu(z) = \sum_{n=2}^{\infty} L\kappa_n(\mu) z^n$. □

21. Convolution semigroups on $U(N)$

In this section, we define and study the convolution semigroups on the unitary group $U(N)$. More precisely, we are interested in computing $\int_{U(N)} g^{\otimes n} \mathrm{d}\mu(g)$ for μ arising from a convolution semigroup. In Proposition 21.2 and Proposition 21.8, we shall express this quantity in two different ways. The technique of proof is in the spirit of [53]. It relies on a detailed comprehension of the generator of a convolution semigroup on $U(N)$ (see [57]), and on the Schur-Weyl duality (see Section 21.4, and [29, 30]).

Let $N \in \mathbb{N}$ and let $M_N(\mathbb{C})$ be the space of matrices of dimension N . If $M \in M_N(\mathbb{C})$, we denote by M^* the adjoint of M. Let us denote by $\mathrm{Tr} : M_N(\mathbb{C}) \to \mathbb{C}$ the usual trace. The identity matrix is denoted by I_N. We consider the unitary group

$$U(N) = \{U \in M_N(\mathbb{C}) : U^*U = I_N\}.$$

The $\circledast$-convolution of two probability measures μ and ν on $U(N)$ is defined to be the unique probability measure $\mu \circledast \nu$ on $U(N)$ such that $\int_{U(N)} f \mathrm{d}(\mu \circledast \nu) = \int_{U(N)} f(gh)\ \mu(\mathrm{d}g)\nu(\mathrm{d}h)$ for all bounded Borel function f on $U(N)$. Let us denote by $\mathcal{ID}(U(N), \circledast)$ the space of infinitely divisible probability measures on $U(N)$ and by $\mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$ the subspace of measures μ in $\mathcal{ID}(U(N), \circledast)$ which are invariant by unitary conjugation, that is, such that for all bounded Borel function f on $U(N)$ and all $g \in U(N)$, we have

$$\int_{U(N)} f \mathrm{d}\mu = \int_{U(N)} f(ghg^*)\mathrm{d}\mu(h).$$

21.1. Generators of semigroups. Let $\mu = (\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$. We define the transition semigroup $(P_t)_{t\in\mathbb{R}^+}$ as follows: for all $t \in \mathbb{R}^+$, all bounded Borel function f on $U(N)$ and all $h \in U(N)$, we set $P_t f(h) = \int_{U(N)} f(hg)\mu_t(\mathrm{d}g)$. The *generator* of μ, is defined to be the linear operator L on $C(U(N))$ such as $Lf = \lim_{t\to 0}(P_t f - f)/t$ whenever this limit exists.

In order to describe the generator of a semigroup, we shall successively introduce in the three next paragraphs the Lie algebra $\mathfrak{u}(N)$ of $U(N)$, a scalar product on $\mathfrak{u}(N)$ and the notion of Lévy measure on $U(N)$.

The unitary group $U(N)$ is a compact real Lie group of dimension N^2, whose Lie algebra $\mathfrak{u}(N)$ is the real vector space of skew-Hermitian matrices: $\mathfrak{u}(N) = \{M \in M_N(\mathbb{C}) : M^* + M = 0\}$. We consider also the special unitary group $SU(N) = \{U \in U(N) : \det U = 1\}$, whose Lie algebra is $\mathfrak{su}(N) = \{M \in \mathfrak{u}(N) : \mathrm{Tr}(U) = 0\}$. We remark that $\mathfrak{u}(N) = \mathfrak{su}(N) \oplus (i\mathbb{R}I_N)$. Any $Y \in \mathfrak{u}(N)$

induces a *left invariant vector field* Y^l on $U(N)$ defined for all $g \in U(N)$ by $Y^l(g) = DL_g(Y)$ where DL_g is the differential map of $h \mapsto gh$.

We consider the following *inner product* on $\mathfrak{u}(N)$:

$$(X,Y) \mapsto \langle X,Y\rangle_{\mathfrak{u}(N)} = \mathrm{Tr}(X^*Y) = -\mathrm{Tr}(XY).$$

It is a real scalar product on $\mathfrak{u}(N)$ which is invariant by unitary conjugation, and its restriction to $\mathfrak{su}(N)$ is also a real scalar product which is invariant by unitary conjugation. Let us fix an orthonormal basis $\{Y_1,\dots,Y_{N^2-1}\}$ of $\mathfrak{su}(N)$ and set $Y_{N^2} = \frac{i}{\sqrt{N}} I_N$. This way, $\{Y_1,\dots,Y_{N^2}\}$ is an orthonormal basis of $\mathfrak{u}(N)$.

It is convenient now to introduce an arbitrary auxiliary set of local coordinates around I_N. Let $\Re,\Im : U(N) \to M_N(\mathbb{C})$ be such that for all $U \in U(N)$, we have $\Re(U) = (U+U^*)/2$ and $\Im(U) = (U-U^*)/2i$. Note that $i\Im$ takes its values in $\mathfrak{u}(N)$. A *Lévy measure* Π on $U(N)$ is a measure on $U(N)$ such that $\Pi(\{I_N\}) = 0$, for all neighborhood V of I_N, we have $\Pi(V^c) < +\infty$ and $\int_{U(N)} \|i\Im(x)\|^2_{\mathfrak{u}(N)}\, \Pi(\mathrm{d}x) < \infty$.

The following theorem gives us a characterization of the generator of such semigroups.

THEOREM 21.1 ([7, 57]). *Let $\mu = (\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$. There exist an element $Y_0 \in \mathfrak{u}(N)$, a symmetric positive semidefinite matrix $(y_{i,j})_{1\le i,j\le N^2}$ and a Lévy measure Π on $U(N)$ such that the generator L of μ is the left-invariant differential operator given, for all $f \in C^2(U(N))$ and all $h \in U(N)$, by*

$$Lf(h) = Y_0^l f(h) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} Y_i^l Y_j^l f(h) + \int_{U(N)} f(hg) - f(h) - (i\Im(g))^l f(h)\, \Pi(\mathrm{d}g). \tag{21.1}$$

Conversely, given such a triplet $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$, it exists a unique weakly continuous convolution semigroup on $U(N)$ starting at δ_e whose generator is given by (21.1)*.*

The triplet $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ is called the *characteristic triplet* of $(\mu_t)_{t\in\mathbb{R}^+}$, or of L. Let $\mu \in \mathcal{ID}(U(N), \circledast)$ be such that it exists a weakly continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}^+}$ with $\mu_1 = \mu$ and $\mu_0 = \delta_e$. In this case, we say that the characteristic triplet of $(\mu_t)_{t\in\mathbb{R}^+}$ is *a characteristic triplet* of μ. It is not unique but it completely characterizes the measure μ. Conversely, every triplet of this form is a characteristic triplet of a unique measure in $\mathcal{ID}(U(N), \circledast)$.

21.2. Expected values of polynomials of the entries. Let $n \in \mathbb{N}^*$. In this section, we give a formula for $\int_{U(N)} g^{\otimes n} \mathrm{d}\mu(g)$ when μ arises from a convolution semigroup. Consider the representation $\rho^n_{U(N)}$ of the Lie group $U(N)$ on the vector space $(\mathbb{C}^N)^{\otimes n}$ given by

$$\rho^n_{U(N)}(g) = \underbrace{g \otimes \cdots \otimes g}_{n \text{ times}} \in \mathrm{End}((\mathbb{C}^N)^{\otimes n}).$$

Set $d\rho^n_{U(N)}(L) = L(\rho^n_{U(N)})(I_N) \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, where $\rho^n_{U(N)}$ is seen as an element of $C^2(U(N)) \otimes \mathrm{End}((\mathbb{C}^N)^{\otimes n})$.

PROPOSITION 21.2. *Let $(\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$ with generator L and characteristic triplet $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$. For all $t \in \mathbb{R}_+$, we have the equality in* $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$

$$\int_{U(N)} g^{\otimes n} \mathrm{d}\mu_t(g) = \exp(t\ d\rho^n_{U(N)}(L))$$

with

$$\begin{aligned} d\rho^n_{U(N)}(L) &= \sum_{1\leq k\leq n} \mathrm{Id}_N^{\otimes k-1}\otimes Y_0\otimes \mathrm{Id}_N^{\otimes n-k} \\ &+\frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\cdot \sum_{1\leq k,l\leq n}\left(\mathrm{Id}_N^{\otimes k-1}\otimes Y_i\otimes \mathrm{Id}_N^{\otimes n-k}\right)\circ\left(\mathrm{Id}_N^{\otimes l-1}\otimes Y_j\otimes \mathrm{Id}_N^{\otimes n-l}\right) \\ &+\int_{U(N)}\left(g^{\otimes n}-\mathrm{Id}_N^{\otimes n}-\sum_{1\leq k\leq n}\mathrm{Id}_N^{\otimes k-1}\otimes i\Im(g)\otimes \mathrm{Id}_N^{\otimes n-k}\right)\Pi(\mathrm{d}g). \end{aligned}$$

Proof. Let denote by $U: U(N)\to M_N(\mathbb{C})$ the identity function of $U(N)$. We compute

$$\begin{aligned} L\left(\rho^n_{U(N)}\right) &= Y_0^l(U^{\otimes n})+\frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l(U^{\otimes n}) \\ &+\int_{U(N)}(Ug)^{\otimes n}-U^{\otimes n}-(i\Im(g))^l\,(U^{\otimes n})\,\Pi(\mathrm{d}g) \end{aligned}$$

and using that, for all $Y\in\mathfrak{u}(N)$, we have $Y^l(U^{\otimes n})=U^{\otimes n}\cdot\sum_{1\leq k\leq n}\mathrm{Id}_N^{\otimes k-1}\otimes Y\otimes \mathrm{Id}_N^{\otimes n-k}$,

$$\begin{aligned} L\left(\rho^n_{U(N)}\right) &= U^{\otimes n}\cdot\sum_{1\leq k\leq n} \mathrm{Id}_N^{\otimes k-1}\otimes Y_0\otimes \mathrm{Id}_N^{\otimes n-k} \\ &+\frac{1}{2}U^{\otimes n}\cdot\sum_{i,j=1}^{N^2} y_{i,j} \sum_{1\leq k,l\leq n}\left(\mathrm{Id}_N^{\otimes k-1}\otimes Y_i\otimes \mathrm{Id}_N^{\otimes n-k}\right)\cdot\left(\mathrm{Id}_N^{\otimes l-1}\otimes Y_j\otimes \mathrm{Id}_N^{\otimes n-l}\right) \\ &+U^{\otimes n}\cdot\int_{U(N)}\left(g^{\otimes n}-\mathrm{Id}_N^{\otimes n}-\sum_{1\leq k\leq n}\mathrm{Id}_N^{\otimes k-1}\otimes i\Im(g)\otimes \mathrm{Id}_N^{\otimes n-k}\right)\Pi(\mathrm{d}g). \end{aligned}$$

Hence, $d\rho^n_{U(N)}(L)=L(\rho^n_{U(N)})(e)$ leads to the expression of $d\rho^n_{U(N)}(L)$ given above. We conclude by remarking that $t\to\int_{U(N)}g^{\otimes n}\mathrm{d}\mu_t(g)=\int_{U(N)}\rho^n_{U(N)}(g)\mathrm{d}\mu_t(g)$ and $t\to\exp(t\ d\rho^n_{U(N)}(L))$ are both the unique solution to the differential equation

$$\begin{cases} y(0) &= I_N^{\otimes n}, \\ y' &= y\cdot d\rho^n_{U(N)}(L). \end{cases}$$

□

We now give an alternative expression of $d\rho^n_{U(N)}(L)$. Let $m\geq 0$. For all $1\leq k_1<\ldots<k_m\leq n$, let us denote by $\iota^{M_N(\mathbb{C})^{\otimes n}}_{k_1,\ldots,k_m}: M_N(\mathbb{C})^{\otimes m}\to M_N(\mathbb{C})^{\otimes n}$ (or more simply $\iota_{k_1,\ldots,k_m}$) the mapping defined by

$$\iota_{k_1,\ldots,k_m}(X_1\otimes\cdots\otimes X_m)=I_N^{\otimes k_1-1}\otimes X_1\otimes I_N^{\otimes k_2-k_1-1}\otimes X_2\otimes\cdots\otimes X_m\otimes I_N^{\otimes n-k_m},$$

that is to say in words that $\iota_{k_1,\ldots,k_m}(X_1\otimes\cdots\otimes X_m)$ is the tensor product of $X_1,\ldots,X_m$ at the places $k_1,\ldots,k_m$ and I_N at the other places.

PROPOSITION 21.3. *Let L be a generator with characteristic triplet $(Y_0,(y_{i,j})_{1\leq i,j\leq N^2},\Pi)$. We have*

$$\begin{aligned} d\rho^n_{U(N)}(L) &= \sum_{1\leq k\leq n}\iota_k\left(Y_0+\int_{U(N)}(\Re(g)-I_N)\Pi(\mathrm{d}g)\right)\\ &\quad+\frac{1}{2}\sum_{i,j=1}^{N^2}y_{i,j}\cdot\sum_{1\leq k,l\leq n}\iota_k(Y_i)\circ\iota_l(Y_j)\\ &\quad+\sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}}\iota_{k_1,\ldots,k_m}\left(\int_{U(N)}(g-I_N)^{\otimes m}\Pi(\mathrm{d}g)\right).\end{aligned}$$

PROOF. Our starting point is the expression of $d\rho^n_{U(N)}(L)$ given by Proposition 21.8. Let us remark that

$$\begin{aligned} g^{\otimes n} &= (g-I_N+I_N)^{\otimes n}\\ &= I_N^{\otimes n}+\sum_{\substack{1\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}}\iota_{k_1,\ldots,k_m}((g-I_N)^{\otimes m}),\end{aligned}$$

from which we deduce that

$$\begin{aligned} &\int_{U(N)}\left(g^{\otimes n}-\mathrm{Id}_N^{\otimes n}-\sum_{1\leq k\leq n}\mathrm{Id}_N^{\otimes k-1}\otimes i\Im(g)\otimes\mathrm{Id}_N^{\otimes n-k}\right)\Pi(\mathrm{d}g)\\ =\ &\int_{U(N)}\left(\sum_{1\leq k\leq n}\iota_k(g-I_N-i\Im(g))+\sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}}\iota_{k_1,\ldots,k_m}((g-I_N)^{\otimes m})\right)\Pi(\mathrm{d}g)\\ =\ &\sum_{1\leq k\leq n}\iota_k\left(\int_{U(N)}(\Re(g)-I_N)\Pi(\mathrm{d}g)\right)+\sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}}\iota_{k_1,\ldots,k_m}\left(\int_{U(N)}(g-I_N)^{\otimes m}\Pi(\mathrm{d}g)\right)\end{aligned}$$

because all the integrand are equivalent to $\|i\Im(g)\|^2_{\mathfrak{u}(N)}$ in a neighborhood of I_N and hence integrable with respect to Π. Replacing the last term by this new expression in Proposition 21.2 yields to the result. □

21.3. Conjugate invariant semigroups on $U(N)$. A weakly continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}^+}$ on $U(N)$ starting at $\mu_0=\delta_e$ is said *conjugate invariant* if all μ_t belong to $\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$.

PROPOSITION 21.4. *Let $(\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup starting at $\mu_0=\delta_e$ which is conjugate invariant. Let $(Y_0,(y_{i,j})_{1\leq i,j\leq N},\Pi)$ be its characteristic triplet. The differential operator $\frac{1}{2}\sum_{i,j=1}^{N^2}y_{i,j}Y_i^lY_j^l$ and the measure Π are both conjugate invariant. Moreover, there exists three constants y_0,α and $\beta\in\mathbb{R}$ such that $Y_0=iy_0I_N$ and*

$$(y_{i,j})_{1\leq i,j\leq N^2}=\begin{pmatrix}\alpha & & & 0\\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta\end{pmatrix}.$$

PROOF. Thanks to [57], if we denote by $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ the characteristic triplet of μ, the differential operator $\frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} Y_i^l Y_j^l$ and the measure Π are both conjugate invariant. The map $i\Im$ is equivariant by a unitary conjugation and following the proof of Proposition 4.2.2 of [55], we deduce that Y_0 is in the center of $\mathfrak{u}(N)$: there exists $y_0 \in \mathbb{R}$ such that $Y_0 = iy_0 \mathrm{Id}_N$.

Because $\{Y_1, \ldots, Y_{N^2-1}\}$ is a basis of the conjugate invariant Lie subalgebra $\mathfrak{su}(N)$, $\{y_{i,N}, y_{N,i} : 1 \leq i \leq N^2\} = \{0\}$, and because $\mathfrak{su}(N)$ is simple, there exists $\alpha \in \mathbb{R}$ such that $(y_{i,j})_{1\leq i,j\leq (N-1)^2} = \alpha I_{N-1}$. We set $\beta = y_{N,N}$. □

Thus, the invariance by conjugation of μ implies that its generator L is a bi-invariant pseudo-differential operator. In this particular case, the expression of $d\rho^n_{U(N)}(L)$ can be described with the help of the symmetric group. It is the objet of the next section to use the Schur-Weyl duality in order to formulate a new expression of $d\rho^n_{U(N)}(L)$.

21.4. Schur-Weyl duality. The Schur-Weyl duality is a deep relation between the actions of $U(N)$ and $\mathfrak{S}_n$ on $(\mathbb{C}^N)^{\otimes n}$ which allows one to transfer some elements relative to $U(N)$ to elements relative to $\mathfrak{S}_n$ (see [29, 30], and also [53] and [32]). Let us spell out this fruitful duality.

Let $n \in \mathbb{N}$. Define the action $\rho_N^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$ on $(\mathbb{C}^N)^{\otimes n}$ as follows: for all $\sigma \in \mathfrak{S}_n$ and $x_1, \ldots, x_n \in \mathbb{C}^N$, we set

$$(\rho_N^{\mathfrak{S}_n}(\sigma))(x_1 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}.$$

Let us denote by $\mathbb{C}[\mathfrak{S}_n]$ the group algebra of $\mathfrak{S}_n$. The action $\rho_N^{\mathfrak{S}_n}$ determines a homomorphism of associative algebra $d\rho_N^{\mathfrak{S}_n} : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}((\mathbb{C}^N)^{\otimes n})$. The Schur-Weyl duality asserts that the subalgebras of $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$ generated by the action of $U(N)$ and $\mathfrak{S}_n$ are each other's commutant. In particular, any element of $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$ which commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$ is an element of the algebra generated by $\rho_N^{\mathfrak{S}_n}(\mathfrak{S}_n)$, that is to say an element of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$.

For all $\mathbf{A} \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, we define

$$E(\mathbf{A}) = \int_{U(N)} g^{\otimes n} \circ \mathbf{A} \circ (g^*)^{\otimes n} \mathrm{d}g \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$$

where the integration is taken with respect to the Haar measure of $U(N)$. Obviously, $E(\mathbf{A})$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$, and due to the Schur-Weyl duality, $E(\mathbf{A})$ has to lie in $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. In Proposition 2.4 of [30], Collins and Śniady answered the question of determining an element of $\mathbb{C}[\mathfrak{S}_n]$ which is mapped on $E(\mathbf{A})$, as follows. Set

$$\Phi(\mathbf{A}) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{Tr}\left(\mathbf{A} \circ \rho_N^{\mathfrak{S}_n}(\sigma^{-1})\right) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_n]$$

and define $\mathrm{Wg} = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{Wg}(\sigma) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_n]$ such that $d\rho_N^{\mathfrak{S}_n}(\Phi(\mathrm{Id}_N^{\otimes n}) \cdot \mathrm{Wg}) = \mathrm{Id}_N^{\otimes n}$. If $n \leq N$, the element $\Phi(\mathrm{Id}_N^{\otimes n})$ is invertible and Wg must be $\Phi(\mathrm{Id}_N^{\otimes n})^{-1}$. If $N < n$, one can choose any pseudo-inverse of the symmetric element $\Phi(\mathrm{Id}_N^{\otimes n})$ to be Wg. Let us insist on the fact that Wg depends on both n and N, even if for convenience, this dependence is not explicit in the notation.

PROPOSITION 21.5 ([30]). *For all* $\mathbf{A} \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, *we have* $E(\mathbf{A}) = d\rho_N^{\mathfrak{S}_n}(\Phi(\mathbf{A})\mathrm{Wg})$.

Very succinctly, the argument is as follows:

$$\rho_N^{\mathfrak{S}_n}(\Phi(\mathbf{A})) = \rho_N^{\mathfrak{S}_n}(\Phi(E(\mathbf{A}))) = \rho_N^{\mathfrak{S}_n}(\Phi(E(\mathbf{A}) \cdot \mathrm{Id}_N^{\otimes n})) = E(\mathbf{A}) \cdot \rho_N^{\mathfrak{S}_n}(\Phi(\mathrm{Id}_N^{\otimes n})).$$

It allows us to write explicitly elements of the commutant of the algebra generated by $\rho^n_{U(N)}$ as elements of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. Indeed, if $\mathbf{A}$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$, we have

$$\mathbf{A} = E(\mathbf{A}) = d\rho_N^{\mathfrak{S}_n}(\Phi(\mathbf{A})\mathrm{Wg}).$$

Moreover, they give an asymptotic of the Weingarten function.

PROPOSITION 21.6 ([30]). *For all $\sigma \in \mathfrak{S}_n$, we have* $\mathrm{Wg}(\sigma) = O(N^{-n-|\sigma|})$ *when N tends to ∞. We have also* $\mathrm{Wg}(1_{\mathfrak{S}_n}) = N^{-n} + O(N^{-n-2})$ *when N tends to ∞.*

EXAMPLE 21.7. (1) for $n = 1$: we have $\mathrm{Wg} = 1_{\mathfrak{S}_n}/N$, and therefore, for all $A \in \mathrm{End}(\mathbb{C}^N)$,

$$E(A) = \frac{1}{N}\operatorname{Tr}(A) I_N;$$

(2) for $n = 2$: we have $\mathrm{Wg} = \frac{1}{N^2-1}\left(1_{\mathfrak{S}_n} - \frac{1}{N}(1,2)\right)$, and thus, for all $A, B \in \mathrm{End}(\mathbb{C}^N)$,

$$E(A\otimes B) = \frac{1}{N^2-1}\Big((\operatorname{Tr}(A)\operatorname{Tr}(B) - \operatorname{Tr}(AB)/N)\cdot I_N^{\otimes 2} + (\operatorname{Tr}(AB) - \operatorname{Tr}(A)\operatorname{Tr}(B)/N)\cdot d\rho_N^{\mathfrak{S}_n}((1,2))\Big).$$

The generator L of a conjugate invariant convolution semigroup is a bi-invariant pseudo-differential operator, and by consequence the element $d\rho^n_{U(N)}(L)$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$. Thus, it is an element of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. Let $\mathcal{T}_n$ be the subset of $\mathfrak{S}_n$ consisting of all the transpositions. For all $1 \leq k_1 < \ldots < k_m \leq n$, let us denote by $\iota^{\mathfrak{S}_n}_{k_1,\ldots,k_m} : \mathfrak{S}_m \to \mathfrak{S}_n$ (or more simply $\iota_{k_1,\ldots,k_m}$) the mapping defined by

$$\iota_{k_1,\ldots,k_m}(\sigma) : \left|\begin{array}{ll} k_i \mapsto & k_{\sigma(i)} \\ i \mapsto & i \qquad\quad \text{for } i \notin \{k_1,\ldots,k_m\}. \end{array}\right.$$

This map is such that $\rho_N^{\mathfrak{S}_n} \circ \iota^{\mathfrak{S}_n}_{k_1,\ldots,k_m} = \iota^{M_N(\mathbb{C})^{\otimes n}}_{k_1,\ldots,k_m} \circ \rho_N^{\mathfrak{S}_m}$. We are now ready to state the main result of this section.

PROPOSITION 21.8. *Let $y_0, \alpha, \beta \in \mathbb{R}$ and Π be a Lévy measure on $U(N)$ which is conjugate invariant. Let $\mu \in \mathcal{ID}(U(N), \circledast)$ with characteristic triplet*

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix}, \Pi\right).$$

We have $\int_{U(N)} g^{\otimes n} \mathrm{d}\mu(g) = d\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}})$, *where*

$$\tilde{L} = \left(niy_0 - \frac{n^2}{N}\frac{\beta}{2} + \left(\frac{n^2}{N} - nN\right)\frac{\alpha}{2} + \frac{n}{N}\int_{U(N)} \operatorname{Tr}\left(\Re(g) - 1\right)\Pi(\mathrm{d}g)\right)1_{\mathfrak{S}_n} - \alpha\sum_{\tau\in\mathcal{T}_n}\tau$$
$$+ \sum_{\substack{2\leq m\leq n \\ 1\leq k_1<\ldots<k_m\leq n}} \sum_{\sigma,\pi\in\mathfrak{S}_m} \mathrm{Wg}(\sigma^{-1}\pi)\cdot\int_{U(N)} \prod_{c \text{ cycle of } \sigma} \operatorname{Tr}\left((g-1)^{\sharp c}\right)\,\Pi(\mathrm{d}g)\cdot\iota_{k_1,\ldots,k_m}(\pi).$$

PROOF. Let $(\mu_t)_{t\in\mathbb{R}^+}$ be the weakly continuous convolution semigroup whose characteristic triplet is

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix}, \Pi\right),$$

and let L be its generator. By definition, $\mu = \mu_1$, and thanks to Proposition 21.2, we know that

$$\int_{U(N)} g^{\otimes n}\mathrm{d}\mu(g) = \exp(d\rho^n_{U(N)}(L)).$$

To conclude, it suffices to prove that $d\rho^n_{U(N)}(L) = d\rho_N^{\mathfrak{S}_n}(\tilde{L})$. We start from Proposition 21.3. We have

$$\begin{aligned} d\rho^n_{U(N)}(L) &= \sum_{1\leq k\leq n} \iota_k \left(iny_0 + \int_{U(N)} (\Re(g) - I_N)\Pi(\mathrm{d}g) \right) \\ &+ \frac{\alpha}{2} \sum_{i=1}^{N^2-1} \cdot \sum_{1\leq k,l\leq n} \iota_k(Y_i)\circ\iota_l(Y_i) + \frac{\beta}{2}\sum_{1\leq k,l\leq n} \iota_k(Y_{N^2})\circ\iota_l(Y_{N^2}) \\ &+ \sum_{\substack{2\leq m\leq n \\ 1\leq k_1<\ldots<k_m\leq n}} \iota_{k_1,\ldots,k_m}\left(\int_{U(N)} (g-I_N)^{\otimes m}\Pi(\mathrm{d}g)\right). \end{aligned}$$

Thanks to the invariance under conjugation of Π and $\sum_{i=1}^{N^2-1} Y_i\otimes Y_i$, we know from Example 21.7 that

$$\int_{U(N)} (\Re(g) - I_N)\Pi(\mathrm{d}g) = \int_{U(N)} E(\Re(g) - I_N)\Pi(\mathrm{d}g) = \int_{U(N)} \frac{1}{N}\operatorname{Tr}(\Re(g)-1)\,\Pi(\mathrm{d}g)$$

and

$$\sum_{i=1}^{N^2-1} Y_i\otimes Y_i = E\left(\sum_{i=1}^{N^2-1} Y_i\otimes Y_i\right) = \frac{1}{N} I_N^{\otimes 2} - \rho_N^{\mathfrak{S}_n}((1,2)).$$

We also deduce from Proposition 21.5 that

$$\begin{aligned} \int_{U(N)} (g-I_N)^{\otimes m}\Pi(\mathrm{d}g) &= \int_{U(N)} E\left((g-I_N)^{\otimes m}\right)\Pi(\mathrm{d}g) \\ &= \int_{U(N)} \sum_{\sigma,\pi\in\mathfrak{S}_m} \mathrm{Wg}(\sigma^{-1}\pi) \prod_{c \text{ cycle of } \sigma} \operatorname{Tr}\left((g-1)^{\sharp c}\right)\cdot d\rho_N^{\mathfrak{S}_m}(\pi)\Pi(\mathrm{d}g). \end{aligned}$$

Thus we have

$$\begin{aligned} \mathrm{d}\rho^n_{U(N)}(L) &= \left(niy_0 - \frac{n^2}{N}\frac{\beta}{2} + \left(\frac{n^2}{N} - nN\right)\frac{\alpha}{2} + \frac{n}{N}\int_{U(N)} \operatorname{Tr}(\Re(g)-1)\,\Pi(\mathrm{d}g)\right) I_N^{\otimes n} \\ &-\alpha \sum_{i=1}^{N^2-1} \cdot \sum_{1\leq k<l\leq n} \iota_{k,l}\circ\rho_N^{\mathfrak{S}_2}((1,2)) \\ &+ \sum_{\substack{2\leq m\leq n \\ 1\leq k_1<\ldots<k_m\leq n}} \sum_{\sigma,\pi\in\mathfrak{S}_m} \mathrm{Wg}(\sigma^{-1}\pi)\cdot \int_{U(N)} \prod_{c \text{ cycle of } \sigma} \operatorname{Tr}\left((g-1)^{\sharp c}\right)\ \Pi(\mathrm{d}g) \\ &\qquad \cdot\iota_{k_1,\ldots,k_m}\circ\rho_N^{\mathfrak{S}_m}(\pi), \end{aligned}$$

from which we deduce that $d\rho^n_{U(N)}(L) = d\rho_N^{\mathfrak{S}_n}(\tilde{L})$. □

22. The stochastic exponential $\mathcal{E}_N$

In this section, we shall describe $\mathcal{E}_N$, a map which connects the infinitely divisible measures on the space of Hermitian matrices $\mathcal{H}_N$ and the infinitely divisible measures on $U(N)$. We start by presenting $\mathcal{E}_N$ in Proposition-Definition 22.2, and the rest of the section is devoted to the proof of Proposition-Definition 22.2.

We consider the Hilbert space of Hermitian matrices

$$\mathcal{H}_N = \{x\in M_N(\mathbb{C}) : x^* = x\}.$$

We denote by $*$ the classical convolution on the vector space $\mathcal{H}_N$: given two probability measures μ and ν on $\mathcal{H}_N$, the convolution $\mu*\nu$ is such that $\int_{\mathcal{H}_N} f\mathrm{d}(\mu*\nu) = \int_{\mathcal{H}_N}\int_{\mathcal{H}_N} f(x+y)\,\mu(\mathrm{d}x)\nu(\mathrm{d}y)$ for all bounded Borel function f on $\mathcal{H}_N$. Let us denote by $\mathcal{ID}(\mathcal{H}_N,*)$ the space of infinitely divisible probability measures on $\mathcal{H}_N$ and by $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ the subspace of measures μ in $\mathcal{ID}(\mathcal{H}_N,*)$ which are invariant by unitary conjugation, that is, such that for all bounded Borel function f on $\mathcal{H}_N$ and all $g\in U(N)$, we have

$$\int_{\mathcal{H}_N} f\mathrm{d}\mu = \int_{\mathcal{H}_N} f(gxg^*)\mathrm{d}\mu(x).$$

22.1. Infinite divisibility on $\mathcal{H}_N$. The advantage of $\mathcal{ID}(\mathcal{H}_N,*)$ is that each infinitely divisible measure arises from a unique convolution semigroup, and by consequence, is characterized by a unique generator. In order to describe this generator, we introduce now an inner product on $\mathcal{H}_N$ and we define the notion of Lévy measure.

We endow $\mathcal{H}_N$ with the following inner product:

$$(x,y)\mapsto \langle x,y\rangle_{\mathcal{H}_N} = \mathrm{Tr}(x^*y) = \mathrm{Tr}(xy).$$

It is a real scalar product on $\mathcal{H}_N$ which is invariant by unitary conjugation. We remark that $i\mathcal{H}_N = \mathfrak{u}(N)$. Thus, the family $\{X_1,\dots,X_{N^2}\} = \{-iY_1,\dots,-iY_{N^2}\}$ is an orthonormal basis of $\mathcal{H}_N$ such that $X_{N^2} = \frac{1}{\sqrt{N}}I_N$. It is now useful to fix one compact neighborhood B of 0: we choose to set $B = B(0,1)$, the closed unit ball of $\mathcal{H}_N$. A *Lévy measure* Π on $\mathcal{H}_N$ is a measure on $\mathcal{H}_N$ such that both $\Pi(\{0\}) = 0$ and such that $\int_B \|x\|^2_{\mathcal{H}_N}\,\Pi(\mathrm{d}x)$ and $\Pi(B^c)$ are finite.

Let $C_b^2(\mathcal{H}_N)$ be the space of function $f\in C^2(\mathcal{H}_N)$ with bounded first and second-order partial derivatives.

THEOREM 22.1 ([57, 64]). *Let $\mu\in\mathcal{ID}(\mathcal{H}_N,*)$. There exists a unique weakly continuous semigroup $(\mu^{*t})_{t\in\mathbb{R}^+}$ such that $\mu^{*0}=\delta_0$ and $\mu^{*1}=\mu$. There exist an element $X_0\in\mathcal{H}_N$, a symmetric positive semidefinite matrix $(y_{i,j})_{1\le i,j\le N^2}$ and a Lévy measure Π on $\mathcal{H}_N$ such that the generator L of $(\mu^{*t})_{t\in\mathbb{R}^+}$ is given for all $f\in C_b^2(\mathcal{H}_N)$ and all $y\in\mathcal{H}_N$ by*

$$Lf(y) = \partial_{X_0}f(y) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\partial_{X_i}\partial_{X_j}f(y) + \int_{\mathcal{H}_N} f(y+x) - f(y) - 1_B(x)\partial_x f(y)\;\Pi(\mathrm{d}x). \tag{22.1}$$

The triplet $(X_0,(y_{i,j})_{1\le i,j\le N^2},\Pi)$ is called the *characteristic triplet* of μ, and its associated generator L is called the generator of μ. Conversely, given such a triplet $(X_0,(y_{i,j})_{1\le i,j\le N^2},\Pi)$, there exists a unique infinitely divisible measure μ whose generator is given by (22.1).

Let us remark that the functions $\mathbf{e}$ and sin make sense on $\mathcal{H}_N$. For all $x\in\mathcal{H}_N$, we have

$$\mathbf{e}(x) = \exp(ix)\in U(N) \text{ and } \sin(x) = \Im\circ\mathbf{e} = (e^{ix}-e^{-ix})/2i\in\mathcal{H}_N.$$

As previously, for all measure Π on $\mathcal{H}_N$, the measure $\mathbf{e}_*(\Pi)$ denotes the push-forward Π on $\mathcal{H}_N$ by the mapping $\mathbf{e}:\mathcal{H}_N\to U(N)$, and the measure $\mathbf{e}_*(\Pi)_{|U(N)\setminus\{I_N\}}$ is the measure on $U(N)\setminus\{I_N\}$ induced by $\mathbf{e}_*(\Pi)$. We are now able to formulate the main result of this section.

PROPOSITION-DEFINITION 22.2. *For all $\mu\in\mathcal{ID}(\mathcal{H}_N,*)$ with characteristic triplet*

$$(X_0,(y_{i,j})_{1\le i,j\le N^2},\Pi),$$

we define $\mathcal{E}_N(\mu)$ to be the measure of $\mathcal{ID}(U(N),\circledast)$ with characteristic triplet

$$\left(iX_0 + i\int_{\mathcal{H}_N}(\sin(x) - 1_B(x)x)\;\Pi(\mathrm{d}x), (y_{i,j})_{1\le i,j\le N^2}, \mathbf{e}_*(\Pi)_{|U(N)\setminus\{I_N\}}\right).$$

The map $\mathcal{E}_N:\mathcal{ID}(\mathcal{H}_N,*)\to\mathcal{ID}(U(N),\circledast)$ is called the stochastic exponential and has the following properties :

(1) *For all* $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$, *the measures* $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ *converge weakly to* $\mathcal{E}_N(\mu)$;
(2) *the stochastic exponential maps* $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$ *to* $\mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$, *and for all* μ, ν *measures of* $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$, *we have*

$$\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu).$$

The tool used to prove this proposition is the Fourier transform of a measure on $U(N)$. Before proving Proposition 22.2 in Section 22.3, let us introduce this notion.

22.2. Fourier transform on $U(N)$. The set $\widehat{U(N)}$ of isomorphism classes of irreducible representations of $U(N)$ is in bijection with the set $\mathbb{Z}^N_\downarrow$ of non-increasing sequences of integers $\alpha = (\alpha_1 \geq \ldots \geq \alpha_N)$. For all $\alpha \in \mathbb{Z}^N_\downarrow$, let $\pi^\alpha \in \widehat{U(N)}$ be a unitary representation in the corresponding class, acting on a vector space E_α, and let χ_α be its character, that is to say the function $\mathrm{Tr} \circ \pi^\alpha$. We will also consider the normalized character $\psi_\alpha(\cdot) = \chi_\alpha(\cdot)/\chi_\alpha(I_N)$.

Let μ be a probability measure on $U(N)$. The Fourier transform $\widehat{\mu}$ of μ is defined for all $\alpha \in \mathbb{Z}^N_\downarrow$ by $\widehat{\mu}(\alpha) = \int_{U(N)} \pi^\alpha(g)\ \mu(\mathrm{d}g) \in \mathrm{End}(E_\alpha)$. Here are three properties of the Fourier transform (see [45, 68]).

(1) For all probability measures μ and ν, and for all $\alpha \in \mathbb{Z}^N_\downarrow$ we have $\widehat{\mu \circledast \nu}(\alpha) = \widehat{\mu}(\alpha)\widehat{\nu}(\alpha)$.
(2) A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a measure μ if and only if for all $\alpha \in \mathbb{Z}^N_\downarrow$, the sequence $(\widehat{\mu_n}(\alpha))_{n\in\mathbb{N}}$ converges to $\widehat{\mu}(\alpha)$.
(3) A probability measure μ is central, or conjugate invariant, if and only if for all $\alpha \in \mathbb{Z}^N_\downarrow$, $\widehat{\mu}(\alpha)$ is a homogeneous dilation, and in this case $\widehat{\mu}(\alpha) = (\int_{U(N)} \psi_\alpha(g)\ \mu(\mathrm{d}g))\mathrm{Id}_{E_\alpha}$.

The following proposition gives the Fourier transform of a measure arising from a convolution semigroup.

Proposition 22.3. *Let* $(\mu_t)_{t\in\mathbb{R}^+}$ *be a weakly continuous convolution semigroup on* $U(N)$ *starting at* $\mu_0 = \delta_e$ *with generator* L. *For all* $t \geq 0$, *and all* $\alpha \in \mathbb{Z}^N_\downarrow$, *we have* $\widehat{\mu_t}(\alpha) = e^{tL\pi^\alpha(I_N)}$. *Moreover, if* μ *is conjugate invariant, we have* $\widehat{\mu_t}(\alpha) = e^{tL\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha}$.

Proof. For all $\alpha \in \mathbb{Z}^N_\downarrow$, we have $\widehat{\mu_t}(\alpha) = \int_{U(N)} \pi^\alpha(g)\ \mu_t(\mathrm{d}g) = \mathrm{Id}_{E_\alpha} + t \cdot L\pi^\alpha(I_N) + o_{t\to 0}(t)$, which implies that $\widehat{\mu_t}(\alpha) = \lim_{s\to 0} \widehat{\mu_s}(\alpha)^{t/s} = e^{tL\pi^\alpha(I_N)}$. If μ is conjugate invariant, then, for all $t \in \mathbb{R}^+$, μ_t is conjugate invariant, and we can replace π^α by ψ_α in the previous computation. ☐

Corollary 22.4. *Let* $(\mu_t)_{t\in\mathbb{R}^+}$ *and* $(\nu_t)_{t\in\mathbb{R}^+}$ *be two weakly continuous conjugate invariant convolution semigroups on* $U(N)$ *starting at* $\mu_0 = \delta_e$, *with respective characteristic triplets*

$$(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi) \text{ and } (Y'_0, (y'_{i,j})_{1\leq i,j\leq N^2}, \Pi').$$

Then, $(\mu_t \circledast \nu_t)_{t\in\mathbb{R}^+}$ *is a weakly continuous convolution semigroup on* $U(N)$ *starting at* $\mu_0 = \delta_e$, *with characteristic triplet*

$$(Y_0 + Y'_0, (y_{i,j} + y'_{i,j})_{1\leq i,j\leq N^2}, \Pi + \Pi').$$

Proof. Remark that $(y_{i,j} + y'_{i,j})_{1\leq i,j\leq N^2}$ is a symmetric positive semidefinite matrix and that $\Pi + \Pi'$ is a Lévy measure. Let L and L' be the respective generators of $(\mu_t)_{t\in\mathbb{R}^+}$ and $(\nu_t)_{t\in\mathbb{R}^+}$ given by (21.1). Thanks to Proposition 22.3 and to the conjugation invariance, for all $\alpha \in \mathbb{Z}^N_\downarrow$, we have

$$\widehat{\mu_t * \nu_t}(\alpha) = \widehat{\mu_t}(\alpha) \cdot \widehat{\nu_t}(\alpha) = e^{tL\psi_\alpha(I_N)}e^{tL'\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha} = e^{t(L+L')\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha}$$

To conclude, observe that, for each time $t \in \mathbb{R}_+$, the measure at time t of the weakly continuous semigroup whose characteristic triplet is $(Y_0 + Y'_0, (y_{i,j} + y'_{i,j})_{1\leq i,j\leq N^2}, \Pi + \Pi')$ has the same Fourier transform as $\mu_t \circledast \nu_t$. ☐

LEMMA 22.5. *Let μ and $\nu \in \mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$ with characteristic triplet $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\leq i,j\leq N^2}, \Pi')$. Then, $(Y_0+Y_0', (y_{i,j}+y_{i,j}')_{1\leq i,j\leq N^2}, \Pi+\Pi')$ is a characteristic triplet of $\mu \circledast \nu$. In particular, for all $k\in\mathbb{Z}$, $(Y_0+2ik\pi I_N, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ is also a characteristic triplet of μ.*

PROOF. The first assertion follows from Corollary 22.4. For the second assertion, we remark that $(\delta_{e^{2ikt\Pi}I_N})_{t\in\mathbb{R}^+}$ is a weakly continuous convolution semigroup with characteristic triplet $(2ik\pi, 0, 0)$. By consequence, $(Y_0+2ik\pi I_N, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ is a characteristic triplet of $\mu \circledast \delta_{e^{2ik\pi}I_N} = \mu$. □

We are now ready to prove Proposition-Definition 22.2.

22.3. Proof of Proposition-Definition 22.2. First of all, we remark that the sine function is bounded and $\sin(x) - x \sim_{x\to 0} x^3/6$, which implies that $\int_{\mathcal{H}_N} (\sin(x) - 1_B(x)x)\, \Pi(\mathrm{d}x)$ exists.

We start by proving the first item. Let $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$. Let us denote by L_μ the generator of μ and by $L_{\mathcal{E}_N(\mu)}$ the generator of $\mathcal{E}_N(\mu)$. Let $\alpha \in \mathbb{Z}^N_\downarrow$. We have

$$\widehat{\mathbf{e}_*(\mu^{*\frac{1}{n}})}(\alpha) = \int_{\mathcal{H}_N} \pi^\alpha(\mathbf{e}(x))\ \mu^{*1/n}(\mathrm{d}x) = \mathrm{Id}_{E_\alpha} + L_\mu(\pi^\alpha\circ\mathbf{e})(0)/n + o_{n\to\infty}(1/n),$$

which implies that $\lim_{n\to\infty} (\widehat{\mathbf{e}_*(\mu^{*\frac{1}{n}})})^{\circledast n}(\alpha) = \lim_{n\to\infty}\left(\widehat{\mathbf{e}_*(\mu^{*\frac{1}{n}})}(\alpha)\right)^n = e^{L_\mu(\pi^\alpha\circ\mathbf{e})(0)}$. Let us compute

$$\begin{aligned} L_\mu(\pi^\alpha\circ\mathbf{e})(0) =& \partial_{X_0}(\pi^\alpha\circ\mathbf{e})(0) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\partial_{X_i}\partial_{X_j}(\pi^\alpha\circ\mathbf{e})(0) \\ &+ \int_{\mathcal{H}_N} \pi^\alpha(e^{i(x+0)}) - \pi^\alpha(e^{i0}) - 1_B(x)\partial_x(\pi^\alpha\circ\mathbf{e})(0)\ \Pi(\mathrm{d}x). \end{aligned}$$

Recall that, for all $Y\in\mathfrak{u}(N)$, Y^l is the left invariant vector field on $U(N)$ induced by Y. Using the fact that, for all $x\in\mathcal{H}_N$, $\partial_x(\pi^\alpha\circ\mathbf{e})(0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\pi^\alpha(e^{itx}) = (ix)^l\pi^\alpha(I_N)$, we infer

$$\begin{aligned} L_\mu(\pi^\alpha\circ\mathbf{e})(0) =& (iX_0)^l(\pi^\alpha)(I_N) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l(\pi^\alpha)(I_N) \\ &+ \int_{\mathcal{H}_N} \pi^\alpha(e^{ix}) - \mathrm{Id}_{E_\alpha} - 1_B(x)(ix)^l\pi^\alpha(I_N)\ \Pi(\mathrm{d}x) \\ =& (iX_0)^l(\pi^\alpha)(I_N) + \int_{\mathcal{H}_N} (i\sin(x) - i1_B(x)x)^l\pi^\alpha(I_N)\ \Pi(\mathrm{d}x) \\ &+ \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l(\pi^\alpha)(I_N) + \int_{\mathcal{H}_N} \pi^\alpha(\mathbf{e}(x)) - \mathrm{Id}_{E_\alpha} - (i\Im(\mathbf{e}(x)))^l\,\pi^\alpha(I_N)\ \Pi(\mathrm{d}x) \\ =& L_{\mathcal{E}_N(\mu)}\pi^\alpha(I_N). \end{aligned}$$

Finally, for all $\alpha\in\mathbb{Z}^N_\downarrow$, the sequence $(\widehat{\mathbf{e}_*(\mu^{*\frac{1}{n}})})^{*n}(\alpha)$ converges to $e^{L_{\mathcal{E}_N(\mu)}\pi^\alpha(I_N)} = \widehat{\mathcal{E}_N(\mu)}(\alpha)$ and consequently the sequence $(\mathbf{e}_*(\mu^{*\frac{1}{n}}))^{\circledast n}$ converges to $\mathcal{E}_N(\mu)$.

For the proof of the second item, we use the Fourier transform of a measure in $\mathcal{ID}(\mathcal{H}_N, *)$, which is given by the following proposition.

Proposition 22.6 ([64]). *Let $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$ and let $(X_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ be its characteristic triplet. We have $\int_{\mathcal{H}_N} e^{i\operatorname{Tr}(xy)}\mu^{*t}(\mathrm{d}x) = \exp(t\varphi_\mu(y))$ with*

$$\varphi_\mu(y) = i\operatorname{Tr}(X_0 y) - \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\operatorname{Tr}(X_i y)\operatorname{Tr}(X_j y) + \int_{\mathcal{H}_N} e^{i\operatorname{Tr}(xy)} - 1 - i1_B(x)\operatorname{Tr}(xy)\ \Pi(\mathrm{d}x).$$

Let $\mu \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$. We claim that, for all $t \ge 0$, $\mu^{*t} \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$. Assuming for a moment that this claim is proved, let us explain how it leads to the result: in this case, each measure $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ is conjugate invariant and so is the limit $\mathcal{E}_N(\mu)$. In addition, for all μ, $\nu \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$, the characteristic triplets of $\mathcal{E}_N(\mu * \nu)$ and of $\mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$ coincide thanks to Corollary 22.4, and thus $\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$.

Thus, it remains to prove that, for all $t \ge 0$, $\mu^{*t} \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$. For this, we prove that the Fourier transform of μ^{*t} is conjugate invariant. Firstly, φ_μ is conjugate invariant. Indeed, for all $g \in U(N)$, we have

$$\exp\circ\varphi_\mu(gyg^*) = \int_{\mathcal{H}_N} e^{i\operatorname{Tr}(xgyg^*)}\mathrm{d}\mu(x) = \int_{\mathcal{H}_N} e^{i\operatorname{Tr}(g^*xgy)}\mathrm{d}\mu(x) = \int_{\mathcal{H}_N} e^{i\operatorname{Tr}(xy)}\mathrm{d}\mu(x) = \exp\circ\varphi_\mu(y).$$

We deduce that φ_μ is conjugate invariant since it is continuous and $\exp\circ\varphi_\mu$ is conjugate invariant. Consequently, $\int_{\mathcal{H}_N} e^{i\operatorname{Tr}(xg\cdot g^*)}\mathrm{d}\mu^{*t}(x) = \exp(t\varphi_\mu(\cdot))$ is conjugate invariant, which is sufficient to conclude.

23. Random matrices

In this last section, we shall define the mappings Π_N and Γ_N. Then we prove Theorem 17.2, and in particular our main result, the weak convergence in expectation of the empirical spectral measures of random matrices distributed over $\Gamma_N(\mu)$ for some $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ (see Theorem 23.6). We finish the section by the proof of Theorem 17.3.

23.1. The matrix model Π_N. Recall that the covariance matrix, which corresponds to the diffuse part of an infinitely divisible measure, depends on the choice of a basis of $\mathcal{H}_N$ (see Section 22.1). In this article, we fixed an orthonormal basis $\{X_1, \dots, X_{N^2}\}$ of $\mathcal{H}_N$ such that $X_{N^2} = \frac{1}{\sqrt{N}} I_N$.

Definition 23.1. Let $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$ and let (η, a, ρ) be its $\boxplus$-characteristic triplet. The distribution $\Pi_N(\mu) \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N, *)$ is defined to be the infinitely divisible measure with characteristic triplet $(\eta I_N, a_N, \rho_N)$, where a_N is the $N^2 \times N^2$-matrix

$$a_N = \begin{pmatrix} \frac{a}{N+1} & & & 0 \\ & \ddots & & \\ & & \frac{a}{N+1} & \\ 0 & & & a \end{pmatrix},$$

and ρ_N is the Lévy measure on $\mathcal{H}_N$ which is the push-forward measure of $N\rho \otimes \mathrm{Haar}$ by the mapping from $\mathbb{R} \times U(N)$ to $\mathcal{H}_N$ defined by

$$(x, g) \mapsto g\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} g^*.$$

The application $\Pi_N : \mathcal{ID}(\mathbb{R},\boxplus) \to \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ is obviously a homomorphism of semigroups and we have $\Pi_1 = \Lambda^{-1}$. Moreover, Π_N is a matricial model for $\mathcal{ID}(\mathbb{R},\boxplus)$ in the sense of the following theorem.

THEOREM 23.2 ([12, 23]). *Let $\mu \in \mathcal{ID}(\mathbb{R},\boxplus)$. For all $N \in \mathbb{N}^*$, let H_N be a random matrix whose law is $\Pi_N(\mu)$, and let μ_{H_N} be its empirical spectral measure, that is to say*

$$\mu_{H_N} = \frac{1}{N} \sum_{\substack{\text{eigenvalue } \lambda \text{ of } H_N \\ \text{(with multiplicity)}}} \delta_\lambda.$$

Then, the measures μ_{H_N} converge weakly to μ in probability when N tends to ∞.

In [12, 23], the model is in fact defined starting from a measure $\mu \in \mathcal{ID}(\mathbb{R},*)$. More precisely, for all $\mu \in \mathcal{ID}(\mathbb{R},*)$ with $*$-characteristic triplet (η, a, ρ) and Lévy exponent

$$\varphi_\mu(\theta) = \left(i\eta\theta - \frac{1}{2}a\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x 1_{[-1,1]}(x)) \mathrm{d}\rho(x) \right),$$

Benaych-Georges and Cabanal-Duvillard defined $\Lambda_N(\mu) \in \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ by its Fourier transform: for $x, y \in \mathcal{H}_N$, we have

$$\int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xy)} \Lambda_N(\mu)(\mathrm{d}x) = \exp\left(\varphi_{\Lambda_N(\mu)}(y) \right)$$

where $\varphi_{\Lambda_N(\mu)}(y) = N\mathbb{E}[\varphi_\mu(\langle u, yu\rangle)]$, with u uniformly distributed on the unit sphere of $\mathbb{C}^N$. More explicitly,

$$\varphi_{\Lambda_N(\mu)}(y) = i\eta \operatorname{Tr}(y) - \frac{a}{2(N+1)} \left(\operatorname{Tr}(y)\operatorname{Tr}(y) + \operatorname{Tr}(y)^2 \right) + \int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xy)} - 1 - i 1_B(x) \operatorname{Tr}(xy)\, \Pi(\mathrm{d}x).$$

Using Proposition 22.6, we see that it is exactly the Fourier transform of the infinitely divisible measure of $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ with characteristic triplet (η, a_N, ρ_N). Consequently, we have $\Lambda_N = \Pi_N \circ \Lambda$, or $\Pi_N = \Lambda_N \circ \Lambda^{-1}$ which can be expressed as the commutativity of the following diagram

$$\mathcal{ID}(\mathbb{R},*) \xrightarrow{\Lambda_N} \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*) \xleftarrow{\Pi_N} \mathcal{ID}(\mathbb{R},\boxplus), \qquad \mathcal{ID}(\mathbb{R},*) \xrightarrow{\Lambda} \mathcal{ID}(\mathbb{R},\boxplus).$$

Nevertheless, we prefer to use Π_N which turns out to be more suitable for our present purposes (see Theorem 17.2). One can consult also [34, 35] for further information about this model.

23.2. The matrix model Γ_N. Here again, observe that the data of a covariance matrix of $\mathfrak{u}(N)$ depends on the basis chosen, and recall that we fixed an orthonormal basis $\{Y_1, \ldots, Y_{N^2}\}$ of $\mathfrak{u}(N)$ such that $Y_{N^2} = \frac{i}{\sqrt{N}} I_N$ (see Section 21.1).

DEFINITION 23.3. Let $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)$ and let (ω, b, υ) be its $\boxtimes$-characteristic triplet. The distribution $\Gamma_N(\mu) \in \mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ is defined to be the infinitely divisible measure with characteristic triplet $(\mathrm{Log}(\omega) I_N, b_N, \upsilon_N)$, where Log is the principal logarithm, b_N is the $N^2 \times N^2$-matrix

$$b_N = \begin{pmatrix} \frac{b}{N+1} & & & 0 \\ & \ddots & & \\ & & \frac{b}{N+1} & \\ 0 & & & b \end{pmatrix},$$

and υ_N is the Lévy measure on $U(N)$ which is the push-forward measure of $N\upsilon \otimes \text{Haar}$ by the mapping from $\mathbb{U} \times U(N)$ to $U(N)$ defined by

$$(\zeta, g) \mapsto g \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} g^*.$$

If λ is the Haar measure on $\mathbb{U}$, then we agree to define $\Gamma_N(\lambda)$ to be the Haar measure of $U(N)$.

From this definition, we deduce right now the second half of Theorem 17.2, as a consequence of the following propositions.

Proposition 23.4. *For all μ and $\nu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$, we have $\Gamma_N(\mu \boxtimes \nu) = \Gamma_N(\mu) \circledast \Gamma_N(\nu)$.*

Proof. Let μ and $\nu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$. If μ or ν is equal to λ, we have $\mu \boxtimes \nu = \lambda$. In this case, $\Gamma_N(\mu)$ or $\Gamma_N(\nu)$ is the Haar measure on $U(N)$ and consequently, $\text{Haar} = \Gamma_N(\mu \boxtimes \nu) = \Gamma_N(\mu) \circledast \Gamma_N(\nu)$.

If $\mu, \nu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$, with respective $\boxtimes$-characteristic triplets $(\omega_1, b_1, \upsilon_1)$ and $(\omega_2, b_2, \upsilon_2)$, the measure $\mu \boxtimes \nu \in \mathcal{M}_*$ is a $\boxtimes$-infinitely divisible measure with $\boxtimes$-characteristic triplet $(\omega_1\omega_2, b_1 + b_2, \upsilon_1 + \upsilon_2)$. We denote by $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\leq i,j\leq N^2}, \Pi')$ the respective characteristic triplets of $\Gamma(\mu \boxtimes \nu)$ and $\Gamma(\mu) \circledast \Gamma(\nu)$. It is straightforward to verify that $((y_{i,j})_{1\leq i,j\leq N^2}, \Pi) = ((y_{i,j}')_{1\leq i,j\leq N^2}, \Pi')$, and it remains to compare Y_0 and Y_0'. We have $Y_0 = \text{Log}(\omega_1\omega_2)I_N$ and $Y_0 = (\text{Log}(\omega_1) + \text{Log}(\omega_2))I_N$. As a consequence, Y_0 and Y_0' differ by a multiple of $2i\pi I_N$. Using Lemma 22.5, we deduce that $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\leq i,j\leq N^2}, \Pi')$ are characteristic triplets of the same measure. In other words, $\Gamma(\mu \boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$. □

Proposition 23.5. *For all $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$, we have $\Gamma_N \circ \mathbf{e}_\boxplus(\mu) = \mathcal{E}_N \circ \Pi_N(\mu)$.*

Proof. Let (η, a, ρ) be the $\boxplus$-characteristic triplet of μ. We denote by $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\leq i,j\leq N^2}, \Pi')$ the respective characteristic triplets of $\Gamma_N \circ \mathbf{e}_\boxplus(\mu)$ and $\mathcal{E}_N \circ \Pi_N(\mu)$. We remark first that, following the definitions,

$$(y_{i,j})_{1\leq i,j\leq N^2} = (y_{i,j}')_{1\leq i,j\leq N^2} = \begin{pmatrix} \frac{a}{N+1} & & & 0 \\ & \ddots & & \\ & & \frac{a}{N+1} & \\ 0 & & & a \end{pmatrix}$$

and $\Pi = \Pi' = M_{|U(N)\setminus\{I_N\}}$ where M is the push-forward measure of $N\rho \otimes \text{Haar}$ by the mapping from $\mathbb{R} \times U(N)$ to $U(N)$ given by

$$(x, g) \mapsto g \begin{pmatrix} e^{ix} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} g^*.$$

To conclude, it remains to compare Y_0 and Y_0'. We have

$$Y_0 = \text{Log} \circ \exp\left(i\eta + i\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x) \ \rho(\mathrm{d}x)\right) I_N$$

and

$$\begin{aligned}
Y_0' &= i\eta I_N + i\int_{\mathcal{H}_N} (\sin(x) - 1_U(x)x)\ \mathrm{d}\rho_N(x) \\
&= i\eta I_N + iN\int_{\mathbb{R}}\int_{U(N)} g \begin{pmatrix} (\sin(x)-1_{[-1,1]}(x)x) & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} g^*\ \mathrm{d}g\rho(\mathrm{d}x) \\
&= i\eta I_N + iN\int_{\mathbb{R}} \frac{1}{N}(\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x) \\
&= \left(i\eta + i\int_{\mathbb{R}}(\sin(x) - 1_{[-1,1]}(x)x)\ \rho(\mathrm{d}x)\right) I_N,
\end{aligned}$$

where we have used that $E(A) = \frac{1}{N}\operatorname{Tr}(A)I_N$ (see Example 21.7) for the integration with respect to the Haar measure of $U(N)$. The difference between Y_0 and Y_0' is a multiple of $2i\pi I_N$. Using Lemma 22.5, we deduce that $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ and $(Y_0', (y'_{i,j})_{1\leq i,j\leq N^2}, \Pi')$ are characteristic triplets of the same measure. In other words, $\Gamma_N \circ \mathbf{e}_{\boxplus}(\mu) = \mathcal{E}_N \circ \Pi_N(\mu)$. □

23.3. The large-N limit. We are now ready to prove the first half of Theorem 17.2, which is a corollary of the following theorem.

THEOREM 23.6. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$. For all $N \in \mathbb{N}^*$, let U_N be a random matrix whose law is $\Gamma_N(\mu)$. For all polynomials $P_1, \dots, P_k \in \mathbb{C}[X]$, we have,*

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}(P_1(U_N))\cdots\frac{1}{N}\operatorname{Tr}(P_k(U_N))\right] = \int_{\mathbb{U}} P_1 \mathrm{d}\mu \cdots \int_{\mathbb{U}} P_k \mathrm{d}\mu.$$

PROOF. If μ is the Haar measure λ of $\mathbb{U}$, then $\Gamma_N(\mu)$ is the Haar measure on $U(N)$ for which the result is well-known. Let us assume that $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$, and let (ω, b, υ) be its $\boxtimes$-characteristic triplet. Thanks to Definition 23.3, we know that a characteristic triplet of $\Gamma_N(\mu)$ is given by

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix}, \Pi \right)$$

where $y_0 = -i\operatorname{Log}(\omega)$, $\alpha = b/(N+1)$, $\beta = b$ and Π is the Lévy measure obtained from υ as in Definition 23.3.

By linearity, it suffices to prove the result for monomials. Let $l_1, \dots, l_k \in \mathbb{N}$. We want to prove that

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr}\left(U_N^{l_1}\right)\cdots\frac{1}{N}\operatorname{Tr}\left(U_N^{l_k}\right)\right] = m_{l_1}(\mu)\cdots m_{l_k}(\mu).$$

We will prove the result under the following form: for all $\sigma \in \mathfrak{S}_n$,

$$\lim_{N\to\infty} \mathbb{E}\left[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \operatorname{Tr}(U_N^{\sharp c})\right] = \prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu).$$

We observe that, for all $U \in U(N)$ and $\sigma \in \mathfrak{S}_n$, we have

$$\prod_{c \text{ cycle of } \sigma} \operatorname{Tr}(U^{\sharp c}) = \operatorname{Tr}_{(\mathbb{C}^N)^{\otimes n}}\left(U^{\otimes n} \circ \rho_N^{\mathfrak{S}_n}(\sigma)\right). \tag{23.1}$$

In order to use Proposition 21.8, we define $\tilde{L}_N \in \mathbb{C}[\mathfrak{S}_n]$ by

$$\begin{aligned}\tilde{L}_N &= \left(niy_0 - \frac{n^2}{N}\frac{\beta}{2} + \left(\frac{n^2}{N} - nN\right)\frac{\alpha}{2} + \frac{n}{N}\int_{U(N)} \mathrm{Tr}\left(\Re(g) - 1\right)\Pi(\mathrm{d}g)\right) 1_{\mathfrak{S}_n} - \alpha \sum_{\tau \in \mathcal{T}_n} \tau \\ &+ \sum_{\substack{2\le m\le n\\ 1\le k_1<\ldots<k_m\le n}} \sum_{\pi',\pi\in\mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi)\int_{U(N)} \prod_{c \text{ cycle of } \pi'} \mathrm{Tr}\left((g-1)^{\sharp c}\right)\ \Pi(\mathrm{d}g)\cdot \iota_{k_1,\ldots,k_m}(\pi) \\ &= \left(n\,\mathrm{Log}(\omega) - \frac{n^2}{N}b + \left(\frac{n^2}{N} - nN\right)\frac{b}{2(N+1)} + n\int_{\mathbb{U}} \left(\Re(\zeta) - 1\right) \upsilon(\mathrm{d}\zeta)\right) 1_{\mathfrak{S}_n} - \frac{b}{N+1}\sum_{\tau\in\mathcal{T}_n}\tau \\ &+ \sum_{\substack{2\le m\le n\\ 1\le k_1<\ldots<k_m\le n}} \sum_{\pi',\pi\in\mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi)N\int_{\mathbb{U}} (\zeta-1)^m\ \upsilon(\mathrm{d}\zeta)\cdot \iota_{k_1,\ldots,k_m}(\pi).\end{aligned}$$

Using Proposition 21.8, we have

$$\begin{aligned}\mathbb{E}\left[N^{-\ell(\sigma)}\prod_{c \text{ cycle of } \sigma}\mathrm{Tr}(U_N^{\sharp c})\right] &= N^{-\ell(\sigma)}\,\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\left(\mathbb{E}\left[U_N^{\otimes n}\right]\circ \rho_N^{\mathfrak{S}_n}(\sigma)\right) \\ &= N^{-\ell(\sigma)}\,\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\left(\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}_N}\sigma)\right).\end{aligned}$$

From (23.1), we deduce also that, for all $\sigma \in \mathfrak{S}_n$, we have

$$\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\left(\rho_N^{\mathfrak{S}_n}(\sigma)\right) = N^{\ell(\sigma)}.$$

We denote by N^{ℓ} (resp. $N^{-\ell}$) the linear operator on $\mathbb{C}[\mathfrak{S}_n]$ defined by $N^{\ell}(\sigma) = N^{\ell(\sigma)}\sigma$ (resp. $N^{-\ell}(\sigma) = N^{-\ell(\sigma)}\sigma$) and by ϕ the linear functional defined by $\phi(\sigma) = 1$. This way, we have $\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}} \circ \rho_N^{\mathfrak{S}_n} = \phi \circ N^{\ell}$. Let us also denote by T_N the linear operator on $\mathbb{C}[\mathfrak{S}_n]$ of multiplication by $\tilde{L}_N$, defined by $T_N(\sigma) = \tilde{L}_N\sigma$. We can rewrite

$$\begin{aligned}\mathbb{E}\left[N^{-\ell(\sigma)}\prod_{c \text{ cycle of } \sigma}\mathrm{Tr}(U_N^{\sharp c})\right] &= \mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\left(\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}_N}N^{-\ell(\sigma)}\sigma)\right) \\ &= \phi\left(N^{\ell}e^{T_N}N^{-\ell}(\sigma)\right) \\ &= \phi\left(e^{N^{\ell}T_N N^{-\ell}}(\sigma)\right).\end{aligned}$$

We take the limit with the help of the following lemma. Recall that $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ are the free log-cumulants of μ (see Section 20), which are given by

(1) $L\kappa_1(\mu) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\,\mathrm{d}\upsilon(\zeta)$,
(2) $L\kappa_2(\mu) = -b + \int_{\mathbb{U}}(\zeta-1)^2\mathrm{d}\upsilon(\zeta)$
(3) and $L\kappa_n(\mu) = \int_{\mathbb{U}}(\zeta-1)^n\mathrm{d}\upsilon(\zeta)$ for all $n \ge 2$.

LEMMA 23.7. *When N tends to ∞, the operator $N^{\ell}T_N N^{-\ell}$ converges to an operator T which is such that, for all $\sigma \in \mathfrak{S}_n$,*

$$T(\sigma) = nL\kappa_1(\mu)\cdot\sigma + \sum_{\substack{2\le m\le n\\ c\ m\text{-cycle of } \mathfrak{S}_n\\ c\sigma\preceq\sigma}} L\kappa_m(\mu)\cdot c\sigma.$$

PROOF. We shall prove that, for a fixed $\sigma \in \mathfrak{S}_n$, $\lim_{N\to\infty} N^{\ell} T_N N^{-\ell}(\sigma) = T(\sigma)$. Let us compute

$$N^{\ell} T_N N^{-\ell}(\sigma) = N^{\ell(\sigma)} N^{\ell}(\tilde{L}\sigma).$$

Replacing $\tilde{L}$ by its value gives us $N^{\ell(\sigma)} N^{\ell}(\tilde{L}\sigma) = (I + II + III)\sigma$, with

$$I = \left(n\,\mathrm{Log}(\omega) - \frac{n^2}{N} b + \left(\frac{n^2}{N} - nN \right) \frac{b}{2(N+1)} + n \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \upsilon(\mathrm{d}\zeta) \right) 1_{\mathfrak{S}_n},$$

$$II = -\frac{b}{N+1} \sum_{\tau \in \mathcal{T}_n} N^{\ell(\tau\sigma) - \ell(\sigma)} \tau,$$

and

$$III = \sum_{\substack{2 \le m \le n \\ 1 \le k_1 < \ldots < k_m \le n}} \sum_{\pi \in \mathfrak{S}_m} \int_{\mathbb{U}} (\zeta - 1)^m \; \upsilon(\mathrm{d}\zeta) \cdot \left(\sum_{\pi' \in \mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) \right) \cdot N^{1 + \ell\left(\iota_{k_1,\ldots,k_m}(\pi)\sigma\right) - \ell(\sigma)} \cdot \iota_{k_1,\ldots,k_m}(\pi).$$

The first limit is immediate:

$$\lim_{N\to\infty} I = \left(n\,\mathrm{Log}(\omega) - \frac{n}{2} b + n \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \upsilon(\mathrm{d}\zeta) \right) 1_{\mathfrak{S}_n} = nL\kappa_1(\mu) 1_{\mathfrak{S}_n}.$$

For the second and the third term, we recall that for all $\pi \in \mathfrak{S}_n$, we have

$$d(1, \sigma) \le d(1, \pi\sigma) + d(\pi\sigma, \sigma)$$

with equality if and only if $\pi\sigma \preceq \sigma$ (see Section 20.1).

Let us focus on II. We fix $\tau \in \mathcal{T}_n$. We know that $d(1,\sigma) \le d(1,\tau\sigma) + d(\tau\sigma,\sigma)$. In term of numbers of cycles, it means that $n - \ell(\sigma) \le n - \ell(\tau\sigma) + n - \ell(\tau)$. Because $\ell(\tau) = n - 1$, we have $\ell(\tau\sigma) - \ell(\sigma) \le 1$ with equality if and only if $\tau\sigma \preceq \sigma$. By consequence,

$$\lim_{N\to\infty} II = -b \sum_{\substack{\tau \in \mathcal{T}_n \\ \tau\sigma \preceq \sigma}} \tau\sigma.$$

A similar reasoning can be made for III. Let us fix $2 \le m \le n, 1 \le k_1 < \ldots < k_m \le n$ and $\pi \in \mathfrak{S}_m$. We denote by c the permutation $\iota_{k_1,\ldots,k_m}(\pi)$. On one hand, Proposition 21.6 gives us $\mathrm{Wg}(\pi'^{-1}\pi) = O(N^{-m-1})$ if $\pi \ne \pi'$ and $\mathrm{Wg}(\pi'^{-1}\pi) = N^{-n} + O(N^{-n-1})$ if $\pi = \pi'$, and by consequence,

$$\sum_{\pi' \in \mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) = N^{-m} + O(N^{-m-1}).$$

On the other hand, we know that $d(1,\sigma) \le d(1,c\sigma) + d(c\sigma,\sigma)$. In terms of numbers of cycles, it means that $n - \ell(\sigma) \le n - \ell(c\sigma) + n - \ell(c)$. Because $\ell(c) = \ell\left(\iota_{k_1,\ldots,k_m}(\pi)\right) = n - m + \ell(\pi)$, we have $1 + \ell\left(c\sigma\right) - \ell(\sigma) \le 1 + m - \ell(\pi)$. Thus, we have,

$$1 + \ell\left(c\sigma\right) - \ell(\sigma) \le m$$

with equality if and only if we have both $c\sigma \preceq \sigma$ and $\ell(\pi) = 1$. Consequently, the term

$$\sum_{\pi' \in \mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) N^{1 + \ell\left(\iota_{k_1,\ldots,k_m}(\pi)\sigma\right) - \ell(\sigma)}$$

is equal to $1+O(N^{-1})$ if we have both $c\sigma \preceq \sigma$ and $\ell(\pi)=1$, but it is $O(N^{-1})$ if not. Finally,

$$\lim_{N\to\infty} \mathit{III} = \sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\ldots<k_m\leq n}} \sum_{\substack{\pi\ m-\text{cycle of } \mathfrak{S}_m\\ \iota_{k_1,\ldots,k_m}(\pi)\sigma\preceq\sigma}} \int_{\mathbb{U}} (\zeta-1)^m\ \upsilon(\mathrm{d}\zeta)\cdot \iota_{k_1,\ldots,k_m}(\pi)\sigma$$
$$= \sum_{2\leq m\leq n} \sum_{\substack{c\ m-\text{cycle of } \mathfrak{S}_n\\ c\sigma\preceq\sigma}} \int_{\mathbb{U}} (\zeta-1)^m\ \upsilon(\mathrm{d}\zeta)\cdot c\sigma.$$

Thus, we have

$$\lim_{N\to\infty} I+\mathit{II}+\mathit{III} = nL\kappa_1(\mu)\cdot\sigma + \sum_{\substack{2\leq m\leq n\\ c\ m\text{-cycle of } \mathfrak{S}_n\\ c\sigma\preceq\sigma}} L\kappa_m(\mu)\cdot c\sigma = T(\sigma).$$ □

As a consequence, we have

$$\lim_{N\to\infty} \mathbb{E}\left[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}(U_N^{\sharp c})\right] = \phi(e^T(\sigma)) = \phi(e^{nL\kappa_1(\mu)} e^{T-nL\kappa_1(\mu)}(\sigma))$$
$$= \phi\left(e^{nL\kappa_1(\mu)} \sum_{\substack{\Gamma \text{ simple chain in } [1,\sigma]\\ \Gamma=(\sigma_0,\ldots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu)\cdot\sigma_0 \right)$$
$$= e^{nL\kappa_1(\mu)} \sum_{\substack{\Gamma \text{ simple chain in } [1,\sigma]\\ \Gamma=(\sigma_0,\ldots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu).$$

Using (20.1) on the right-hand side, we conclude that

$$\lim_{N\to\infty} \mathbb{E}\left[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}(U_N^{\sharp c})\right] = \prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu).$$ □

Corollary 23.8. *Let $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$. For all $N\in\mathbb{N}^*$, let U_N be a random matrix whose law is $\Gamma_N(\mu)$, and whose empirical spectral measure is*

$$\mu_{U_N} = \frac{1}{N} \sum_{\substack{\text{eigenvalue } \lambda \text{ of } U^N\\ \text{(with multiplicity)}}} \delta_\lambda.$$

Then, the measures $\mathbb{E}[\mu_{U_N}]$ converge weakly to μ when N tends to ∞.

Proof. We verify the convergence of moments. Let $n\in\mathbb{N}$. We have

$$\int_{\mathbb{U}} \zeta^n \mathrm{d}\mathbb{E}\left[\mu_{U_N}\right] = \mathbb{E}\left[\int_{\mathbb{U}} \zeta^n \mathrm{d}\mu_{U_N}\right] = \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}((U_N)^n)\right]$$

which tends to $m_n(\mu)$ as N tends to ∞. □

Remark 23.9. In fact, the proof can be easily extended to a more general situation. Let $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$ and let (ω,b,υ) be its $\boxtimes$-characteristic triplet. For all $N\in\mathbb{N}^*$, let $y_0^{(N)},\alpha^{(N)},\beta^{(N)}\in\mathbb{R}$ and $\Pi^{(N)}$ be a Lévy measure on $U(N)$ which is conjugate invariant. We suppose that

(1) $\lim_{N\to\infty} e^{iy_0^{(N)}}=\omega$, $\alpha^{(N)}\sim_{N\to\infty} b/N$ and $\beta^{(N)}=O(1)$ as N tends to ∞;

(2) for all $k \geq 2$,

$$\lim_{N\to\infty} \frac{1}{N} \int_{U(N)} \operatorname{Tr}((g - I_N)^k)\Pi^{(N)}(\mathrm{d}g) = \int_{\mathbb{U}} (\zeta - 1)^k \upsilon(\mathrm{d}\zeta);$$

(3) for all $k_1, \ldots, k_n \in \mathbb{N}^*$ such that $k_1 + \ldots + k_n \geq 2$, we have, as N tends to infinity,

$$\frac{1}{N} \int_{U(N)} \operatorname{Tr}((g - I_N)^{k_1}) \cdots \operatorname{Tr}((g - I_N)^{k_n})\Pi^{(N)}(\mathrm{d}g) = O(1).$$

Then, the conclusions of Theorem 23.6 and Corollary 23.8 are still true whenever U_N is a random matrix whose law is an infinitely divisible measure which admits

$$\left(iy_0^{(N)} I_N, \begin{pmatrix} \alpha^{(N)} & & & 0 \\ & \ddots & & \\ & & \alpha^{(N)} & \\ 0 & & & \beta^{(N)} \end{pmatrix}, \Pi^{(N)} \right)$$

as a characteristic triplet.

23.4. Proof of Theorem 17.3. We refer the reader to [74] for the main definitions of free probability spaces. We call *free unitary multiplicative Lévy process* a family $(U_t)_{t\in\mathbb{R}_+}$ of unitary elements of a non-commutative probability space $(\mathcal{A}, \tau)$ such that

(1) $U_0 = 1_{\mathcal{A}}$;
(2) For all $0 \leq s \leq t$, the distribution of $U_t U_s^{-1}$ depends only on $t - s$;
(3) For all $0 \leq t_1 < \ldots < t_n$, the elements $U_{t_1}, U_{t_2}U_{t_1}^{-1}, \ldots U_{t_n}U_{t_{n-1}}^{-1}$ are freely independent;
(4) The distribution of U_t converge weakly to δ_1 as t tends to 0.

Notice that this definition differs from the definition in [18] by the first and the fourth items.

Let $(U_t)_{t\in\mathbb{R}_+}$ be a free unitary multiplicative Lévy process with marginal distributions $(\mu_t)_{t\in\mathbb{R}_+}$ in $\mathcal{M}_*$. Then, $(\mu_t)_{t\in\mathbb{R}_+}$ is a weakly continuous semigroup of measures for the convolution $\boxtimes$ on $\mathbb{U}$. Moreover, there exists $\alpha \in \mathbb{R}$ and $b \geq 0$ and υ a Lévy measure on $\mathbb{U}$ such that, for all $t \geq 0$, $(e^{i\alpha t}, tb, t\upsilon)$ is a $\boxtimes$-characteristic triplet of U_t (see [15]). Using Lemma 22.5, it is straightforward to verify that the weakly continuous semigroup whose characteristic triplet is $(i\alpha I_N, b_N, \upsilon_N)$ coincides with $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$. Therefore, there exists a Lévy process $(U_t^{(N)})_{t\in\mathbb{R}_+}$ in $U(N)$ such that $\Gamma_N(\mu_t)$ is the distribution of $U_t^{(N)}$ for each $t \in \mathbb{R}_+$ (see [57]). We already know that, for each fixed $t \in \mathbb{R}_+$, the element $U_t^{(N)}$ converges to U_t in non-commutative $*$-distribution, in the sense that, for each non-commutative polynomial P in two variables, one has the convergence

$$\lim_{N\mapsto\infty} \frac{1}{N} \mathbb{E}\left[\operatorname{Tr}\left(P\left(U_t^{(N)}, U_t^{(N)*} \right) \right) \right] = \tau(P(U_t, U_t^*)).$$

Since the increments of $(U_t)_{t\in\mathbb{R}_+}$ are freely independent, to prove the convergence of the whole process, it suffices to prove that the increments of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ are asymptotically free. This is a well-known consequence of the factorization property of Theorem 23.6 and the fact that the increments of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ are independent and invariant under conjugation by unitary matrices (see for example [29, 74, 75, 80], or the appendix of [54] for a concise treatment).

Bibliographie

[1] L. Accardi et C. Cecchini, Conditional expectations in von Neumann algebras and a theorem of Takesaki, *Journal of Functional Analysis* 45.2 (Feb. 1982), pages 245–273.

[2] G. W. Anderson, A. Guionnet, et O. Zeitouni, *An introduction to random matrices*, vol. 118, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2010, pages xiv+492.

[3] M. Anshelevich, Free martingale polynomials, *Journal of Functional Analysis* 201.1 (June 2003), pages 228–261.

[4] M. Anshelevich, Generators of some non-commutative stochastic processes, *Probability Theory and Related Fields* (Dec. 2012).

[5] M. Anshelevich, Itô Formula for Free Stochastic Integrals, *Journal of Functional Analysis* 188.1 (Jan. 2002), pages 292–315.

[6] M. Anshelevich, J.-C. Wang, et P. Zhong, Local limit theorems for multiplicative free convolutions, *arXiv:1312.2487* (2013).

[7] D. Applebaum et H. Kunita, Lévy flows on manifolds and Lévy processes on Lie groups, EN, *Kyoto Journal of Mathematics* 33.4 (1993), pages 1103–1123.

[8] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform. Part I, *Communications on Pure and Applied Mathematics* 14.3 (1961), pages 187–214.

[9] V. Bargmann, On a Hilbert Space of Analytic Functions and an Associated Integral Transform. Part II. A Family of Related Function Spaces Application to Distribution Theory, *Communications on Pure and Applied Mathematics* 20.1 (1967), pages 187–214.

[10] V. Bargmann, Remarks on a Hilbert space of analytic functions., *Proceedings of the National Academy of Sciences of the United States of America* 48.2 (1962), pages 199–204.

[11] O. E. Barndorff-Nielsen et S. Thorbjørnsen, Classical and Free Infinite Divisibility and Lévy Processes, *Quantum Independent Increment Processes II*, ed. by M. Schüermann et U. Franz, vol. 1866, Lecture Notes in Mathematics, Berlin Heidelberg: Springer-Verlag, 2005, pages 33–159.

[12] F. Benaych-Georges, Classical and free infinitely divisible distributions and random matrices, *The Annals of Probability* 33.3 (May 2005), pages 1134–1170.

[13] H. Bercovici, V. Pata, et P. Biane, Stable laws and domains of attraction in free probability theory, *Annals of Mathematics* 149.3 (1999), pages 1023–1060.

[14] H. Bercovici et D. Voiculescu, Free convolution of measures with unbounded support, *Indiana Univ. Math. J.* 42.3 (1993), pages 733–773.

[15] H. Bercovici et D. Voiculescu, Lévy-Hinčin type theorems for multiplicative and additive free convolution., *Pacific Journal of Mathematics* 153.2 (1992), pages 217–248.

[16] H. Bercovici et J.-C. Wang, Limit theorems for free multiplicative convolutions, *Trans. Amer. Math. Soc.* 360.11 (2008), pages 6089–6102.

[17] P. Biane, Free Brownian motion, free stochastic calculus and random matrices, *Free probability theory, (Waterloo ON, 1995)*, vol. 12, Fields Inst. Commun. Providence, RI: Amer. Math. Soc., 1997, pages 1–19.

[18] P. Biane, Processes with free increments, *Mathematische Zeitschrift* 227.1 (Jan. 1998), pages 143–174.

[19] P. Biane, Segal–Bargmann Transform, Functional Calculus on Matrix Spaces and the Theory of Semi-circular and Circular Systems, *Journal of Functional Analysis* 144.1 (Feb. 1997), pages 232–286.

[20] P. Biane, Some properties of crossings and partitions, *Discrete Mathematics* 175.1 (1997), pages 41–53.

[21] P. Biane et R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probab Theory Related Fields* 112.3 (1998), pages 373–409.

[22] N. Bourbaki, *Éléments de mathématique. Groupes et Algèbres de Lie.* Springer-V, Berlin.

[23] T. Cabanal-Duvillard, A Matrix Representation of the Bercovici-Pata Bijection, *Electronic Journal of Probability* 10 (June 2005), pages 632–661.

[24] T. Cabanal-Duvillard, Fluctuations de la loi empirique de grandes matrices aléatoires, *Annales de l'Institut Henri Poincare (B) Probability and Statistics* 37.3 (June 2001), pages 373–402.

[25] G. Cébron, Free convolution operators and free Hall transform, *Journal of Functional Analysis* 265.11 (2013), pages 2645–2708.

[26] G. Cébron, Matricial model for the free multiplicative convolution, *arXiv:1402.5286* (2014).

[27] G. Cébron et T. Kemp, Fluctuations of Brownian Motions on GL_N, *arXiv:1409.5624* (2014).

[28] G. P. Chistyakov et F. Götze, Limit theorems in free probability theory II, *Central European Journal of Mathematics* 6.1 (Feb. 2008), pages 87–117.

[29] B. Collins, Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability, *Int. Math. Res. Not.* 17 (2003), pages 953–982.

[30] B. Collins et P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic group, *Communications in Mathematical Physics* 264.3 (2004), pages 773–795.

[31] A. Dahlqvist, Dualité de Schur-Weyl, mouvement brownien sur les groupes de Lie compacts classiques et étude asymptotique de la mesure de Yang-Mills, PhD thesis, Pierre et Marie Curie, Feb. 2014.

[32] A. Dahlqvist, Integration formula for Brownian motion on classical compact Lie groups, *arXiv:1212.5107* (Dec. 2012).

[33] P. Diaconis et S. Evans, Linear functionals of eigenvalues of random matrices, *Transactions of the American Mathematical Society* 353.7 (2001), pages 2615–2633.

[34] J. A. Domínguez-Molina et A. Rocha-Arteaga, Random matrix models of stochastic integrals type for free infinitely divisible distributions, *Periodica Mathematica Hungarica* 64.2 (2012), pages 145–160.

[35] J. A. Domínguez-Molina, V. Pérez-Abreu, et A. Rocha-Arteaga, Covariation representations for Hermitian Lévy process ensembles of free infinitely divisible distributions, en, *Electronic Communications in Probability* 18 (Jan. 2013), pages 1–14.

[36] B. Driver, On the Kakutani-Itô-Segal-Gross and Segal-Bargmann-Hall Isomorphisms, *Journal of Functional Analysis* 133.1 (Oct. 1995), pages 69–128.

[37] B. K. Driver et B. C. Hall, Yang-Mills theory and the Segal-Bargmann transform, *Communications in Mathematical Physics* 201.2 (1999), pages 249–290.

[38] B. K. Driver, B. C. Hall, et T. Kemp, The Large-N Limit of the Segal–Bargmann Transform on $\mathbb{U}_N$, *Journal of Functional Analysis* 265.11 (May 2013), pages 2585–2644.

[39] A. Estrade, Exponentielle stochastique et intégrale multiplicative discontinues., French, *Ann. Inst. Henri Poincaré, Probab. Stat.* 28.1 (1992), pages 107–129.

[40] R. Friedrich et J. McKay, Formal Groups, Witt vectors and Free Probability, *arXiv: 1204.6522* (Apr. 2012).

[41] R. Friedrich et J. McKay, The S-transform in arbitrary dimensions, *arXiv: 1308.0733* (Aug. 2013).

[42] L. Gross et P. Malliavin, Hall's transform and the Segal-Bargmann map, *Itô's stochastic calculus and probability theory*, Tokyo: Springer, 1996, pages 73–116.

[43] B. C. Hall, Geometric Quantization and the Generalized Segal–Bargmann Transform for Lie Groups of Compact Type, *Communications in Mathematical Physics* 226.2 (2002), pages 233–268.

[44] B. C. Hall, The Segal-Bargmann "Coherent State" Transform for Compact Lie Groups, *Journal of Functional Analysis* 122.1 (1994), pages 103–151.

[45] H. Heyer, L'analyse de Fourier non-commutative et applications à la théorie des probabilités, *Annales de l'I.H.P. Probabilités et statistiques* 4.2 (1968), pages 143–164.

[46] S. Janson, *Gaussian Hilbert Spaces*, Cambridge University Press, 1997, pages 340.

[47] K. Johansson, On fluctuations of eigenvalues of random Hermitian matrices, *Duke Mathematical Journal* 91.1 (Jan. 1998), pages 151–204.

[48] O. Kallenberg, *Foundations of modern probability*, Second, Probability and its Applications (New York), New York: Springer-Verlag, 2002, pages xx+638.

[49] V. Kargin, On Free Stochastic Differential Equations, *Journal of Theoretical Probability* 24.3 (Jan. 2011), pages 821–848.

[50] T. Kemp, Heat Kernel Empirical Laws on $\mathbb{U}_N$ and $\mathbb{GL}_N$, *arXiv:1306.2140* (June 2013).

[51] T. Kemp, The Large-N Limits of Brownian Motions on $\mathbb{GL}_N$, *arXiv:1306.6033* (June 2013).

[52] G. Kreweras, Sur les partitions non croisées d'un cycle, *Discrete Mathematics* 1.4 (1972), pages 333–350.

[53] T. Lévy, Schur–Weyl duality and the heat kernel measure on the unitary group, *Advances in Mathematics* 218.2 (June 2008), pages 537–575.

[54] T. Lévy, The master field on the plane, *arXiv:1112.2452* (Dec. 2011).

[55] T. Lévy, *Two-dimensional Markovian holonomy fields*, 329th ed., Astérisque, 2010, pages 172.

[56] T. Lévy et M. Maïda, Central limit theorem for the heat kernel measure on the unitary group, *Journal of Functional Analysis* 259.12 (Dec. 2010), pages 3163–3204.

[57] M. Liao, *Lévy Processes in Lie Groups*, Cambridge University Press, 2004, pages 266.

[58] M. Mastnak et A. Nica, Hopf algebras and the logarithm of the S-transform in free probability, *Transactions of the American Mathematical Society* 362.07 (Feb. 2010), pages 3705–3743.

[59] E. Nelson, Notes on non-commutative integration, *Journal of Functional Analysis* 15.2 (Feb. 1974), pages 103–116.

[60] A. Nica et R. Speicher, *Lectures on the Combinatorics of Free Probability*, vol. 335, London Mathematical Society Lecture Note Series, Cambridge University Press, 2006, pages 1–417.

[61] J.-M. Nicolas, Introduction aux Statistiques de deuxième espèce : applications des Logs-moments et des Logs-cumulants à l'analyse des lois d'images radar, fre, *TS. Traitement du signal* 19.3 (2002), pages 139–167.

[62] K. R. Parthasarathy, *Probability measures on metric spaces*, New York, London: Academic Press, 1967.

[63] E. Rains, Combinatorial Properties of Brownian Motion on the Compact Classical Groups, *Journal of Theoretical Probability* 10.3 (1997), pages 21.

[64] K. I. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge University Press, 1999.

[65] P. Schatte, On sums modulo 2π of independent random variables, *Mathematische Nachrichten* 110.1 (1983), pages 243–262.

[66] I. E. Segal, Mathematical characterization of the physical vacuum for a linear Bose-Einstein field, EN, *Illinois Journal of Mathematics* 6.3 (Sept. 1962), pages 500–523.

[67] A. Sengupta, *Traces in two-dimensional QCD: the large N-limit*, Aspects of Mathematics, E38, Wiesbaden: Friedr. Vieweg & Sohn, 2008, pages 193–212.

[68] E. Siebert, Fourier analysis and limit theorems for convolution semigroups on a locally compact group, *Advances in Mathematics* 39.2 (Feb. 1981), pages 111–154.

[69] R. Speicher, Combinatorial theory of the free product with amalgamation and operator-valued free probability theory, *Mem Amer Math Soc* 132.627 (1998), pages x+88.

[70] R. Speicher, Free probability theory and non-crossing partitions, *Sém Lothar Combin* 39 (1997), pages 1–17.

[71] R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Mathematische Annalen* 298.4 (1994), pages 611–628.

[72] R. P. Stanley, *Enumerative combinatorics. Volume 1*, Second, vol. 49, Cambridge Studies in Advanced Mathematics, Cambridge: Cambridge University Press, 2011, pages xiv+626.

[73] M. Takesaki, Conditional expectations in von Neumann algebras, *Journal of Functional Analysis* 9.3 (Mar. 1972), pages 306–321.

[74] D. V. Voiculescu, K. J. Dykema, et A Nica, *Free Random Variables*, vol. 1, CRM Monograph Series X, American Mathematical Society, 1992, pages 1–14.

[75] D. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* 104.1 (1991), pages 201–220.

[76] D.-V. Voiculescu, Symmetries of some reduced free product C*-algebras, *Operator Algebras and their Connections with Topology and Ergodic Theory*, ed. by H. Araki et al., Springer Berlin Heidelberg, 1985, pages 556–588.

[77] J.-C. Wang, Limit laws for Boolean convolutions, *Pacific Journal of Mathematics* 237.2 (2008), pages 349–371.

[78] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *Journal of Mathematical Physics* 19.5 (1978), pages 999–1001.

[79] E. P. Wigner, On the Distribution of the Roots of Certain Symmetric Matrices, *The Annals of Mathematics* 67.2 (Mar. 1958), pages 325.

[80] F. Xu, A random matrix model from two-dimensional Yang-Mills theory, *Comm. Math. Phys.* 190.2 (1997), pages 287–307.