Nonlinear orbital stability of stationary shock profiles for the Lax-Wendroff scheme

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Abstract

In this article we study the spectral, linear and nonlinear stability of stationary shock profile solutions to the Lax-Wendroff scheme for hyperbolic conservation laws. We first clarify the spectral stability of such solutions depending on the convexity of the flux for the underlying conservation law. The main contribution of this article is a detailed study of the so-called Green's function for the linearized numerical scheme. As evidenced on numerical simulations, the Green's function exhibits a highly oscillating behavior ahead of the leading wave before this wave reaches the shock location. One of our main results gives a quantitative description of this behavior. Because of the existence of a one-parameter family of stationary shock profiles, the linearized numerical scheme admits the eigenvalue 1 that is embedded in its continuous spectrum, which gives rise to several contributions in the Green's function. Our detailed analysis of the Green's function describes these contributions by means of a so-called activation function. For large times, the activation function describes how the mass of the initial condition accumulates along the eigenvector associated with the eigenvalue 1 of the linearized numerical scheme. We can then obtain sharp decay estimates for the linearized numerical scheme in polynomially weighted spaces, which in turn yield a nonlinear orbital stability result for spectrally stable stationary shock profiles. This nonlinear result is obtained despite the lack of uniform ℓ^1 estimates for the Green's function of the linearized numerical scheme, the lack of such estimates being linked with the dispersive nature of the numerical scheme. This dispersive feature is in sharp contrast with previous results on the orbital stability of traveling waves or discrete shock profiles for parabolic perturbations of conservation laws.

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Contents

Notation

Throughout this article, we let $\mathbb{N}^*$ denote the set of positive integers $\{1, 2, 3, \dots\}$ and $\mathbb{N}$ denote the set of integers (including 0). We let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^+ := [0, +\infty)$ the set of nonnegative real numbers and $\mathbb{R}^{+*} := (0, +\infty)$ the set of positive numbers. We also let $\mathbb{C}$ denote the set of complex numbers. We shall use the notation:

$$\begin{aligned} &\mathscr{U} := \{\zeta \in \mathbb{C} \,|\, |\zeta| > 1\}\,, \quad \mathbb{D} := \{\zeta \in \mathbb{C} \,|\, |\zeta| < 1\}\,, \quad \mathbb{S}^1 := \{\zeta \in \mathbb{C} \,|\, |\zeta| = 1\}\,, \\ &\overline{\mathscr{U}} := \mathscr{U} \cup \mathbb{S}^1\,, \quad \overline{\mathbb{D}} := \mathbb{D} \cup \mathbb{S}^1\,. \end{aligned}$$

If w is a complex number, the notation $B_r(w)$ stands for the (round) open ball in $\mathbb{C}$ centered at w and with radius $r > 0$, that is $B_r(w) := \{z \in \mathbb{C} \,|\, |z - w| < r\}$. We shall also use "square balls" in the complex plane. Namely, for any $r > 0$, the notation $\mathbf{B}_r(w)$ stands for the open set:

$$\mathbf{B}_r(w) := \Big\{ z \in \mathbb{C} \,\Big|\, \max\big(|\mathrm{Re}(z - w)|, |\mathrm{Im}(z - w)|\big) < r \Big\}\,.$$

The notation $\overline{B_r(w)}$, resp. $\overline{\mathbf{B}_r(w)}$, refers to the closure of $B_r(w)$, resp. $\mathbf{B}_r(w)$, in $\mathbb{C}$. For any positive number r, we let $r\,\mathbb{S}^1$ denote the circle in $\mathbb{C}$ centered at the origin and with radius r. We also let $\mathscr{M}_{n,k}(\mathbb{C})$ denote the set of $n \times k$ matrices with complex entries. If $n = k$, we simply write $\mathscr{M}_n(\mathbb{C})$.

For $q \in [1, +\infty)$, we let $\ell^q(\mathbb{Z};\mathbb{C})$ denote the space of complex valued sequences $\mathbf{v} = (v_j)_{j \in \mathbb{Z}}$ indexed by $\mathbb{Z}$ and such that the quantity:

$$\sum_{j \in \mathbb{Z}} |v_j|^q$$

is finite. The $1/q$-th power of this quantity defines a norm with which $\ell^q(\mathbb{Z};\mathbb{C})$ becomes a Banach space. This norm is denoted $\| \cdot \|_{\ell^q}$. For $q = +\infty$, we let $\ell^\infty(\mathbb{Z};\mathbb{C})$ denote the space of bounded complex valued sequences indexed by $\mathbb{Z}$. This space is equipped with the norm:

$$\sup_{j \in \mathbb{Z}} |v_j|\,,$$

which we shall refer to as $\| \cdot \|_{\ell^\infty}$. When equipped with this norm, the space $\ell^\infty(\mathbb{Z};\mathbb{C})$ is a Banach algebra (the product between two sequences being here understood in the pointwise sense $(v\,w)_j := v_j\,w_j$ for each $j \in \mathbb{Z}$). Sequences will be denoted with bold letters, while their evaluation at a given integer will be denoted with standard letters.

We let C, resp. c, denote some (large, resp. small) positive constants that may vary throughout the text (sometimes within the same line). A typical example is the inequality:

$$\forall\, x \in \mathbb{R}^+\,, \quad x\, \exp(-c\,x) \le C\, \exp(-c\,x)\,,$$

where, of course, the constant c on the right-hand side of the inequality is not the same as in the left-hand side. The dependance of the constants on the various involved parameters is made precise throughout the article.

In order to avoid overloading some expressions, we sometimes write:

$$\sum_{m=0}^{x}$$

for a sum that runs over the integer m from $m = 0$ to $m = x$, even when x is a positive real number that is not an integer. In that case, it is understood that the sum runs up to the largest number that is less than x. This will allow us to avoir using the integer part of several quantities.

Chapter 1

Introduction

This article is devoted to a detailed stability analysis of stationary shock profiles for the Lax-Wendroff scheme. The Lax-Wendroff scheme is a finite difference approximation of hyperbolic conservation laws that is formally second order accurate (at least for smooth solutions), see [19, 21]. As any second order accurate numerical scheme, the Lax-Wendroff scheme gives rise to spurious oscillations once the solution to the conservation law develops discontinuities, that is, once shock waves have appeared. This is evidenced for instance in the numerical simulation reported in Figure 2.4 below. This undesirable numerical feature is usually corrected by introducing flux limiters or other more involved numerical treatments (essentially non-oscillatory schemes, weighted non-oscillatory schemes and so on). Our goal here is to show that despite the formation of oscillatory wave trains in the numerical computation of shock waves, the Lax-Wendroff scheme gives rise to *stable* stationary shock profiles for *convex* or *concave* fluxes, and for even other situations. The whole point is to clarify the meaning of "stable" in the previous sentence.

Following a long line of research, we aim at studying this stability problem by first showing a *spectral* stability result, then turning this spectral stability result into a *linear* stability result by proving sharp decay estimates for the linearized numerical scheme. Obtaining these sharp linear decay estimates is the cornerstone of our work. The final step of the analysis is to use the linear decay estimates in order to obtain an *orbital* stability result for the nonlinear dynamics. The approach is definitely not new. For the Lax-Wendroff scheme, it was followed by Smyrlis [28] who dealt with the exact same problem as the one we are looking at here. However, the functional framework that we use is much larger than the one in [28] where the introduction of *exponential* weights stabilizes the linearized operator. Actually, the introduction of exponential weights gives rise to a spectral gap for some carefully chosen parameters, thus bypassing a detailed stability analysis that would take the flux properties into account. We consider here a larger and, probably, more natural functional framework in which the spectrum of the linearized operator depends on the flux of the conservation law.

The overall strategy is the same as the one followed in the contributions [14, 6, 7]. We start from a given discrete shock profile, which is quite easy for the Lax-Wendroff scheme since this numerical scheme exhibits *exact*, piecewise constant, stationary discrete shock profiles that are the mere projections on the numerical grid of the continuous step shock. We linearize the Lax-Wendroff scheme around this stationary solution and try to locate the spectrum of the linearized evolution operator. Enlarging the functional framework has some major impacts, the first of which being that we have to take into account the (neutral) eigenvalue 1 for the linearized numerical scheme. This eigenvalue arises by "translation invariance", as explained in [28] (see also [27] for a general overview on discrete shock profiles). Translation invariance means here that there exists a one-parameter smooth family of stationary discrete shock profiles[1]. The rest of the spectrum is located by analyzing the so-called Evans function associated with the shock profile. The Evans function plays the role of a characteristic polynomial for the linearized operator. The general construction of the Evans function is detailed in [27], see also [14, 6]. In our case, the construction is far easier because our reference discrete shock is piecewise constant so the Evans function reduces to what could be referred to as a Lopatinskii determinant (by analogy with the stability analysis of shock waves [2]). Our problem is similar to the stability analysis of the discrete shock profiles for the Godunov scheme [5] in which the shock profiles are also piecewise constant. For scalar equations, as we consider here, the expression of the Lopatinskii determinant is simple enough so that we can analyze the location of its zeroes in the case where the flux of the conservation law is either convex or concave. Namely, we shall show that for convex or concave fluxes, the Lopatinskii determinant has only 1 as a (simple) zero in the region $\overline{\mathscr{U}}$ of the complex plane. This corresponds, in the terminology of [27] to a *spectrally stable* situation. Our analysis encompasses a particular (symmetric) case that was considered in the seminal work [15].

Once we have located the spectrum of the operator, we can construct the so-called spatial Green's function, that is the fundamental solution of the resolvent equation. For spectrally stable configurations, we show that the spatial Green's function has a meromorphic extension near 1 and it can also be holomorphically extended near any other point of the unit circle. This is a key step towards the decomposition and the proof of sharp decay bounds for the Green's function of the linearized evolution operator. In the parabolic case, the Green's function satisfies uniform (in time) ℓ^1 estimates (in space) and it also satisfies decaying (in time) ℓ^∞ estimates (in space), just like the heat kernel. For the Lax-Wendroff scheme, there is unfortunately no hope to obtain such favorable uniform bounds. Indeed, it is already known that when applied to the transport equation, the Green's function of the Lax-Wendroff scheme does not enjoy uniform ℓ^1 estimates. This failure of ℓ^1 stability, that is linked to the *dispersive* behavior of the Lax-Wendroff scheme, has been identified in a general context by Thomée [29] for convolution operators. The analysis of ℓ^1 instability was later refined in [16, 17] in order to make the instability growth rates percise. Rather than following [16, 17] for the Lax-Wendroff scheme, we shall build here on the recent work [9] by one of the authors. The analysis in [9] gives a precise description of the Green's function for this numerical scheme in the context of the Cauchy problem for the transport equation[2]. We extend here the analysis of [9] to the context of the shock profile stability analysis in which Fourier analysis is no longer available due to spatial variations. We therefore substitute the so-called spatial dynamics approach in place of Fourier analysis and use the inverse Laplace transform to obtain a representation formula for the Green's function of the linearized operator. The main difficulty that we shall face arises from the singularity of the spatial Green's function at the eigenvalue 1 (that is imbedded in the continuous spectrum). This pole gives rise to a leading contribution in the Green's function which, following [7], we refer to as an *activation function*. This function can be thought of as a primitive of the Green's function for the Cauchy problem. Detailed bounds on this activation function are given in Appendix A at the end of this article. In particular, a lengthy -though crucial- argument here is to obtain a uniform bound for the activation function. This is rather trivial in the parabolic setting thanks to uniform ℓ^1 estimates for the Green's function but such uniform ℓ^1 estimates are known to fail in the dispersive setting. Our uniform bound for the activation function is reminiscent of the main result in [13] where such a uniform bound is also the cornerstone of

[1] For continuous problems, such as parabolic perturbations of conservation laws, the one-parameter family of profiles is obtained by simply translating in space a given profile, but the construction of shock profiles is unfortunately more complicated for discrete problems.

[2] The quantitative estimates in [9] are more accurate in many regimes than the ones in [16, 17] and this is crucial for several arguments that we use below.

the argument.

Once we have an accurate description of the Green's function with sharp bounds for each term in its decomposition, the nonlinear orbital stability result follows more or less by using standard tools as in [1]. Since our Green's function does not satisfy uniform ℓ^1 estimates and exhibits a dispersive instability[3], we can not complete an orbital stability for small ℓ^1 perturbations. We need to work in *polynomially weighted* ℓ^p spaces and carefully play with the weights in the definition of the norms in order to recover some integrability in time, which is a crucial feature of bootstrap arguments. This is less favorable than the situation for semi-discrete shocks that is dealt with in [1] where the time translation invariance also allows to use a larger functional framework. The fully discrete situation does not give as many tools to deal with the nonlinear argument.

The plan of this article is the following. In Chapter 2 we introduce the numerical scheme, we give the expression of the reference shock profile that we consider and state our main results. For the sake of clarity, we have split the main results in four theorems, the first two being devoted to *spectral* stability, the third one being devoted to *linear* stability, and the fourth and main one being devoted to *nonlinear orbital* stability. We also report on some numerical simulations that illustrate these results. Chapter 3 is devoted to the spectral analysis. The linear decay estimates are proved in Chapter 4 with the help of several key estimates that are given in Appendix A. Eventually, the nonlinear analysis is detailed in Chapter 5.

[3] The ℓ^∞ decay of the Green's function for the Lax-Wendroff scheme is also slower than for the parabolic situation that is considered in [1].

Chapter 2

The Lax-Wendroff scheme. Notation and main results

2.1 The Lax-Wendroff scheme and its stationary shock profiles

We consider in this article a scalar conservation law of the form:

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, \tag{2.1}$$

with a smooth flux $f \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R})$. Since the solutions to the Cauchy problem for (2.1) generically develop singularities in finite time [12, 26], we directly consider piecewise smooth solutions to (2.1), which we assume to be stationary for simplicity:

$$u(t,x) = \begin{cases} u_\ell, & x < 0, \\ u_r, & x > 0. \end{cases} \tag{2.2}$$

For (2.2) to be a weak solution to (2.1), the so-called Rankine-Hugoniot relation must hold:

$$f(u_\ell) = f(u_r)\,, \tag{2.3}$$

and we also enforce the so-called entropy criterion [26]:

$$f'(u_r) < 0 < f'(u_\ell)\,. \tag{2.4}$$

The Lax entropy inequalities (2.4) imply that the characteristics stemming from either side of the shock wave (that is located here at $x = 0$) enter the shock in positive time, as depicted in Figure 2.1.

Our goal here is to understand the influence of a high order discretization procedure on the stability of the shock (2.2) and more specifically whether dispersion in a numerical scheme may rule out linear or nonlinear stability. Let us recall that a rather complete existence and stability theory for monotone schemes has been developed in [18] but monotone schemes are at most first order accurate [19, 21]. We therefore pursue the analysis of [28] and try to develop a rather complete stability analysis of discrete shock profiles for the Lax-Wendroff scheme. This is a model situation for a high order finite difference scheme and we hope that some of our arguments below may prove useful for other discretization procedures of conservation laws.

We therefore introduce a space step $\Delta x > 0$ and a time step $\Delta t > 0$ that are always chosen such that the ratio $\lambda := \Delta t/\Delta x$ is kept constant. For any couple of integers $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, the solution to (2.1)

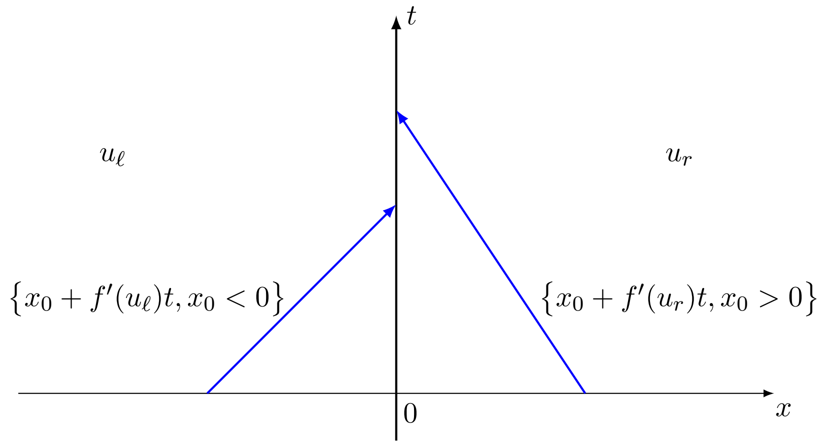

Figure 2.1: *The characteristics on either side of the shock.*

is approximated, on the time-space cell $[n\,\Delta t, (n+1)\,\Delta t) \times [j\,\Delta x, (j+1)\,\Delta x)$ by a constant value u_j^n, that is iteratively defined with respect to n according to the formula:

$$\forall\,(n,j) \in \mathbb{N} \times \mathbb{Z}\,, \quad u_j^{n+1} = u_j^n - \lambda \left(\mathscr{F}_\lambda(u_j^n, u_{j+1}^n) - \mathscr{F}_\lambda(u_{j-1}^n, u_j^n) \right)\,, \tag{2.5}$$

with a *numerical flux* $\mathscr{F}_\lambda$ that is defined by:

$$\forall\,(u,v) \in \mathbb{R}^2\,, \quad \mathscr{F}_\lambda(u,v) := \frac{1}{2}\,(f(u) + f(v)) - \frac{\lambda}{2} f'\left(\frac{u+v}{2}\right)(f(v) - f(u))\,. \tag{2.6}$$

The numerical scheme (2.5)-(2.6), that is referred to as the Lax-Wendroff scheme and dates back to [20], is a formally second order accurate approximation procedure for (2.1), see [19, 21]. The initial condition $(u_j^0)_{j\in\mathbb{Z}}$ for (2.5) will be chosen as a small perturbation of a discretized version of the shock wave (2.2).

A very specific feature of the numerical scheme (2.5)-(2.6) is the fact that it captures *exactly* the stationary piecewise constant solutions to (2.1). Namely, if we consider the weak solution (2.2) to (2.1), then the following sequence:

$$\overline{u}_j^n := \begin{cases} u_\ell, & j \leq 0, \\ u_r, & j \geq 1, \end{cases} \qquad n \in \mathbb{N}\,, \tag{2.7}$$

defines a stationary solution to (2.5)-(2.6) since it does not depend on n and satisfies (we omit the index n in $\overline{u}_j$ from now on):

$$\forall\,j \in \mathbb{Z}\,, \quad \mathscr{F}_\lambda(\overline{u}_j, \overline{u}_{j+1}) = \mathscr{F}_\lambda(\overline{u}_{j-1}, \overline{u}_j) = f(u_\ell) = f(u_r)\,,$$

the final equality coming from the Rankine-Hugoniot relation (2.3). The aim of this article is to study the stability of the stationary solution $\overline{\mathbf{u}} = (\overline{u}_j)_{j\in\mathbb{Z}}$ defined in (2.7) with respect to the dynamics of (2.5)-(2.6). Some numerical experiments are reported below in Section 2.3. Let us recall that the piecewise constant discrete shock (2.7) is not the only discrete shock for the Lax-Wendroff scheme (2.5) associated with (2.2). As explained in [28], a stationary discrete shock profile for (2.2) is a real valued sequence $\mathbf{u} = (u_j)_{j\in\mathbb{Z}}$ such that:

$$\forall\,j \in \mathbb{Z}\,, \quad \mathscr{F}_\lambda(u_j, u_{j+1}) = \mathscr{F}_\lambda(u_{j-1}, u_j)\,,$$

and

$$\lim_{j \to -\infty} u_j \,=\, u_\ell \,, \quad \lim_{j \to +\infty} u_j \,=\, u_r \,.$$

The sequence in (2.7) is a particular case of a stationary discrete shock profile but there are many other ones. We refer to [27] and references therein for a thorough description of the existence theory (both in the scalar and system cases), and to [28] for the specific case of the Lax-Wendroff scheme (2.5)-(2.6) in the scalar case. We recall the following result that is one of the achievements in [28]:

Theorem 2.1 (Smyrlis [28])**.** *Let the shock* (2.2) *satisfy the Rankine-Hugoniot condition* (2.3) *and the entropy condition* (2.4)*. Let* λ *satisfy the so-called CFL condition:*

$$\max \left(\lambda \, f'(u_\ell), \lambda \, |f'(u_r)| \right) \,<\, 1 \,. \tag{2.8}$$

Then there exist $\underline{\theta} > 0$ *and a one-parameter family of stationary discrete shock profiles* $\mathbf{v}^\theta = (v_j^\theta)_{j \in \mathbb{Z}}$*,* $\theta \in (-\underline{\theta}, \underline{\theta})$*, that satisfies the following properties:*

- *(i)* $\mathbf{v}^0 = \overline{\mathbf{u}}$ *is the piecewise constant discrete shock* (2.7)*,*
- *(ii) for every* $j \in \mathbb{Z}$*, the map* $\theta \mapsto v_j^\theta$ *is* $\mathscr{C}^\infty$ *on the interval* $(-\underline{\theta}, \underline{\theta})$*,*
- *(iii) there exist* $\delta > 0$ *and* $C > 0$ *such that for any* $\theta \in (-\underline{\theta}, \underline{\theta})$*, the discrete shock profile* $\mathbf{v}^\theta$ *converges towards its end states exponentially fast at rate* δ*, namely:*

$$\forall \, j \in \mathbb{N} \,, \quad \left| v_j^\theta - u_r \right| + \left| v_{-j}^\theta - u_\ell \right| \,\le\, C \, \mathrm{e}^{-\delta \, j} \,,$$

and furthermore

$$\forall \, j \in \mathbb{Z} \,, \quad \left| v_j^\theta - v_j^0 \right| \,\le\, C \, |\theta| \, \mathrm{e}^{-\delta \, |j|} \,,$$

- *(iv) for every* θ *in the interval* $(-\underline{\theta}, \underline{\theta})$*, the "excess mass" of* $\mathbf{v}^\theta$ *equals* θ*, namely*[1]*:*

$$\sum_{j \in \mathbb{Z}} v_j^\theta - v_j^0 \,=\, \theta \,.$$

Since the numerical scheme (2.5)-(2.6) is conservative, it preserves mass. More precisely, given an initial sequence $\mathbf{h}$ in $\ell^1(\mathbb{Z};\mathbb{R})$, we want to consider the dynamics of (2.5)-(2.6) for the initial condition $\mathbf{v}^0 + \mathbf{h}$ (see the numerical experiments in Section 2.3). Let $(u_j^n)_{j \in \mathbb{Z}, n \in \mathbb{N}}$ denote the corresponding solution to (2.5)-(2.6). By mass conservation, we have:

$$\forall \, n \in \mathbb{N} \,, \quad \sum_{j \in \mathbb{Z}} u_j^n - v_j^0 \,=\, \sum_{j \in \mathbb{Z}} u_j^0 - v_j^0 \,=\, \sum_{j \in \mathbb{Z}} h_j \,.$$

In particular, if we can prove that $(u_j^n)_{j \in \mathbb{Z}, n \in \mathbb{N}}$ converges, as n tends to $+\infty$, towards a stationary discrete shock profile, then this limit can only be $\mathbf{v}^\theta$ where θ denotes the mass of the initial perturbation $\mathbf{h}$. In other words, we rather intend to show that the whole curve $\{\mathbf{v}^\theta, \theta \in (-\underline{\theta}, \underline{\theta})\}$ of stationary discrete shock profiles is *orbitally stable*.

To study the stability of (2.7), or rather the orbital stability of the curve $\{\mathbf{v}^\theta, \theta \in (-\underline{\theta}, \underline{\theta})\}$, we follow a common approach that is based on first studying the spectral stability of (2.7) and then on proving

[1] The series is indeed convergent because stationary shock profiles converge exponentially fast towards their end states.

linear and nonlinear decay estimates. We therefore introduce the linearized numerical scheme that is obtained by linearizing (2.5)-(2.6) around the constant solution $\mathbf{v}^0$ that is given by (2.7). For future use, we introduce the notation:

$$\alpha_{\ell,r} := \lambda\, f'(u_{\ell,r})\,, \quad \alpha_m := \lambda\, f'\left(\frac{u_\ell + u_r}{2}\right)\,. \tag{2.9}$$

The linearization of (2.5)-(2.6) around the discrete shock profile (2.7) leads to the iteration:

$$\mathbf{w}^{n+1} = \mathscr{L}\, \mathbf{w}^n, \quad \mathbf{w}^n = (w_j^n)_{j\in\mathbb{Z}}, \tag{2.10}$$

and the bounded linear operator $\mathscr{L} : \ell^q(\mathbb{Z};\mathbb{C}) \to \ell^q(\mathbb{Z};\mathbb{C})$ is defined by:

$$(\mathscr{L}\, \mathbf{w})_j := \begin{cases} w_j - \dfrac{\alpha_r}{2}\,(w_{j+1} - w_{j-1}) + \dfrac{\alpha_r^2}{2}\,(w_{j+1} - 2\,w_j + w_{j-1})\,, & j \geq 2, \\ w_j - \dfrac{\alpha_\ell}{2}\,(w_{j+1} - w_{j-1}) + \dfrac{\alpha_\ell^2}{2}\,(w_{j+1} - 2\,w_j + w_{j-1})\,, & j \leq -1, \\ w_0 - \dfrac{1}{2}\,(\alpha_r\, w_1 - \alpha_\ell\, w_{-1}) + \dfrac{1}{2}\,\left(\alpha_r\, \alpha_m\, w_1 - \alpha_\ell\,(\alpha_\ell + \alpha_m)\, w_0 + \alpha_\ell^2\, w_{-1}\right)\,, & j = 0, \\ w_1 - \dfrac{1}{2}\,(\alpha_r\, w_2 - \alpha_\ell\, w_0) + \dfrac{1}{2}\,\left(\alpha_r^2\, w_2 - \alpha_r\,(\alpha_r + \alpha_m)\, w_1 + \alpha_\ell\, \alpha_m\, w_0\right)\,, & j = 1. \end{cases} \tag{2.11}$$

Let us observe that the values $\alpha_\ell = -\alpha_r$ and $\alpha_m = 0$ correspond to the particular "symmetric" case studied in [15]. This case corresponds for instance to an even function f with respect to the mid-point $(u_\ell + u_r)/2$. Actually, the spectral stability result of [15] is, to some extent, the starting point for our analysis in Chapter 3 below.

2.2 Main results

Our first main result is a spectral stability result for the stationary discrete shock (2.7) when the flux f is either convex or concave.

Theorem 2.2 (Spectral stability for convex or concave fluxes)**.** *Let the flux f in (2.1) be either convex or concave, and let the weak solution (2.2) satisfy the Rankine-Hugoniot relation (2.3) and the entropy inequalities (2.4). Then under the condition[2]:*

$$\max\,(\alpha_\ell, |\alpha_r|) < 1\,, \tag{2.12}$$

the operator $\mathscr{L}$ has no spectrum in $\overline{\mathscr{U}} \setminus \{1\}$. Moreover, 1 is an eigenvalue of $\mathscr{L}$ in $\ell^q(\mathbb{Z};\mathbb{C})$ for any $q \in [1, +\infty]$.

Actually, it will turn out in Chapter 3 that in the case of a convex or concave flux f, the operator $\mathscr{L}$ will have no spectrum in the exterior of the curve[3]:

$$\left\{1 - 2\,\alpha^2 \sin^2 \frac{\xi}{2} + \mathbf{i}\,\alpha \sin \xi \,|\, \xi \in \mathbb{R}\right\}\,, \quad \alpha := \max(\alpha_\ell, |\alpha_r|)\,. \tag{2.13}$$

[2] This condition is a mere restriction on the ratio $\lambda = \Delta t/\Delta x$. We recall that α_ℓ and α_r are defined in (2.9). The condition (2.12) is the exact same one as in the existence theorem for stationary discrete shock profiles, see (2.8).

[3] This curve is actually an ellipse that is centered at $1 - \alpha^2$.

The exterior of this curve will always be denoted $\mathscr{O}$ in Chapter 3. It can be parametrized as:

$$\mathscr{O} \,=\, \left\{ z = x + \mathbf{i}\, y \in \mathbb{C} \, | \, (x - 1 + \alpha^2)^2 \,+\, \alpha^2\, y^2 \,>\, \alpha^4 \right\} ,$$

see for instance Figure 3.1 in Chapter 3 below where $\mathscr{O}$ corresponds to the complement of the grey shaded area.

We wish to encompass slightly more general situations than the sole case of convex (or concave) fluxes. Indeed, our goal is to show that spectral stability of a stationary discrete shock is a *sufficient* condition for linear and orbital nonlinear stability. In practice, this requires having at our disposal a convenient tool that locates the spectrum of $\mathscr{L}$ just as the characteristic polynomial does so for a matrix. Constructing such a tool, which we shall refer to as a *Lopatinskii determinant* (in analogy with the shock wave theory for systems of conservation laws, see e.g. [2]), is the purpose of part of Chapter 3 below. Examples of numerical calculations of such Lopatinskii determinants can be found in [4] in the context of numerical boundary conditions for transport equations but the analysis is entirely similar here.

Going beyond the case of a convex or concave flux requires making a stability assumption that is given as Assumption 1 in Chapter 3 below. This stability assumption is meant to exclude the possibility of having some spectrum of $\mathscr{L}$ in $\overline{\mathscr{U}} \setminus \{1\}$. Our hope is that Assumption 1 can be "easily" verified (or proved not to hold) in specific situations. The generalization of Theorem 2.2 is the following result.

Theorem 2.3. *Let the weak solution* (2.2) *satisfy the Rankine-Hugoniot relation* (2.3) *and the entropy inequalities* (2.4)*. Then under the condition* (2.12) *on the parameter λ, we can define a function $\underline{\Delta}$ that is holomorphic on a neighborhood of $\overline{\mathscr{U}}$ (see* (3.3) *in Chapter 3). Then 1 is an eigenvalue of $\mathscr{L}$ in $\ell^q(\mathbb{Z};\mathbb{C})$ for any $q \in [1,+\infty]$. Moreover, under Assumption 1 below on the location of the zeroes of $\underline{\Delta}$, the operator $\mathscr{L}$ has no spectrum in $\overline{\mathscr{U}} \setminus \{1\}$.*

Once we know that spectral stability holds, we expect that the favorable localization of the spectrum of $\mathscr{L}$ will imply some decay estimates for the semigroup of operators $\{\mathscr{L}^n \, | \, n \in \mathbb{N}\}$. We gather here these estimates but, to some extent, our main result is rather Theorem 4.1 that will be stated in Chapter 4 below. Theorem 4.1 gives accurate bounds for the Green's function of the operator $\mathscr{L}$ so that, with classical convolution estimates, we can infer estimates on the norm of $\mathscr{L}^n$ in some polynomially weighted spaces. These spaces are defined as follows. We recall that $\ell^q(\mathbb{Z};\mathbb{C})$ denotes the space of complex valued sequences that are indexed by $\mathbb{Z}$ and such that their q-th power is integrable (bounded sequences if $q = +\infty$). The norm is denoted $\| \cdot \|_{\ell^q}$. Given a real number $\gamma \geq 0$, we then introduce the polynomially weighted ℓ^q space:

$$\ell^q_\gamma(\mathbb{Z};\mathbb{C}) \,:=\, \left\{ \mathbf{h} \in \ell^q(\mathbb{Z};\mathbb{C}) \,\mid\, \big((1+|j|^\gamma)\, h_j \big)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C}) \right\} . \tag{2.14}$$

For $\mathbf{h} \in \ell^q_\gamma$, we denote:

$$\|\mathbf{h}\|_{\ell^q_\gamma} \,:=\, \left\| \big((1+|j|^\gamma)\, h_j \big)_{j\in\mathbb{Z}} \right\|_{\ell^q} ,$$

the norm of $\mathbf{h}$, so that $\ell^q_\gamma(\mathbb{Z};\mathbb{C})$ is endowed with a Banach space structure. Our main decay estimates for the operator $\mathscr{L}$ read as follows.

Theorem 2.4. *Let the weak solution* (2.2) *satisfy the Rankine-Hugoniot relation* (2.3) *and the entropy inequalities* (2.4)*. Let the parameter λ satisfy the condition* (2.12) *and let Assumption 1 on the zeroes* *of $\underline{\Delta}$ be satisfied. Then for any real numbers $\gamma_2 \geq \gamma_1 \geq 0$, there exists a constant C such that, for any $\mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{C})$ that satisfies:*

$$\sum_{j\in\mathbb{Z}} h_j \, = \, 0 \, ,$$

there holds:

$$\forall\, n \geq 1 \, , \quad \|\mathscr{L}^n \, \mathbf{h}\|_{\ell^1_{\gamma_1}} \, \leq \, \frac{C}{n^{\gamma_2-\gamma_1-1/8}} \, \|\mathbf{h}\|_{\ell^1_{\gamma_2}} \, ,$$
$$\|\mathscr{L}^n \, \mathbf{h}\|_{\ell^\infty_{\gamma_1}} \, \leq \, \frac{C}{n^{\gamma_2-\gamma_1+\min(1/3,\gamma_1)}} \, \|\mathbf{h}\|_{\ell^1_{\gamma_2}} \, .$$

The first estimate in Theorem 2.4 is consistent with all known results on the behavior of the Lax-Wendroff scheme for the transport equation. Indeed, if one chooses $\gamma_2 = \gamma_1 = 0$, the first estimate in Theorem 2.4 corresponds to the well-known (weak) instability of the Lax-Wendroff scheme in either the ℓ^1 or ℓ^∞ norm. The growth $n^{1/8}$ is known to be sharp, see e.g. [16, 17, 9]. Since we wish to obtain a nonlinear orbital stability result by means of a bootstrap argument, some decay should be gained one way or another, and the introduction of the polynomial weight is a way to compensate for the weak instability of the numerical scheme (and to gain enough time integrability as well). If we compare with the more standard Cauchy problem on $\mathbb{Z}$ for the transport equation, the main new difficulty that we are facing here is the eigenvalue 1 for the operator $\mathscr{L}$, this eigenvalue being imbedded into the continuous spectrum. Nevertheless, our decay estimates in Theorem 2.4 (and some complementary estimates that are detailed in Chapter 4) are strong enough to yield the following *orbital nonlinear stability* result for the family of stationary discrete shock profiles exhibited in Theorem 2.1.

Theorem 2.5. *Let the weak solution (2.2) satisfy the Rankine-Hugoniot relation (2.3) and the entropy inequalities (2.4). Let the parameter λ satisfy the condition (2.12) and let Assumption 1 on the zeroes of $\underline{\Delta}$ be satisfied. Let now $\beta, \sigma \in \mathbb{R}^+$ satisfy $\beta + \sigma \geq \frac{5}{12}$ and $0 \leq \sigma < \beta + \frac{1}{8}$. We define the constant:*

$$\gamma \, := \, \sigma + \beta + \frac{1}{8} \, . \tag{2.15}$$

Then there exist some positive real numbers $C_0, \epsilon > 0$ such that for any sequence $\mathbf{h} \in \ell^1_\gamma(\mathbb{Z};\mathbb{R})$ satisfying

$$\|\mathbf{h}\|_{\ell^1_\gamma} < \epsilon \, ,$$

then one has

$$\theta \, := \, \sum_{j\in\mathbb{Z}} h_j \in (-\underline{\theta}, \underline{\theta}) \, ,$$

and the solution $(\mathbf{u}^n)_{n\in\mathbb{N}}$ of the Lax-Wendroff scheme (2.5)-(2.6) with the initial condition $\mathbf{u}^0 := \overline{\mathbf{u}} + \mathbf{h}$ is well-defined. Furthermore, if we introduce the sequence $(\mathbf{p}^n)_{n\in\mathbb{N}}$ defined as

$$\forall\, n \in \mathbb{N} \, , \quad \mathbf{p}^n \, := \, \mathbf{u}^n - \mathbf{v}^\theta \, ,$$

then for all $n \in \mathbb{N}$ one has $\mathbf{p}^n \in \ell^1_\beta(\mathbb{Z};\mathbb{R})$ together with the estimates:

$$\forall\, n \in \mathbb{N} \, , \quad \|\mathbf{p}^n\|_{\ell^1_\beta} \, \leq \, \frac{C_0}{(1+n)^\sigma} \, \|\mathbf{h}\|_{\ell^1_\gamma} \, , \ \text{and} \ \|\mathbf{p}^n\|_{\ell^\infty_\beta} \, \leq \, \frac{C_0}{(1+n)^{\sigma+\frac{11}{24}}} \, \|\mathbf{h}\|_{\ell^1_\gamma} \, ,$$

so that $\mathbf{u}^n$ tends to $\mathbf{v}^\theta$ in $\ell^\infty_\beta(\mathbb{Z};\mathbb{R})$ (and also in $\ell^1_\beta(\mathbb{Z};\mathbb{R})$ if σ is positive).

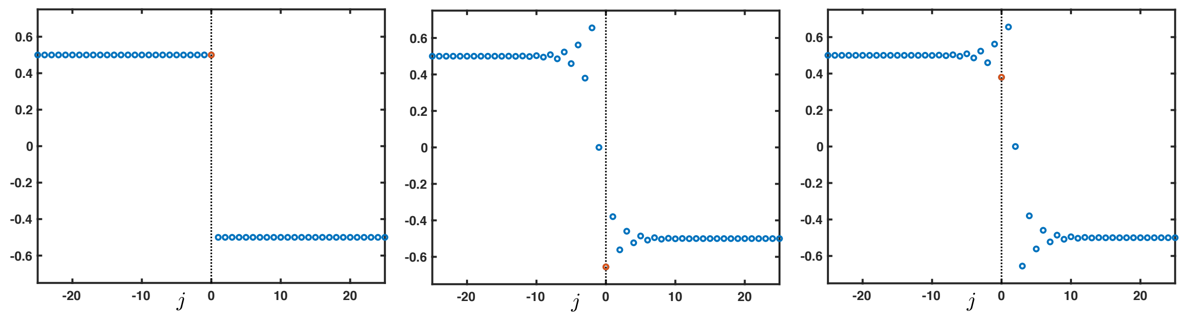

Figure 2.2: *Discrete shock profiles for the Lax-Wendroff scheme applied to the Burgers equation. Left: the reference shock* (2.2)*. Middle: a discrete shock profile with same end states but negative mass difference ($\theta < 0$ in Theorem* 2.1*). Right: a discrete shock profile with same end states but positive mass difference ($\theta > 0$ in Theorem* 2.1*).*

2.3 Numerical experiments

We first present some numerical computations of stationary shock profiles. We consider for simplicity the Burgers equation, which corresponds to the convex flux function $f(u) = u^2/2$. By Theorem 2.2 above, every stationary discrete shock of the form (2.7) with $u_\ell > 0$ and $u_r = -u_\ell$ is spectrally and therefore nonlinearly orbitally stable (in the sense of Theorem (2.5)) if the parameter λ is chosen to satisfy the CFL stability condition. Figure 2.2 displays three possible stationary discrete shock profiles for the Lax-Wendroff scheme with the same end states $u_\ell = 1/2$, $u_r = -1/2$. The Rankine-Hugoniot conditions (2.3) and the Lax entropy condition (2.4) are satisfied. The CFL parameter λ is chosen to be $1/2$ so that the CFL condition (2.12) is also satisfied.

As follows from Theorem 2.1, discrete shock profiles can be parametrized, at least for small enough mass perturbations, by their mass difference with respect to the reference discrete shock profile (2.7). The first graph on the left of Figure 2.2 corresponds to the reference shock (2.2) with end states $u_\ell = 1/2$, $u_r = -1/2$. The value of that discrete shock at $j = 0$ is highlighted in red. The middle and right graphs in Figure 2.2 correspond to stationary discrete shock profiles as given by Theorem 2.1, one being with negative mass difference (middle graph), and the other one being with positive mass difference (on the right of Figure 2.2).

It turns out that the family $\{\mathbf{v}^\theta\}$ of stationary discrete shock profiles given in Theorem 2.1 can be parametrized *globally* for the Burgers equation. This was already mentioned in [28] and we report here on the numerical computation of the whole family. As a first observation, we remark that the translation of the shock profile (2.7), namely:

$$\begin{cases} u_\ell\,, & j \leq 1\,, \\ u_r\,, & j \geq 2\,, \end{cases} \tag{2.16}$$

is also a stationary discrete shock profile for (2.5) with mass difference $u_\ell - u_r = 1$ with respect to (2.7) (the only difference with (2.7) is in the cell labeled with the index $j = 1$). In other words, if we can parametrize the family $\mathbf{v}^\theta$ of Theorem 2.1 for $\theta \in [0, 1]$ with $\mathbf{v}^0$ being equal to the stationary discrete shock profile (2.7) and $\mathbf{v}^1$ being equal to the translated discrete shock profile (2.16), then by repeated translations -either to the left or to the right- one can parametrize a *global* family $\{\mathbf{v}^\theta \,|\, \theta \in \mathbb{R}\}$ where θ still refers to the mass perturbation. This is illustrated in Figure 2.3 where we plot the evaluation at $j = 0$ and $j = 1$ of the family of stationary discrete shock profiles with $\theta \in [-2, 2]$. For $\theta = 1$, both v_0^1 and v_1^1 equal $1/2$ in agreement with (2.16). For $\theta = -1$, the translation of the reference discrete shock is to the left. Both curves that are plotted in Figure 2.3 are translates one of the other.

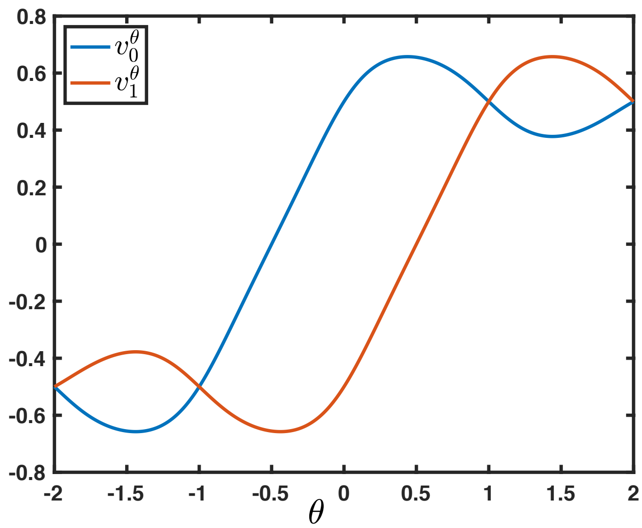

Figure 2.3: *The evaluation at $j = 0$ and $j = 1$ of the family of stationary discrete shock profiles $\mathbf{v}^\theta$, $\theta \in \mathbb{R}$.*

We now report on some computations that illustrate the dynamics of the numerical scheme (2.5). In Figure 2.4, we give several plots corresponding to the evolution of a zero mass perturbation of the reference discrete shock (2.7). By appealing to Theorem 2.5, we expect that the numerical solution converges asymptotically towards (2.7) and this is what we observe. The convergence is illustrated in Figure 2.4 where two waves, one emanating from the right and one from the left of the shock, hit the shock one after the other, giving first rise to a translation of the initial shock to the right (due to positive mass excess) and then going back to the reference discrete shock (2.7). Another view of this computation is given in Figure 2.5 where the plot is in the (j, n) plane. The translation of the shock to the right after the first wave hit the shock is more visible.

As a conclusion, we illustrate the behavior of the Lax-Wendroff scheme (2.5) for a positive mass initial perurbation, see Figures 2.6 and 2.7. The asymptotic state is a stationary discrete shock that corresponds to a non-integer value of $\theta > 0$, thus displaying the typical oscillations associated with the Lax-Wendroff scheme. Despite these oscillations, these solutions are spectrally and nonlinearly stable.

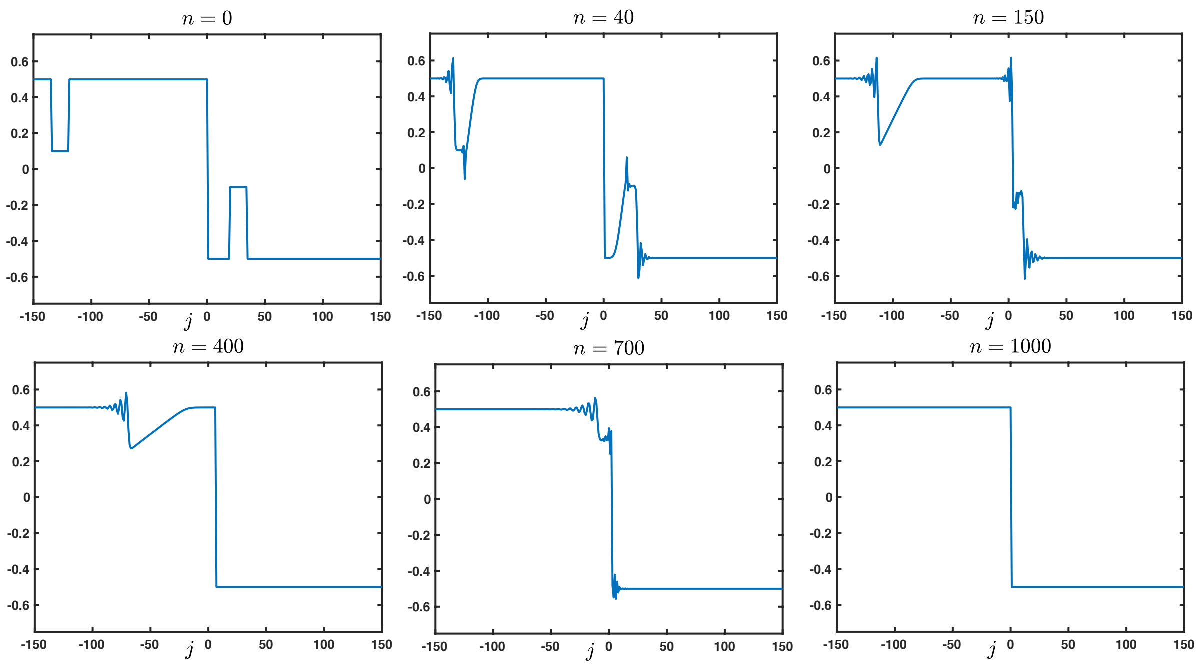

Figure 2.4: *Evolution of a perturbation with zero mass of the reference discrete shock* (2.2)*. First line (from left to right): the initial condition, the solution at $n = 40$, the solution at $n = 150$. Second line (from left to right): the solution at $n = 400$, the solution at $n = 700$, the solution at $n = +\infty$ (convergence towards the reference shock* (2.2)*).*

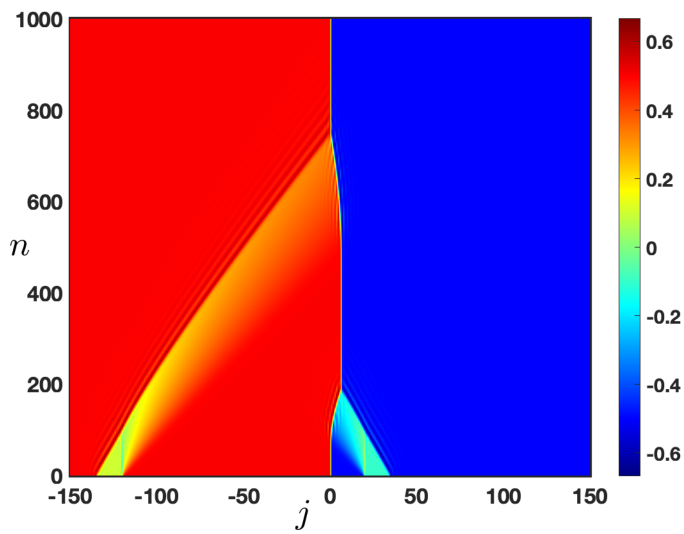

Figure 2.5: *Evolution of a perturbation with zero mass of the reference discrete shock* (2.2)*.*

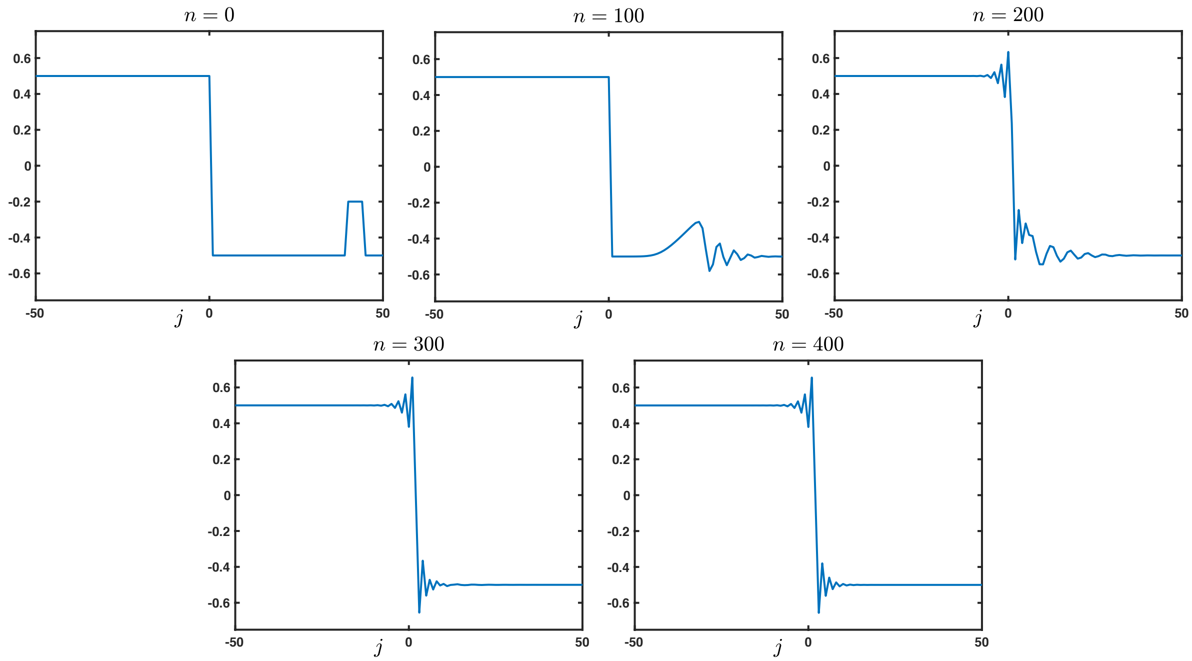

Figure 2.6: *Evolution of a perturbation with positive mass of the reference discrete shock* (2.2)*. First line (from left to right): the initial condition, the solution at $n = 100$, the solution at $n = 200$. Second line (from left to right): the solution at $n = 300$, the solution at $n = 400$ (convergence towards a discrete shock profile).*

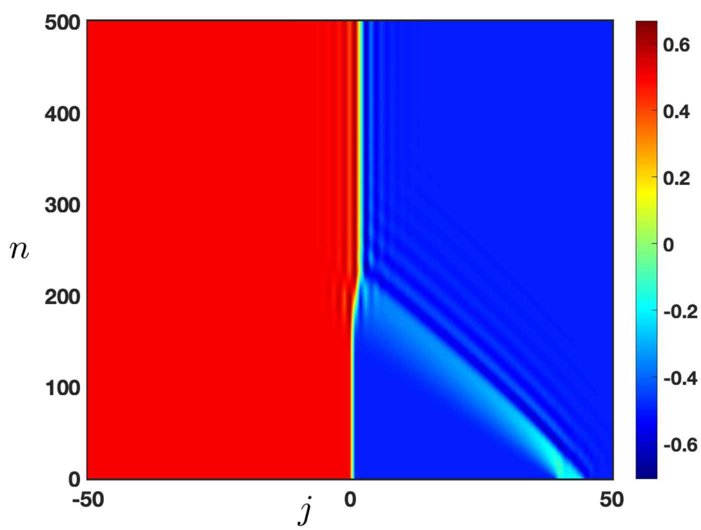

Figure 2.7: *Evolution of a perturbation with positive mass of the reference discrete shock* (2.2).

Chapter 3

Spectral stability

Localizing the spectrum of the linearized operator $\mathscr{L}$ proceeds in several steps. In the first Section below, we analyze the eigenvalue problem[1] $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$ for z in the exterior $\mathscr{O}$ of the curve (2.13). We show that the existence of a nonzero eigensequence $\mathbf{v} \in \ell^q(\mathbb{Z};\mathbb{C})$ is equivalent to a scalar equation $\underline{\Delta}(z) = 0$, where the so-called Lopatinskii determinant $\underline{\Delta}$ is a holomorphic function on $\mathscr{O}$, and, actually, even on a larger region of the complex plane. This result is independent of the considered space $\ell^q(\mathbb{Z};\mathbb{C})$, $1 \le q \le +\infty$, and the Lopatinskii determinant $\underline{\Delta}$ does not depend on q. Since the discrete shock profile $\overline{\mathbf{u}}$ in (2.7) is piecewise constant and the numerical scheme (2.5) only involves a three point stencil, the Lopatinskii determinant $\underline{\Delta}$ is explicitly computable[2]. We analyze the zeroes of the function $\underline{\Delta}$ in the case of a convex (or concave) flux f in (2.1) which seems to be new, up to our knowledge. The symmetric case $\alpha_r = -\alpha_\ell$, $\alpha_m = 0$ is dealt with in [15]. We shall also show in Section 3.4 that for a non-convex flux, the Lopatinskii determinant $\underline{\Delta}$ can have zeroes in the unstable region $\mathscr{U} = \{z \in \mathbb{C} \,|\, |z| > 1\}$ or that it can have a double root at 1. In Section 3.2, we use our knowledge on $\underline{\Delta}$ to compute the so-called spatial Green's function, that is, the solution to the resolvent problem:

$$(z\,\mathrm{Id} - \mathscr{L})\,\mathcal{G}^{j_0}(z) \;=\; \boldsymbol{\delta}_{j_0}\,,$$

where $\boldsymbol{\delta}_{j_0}$ denotes the discrete Dirac mass located at the index $j_0 \in \mathbb{Z}$:

$$\forall\, j \in \mathbb{Z}\,, \quad (\boldsymbol{\delta}_{j_0})_j \;:=\; \begin{cases} 1\,, & \text{if } j = j_0, \\ 0\,, & \text{otherwise.} \end{cases}$$

The construction of the spatial Green's function and the estimates we find on it will directly show that the operator $\mathscr{L}$ has no spectrum outside the curve (2.13) as long as f is convex or concave, as claimed in Theorem 2.2. Theorem 2.3, that considers more general flux functions f, will follow from similar arguments. The spatial Green's function is our starting point for the analysis of the large time behavior of the linearized numerical scheme (2.10).

[1] Since the only discrete shock profile that will appear in this Chapter is the piecewise constant one in (2.7), we feel free to use the notation $\mathbf{v}$ for generic sequences. No possible confusion can be made with the family of discrete shock profiles $\mathbf{v}^\theta$ that will not be mentioned in this Chapter.

[2] The situation is opposite to the case of the Lax-Friedrichs scheme, for instance, where the discrete shock profiles depend on the spatial variable and the localization of the spectrum of the linearized operator involves an Evans function that is not accessible analytically, see [14, 6] or [27]. Shock profiles such as (2.2) for the Lax-Wendroff scheme rather look like the stationary shock profiles for the Godunov scheme that are studied in [5].

3.1 The Lopatinskii determinant

We consider a complex number z in the exterior $\mathscr{O}$ of the curve (2.13) and we shall be first looking for solutions $\mathbf{v} = (v_j)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C})$ to the eigenvalue problem $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$. Our goal is to reduce the existence of a non-trivial solution $\mathbf{v}$ to an equation of the form $\underline{\Delta}(z) = 0$ where the holomorphic function $\underline{\Delta}$ plays the role of a characteristic polynomial for the operator $\mathscr{L}$. Specifying the relation $(\mathscr{L}\,\mathbf{v})_j = z\,v_j$ to those indices $j \leq -1$ or $j \geq 2$ (see (2.11) for the definition of the operator $\mathscr{L}$), we are led to solving the dispersion relations:

$$\frac{\alpha_\ell\,(\alpha_\ell-1)}{2}\,\kappa^2 + (1-\alpha_\ell^2 - z)\,\kappa + \frac{\alpha_\ell\,(\alpha_\ell+1)}{2} = 0\,, \tag{3.1a}$$

$$\frac{\alpha_r\,(\alpha_r-1)}{2}\,\kappa^2 + (1-\alpha_r^2 - z)\,\kappa + \frac{\alpha_r\,(\alpha_r+1)}{2} = 0\,. \tag{3.1b}$$

The behavior of the roots of the above dispersion relations (3.1) is encoded in the following result.

Lemma 3.1. *Let the conditions (2.4) and (2.12) be satisfied, and let $z \in \mathbb{C}$ belong to the exterior $\mathscr{O}$ of the curve (2.13). Then the dispersion relation (3.1a) has one solution $\kappa_\ell(z) \in \mathscr{U}$ and one solution $\kappa_\ell^u(z) \in \mathbb{D}\setminus\{0\}$. Both functions depend holomorphically on z over $\mathscr{O}$, and they can be holomorphically extended to the set:*

$$\mathbb{C} \setminus \left\{ 1-\alpha_\ell^2 + \mathbf{i}\,t\,\alpha_\ell\,\sqrt{1-\alpha_\ell^2} \,\Big|\, t \in [-1,1] \right\}\,,$$

on which they satisfy $\kappa_\ell(z) \neq \kappa_\ell^u(z)$ and $\kappa_\ell(z)\,\kappa_\ell^u(z) \neq 0$.

Furthermore, assuming still that $z \in \mathbb{C}$ belongs to the exterior $\mathscr{O}$ of the curve (2.13), the dispersion relation (3.1b) has one solution $\kappa_r(z) \in \mathbb{D}\setminus\{0\}$ and one solution $\kappa_r^u(z) \in \mathscr{U}$. Both functions depend holomorphically on z over $\mathscr{O}$, and they can be holomorphically extended to the set:

$$\mathbb{C} \setminus \left\{ 1-\alpha_r^2 + \mathbf{i}\,t\,\alpha_r\,\sqrt{1-\alpha_r^2} \,\Big|\, t \in [-1,1] \right\}\,,$$

on which they satisfy $\kappa_r(z) \neq \kappa_r^u(z)$ and $\kappa_r(z)\,\kappa_r^u(z) \neq 0$.

Let us quickly observe that, for $\beta \in [-1,1]$, the compact domain D_β that is delimited by the ellipse:

$$\left\{ 1 - 2\,\beta^2 \sin^2\frac{\xi}{2} + \mathbf{i}\,\beta\sin\xi \,\Big|\, \xi \in \mathbb{R} \right\} = \left\{ 1-\beta^2+\beta^2\,\cos\xi + \mathbf{i}\,\beta\sin\xi \,\big|\, \xi \in \mathbb{R} \right\},$$

satisfies $D_{\beta_1} \subset D_{\beta_2}$ as long as $|\beta_1| \leq |\beta_2|$. This is the reason why the exterior $\mathscr{O}$ of the curve (2.13) does not contain any element of the two curves:

$$\left\{ 1 - 2\,\alpha_\ell^2 \sin^2\frac{\xi}{2} + \mathbf{i}\,\alpha_\ell\sin\xi \,\Big|\, \xi \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ 1 - 2\,\alpha_r^2 \sin^2\frac{\xi}{2} + \mathbf{i}\,\alpha_r\sin\xi \,\Big|\, \xi \in \mathbb{R} \right\}.$$

We also observe that the segment:

$$\left\{ 1-\alpha_\ell^2 + \mathbf{i}\,t\,\alpha_\ell\,\sqrt{1-\alpha_\ell^2} \,\Big|\, t \in [-1,1] \right\}\,,$$

is located in the closed ball of $\mathbb{C}$ centered at 0 and with radius $\sqrt{1-\alpha_\ell^2}$. Moreover, it is located within the curve (2.13). Same for the analogous segment associated with the "right" state u_r rather than with

u_ℓ (with obvious modifications). In what follows, we shall mainly be interested in the fact that some quantities can be holomorphically extended through the unit circle $\mathbb{S}^1$. The exterior $\mathscr{O}$ of the curve (2.13) contains some elements of $\mathbb{D}$ far from the point 1 where it is tangent to $\mathbb{S}^1$, and this is the reason why, sometimes, we need to consider the set:

$$\mathscr{O} \cup \left\{ \zeta \in \mathbb{C} \,|\, |\zeta| > \max\left(\sqrt{1-\alpha_\ell^2}, \sqrt{1-\alpha_r^2}\right) \right\} .$$

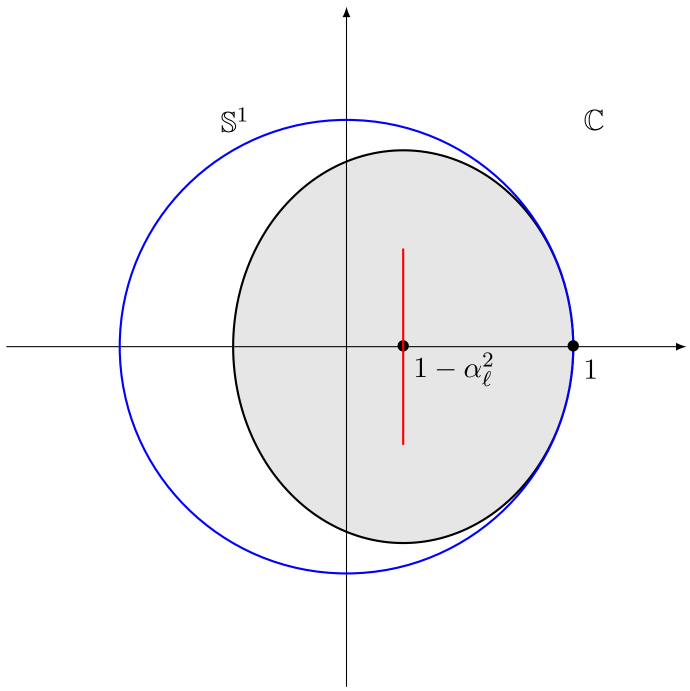

Figure 3.1: *Locating the spectrum of the operator $\mathscr{L}$. In blue: the unit circle. In black: the curve (2.13). The region $\mathscr{O}$ is the complement of the grey shaded area. In red: the segment $[1-\alpha_\ell^2 - \mathbf{i}\,\alpha_\ell\sqrt{1-\alpha_\ell^2}, 1-\alpha_\ell^2 + \mathbf{i}\,\alpha_\ell\sqrt{1-\alpha_\ell^2}]$ outside of which one can holomorphically extend κ_ℓ and κ_ℓ^u. The chosen parameter is $\alpha_\ell = \sqrt{3}/2$ with $\alpha_\ell \geq |\alpha_r|$ so that $\alpha = \alpha_\ell$.*

Proof of Lemma 3.1. We shall only give the proof of Lemma 3.1 for the case of Equation (3.1a), the case of Equation (3.1b) being entirely similar. We begin with some preliminary observations. First of all, the curve (2.13) is a closed curve that is enclosed within the closed unit disk $\overline{\mathbb{D}}$, and that encompasses a strictly convex region. It is actually an ellipse that is centered at $1-\alpha^2$ with axis of half-length α^2 and α. This ellipse is tangent to the unit circle $\mathbb{S}^1$ from within at 1, see Figure 3.1 for an illustration. As we have pointed out above, thanks to our choice for α, the curve (2.13) encompasses both curves:

$$\left\{ 1 - 2\,\alpha_{\ell,r}^2 \sin^2\frac{\xi}{2} - \mathbf{i}\,\alpha_{\ell,r}\sin\xi \,\middle|\, \xi \in \mathbb{R} \right\} .$$

Let us also note that the exterior $\mathscr{O}$ of the curve (2.13) is a connected set and that, when z belongs to $\mathscr{O}$, Equation (3.1a) has no root $\kappa \in \mathbb{S}^1$, for otherwise we would have:

$$z = 1 - 2\,\alpha_\ell^2 \sin^2\frac{\theta}{2} - \mathbf{i}\,\alpha_\ell \sin\theta$$

for some $\theta \in \mathbb{R}$, and this fact would imply that z belongs either to the interior of the curve (2.13) or to its boundary (depending whether $\alpha = \alpha_\ell$ or $\alpha = \alpha_r$), which is precluded here by our assumption $z \in \mathscr{O}$. Consequently, the number of roots of Equation (3.1a) in $\mathscr{U}$, resp. in $\mathbb{D}$, does not depend on $z \in \mathscr{O}$, and these two numbers (whose sum is 2) are determined by letting z tend to infinity. In that case, one root of Equation (3.1a) tends to zero and the other one tends to infinity, which gives the first half of Lemma 3.1.

Once the functions κ_ℓ and κ_ℓ^u have been defined on the exterior $\mathscr{O}$ of the curve (2.13), it only remains to determine how they can be holomorphically extended. The product $\kappa_\ell(z)\,\kappa_\ell^u(z)$ equals $-(1+\alpha_\ell)/(1-\alpha_\ell)$. It is then rather easy to observe that the two roots of (3.1a) have same modulus if and only if z belongs to the segment:

$$\left\{ 1-\alpha_\ell^2 \, + \mathbf{i}\, t\, \alpha_\ell \sqrt{1-\alpha_\ell^2} \, \Big| \, t \in [-1,1] \right\} ,$$

and that segment is located within the interior of the curve (2.13). If $\alpha = \alpha_\ell$, the segment is actually part of one axis of the ellipse. Away from that segment, we can therefore always extend $\kappa_\ell(z)$ as the root of largest modulus to (3.1a), and since that root is necessarily simple[3], it depends locally holomorphically on z. The proof of Lemma 3.1 is thus complete. □

Remark 3.1. *We observe from (3.1) that in the case $\alpha_r = -\alpha_\ell$, there holds $\kappa_r(z) = 1/\kappa_\ell(z)$ for any z in the exterior $\mathscr{O}$ of the curve (2.13).*

Specifying the relation $(\mathscr{L}\,\mathbf{v})_j = z\, v_j$ to the indices $j \geq 2$, we obtain:

$$\forall\, j \geq 2\,, \quad \frac{\alpha_r\,(\alpha_r - 1)}{2}\, v_{j+1} + (1-\alpha_r^2 - z)\, v_j + \frac{\alpha_r\,(\alpha_r+1)}{2}\, v_{j-1} = 0\,,$$

and because the sequence $\mathbf{v} = (v_j)_{j\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z};\mathbb{C})$, with $1 \leq q \leq +\infty$, we obtain, thanks to Lemma 3.1, the expression:

$$\forall\, j \geq 1\,, \quad v_j = v_1\, \kappa_r(z)^{j-1}\,.$$

Since $\kappa_r(z)$ belongs to the unit disk $\mathbb{D}$, the sequence $(\kappa_r(z)^{j-1})_{j\geq 1}$ has exponential decay and therefore belongs to any of the spaces $\ell^p(\mathbb{Z};\mathbb{C})$, $1 \leq p \leq \infty$. Specifying now to the indices $j \leq -1$, we obtain in a similar way:

$$\forall\, j \leq 0\,, \quad v_j = v_0\, \kappa_\ell(z)^j\,.$$

There now remains to determine whether we can find a nonzero pair $(v_0, v_1) \in \mathbb{C}^2$ such that, with the sequence $\mathbf{v} = (v_j)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C})$ defined by:

$$v_j := \begin{cases} v_1\, \kappa_r(z)^{j-1}\,, & \text{if } j \geq 1, \\ v_0\, \kappa_\ell(z)^j\,, & \text{if } j \leq 0, \end{cases}$$

then there holds $(\mathscr{L}\,\mathbf{v})_0 = z\, v_0$ and $(\mathscr{L}\,\mathbf{v})_1 = z\, v_1$, which would provide us with a nonzero solution to the eigenvalue problem $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$. Substituting the values of v_2 and v_{-1} and collecting the terms, we find that the eigenvalue problem $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$ amounts to solving the 2×2 linear system:

$$\begin{bmatrix} \dfrac{\alpha_\ell}{2}\big(\alpha_\ell - \alpha_m + (1-\alpha_\ell)\,\kappa_\ell(z)\big) & -\dfrac{\alpha_r}{2}\,(1-\alpha_m) \\ \dfrac{\alpha_\ell}{2}\,(1+\alpha_m) & \dfrac{\alpha_r}{2}\left(\alpha_r - \alpha_m - \dfrac{1+\alpha_r}{\kappa_r(z)}\right) \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \end{bmatrix} = 0\,. \tag{3.2}$$

[3] The endpoints of the segment correspond precisely to the two values of z at which (3.1a) has a double root, the so-called "glancing" points. These two glancing points lie here in the unit disk $\mathbb{D}$ since the Lax-Wendroff scheme is dissipative.

Here we have used the relations (3.1a) and (3.1b) in order to simplify some coefficients of the linear system (3.2) and also used the fact that $\kappa_r(z)$ is nonzero. Up to the harmless nonzero factor $\alpha_\ell\,\alpha_r/4$, we are led to study the so-called Lopatinskii determinant $\underline{\Delta}$ associated with the linear system (3.2). Its expression is explicitly given here by:

$$\underline{\Delta}(z) := 1-\alpha_m^2+\Big(\alpha_\ell-\alpha_m+(1-\alpha_\ell)\,\kappa_\ell(z)\Big)\left(\alpha_r-\alpha_m-\frac{1+\alpha_r}{\kappa_r(z)}\right). \tag{3.3}$$

If $\underline{\Delta}(z)=0$, then we can find a nonzero solution to (3.2), which means that we can find a nonzero solution to the eigenvalue problem $\mathscr{L}\,\mathbf{v}=z\,\mathbf{v}$. The converse is also true. We have therefore reduced, for $z\in\mathscr{O}$, the eigenvalue problem for the operator $\mathscr{L}$ to determining the zeroes of the function $\underline{\Delta}$.

Remark 3.2. *In the symmetric case $\alpha_r=-\alpha_\ell$ (and therefore $\kappa_r(z)=1/\kappa_\ell(z)$), that is considered in [15], the expression (3.3) of the Lopatinskii determinant reduces to:*

$$\underline{\Delta}(z)=(1-\alpha_\ell)\,(1-\kappa_\ell(z))\,\big(1+\alpha_\ell+(1-\alpha_\ell)\,\kappa_\ell(z)\big)\,,$$

independently *of the value of α_m. For $z\in\mathscr{O}$, we have $\kappa_\ell(z)\in\mathscr{U}$ so $\kappa_\ell(z)$ can not equal 1. Hence, for $z\in\mathscr{O}$, we have $\underline{\Delta}(z)=0$ if and only if:*

$$\kappa_\ell(z)\;=\;-\frac{1+\alpha_\ell}{1-\alpha_\ell}\,.$$

Plugging this value in (3.1a), we obtain $z=1\notin\mathscr{O}$, meaning that $\underline{\Delta}$ does not vanish on $\mathscr{O}$. This spectral stability result is independent *of the convexity or concavity properties of the flux f.*

We explain below how to analyze the general case of a convex (or concave) flux f. We summarize the properties of the Lopatinskii determinant $\underline{\Delta}$ in the following result. It is important to observe that the first part of Lemma 3.2, namely the holomorphy properties of $\underline{\Delta}$ is independent of f.

Lemma 3.2. *Let the weak solution (2.2) satisfy the Rankine-Hugoniot relation (2.3) and the entropy inequalities (2.4). Then under the condition (2.12) on the parameter λ, the Lopatinskii determinant $\underline{\Delta}$ defined in (3.3) is holomorphic on the open set:*

$$\mathscr{O}\cup\left\{\zeta\in\mathbb{C}\,\middle|\,|\zeta|>\max\left(\sqrt{1-\alpha_\ell^2},\sqrt{1-\alpha_r^2}\right)\right\},$$

and $\underline{\Delta}(1)=0$. Furthermore, if the flux f in (2.1) is either convex or concave, then $\underline{\Delta}$ satisfies:

- *$\underline{\Delta}'(1)\neq 0$,*
- *$\underline{\Delta}$ does not vanish on $\mathscr{O}$.*

The fact that $\underline{\Delta}$ vanishes at 1 is associated with the presence of an eigenvalue at 1 for the operator $\mathscr{L}$. This is discussed in more details below (we refer to [27] for a general discussion on this fact). In the terminology of [27, Definition 4.1], if the flux f is either convex or concave, the stationary discrete shock profile (2.2) is thus *spectrally stable* since the non-vanishing of $\underline{\Delta}$ on $\mathscr{O}$ will imply that the spectrum of $\mathscr{L}$ is located within $\mathbb{D}\cup\{1\}$ (see the final argument below in Section 3.3 after our construction of the spatial Green's function).

Proof of Lemma 3.2. The holomorphy of $\underline{\Delta}$ follows from that of κ_ℓ and κ_r^{-1} as given in Lemma 3.1. Indeed, κ_ℓ is holomorphic on:

$$\mathbb{C} \setminus \left\{ 1-\alpha_\ell^2 + \mathbf{i}\, t\, \alpha_\ell \sqrt{1-\alpha_\ell^2} \,\middle|\, t \in [-1,1] \right\},$$

and the segment that has to be excluded is contained in the closed ball $\overline{B_{\sqrt{1-\alpha_\ell^2}}(0)}$ and within the spectral curve (2.13). Arguing similarly with κ_r^{-1}, one sees that $\underline{\Delta}$ is holomorphic on $\mathscr{O} \cup \{\zeta \in \mathbb{C} \,|\, |\zeta| > \max(\sqrt{1-\alpha_\ell^2}, \sqrt{1-\alpha_r^2})\}$.

For the behavior of $\underline{\Delta}$ at 1, we have:

$$\kappa_\ell(1) = -\frac{1+\alpha_\ell}{1-\alpha_\ell} \in \mathscr{U}\,, \quad \kappa_r(1) = -\frac{1+\alpha_r}{1-\alpha_r} \in \mathbb{D}\,,$$

and we thus compute $\underline{\Delta}(1) = 0$ (plug the above values at $z = 1$ in (3.3)). This proves the first part of Lemma 3.2, which holds independently of the convexity or concavity of f.

Differentiating $\underline{\Delta}$ with respect to z, we also get:

$$\underline{\Delta}'(1) = (1-\alpha_\ell)\,(1-\alpha_m)\kappa_\ell'(1) - (1+\alpha_m)\,\frac{(1-\alpha_r)^2}{1+\alpha_r}\,\kappa_r'(1)\,,$$

and the derivatives $\kappa_\ell'(1), \kappa_r'(1)$ are obtained by differentiating (3.1a) and (3.1b) and evaluating at $z = 1$:

$$\kappa_\ell'(1) = -\frac{1+\alpha_\ell}{\alpha_\ell\,(1-\alpha_\ell)}\,, \quad \kappa_r'(1) = -\frac{1+\alpha_r}{\alpha_r\,(1-\alpha_r)}\,.$$

We end up with the expression:

$$\underline{\Delta}'(1) = -\frac{(1+\alpha_\ell)(1-\alpha_m)}{\alpha_\ell} + \frac{(1-\alpha_r)(1+\alpha_m)}{\alpha_r}\,. \tag{3.4}$$

Let us from now on assume that the flux f is either convex or concave. The crucial consequence is that we have $\alpha_m \in [\alpha_r, \alpha_\ell]$, and we recall the entropy inequalities (2.4) as well as the stability restriction (2.12). This implies that α_m belongs to the open interval $(-1,1)$ and therefore $\underline{\Delta}'(1)$ is the sum of two negative quantities. It therefore does not vanish.

It remains to study the other possible zeroes of $\underline{\Delta}$ and specifically to show that $\underline{\Delta}$ does not vanish on $\mathscr{O}$. We are first going to expand the expression of $\underline{\Delta}$ in (3.3). Because of the form of the dispersion relations (3.1a) and (3.1b), we first write:

$$\kappa_\ell(z) = \frac{z-1+\alpha_\ell^2+\mathcal{W}_\ell(z)}{\alpha_\ell\,(\alpha_\ell-1)}\,, \quad \mathcal{W}_\ell(z)^2 = (z-1+\alpha_\ell^2)^2 + \alpha_\ell^2\,(1-\alpha_\ell^2)\,,$$
$$\kappa_r(z) = \frac{z-1+\alpha_r^2+\mathcal{W}_r(z)}{\alpha_r\,(\alpha_r-1)}\,, \quad \mathcal{W}_r(z)^2 = (z-1+\alpha_r^2)^2 + \alpha_r^2\,(1-\alpha_r^2)\,,$$

where we do not mind at this stage which square root should be picked for $\mathcal{W}_\ell(z)$ and $\mathcal{W}_r(z)$. Plugging these expressions for $\kappa_\ell(z)$ and $\kappa_r(z)$ in (3.3), we obtain the expression:

$$\alpha_\ell\,\alpha_r\,\underline{\Delta}(z) = Z^2 + \alpha_m(\alpha_\ell+\alpha_r)\,Z + \alpha_\ell\,\alpha_r + (Z+\alpha_m\alpha_r)\,\mathcal{W}_\ell - (Z+\alpha_m\alpha_\ell)\,\mathcal{W}_r - \mathcal{W}_\ell\,\mathcal{W}_r\,, \tag{3.5}$$

where here and from now on, we use the notation $Z := z - 1$ and we rather consider $\mathcal{W}_\ell$ and $\mathcal{W}_r$ as functions of Z.

Let us now assume that $z \in \mathscr{O}$ is a point where the Lopatinskii determinant $\underline{\Delta}$ vanishes. Then (3.5) provides us with an expression for the product $\mathcal{W}_\ell \, \mathcal{W}_r$ which we can raise to the square (this is the reason why we do not care at this stage about which square root should be preferred). Namely, we introduce the polynomials:

$$\mathcal{Q}(Z) := Z^2 + \alpha_m(\alpha_\ell + \alpha_r)\, Z + \alpha_\ell\, \alpha_r \,, \quad \mathcal{P}_{\ell,r}(Z) := Z + \alpha_m \alpha_{\ell,r} \,,$$

and we see from (3.5) that if $\underline{\Delta}$ vanishes at $z \in \mathscr{O}$, then $Z = z - 1$ satisfies:

$$\mathcal{W}_\ell \, \mathcal{W}_r \;=\; \mathcal{Q}(Z) + \mathcal{P}_r(Z)\, \mathcal{W}_\ell - \mathcal{P}_\ell(Z)\, \mathcal{W}_r \,.$$

Squaring both left and right-hand sides, and then substituting the expression:

$$\mathcal{P}_r(Z)\, \mathcal{W}_\ell - \mathcal{P}_\ell(Z)\, \mathcal{W}_r = \mathcal{W}_\ell \, \mathcal{W}_r - \mathcal{Q}(Z),$$

we obtain the relation:

$$2\,(\mathcal{Q} - \mathcal{P}_\ell \, \mathcal{P}_r)\, \mathcal{W}_\ell \, \mathcal{W}_r \;=\; \mathcal{W}_\ell^2 \, \mathcal{W}_r^2 + \mathcal{Q}^2 + \mathcal{P}_r^2 \, \mathcal{W}_\ell^2 + \mathcal{P}_\ell^2 \, \mathcal{W}_r^2 \,.$$

Expanding each side with respect to Z and simplyfing by the nonzero factor $\alpha_\ell \alpha_r$, we obtain:

$$(1 - \alpha_m^2)\, \mathcal{W}_\ell \, \mathcal{W}_r \;=\; \Big\{ 1 + \alpha_m^2 + 2\, \alpha_\ell \alpha_r - 2\, \alpha_m(\alpha_\ell + \alpha_r) \Big\}\, Z^2 + \alpha_\ell \alpha_r (1 - \alpha_m^2)\, (2\, Z + 1) \,.$$

Raising one last time to the square and collecting the various terms, we end up with the following polynomial equation for Z:

$$Z^2 \, \Big(4\, \gamma \, Z^2 + 2\, \beta \, Z + \beta \Big) = 0 \,, \tag{3.6}$$

with the parameters β and γ being defined by:

$$\beta := (1 - \alpha_m^2)\, \Big\{ (\alpha_\ell - \alpha_r)^2 - \big(\alpha_m(\alpha_\ell + \alpha_r) - 2\, \alpha_\ell \alpha_r \big)^2 \Big\} \,, \tag{3.7}$$

$$\gamma := (\alpha_m - \alpha_r)(\alpha_\ell - \alpha_m)\, \big(1 + \alpha_\ell \alpha_r - \alpha_m(\alpha_\ell + \alpha_r) \big) \,. \tag{3.8}$$

Let us summarize where we are at this stage. We have shown that that if $z \in \mathscr{O}$ is a point where the Lopatinskii determinant $\underline{\Delta}$ vanishes, then $Z = z - 1$ is a root of (3.6), with coefficients β, γ given by (3.7)-(3.8). Since the function:

$$\alpha_m \in [\alpha_r, \alpha_\ell] \longmapsto (\alpha_\ell - \alpha_r)^2 - \big(\alpha_m(\alpha_\ell + \alpha_r) - 2\, \alpha_\ell \alpha_r \big)^2$$

is concave, its minimum on the segment $[\alpha_r, \alpha_\ell]$ is attained either at α_ℓ or α_r (that is, at one of the endpoints) and this minimum is therefore necessarily positive. In other words, we have $\beta > 0$ for all relevant values of α_m (let us recall that $\alpha_m \in [\alpha_r, \alpha_\ell]$ follows from the convexity or concavity of f).

In the cases $\alpha_m = \alpha_r$ or $\alpha_m = \alpha_\ell$, the coefficient γ vanishes. In these two extreme cases, it is clear that all the roots of (3.6) are real. If α_m belongs to the open interval (α_r, α_ℓ), then γ is also positive, and we need to compute the discriminant of the second order factor in (3.6) in order to determine the location of its two roots. Since β is positive, the discriminant has the same sign as the quantity[4]:

$$\beta - 4\, \gamma = \Big((\alpha_\ell + \alpha_r)(1 + \alpha_m^2) - 2\,(1 + \alpha_\ell \alpha_r) \alpha_m \Big)^2 \geq 0 \,.$$

[4]The key final argument is that the quantity $\beta - 4\, \gamma$ can be explicitly factorized !

This means that for $\alpha_m \in [\alpha_r, \alpha_\ell]$, all the roots of (3.6) are real, which means that the only possible roots of $\underline{\Delta}$ in $\mathscr{O}$ are real.

To complete the proof of Lemma 3.2, we thus restrict to real values of the parameter $z \in \mathscr{O}$, that is we consider either $z \in (-\infty, 1-2\,\alpha^2)$ or $z \in (1, +\infty)$ (see Figure 3.1 for visualizing the set $\mathscr{O} \cap \mathbb{R}$). Let us first consider the case $z > 1$, that is $Z = z - 1 > 0$ for which we have, with the standard determination of the square root[5]:

$$\kappa_\ell(z) = \frac{Z + \alpha_\ell^2 + \sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}}{\alpha_\ell\,(\alpha_\ell - 1)}\,, \quad \kappa_r(z) = \frac{Z + \alpha_r^2 - \sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}}{\alpha_r\,(\alpha_r - 1)}\,.$$

We then compute, for $z > 1$, the expression:

$$\begin{aligned}\alpha_\ell\,\alpha_r\,\underline{\Delta}(z) = &\, Z^2 + \alpha_m(\alpha_\ell + \alpha_r)\,Z + \alpha_\ell\,\alpha_r + (Z + \alpha_m\alpha_r)\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2} \\ &+ (Z + \alpha_m\alpha_\ell)\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2} + \sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}\,\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}\,,\end{aligned}$$

where we recall that α_ℓ, α_m and α_r satisfy:

$$-1 < \alpha_r \le \alpha_m \le \alpha_\ell < 1\,, \quad \alpha_r < 0 < \alpha_\ell\,.$$

The function $\underline{\Delta}$ is real valued and smooth (that is, analytic) on $[1, +\infty)$. It vanishes at 1, and its derivative is given, for $z \ge 1$, by:

$$\begin{aligned}\alpha_\ell\,\alpha_r\,\underline{\Delta}'(z) = &\, 2\,Z + \alpha_m(\alpha_\ell + \alpha_r) + \sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2} + \sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2} \\ &+ (Z + \alpha_m\alpha_r)\frac{Z + \alpha_\ell^2}{\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}} + (Z + \alpha_m\alpha_\ell)\frac{Z + \alpha_r^2}{\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}} \\ &+ (Z + \alpha_\ell^2)\frac{\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}}{\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}} + (Z + \alpha_r^2)\frac{\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}}{\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}}\,.\end{aligned} \tag{3.9}$$

For $z \ge 1$, we have $Z \ge 0$, and this implies:

$$\frac{Z + \alpha_\ell^2}{\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}} \ge \alpha_\ell\,, \quad \frac{Z + \alpha_r^2}{\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}} \ge |\alpha_r|\,,$$

which gives (deriving lower bounds for the above three blue terms in (3.9)), for $z \ge 1$:

$$\begin{aligned}\alpha_\ell\,\alpha_r\,\underline{\Delta}'(z) \ge D(\alpha_m, Z) := &\, \alpha_m(\alpha_\ell + \alpha_r) + 2\,\alpha_\ell\,|\alpha_r| + \sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2} + \sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2} \\ &+ (Z + \alpha_m\alpha_r)\frac{Z + \alpha_\ell^2}{\sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}} + (Z + \alpha_m\alpha_\ell)\frac{Z + \alpha_r^2}{\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2}}\,.\end{aligned} \tag{3.10}$$

[5]Since Z is now restrained to real values only, we shall only take square roots of positive real numbers.

We now consider the quantity $D(\alpha_m, Z)$ that is defined in (3.10) and try to show that it does not vanish for $\alpha_m \in [\alpha_r, \alpha_\ell]$ and $Z > 0$. This is shown by deriving a positive lower bound. Indeed, the quantity $D(\alpha_m, Z)$ for $\alpha_\ell \, \alpha_r \, \underline{\Delta}'(z)$ is an affine function with respect to $\alpha_m \in [\alpha_r, \alpha_\ell]$. Its value is therefore not smaller than its values at the endpoints of the interval $[\alpha_r, \alpha_\ell]$. For $\alpha_m = \alpha_r$, we compute:

$$\begin{aligned}
D(\alpha_r, Z) = \, & \alpha_\ell \, |\alpha_r| + \alpha_r^2 + \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} + \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} \\
& + (Z + \alpha_r^2) \, \frac{Z + \alpha_\ell^2}{\sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2}} + (Z - \alpha_\ell \, |\alpha_r|) \, \frac{Z + \alpha_r^2}{\sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2}} \\
& \geq \alpha_\ell \, |\alpha_r| \underbrace{\left(1 - \frac{Z + \alpha_r^2}{\sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2}} \right)}_{\geq 0} + \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} + \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} \, .
\end{aligned}$$

We thus obtain the (far from optimal but nevertheless sufficient !) uniform lower bound:

$$\forall \, Z \geq 0 \, , \quad D(\alpha_r, Z) \geq \alpha_\ell + |\alpha_r| > 0 \, ,$$

and similar arguments lead to the analogous estimate:

$$\forall \, Z \geq 0 \, , \quad D(\alpha_\ell, Z) \geq \alpha_\ell + |\alpha_r| > 0 \, .$$

For $\alpha_m \in [\alpha_r, \alpha_\ell]$, we have thus obtained the lower bound:

$$\forall \, z \geq 1 \, , \quad \alpha_\ell \, \alpha_r \, \underline{\Delta}'(z) \geq \alpha_\ell + |\alpha_r| > 0 \, ,$$

which implies that $\underline{\Delta}$ does not vanish on the open interval $(1, +\infty)$. (Let us recall that $\underline{\Delta}$ vanishes at 1.)

It remains to examine the case $z \in (-\infty, 1 - 2\,\alpha^2)$, for which we now have $Z \leq -2\,\alpha^2$ and we get the expressions:

$$\kappa_\ell(z) = \frac{Z + \alpha_\ell^2 - \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2}}{\alpha_\ell \, (\alpha_\ell - 1)} \, , \quad \kappa_r(z) = \frac{Z + \alpha_r^2 + \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2}}{\alpha_r \, (\alpha_r - 1)} \, ,$$

which yields:

$$\begin{aligned}
\alpha_\ell \, \alpha_r \, \underline{\Delta}(z) = \, & Z^2 + \alpha_m(\alpha_\ell + \alpha_r) \, Z + \alpha_\ell \, \alpha_r - (Z + \alpha_m \alpha_r) \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} \\
& - (Z + \alpha_m \alpha_\ell) \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} + \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} \, \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} \, .
\end{aligned}$$

Once again, the right-hand side of the latter equality, which we denote $\mathbb{D}(\alpha_m, Z)$, is an affine function with respect to α_m and we are going to derive a lower bound for either $\alpha_m = \alpha_r$ or $\alpha_m = \alpha_\ell$, which will give a lower bound for any value of $\alpha_m \in [\alpha_r, \alpha_\ell]$. For $\alpha_m = \alpha_r$ and $Z \leq -2\,\alpha^2$, we have:

$$\begin{aligned}
\mathbb{D}(\alpha_r, Z) = \, & Z^2 + \alpha_r^2 \, Z + \alpha_\ell \, \alpha_r \, Z \\
& + (|Z| - \alpha_r^2) \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} + (|Z| + \alpha_\ell \, |\alpha_r|) \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} \\
& + \sqrt{Z^2 + 2\,\alpha_\ell^2 \, Z + \alpha_\ell^2} \, \sqrt{Z^2 + 2\,\alpha_r^2 \, Z + \alpha_r^2} - \alpha_\ell \, |\alpha_r| \, .
\end{aligned}$$

Recalling that α stands for the maximum of $|\alpha_r|$ and α_ℓ, we find that both functions:

$$Z \longmapsto Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2\,, \quad Z \longmapsto Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2$$

are decreasing on $(-\infty, -2\,\alpha^2]$, and we thus derive the lower bounds:

$$\forall\,Z \le -2\,\alpha^2\,, \quad Z^2 + 2\,\alpha_{\ell,r}^2\,Z + \alpha_{\ell,r}^2 \ge \alpha_{\ell,r}^2\,.$$

Using these lower bounds in the expression of $\mathbb{D}(\alpha_r, Z)$ leads to:

$$\begin{aligned}\forall\,Z \le -2\,\alpha^2\,, \quad \mathbb{D}(\alpha_r, Z) \ge &\, Z^2 + \alpha_r^2\,Z + \alpha_\ell\,|\alpha_r|\,|Z| \\ &+ \sqrt{Z^2 + 2\,\alpha_\ell^2\,Z + \alpha_\ell^2}\,\sqrt{Z^2 + 2\,\alpha_r^2\,Z + \alpha_r^2} - \alpha_\ell\,|\alpha_r| \\ \ge &\, Z^2 + \alpha_r^2\,Z \ge 2\,\alpha^2\,(2\,\alpha^2 - \alpha_r^2) \ge 2\,\alpha^4 > 0\,.\end{aligned}$$

Similar arguments lead to:

$$\forall\,Z \le -2\,\alpha^2\,, \quad \mathbb{D}(\alpha_\ell, Z) \ge 2\,\alpha^4 > 0\,,$$

and we have therefore proved that $\underline{\Delta}(z)$ does not vanish on the interval $z \in (-\infty, 1 - 2\,\alpha^2]$. In other words, the Lopatinskii determinant $\underline{\Delta}$ does not vanish on $\mathscr{O}$. The conclusion of Lemma 3.2 follows. □

Lemma 3.2 has an immediate consequence regarding the solvability of the eigenvalue problem. The proof is a straightforward application of all above results on the Lopatinskii determinant $\underline{\Delta}$.

Corollary 3.1. *Let the flux f in (2.1) be either convex or concave. Let the weak solution (2.2) satisfy the Rankine-Hugoniot relation (2.3) and the entropy inequalities (2.4). Then, under the CFL condition (2.12), for any $z \in \mathscr{O}$ and $1 \le q \le +\infty$, the only solution $\mathbf{v} \in \ell^q(\mathbb{Z};\mathbb{C})$ to the eigenvalue problem $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$ is the zero sequence.*

3.2 The spatial Green's function

In this section, we intend to construct the so-called Green's function, which amounts to inverting the operator $z\,\mathrm{Id} - \mathscr{L}$. We have already seen that the Lopatinskii determinant $\underline{\Delta}$ plays a crucial role in the location of the eigenvalues of the operator $\mathscr{L}$. We are going to show below that the condition $\underline{\Delta}(z) \neq 0$ is actually necessary and sufficient for a complex number $z \in \overline{\mathscr{U}} \setminus \{1\}$ to lie in the resolvent set of $\mathscr{L}$. We do not wish any longer to assume that the flux f is either convex or concave. We rather wish to deal more generally with *spectrally stable* configurations. We shall therefore substitute Assumption 1 below in place of the convexity (or concavity) assumption on f. In particular, there is now no obvious reason for the parameter α_m in (2.9) to belong to the interval $[\alpha_r, \alpha_\ell]$. We make from now on the following assumption.

Assumption 1. *The Lopatinskii determinant $\underline{\Delta}$ in (3.3) associated with the discrete shock (2.7) satisfies:*

- $\underline{\Delta}'(1) \neq 0$,
- $\underline{\Delta}$ *does not vanish on* $\overline{\mathscr{U}} \setminus \{1\}$.

As we have seen in the proof of Lemma 3.2, the holomorphy of $\underline{\Delta}$ on an open set that contains $\overline{\mathscr{U}}$ always holds as long as the CFL condition (2.12) is satisfied (the CFL condition (2.12) allows us to define the modes $\kappa_{r,\ell}$ by Lemma 3.1, and therefore the function $\underline{\Delta}$). The property $\underline{\Delta}(1) = 0$ also holds independently of the convexity properties of f. Moreover, Lemma 3.2 shows that Assumption 1 is satisfied whenever the flux f is either convex or concave. There is no real difficulty to generalize the analysis of the previous Section in order to consider a slightly larger context than the sole case of a convex or concave flux f. We can indeed extend Corollary 3.1 and obtain the following result.

Corollary 3.2. *Let the weak solution* (2.2) *satisfy the entropy inequalities* (2.4)*. Let the parameter* λ *satisfy the CFL condition* (2.12) *and let Assumption 1 be satisfied. Then for any* $z \in \overline{\mathscr{U}} \setminus \{1\}$ *and* $1 \leq q \leq +\infty$*, the only solution* $\mathbf{v} \in \ell^q(\mathbb{Z};\mathbb{C})$ *to the eigenvalue problem* $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$ *is the zero sequence.*

For z in the resolvent set of the operator $\mathscr{L}$, we denote by $\mathcal{G}^{j_0}(z) = \left(\mathcal{G}_j^{j_0}(z)\right)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C})$ the solution to the resolvent problem:

$$(z\,\mathrm{Id} - \mathscr{L})\,\mathcal{G}^{j_0}(z) \;=\; \boldsymbol{\delta}_{j_0}\,, \tag{3.11}$$

where $\boldsymbol{\delta}_{j_0} = (\delta_{j_0}(j))_{j\in\mathbb{Z}}$ stands for the discrete Dirac mass located at the index $j_0 \in \mathbb{Z}$. The following result gives an explicit expression for the spatial Green's function $\mathcal{G}^{j_0}(z)$ for $z \in \overline{\mathscr{U}} \setminus \{1\}$. This makes use of the fact that the Lopatinskii determinant does not vanish (which is the reason for Assumption 1). Corollary 3.2 above shows that the solution to (3.11) is necessarily unique for $z \in \overline{\mathscr{U}} \setminus \{1\}$.

Proposition 3.1. *Let the weak solution* (2.2) *satisfy the entropy inequalities* (2.4)*. Let the parameter* λ *satisfy the CFL condition* (2.12) *and let Assumption 1 be satisfied. Then for any* $z \in \overline{\mathscr{U}} \setminus \{1\}$ *and for any* $j_0 \in \mathbb{Z}$*, there exists a unique solution* $\mathcal{G}^{j_0}(z) \in \ell^q(\mathbb{Z};\mathbb{C})$ *to the equation* (3.11)*.*

For $j_0 \geq 1$*, this sequence* $\mathcal{G}^{j_0}(z) = \left(\mathcal{G}_j^{j_0}(z)\right)_{j\in\mathbb{Z}}$ *is explicitly given by:*

$$\mathcal{G}_j^{j_0}(z) \;=\; \begin{cases} -\dfrac{2\,(1-\alpha_m)}{\alpha_\ell\,\underline{\Delta}(z)}\,\kappa_r^u(z)^{1-j_0}\,\kappa_\ell(z)^j, & \text{if } j \leq 0, \\[2ex] -\dfrac{2\,(\alpha_\ell-\alpha_m+(1-\alpha_\ell)\,\kappa_\ell(z))}{\alpha_r\,\underline{\Delta}(z)}\,\kappa_r^u(z)^{1-j_0}\,\kappa_r(z)^{j-1} & \\[2ex] \qquad +\dfrac{2\left(\kappa_r^u(z)^{1-j_0}\,\kappa_r(z)^{j-1}-\kappa_r^u(z)^{j-j_0}\right)}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}, & \text{if } 1 \leq j \leq j_0, \\[2ex] -\dfrac{2\,(\alpha_\ell-\alpha_m+(1-\alpha_\ell)\,\kappa_\ell(z))}{\alpha_r\,\underline{\Delta}(z)}\,\kappa_r^u(z)^{1-j_0}\,\kappa_r(z)^{j-1} & \\[2ex] \qquad +\dfrac{2\left(\kappa_r^u(z)^{1-j_0}\,\kappa_r(z)^{j-1}-\kappa_r(z)^{j-j_0}\right)}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}, & \text{if } j > j_0, \end{cases} \tag{3.12}$$

and for $j_0 \leq 0$, $\mathcal{G}^{j_0}(z)$ is explicitly given by:

$$\mathcal{G}_j^{j_0}(z) \,=\, \begin{cases} \dfrac{2\,(1+\alpha_m)}{\alpha_r\,\underline{\Delta}(z)}\,\kappa_\ell^u(z)^{-j_0}\,\kappa_r(z)^{j-1}, & \text{if } j \geq 1, \\[2ex] -\dfrac{2\,(\alpha_r-\alpha_m-(1+\alpha_r)\,\kappa_r(z)^{-1})}{\alpha_\ell\,\underline{\Delta}(z)}\,\kappa_\ell^u(z)^{-j_0}\,\kappa_\ell(z)^j & \\[2ex] \qquad +\dfrac{2\left(\kappa_\ell^u(z)^{-j_0}\,\kappa_\ell(z)^j-\kappa_\ell^u(z)^{j-j_0}\right)}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}, & \text{if } j_0 \leq j \leq 0, \\[2ex] -\dfrac{2\,(\alpha_r-\alpha_m-(1+\alpha_r)\,\kappa_r(z)^{-1})}{\alpha_\ell\,\underline{\Delta}(z)}\,\kappa_\ell^u(z)^{-j_0}\,\kappa_\ell(z)^j & \\[2ex] \qquad +\dfrac{2\left(\kappa_\ell^u(z)^{-j_0}\,\kappa_\ell(z)^j-\kappa_\ell(z)^{j-j_0}\right)}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}, & \text{if } j < j_0 \leq 0. \end{cases} \tag{3.13}$$

The result of Proposition 3.1 is independent of the space $\ell^q(\mathbb{Z},\mathbb{C})$ that is considered since for any $j_0 \in \mathbb{Z}$ and $z \in \overline{\mathscr{U}} \setminus \{1\}$, the sequence $\mathcal{G}^{j_0}(z)$ belongs to the intersection of all spaces $\ell^p(\mathbb{Z},\mathbb{C})$, $p \in [1,+\infty]$ (it has exponential decay at infinity with respect to j).

Proof of Proposition 3.1. We shall give the proof of Proposition 3.1 in the case $\alpha_m \neq 1$. This assumption is used below to rewrite the second order scalar recurrence relation (3.11) as a first order recurrence relation for a vector V_{j+1} in terms of V_j. In the case $\alpha_m = 1$, which is left to the interested reader, one should rather write a first order recurrence relation for a vector V_{j-1} in terms of V_j, that is, going backwards.

Let us, for a moment, consider the slightly more general problem where we let $\mathbf{h} = (h_j)_{j\in\mathbb{Z}}$ be a given sequence in $\ell^q(\mathbb{Z},\mathbb{C})$ and take $z \in \overline{\mathscr{U}} \setminus \{1\}$. We then wish to construct a solution $\mathbf{v}(z) \in \ell^q(\mathbb{Z};\mathbb{C})$ to the resolvent equation:

$$(z\,\mathrm{Id} - \mathscr{L})\,\mathbf{v}(z) \,=\, \mathbf{h}\,.$$

To do so, we first introduce the augmented vectors

$$\forall\, j \in \mathbb{Z}\,,\quad V_j(z) := \begin{pmatrix} v_{j-1}(z) \\ v_j(z) \end{pmatrix} \in \mathbb{C}^2, \quad \text{and} \quad H_j := \begin{pmatrix} 0 \\ h_j \end{pmatrix} \in \mathbb{C}^2\,.$$

Using the definition of $\mathscr{L}$, we obtain that

$$\begin{cases} V_{j+1}(z) = \mathbb{M}_r(z)V_j(z) + \mathbb{A}_r H_j, & j \geq 2, \\ V_{j+1}(z) = \mathbb{M}_\ell(z)V_j(z) + \mathbb{A}_\ell H_j, & j \leq -1, \\ V_2(z) = \mathbb{M}_{2,1}(z)V_1(z) + \mathbb{A}_r H_1, & \\ V_1(z) = \mathbb{M}_{1,0}(z)V_0(z) + \mathbb{A}_{r,m} H_0, & \end{cases} \tag{3.14}$$

where the above matrices in (3.14) are defined as

$$\mathbb{M}_k(z) := \begin{pmatrix} 0 & 1 \\ \dfrac{1+\alpha_k}{1-\alpha_k} & \dfrac{2(1-\alpha_k^2-z)}{\alpha_k(1-\alpha_k)} \end{pmatrix}\,, \quad \mathbb{A}_k := \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2}{\alpha_k(1-\alpha_k)} \end{pmatrix}\,, \quad k \in \{r,\ell\},$$

and[6]

$$
\mathbb{M}_{2,1}(z) := \begin{pmatrix} 0 & 1 \\ \dfrac{\alpha_\ell(1+\alpha_m)}{\alpha_r(1-\alpha_r)} & \dfrac{2-\alpha_r(\alpha_r+\alpha_m)-2z}{\alpha_r(1-\alpha_r)} \end{pmatrix} ,
$$

$$
\mathbb{M}_{1,0}(z) := \begin{pmatrix} 0 & 1 \\ \dfrac{\alpha_\ell(1+\alpha_\ell)}{\alpha_r(1-\alpha_m)} & \dfrac{2-\alpha_\ell(\alpha_\ell+\alpha_m)-2z}{\alpha_r(1-\alpha_m)} \end{pmatrix} ,
$$

$$
\mathbb{A}_{r,m} := \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{2}{\alpha_r(1-\alpha_m)} \end{pmatrix} .
$$

Keeping the notation of Lemma 3.1, we denote, for $z \in \overline{\mathscr{U}} \setminus \{1\}$, by $\kappa_r(z) \in \mathbb{D}$ and $\kappa_r^u(z) \in \mathscr{U}$ the two eigenvalues of $\mathbb{M}_r(z)$ (these are the roots of the dispersion relation (3.1b)), while we denote by $\kappa_\ell(z) \in \mathscr{U}$ and $\kappa_\ell^u(z) \in \mathbb{D}$ the two eigenvalues of $\mathbb{M}_\ell(z)$ (these are the roots of the dispersion relation (3.1a)). Upon denoting $\Pi_{r,\ell}^{s,u}(z)$ the corresponding stable/unstable spectral projections together with $\mathbb{E}_{r,\ell}^{s,u}(z)$ the associated eigenspaces, we have that[7]:

$$
\mathbb{C}^2 = \mathbb{E}_k^s(z) \oplus \mathbb{E}_k^u(z), \quad k \in \{r, \ell\} ,
$$

with

$$
\mathbb{E}_k^s(z) := \mathrm{Span} \begin{pmatrix} 1 \\ \kappa_k(z) \end{pmatrix}, \quad \mathbb{E}_k^u(z) := \mathrm{Span} \begin{pmatrix} 1 \\ \kappa_k^u(z) \end{pmatrix}, \quad k \in \{r, \ell\} .
$$

Integrating the stable and unstable parts in (3.14), we have that

$$
\forall\, j \geq 2 , \quad \begin{cases} \Pi_r^u(z) V_j(z) = -\displaystyle\sum_{p=0}^{+\infty} \kappa_r^u(z)^{-1-p}\, \Pi_r^u(z) \mathbb{A}_r H_{j+p}, \\ \Pi_r^s(z) V_j(z) = \kappa_r(z)^{j-2}\, \Pi_r^s(z) V_2(z) + \displaystyle\sum_{p=2}^{j-1} \kappa_r(z)^{j-p-1}\, \Pi_r^s(z) \mathbb{A}_r H_p, \end{cases} \tag{3.15}
$$

from which we already deduce that

$$
\Pi_r^u(z) V_2(z) = -\sum_{p=0}^{+\infty} \kappa_r^u(z)^{-1-p}\, \Pi_r^u(z) \mathbb{A}_r H_{2+p}.
$$

On the other hand, we get

$$
\forall\, j \leq 0 , \quad \begin{cases} \Pi_\ell^u(z) V_j(z) = \displaystyle\sum_{p=0}^{+\infty} \kappa_\ell^u(z)^{p}\, \Pi_\ell^u(z) \mathbb{A}_\ell H_{j-p-1}, \\ \Pi_\ell^s(z) V_j(z) = \kappa_\ell(z)^{j}\, \Pi_\ell^s(z) V_0(z) - \displaystyle\sum_{p=j}^{-1} \kappa_\ell(z)^{p}\, \Pi_\ell^s(z) \mathbb{A}_\ell H_{j-p-1}, \end{cases} \tag{3.16}
$$

[6] The definition of $\mathbb{M}_{1,0}(z)$ and of $\mathbb{A}_{r,m}$ uses the assumption $\alpha_m \neq 1$.

[7] It should be kept in mind that the stable eigenvalue for the "right" state is $\kappa_r(z) \in \mathbb{D}$ since it corresponds to the dynamics for $j \geq 1$, while the stable eigenvalue for the "left" state is $\kappa_\ell(z) \in \mathscr{U}$ since it corresponds to the dynamics for $j \leq 0$.

such that

$$\Pi_\ell^u(z) V_0(z) = \sum_{p=0}^{+\infty} \kappa_\ell^u(z)^p \, \Pi_\ell^u(z) \mathbb{A}_\ell H_{-p-1}.$$

We notice that both vectors $\Pi_r^s(z)V_2(z)$ and $\Pi_\ell^s(z)V_0(z)$ still need to be determined, and if we are able to do so, then we shall have a solution to (3.14) at our disposal. First, we use the remaining two equations in (3.14) to obtain that

$$V_2(z) = \mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)V_0(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0 + \mathbb{A}_r H_1,$$

which we write instead as:

$$\begin{aligned}\Pi_r^s(z)V_2(z) - \mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^s(z)V_0(z)& \\ = \mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)V_0(z) - \Pi_r^u(z)V_2(z) &+ \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0 + \mathbb{A}_r H_1 \,.\end{aligned}$$

Upon writing

$$\Pi_r^s(z)V_2(z) = \underbrace{\chi_2(z)}_{\in\mathbb{C}} \, E_r^s(z), \quad E_r^s(z) := \begin{pmatrix} 1 \\ \kappa_r(z) \end{pmatrix},$$

and

$$\Pi_\ell^s(z)V_0(z) = \underbrace{\chi_0(z)}_{\in\mathbb{C}} \, E_\ell^s(z), \quad E_\ell^s(z) := \begin{pmatrix} 1 \\ \kappa_\ell(z) \end{pmatrix},$$

with the vector $(\chi_2(z), \chi_0(z)) \in \mathbb{C}^2$ still to be determined, we have that

$$\Pi_r^s(z)V_2(z) - \mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^s(z)V_0(z) = \underbrace{\begin{pmatrix} E_r^s(z) & -\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)E_\ell^s(z) \end{pmatrix}}_{:=\mathbb{B}(z)\in\mathscr{M}_2(\mathbb{C})} \begin{pmatrix} \chi_2(z) \\ \chi_0(z) \end{pmatrix}. \tag{3.17}$$

We thus obtain the last two relations:

$$\begin{aligned}\chi_2(z) &= \mathrm{e}_1^t \, \mathbb{B}(z)^{-1} \left(\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)V_0(z) - \Pi_r^u(z)V_2(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0 + \mathbb{A}_r H_1 \right), \\ \chi_0(z) &= \mathrm{e}_2^t \, \mathbb{B}(z)^{-1} \left(\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)V_0(z) - \Pi_r^u(z)V_2(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0 + \mathbb{A}_r H_1 \right),\end{aligned}$$

with $(\mathrm{e}_1, \mathrm{e}_2)$ the canonical basis of $\mathbb{C}^2$, at least as long as the matrix $\mathbb{B}(z) \in \mathscr{M}_2(\mathbb{C})$ in (3.17) is invertible. The invertibility property of the matrix $\mathbb{B}(z)$ is summarized in the following result which is a mere consequence of the analysis in the previous Section and of Assumption 1.

Lemma 3.3. *Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. Then, for any $z \in \overline{\mathscr{U}} \setminus \{1\}$, the matrix $\mathbb{B}(z)$ defined in (3.17) is invertible and its determinant $\Delta(z)$ satisfies:*

$$\forall\, z \in \overline{\mathscr{U}} \setminus \{1\}, \quad \Delta(z) \; = \; -\frac{\alpha_\ell \, \kappa_\ell(z)}{\alpha_r(1-\alpha_r)(1-\alpha_m)} \, \underline{\Delta}(z)\,,$$

with $\underline{\Delta}(z)$ *the Lopatinskii determinant given in* (3.3)*. Moreover, the matrix* $\mathbb{B}(z)$ *is given for* $z \in \overline{\mathscr{U}} \setminus \{1\}$ *by:*

$$\mathbb{B}(z) = \begin{pmatrix} 1 & b_1(z) \\ \kappa_r(z) & b_2(z) \end{pmatrix},$$

$$\text{with} \quad b_1(z) = -\frac{\alpha_\ell\,\kappa_\ell(z)\,(\alpha_\ell - \alpha_m + (1-\alpha_\ell)\,\kappa_\ell(z))}{\alpha_r\,(1-\alpha_m)},$$

$$b_2(z) = -\frac{\alpha_\ell\,\kappa_\ell(z)\,\big(\alpha_r\,(1-\alpha_m^2) - (2\,(z-1) + \alpha_r(\alpha_r+\alpha_m))\,(\alpha_\ell - \alpha_m + (1-\alpha_\ell)\,\kappa_\ell(z))\big)}{\alpha_r^2\,(1-\alpha_r)\,(1-\alpha_m)}.$$

We omit the proof of Lemma 3.3, which is a mere algebra exercise that uses the definition of the matrices $\mathbb{M}_{2,1}(z)$, $\mathbb{M}_{1,0}(z)$ and the dispersion relations (3.1). The above methodology shows how to construct a solution $\mathbf{v}(z)$ to the resolvent equation for any source term $\mathbf{h}$ and $z \in \overline{\mathscr{U}} \setminus \{1\}$. From Corollary 3.2, we know that the solution to the resolvent equation is necessarily unique in $\ell^q(\mathbb{Z};\mathbb{C})$ for any $z \in \overline{\mathscr{U}} \setminus \{1\}$ since the eigenvalue problem does not have any nontrivial solution[8]. We are now going to specify the above calculations to the case $\mathbf{h} = \boldsymbol{\delta}_{j_0}$ (the discrete Dirac mass located at $j_0 \in \mathbb{Z}$) and therefore derive the expression of the spatial Green's function $\mathcal{G}^{j_0}(z)$. We split the calculations according to the location of j_0 with respect to the discontinuity in the discrete shock $\overline{\mathbf{u}}$.

Let us recall that we denote by $\mathcal{G}^{j_0}(z) = \left(\mathcal{G}_j^{j_0}(z)\right)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z},\mathbb{C})$ the solution to the resolvent equation:

$$(z\mathrm{Id} - \mathscr{L})\,\mathcal{G}^{j_0}(z) = \boldsymbol{\delta}_{j_0}.$$

This solution exists and is unique for any $z \in \overline{\mathscr{U}} \setminus \{1\}$. We also introduce the corresponding augmented vectors

$$\forall\, j \in \mathbb{Z}, \quad G_j^{j_0}(z) := \begin{pmatrix} \mathcal{G}_{j-1}^{j_0}(z) \\ \mathcal{G}_j^{j_0}(z) \end{pmatrix} \in \mathbb{C}^2, \qquad \text{and} \qquad H_j^{j_0} := \begin{pmatrix} 0 \\ \delta_{j_0}(j) \end{pmatrix} \in \mathbb{C}^2.$$

Case I: $j_0 \geq 2$. We first recall that for $j \geq 2$ we have

$$\Pi_r^u(z) G_j^{j_0}(z) = -\sum_{p=0}^{+\infty} \kappa_r^u(z)^{-1-p}\,\Pi_r^u(z)\mathbb{A}_r H_{j+p}^{j_0},$$

which yields two cases:

- if $2 \leq j \leq j_0$ then
$$\Pi_r^u(z) G_j^{j_0}(z) = -\kappa_r^u(z)^{-1+j-j_0}\,\Pi_r^u(z)\mathbb{A}_r \mathrm{e}_2;$$
- if $j > j_0$ then
$$\Pi_r^u(z) G_j^{j_0}(z) = 0.$$

Next, for $j \leq 0$, we readily get that

$$\Pi_\ell^u(z) G_j^{j_0}(z) = \sum_{p=0}^{+\infty} \kappa_\ell^u(z)^p\,\Pi_\ell^u(z)\mathbb{A}_\ell H_{j-p-1}^{j0} = 0.$$

[8]Let us recall that the existence of a nontrivial solution to the eigenvalue problem $\mathscr{L}\,\mathbf{v} = z\,\mathbf{v}$ is equivalent to $\underline{\Delta}(z) = 0$, which is precluded for $z \in \overline{\mathscr{U}} \setminus \{1\}$ by Assumption 1.

Using the above results, we deduce that

$$\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)G_0^{j_0}(z) - \Pi_r^u(z)G_2^{j_0}(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0^{j_0} + \mathbb{A}_r H_1^{j_0} = -\Pi_r^u(z)G_2^{j_0}(z) = \kappa_r^u(z)^{1-j_0}\,\Pi_r^u(z)\mathbb{A}_r \mathrm{e}_2,$$

which implies that (the matrix $\mathbb{B}(z)$ is invertible for $z \in \overline{\mathscr{U}} \setminus \{1\}$ by Lemma 3.3):

$$\begin{aligned}\Pi_r^s(z)G_2^{j_0}(z) &= \kappa_r^u(z)^{1-j_0}\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_r^s(z)\,,\\ \Pi_\ell^s(z)G_0^{j_0}(z) &= \kappa_r^u(z)^{1-j_0}\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_\ell^s(z)\,.\end{aligned}$$

Now, for $j \geq 2$, we observe that

$$\Pi_r^s(z)G_j^{j_0}(z) = \kappa_r(z)^{j-2}\,\Pi_r^s(z)G_2^{j_0}(z) + \sum_{p=2}^{j-1}\kappa_r(z)^{j-p-1}\,\Pi_r^s(z)\mathbb{A}_r H_p^{j_0},$$

which yields two cases:

- if $2 \leq j \leq j_0$ then

$$\Pi_r^s(z)G_j^{j_0}(z) = \kappa_r(z)^{j-2}\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_r^s(z);$$

- if $j > j_0$ then

$$\Pi_r^s(z)G_j^{j_0}(z) = \kappa_r(z)^{j-2}\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_r^s(z) + \kappa_r(z)^{j-j_0-1}\,\Pi_r^s(z)\mathbb{A}_r\mathrm{e}_2.$$

Finally, for $j \leq 0$, we have

$$\begin{aligned}\Pi_\ell^s(z)G_j^{j_0}(z) &= \kappa_\ell(z)^j\,\Pi_\ell^s(z)G_0^{j_0}(z) - \sum_{p=j}^{-1}\kappa_\ell(z)^p\,\Pi_\ell^s(z)\mathbb{A}_\ell H_{j-p-1}^{j_0}\\ &= \kappa_\ell(z)^j\,\Pi_\ell^s(z)G_0^{j_0}(z) = \kappa_\ell(z)^j\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_\ell^s(z)\,.\end{aligned}$$

As a consequence, summarizing the above results, we have obtained for $j_0 \geq 2$ that

$$G_j^{j_0}(z) = \begin{cases}\kappa_r(z)^{j-2}\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_r^s(z) + \kappa_r(z)^{j-j_0-1}\,\Pi_r^s(z)\mathbb{A}_r E_2, & j > j_0,\\ \kappa_r(z)^{j-2}\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_r^s(z) - \kappa_r^u(z)^{j-j_0-1}\,\Pi_r^u(z)\mathbb{A}_r E_2, & 2 \leq j \leq j_0,\\ \kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]\mathbb{M}_{1,0}(z)E_\ell^s(z), & j = 1,\\ \kappa_\ell(z)^j\,\kappa_r^u(z)^{1-j_0}\,\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2\right]E_\ell^s(z), & j \leq 0.\end{cases}$$

To recover the expression for the spatial Green's function $\mathcal{G}_j^{j_0}(z)$, one just notes that

$$\mathcal{G}_j^{j_0}(z) = \mathrm{e}_2^t\,G_j^{j_0}(z)\,,$$

meaning that $\mathcal{G}_j^{j_0}(z) \in \mathbb{C}$ is the second coordinate in the vector $G_j^{j_0}(z) \in \mathbb{C}^2$. Further computations give

$$\mathrm{e}_2^t\,\Pi_r^s(z)\mathbb{A}_r\mathrm{e}_2 = \frac{2\,\kappa_r(z)}{\alpha_r(1-\alpha_r)(\kappa_r(z)-\kappa_r^u(z))}\,,\quad \mathrm{e}_2^t\,\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2 = \frac{2\,\kappa_r^u(z)}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}\,,$$

and

$$\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2=\frac{2\,(b_2(z)-b_1(z)\kappa_r^u(z))}{\alpha_r(1-\alpha_r)\Delta(z)(\kappa_r^u(z)-\kappa_r(z))},\quad \mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\Pi_r^u(z)\mathbb{A}_r\mathrm{e}_2=\frac{2}{\alpha_r(1-\alpha_r)\Delta(z)}.$$

This yields for $j_0\geq 2$ that

$$\mathcal{G}_j^{j_0}(z)=\begin{cases}\kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2\,(b_2(z)-b_1(z)\kappa_r^u(z))}{\alpha_r(1-\alpha_r)\Delta(z)(\kappa_r^u(z)-\kappa_r(z))} \\ \qquad +\kappa_r(z)^{j-j_0}\,\dfrac{2}{\alpha_r(1-\alpha_r)(\kappa_r(z)-\kappa_r^u(z))}, & j\geq j_0,\\ \kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2\,(b_2(z)-b_1(z)\kappa_r^u(z))}{\alpha_r(1-\alpha_r)\Delta(z)(\kappa_r^u(z)-\kappa_r(z))} \\ \qquad -\kappa_r^u(z)^{j-j_0}\,\dfrac{2}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}, & 2\leq j\leq j_0,\\ \kappa_r^u(z)^{1-j_0}\,\dfrac{2\,\left(\alpha_\ell(1+\alpha_\ell)+\kappa_\ell(z)(2-\alpha_\ell(\alpha_\ell+\alpha_m)-2z)\right)}{\alpha_r^2(1-\alpha_r)(1-\alpha_m)\Delta(z)}, & j=1,\\ \kappa_\ell(z)^{1+j}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2}{\alpha_r(1-\alpha_r)\Delta(z)}, & j\leq 0.\end{cases}$$

Next, we remark two points (by using Lemma 3.3):

$$\frac{2\,\left(\alpha_\ell(1+\alpha_\ell)+\kappa_\ell(z)(2-\alpha_\ell(\alpha_\ell+\alpha_m)-2z)\right)}{\alpha_r^2(1-\alpha_r)(1-\alpha_m)\Delta(z)}=-\frac{2\,b_1(z)}{\alpha_r(1-\alpha_r)\Delta(z)},$$

and

$$\frac{2\,(b_2(z)-b_1(z)\kappa_r^u(z))}{\alpha_r(1-\alpha_r)\Delta(z)(\kappa_r^u(z)-\kappa_r(z))}=-\frac{2\,b_1(z)}{\alpha_r(1-\alpha_r)\Delta(z)}+\frac{2}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))},$$

so that the above expressions for $\mathcal{G}_j^{j_0}(z)$ simplify to

$$\mathcal{G}_j^{j_0}(z)=\begin{cases}-\kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2\,b_1(z)}{\alpha_r(1-\alpha_r)\Delta(z)}+\dfrac{2\,\left(\kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}-\kappa_r(z)^{j-j_0}\right)}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}, & j\geq j_0,\\ -\kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2\,b_1(z)}{\alpha_r(1-\alpha_r)\Delta(z)}+\dfrac{2\,\left(\kappa_r(z)^{j-1}\,\kappa_r^u(z)^{1-j_0}-\kappa_r^u(z)^{j-j_0}\right)}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}, & 1\leq j\leq j_0,\\ \kappa_\ell(z)^{1+j}\,\kappa_r^u(z)^{1-j_0}\,\dfrac{2}{\alpha_r(1-\alpha_r)\Delta(z)}, & j\leq 0.\end{cases}$$

To conclude the analysis of this first case ($j_0\geq 2$), we recall the expression of the coefficient $b_1(z)$ and the link between the determinant $\Delta(z)$ of $\mathbb{B}(z)$ and the Lopatinskii determinant $\underline{\Delta}(z)$ (see Lemma 3.3). This gives, for any $j_0\geq 2$, the expression (3.12) for $\mathcal{G}_j^{j_0}(z)$ as given in Proposition 3.1.

Case II: $j_0=1$**.** We directly notice that for $j\geq 2$ we have

$$\Pi_r^u(z)G_j^1(z)=-\sum_{p=0}^{+\infty}\kappa_r^u(z)^{-1-p}\,\Pi_r^u(z)\mathbb{A}_rH_{j+p}^1=0,$$

together with

$$\forall\,j\leq 0\,,\quad \Pi_\ell^u(z)G_j^1(z)=\sum_{p=0}^{+\infty}\kappa_\ell^u(z)^p\,\Pi_\ell^u(z)\mathbb{A}_\ell H_{j-p-1}^1=0.$$

Using the above two results, we deduce that

$$\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)G_0^1(z) - \Pi_r^u(z)G_2^1(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0^1 + \mathbb{A}_r H_1^1 = \mathbb{A}_r \mathrm{e}_2,$$

which implies that

$$\begin{aligned}\Pi_r^s(z)G_2^1(z) &= \left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_r^s(z),\\ \Pi_\ell^s(z)G_0^1(z) &= \left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_\ell^s(z).\end{aligned}$$

Now, for $j \geq 2$, we get that

$$\begin{aligned}\Pi_r^s(z)G_j^1(z) &= \kappa_r(z)^{j-2}\,\Pi_r^s(z)G_2^1(z) + \sum_{p=2}^{j-1}\kappa_r(z)^{j-p-1}\,\Pi_r^s(z)\mathbb{A}_r H_p^1 = \kappa_r(z)^{j-2}\,\Pi_r^s(z)G_2^1(z)\\ &= \kappa_r(z)^{j-2}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_r^s(z).\end{aligned}$$

Finally, for $j \leq 0$, we have

$$\begin{aligned}\Pi_\ell^s(z)G_j^1(z) &= \kappa_\ell(z)^{j}\,\Pi_\ell^s(z)G_0^1(z) - \sum_{p=j}^{-1}\kappa_\ell(z)^{p}\,\Pi_\ell^s(z)\mathbb{A}_\ell H_{j-p-1}^1 = \kappa_\ell(z)^{j}\,\Pi_\ell^s(z)G_0^1(z)\\ &= \kappa_\ell(z)^{j}\,\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_\ell^s(z).\end{aligned}$$

As a consequence, when $j_0 = 1$, the spatial Green's function reads (in vector form):

$$G_j^1(z) = \begin{cases} \kappa_r(z)^{j-2}\,\left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_r^s(z), & j \geq 2,\\ \kappa_\ell(z)^{j}\,\left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\mathbb{A}_r\mathrm{e}_2\right] E_\ell^s(z), & j \leq 0.\end{cases}$$

Retaining only the second coordinate in each vector (or the first for $j = 2$, which gives the expression of $\mathcal{G}_1^1(z)$), we obtain the expressions:

$$\mathcal{G}_j^1(z) = \begin{cases} -\kappa_r(z)^{j-1}\,\dfrac{2\,b_1(z)}{\alpha_r(1-\alpha_r)\Delta(z)}, & j \geq 1,\\ \kappa_\ell(z)^{j+1}\,\dfrac{2}{\alpha_r(1-\alpha_r)\Delta(z)}, & j \leq 0.\end{cases}$$

The expression (3.12) (with $j_0 = 1$) for $\mathcal{G}_j^1(z)$ as given in Proposition 3.1 follows from the expression of $b_1(z)$ and of the determinant $\Delta(z)$, see Lemma 3.3.

Case III: $j_0 = 0$**.** We now feel free to shorten some details of the computations since many steps in Case III are similar to those in Case II ($j_0 = 1$). Since the Dirac mass is located at $j_0 = 0$, we have $\Pi_r^u(z)G_j^0(z) = 0$ for $j \geq 2$ and $\Pi_\ell^u(z)G_j^0(z) = 0$ for $j \leq 0$. Using these two facts, we deduce that

$$\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)G_0^0(z) - \Pi_r^u(z)G_2^0(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H_0^0 + \mathbb{A}_r H_1^0 = \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2,$$

which implies that

$$\begin{aligned}\Pi_r^s(z)G_2^0(z) &= \left[\mathrm{e}_1^t\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2\right] E_r^s(z),\\ \Pi_\ell^s(z)G_0^0(z) &= \left[\mathrm{e}_2^t\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2\right] E_\ell^s(z).\end{aligned}$$

Now, for $j \geq 2$, we get that

$$\Pi_r^s(z)G_j^0(z) = \kappa_r(z)^{j-2} \left[\mathrm{e}_1^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 \right] E_r^s(z).$$

and for $j \leq 0$, we have

$$\Pi_\ell^s(z)G_j^0(z) = \kappa_\ell(z)^{j} \left[\mathrm{e}_2^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 \right] E_\ell^s(z).$$

At this stage, the spatial Green's function (in vector form) for $j_0 = 0$ reads:

$$G_j^0(z) = \begin{cases} \kappa_r(z)^{j-2} \left[\mathrm{e}_1^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 \right] E_r^s(z), & j \geq 2, \\ \kappa_\ell(z)^{j} \left[\mathrm{e}_2^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 \right] E_\ell^s(z), & j \leq 0. \end{cases}$$

We compute the expressions:

$$\begin{aligned} \mathrm{e}_1^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 &= \frac{2}{\alpha_r(1-\alpha_m)\Delta(z)} \left(b_2(z) - b_1(z)\frac{2(1-z)-\alpha_r(\alpha_r+\alpha_m)}{\alpha_r(1-\alpha_r)} \right), \\ \mathrm{e}_2^t \, \mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}\mathrm{e}_2 &= \frac{2}{\alpha_r(1-\alpha_m)\Delta(z)} \left(-\kappa_r(z) + \frac{2(1-z)-\alpha_r(\alpha_r+\alpha_m)}{\alpha_r(1-\alpha_r)} \right). \end{aligned}$$

Looking at either the first or second coordinate of the vector $G_j^0(z)$ for $j \geq 2$, we obtain the expression of $\mathcal{G}_j^0(z)$ for any $j \geq 1$. Looking then at the second coordinate of $G_j^0(z)$ for $j \leq 0$, we obtain the expression of $\mathcal{G}_j^0(z)$ for any $j \leq 0$. In the end, we obtain that for $j_0 = 0$, the Green's function reads:

$$\mathcal{G}_j^0(z) = \begin{cases} \kappa_r(z)^{j-1} \dfrac{2}{\alpha_r(1-\alpha_m)\Delta(z)} \left(b_2(z) - b_1(z)\dfrac{2(1-z)-\alpha_r(\alpha_r+\alpha_m)}{\alpha_r(1-\alpha_r)} \right), & j \geq 1, \\ \kappa_\ell(z)^{j+1} \dfrac{2}{\alpha_r(1-\alpha_m)\Delta(z)} \left(-\kappa_r(z) + \dfrac{2(1-z)-\alpha_r(\alpha_r+\alpha_m)}{\alpha_r(1-\alpha_r)} \right), & j \leq 0. \end{cases}$$

The expression of $\mathcal{G}_j^0(z)$ for $j \geq 1$ is then simplified one last time by using the expressions of $b_1(z)$ and $b_2(z)$ in Lemma 3.3 and by using the relation between $\Delta(z)$ and $\underline{\Delta}(z)$, while the expression of $\mathcal{G}_j^0(z)$ for $j \leq 0$ is simplified by using the dispersion relation (3.1b) as well as the relation between $\Delta(z)$ and $\underline{\Delta}(z)$. We are then led to the expression (3.13) (with $j_0 = 0$) of $\mathcal{G}_j^0(z)$ given in Proposition 3.1.

Case IV: $j_0 \leq -1$. We directly notice that we have $\Pi_r^u(z)G_j^{j_0}(z) = 0$ for $j \geq 2$, while for $j \leq 0$ the expression

$$\Pi_\ell^u(z)G_j^{j_0}(z) = \sum_{p=0}^{+\infty} \kappa_\ell^u(z)^p \, \Pi_\ell^u(z)\mathbb{A}_\ell H_{j-p-1}^{j_0},$$

gives two cases:

- if $j_0 < j \leq 0$ then

$$\Pi_\ell^u(z)G_j^{j_0}(z) = \kappa_\ell^u(z)^{j-j_0-1} \, \Pi_\ell^u(z)\mathbb{A}_\ell \mathrm{e}_2;$$

- if $j \leq j_0$ then

$$\Pi_\ell^u(z)G_j^{j_0}(z) = 0.$$

Now, using the above results, we deduce that

$$\begin{aligned}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)G^{j_0}_0(z) - \Pi^u_r(z)G^{j_0}_2(z) + \mathbb{M}_{2,1}(z)\mathbb{A}_{r,m}H^{j_0}_0 + \mathbb{A}_r H^{j_0}_1 \\ = \mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)G^{j_0}_0(z) = \kappa^u_\ell(z)^{-j_0-1}\,\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell \mathrm{e}_2,\end{aligned}$$

which implies that

$$\begin{aligned}\Pi^s_r(z)G^{j_0}_2(z) &= \kappa^u_\ell(z)^{-j_0-1}\,\left[\mathrm{e}^t_1\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_r(z),\\ \Pi^s_\ell(z)G^{j_0}_0(z) &= \kappa^u_\ell(z)^{-j_0-1}\,\left[\mathrm{e}^t_2\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_\ell(z).\end{aligned}$$

Now, for $j \geq 2$, we observe that

$$\begin{aligned}\Pi^s_r(z)G^{j_0}_j(z) &= \kappa_r(z)^{j-2}\,\Pi^s_r(z)G^{j_0}_2(z)\\ &= \kappa_r(z)^{j-2}\,\kappa^u_\ell(z)^{-j_0-1}\,\left[\mathrm{e}^t_1\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_r(z).\end{aligned}$$

Finally, for $j \leq 0$, we have

$$\Pi^s_\ell(z)G^{j_0}_j(z) = \kappa_\ell(z)^j\,\Pi^s_\ell(z)G^{j_0}_0(z) - \sum_{p=j}^{-1}\kappa_\ell(z)^p\,\Pi^s_\ell(z)\mathbb{A}_\ell H^{j_0}_{j-p-1},$$

which yields two cases:

- if $j \leq j_0$ then

$$\begin{aligned}\Pi^s_\ell(z)G^{j_0}_j(z) = &\kappa_\ell(z)^j\,\kappa^u_\ell(z)^{-j_0-1}\left[\mathrm{e}^t_2\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_\ell(z)\\ &- \kappa_\ell(z)^{j-j_0-1}\,\Pi^s_\ell(z)\mathbb{A}_\ell\mathrm{e}_2;\end{aligned}$$

- if $j_0 < j \leq -1$ then

$$\Pi^s_\ell(z)G^{j_0}_j(z) = \kappa_\ell(z)^j\,\kappa^u_\ell(z)^{-j_0-1}\left[\mathrm{e}^t_2\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_\ell(z).$$

As a consequence, summarizing the above results, we have obtained for $j_0 \leq -1$ that

$$G^{j_0}_j(z) = \begin{cases}\kappa_r(z)^{j-2}\,\kappa^u_\ell(z)^{-j_0-1}\left[\mathrm{e}^t_1\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_r(z), & j \geq 2,\\ \kappa_\ell(z)^j\,\kappa^u_\ell(z)^{-j_0-1}\,\left[\mathrm{e}^t_2\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_\ell(z) & \\ \quad +\kappa^u_\ell(z)^{j-j_0-1}\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2, & j_0 < j \leq 0,\\ \kappa_\ell(z)^j\,\kappa^u_\ell(z)^{-j_0-1}\,\left[\mathrm{e}^t_2\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2\right] E^s_\ell(z) & \\ \quad -\kappa_\ell(z)^{j-j_0-1}\,\Pi^s_\ell(z)\mathbb{A}_\ell\mathrm{e}_2, & j \leq j_0.\end{cases} \tag{3.18}$$

For later use, we introduce the notation:

$$\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z) = \begin{pmatrix} m_1(z) & m_2(z)\\ m_3(z) & m_4(z)\end{pmatrix},$$

We then compute:

$$\mathrm{e}^t_2\,\Pi^s_\ell(z)\mathbb{A}_\ell\mathrm{e}_2 = \frac{2\,\kappa_\ell(z)}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa^u_\ell(z))},\quad \mathrm{e}^t_2\,\Pi^u_\ell(z)\mathbb{A}_\ell\mathrm{e}_2 = \frac{2\,\kappa^u_\ell(z)}{\alpha_\ell(1-\alpha_\ell)(\kappa^u_\ell(z)-\kappa_\ell(z))},$$

as well as

$$e_1^t\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)\mathbb{A}_\ell e_2 = \frac{2\Big(b_2(z)(m_1(z)+\kappa_\ell^u(z)m_2(z)) - b_1(z)(m_3(z)+\kappa_\ell^u(z)m_4(z))\Big)}{\alpha_\ell(1-\alpha_\ell)\Delta(z)(\kappa_\ell^u(z)-\kappa_\ell(z))},$$

$$e_2^t\,\mathbb{B}(z)^{-1}\mathbb{M}_{2,1}(z)\mathbb{M}_{1,0}(z)\Pi_\ell^u(z)\mathbb{A}_\ell e_2 = \frac{2\Big(-\kappa_r(z)(m_1(z)+\kappa_\ell^u(z)m_2(z)) + m_3(z)+\kappa_\ell^u(z)m_4(z)\Big)}{\alpha_\ell(1-\alpha_\ell)\Delta(z)(\kappa_\ell^u(z)-\kappa_\ell(z))}.$$

Looking at either the first or second coordinate of the vector $G_j^{j_0}(z)$ for $j \geq 2$, we obtain the expression of $\mathcal{G}_j^{j_0}(z)$ for any index $j \geq 1$:

$$\mathcal{G}_j^{j_0}(z) = \kappa_r(z)^{j-1}\,\kappa_\ell^u(z)^{-j_0-1}\,\frac{2\Big(b_2(z)(m_1(z)+\kappa_\ell^u(z)m_2(z)) - b_1(z)(m_3(z)+\kappa_\ell^u(z)m_4(z))\Big)}{\alpha_\ell(1-\alpha_\ell)\Delta(z)(\kappa_\ell^u(z)-\kappa_\ell(z))}.$$

Because of the definition of the matrix $\mathbb{B}(z)$, we have the relations:

$$m_1(z)+\kappa_\ell^u(z)m_2(z) = -b_1(z) + (\kappa_\ell^u(z)-\kappa_\ell(z))\,m_2(z)\,,$$
$$m_3(z)+\kappa_\ell^u(z)m_4(z) = -b_2(z) + (\kappa_\ell^u(z)-\kappa_\ell(z))\,m_4(z)\,,$$

from which we deduce:

$$\forall\, j \geq 1\,,\quad \mathcal{G}_j^{j_0}(z) = \kappa_r(z)^{j-1}\,\kappa_\ell^u(z)^{-j_0-1}\,\frac{2\,(b_2(z)m_2(z) - b_1(z)m_4(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)}\,,$$

and we now use the relations:

$$b_1(z) = -m_1(z) - \kappa_\ell(z)m_2(z)\,,\quad b_2(z) = -m_3(z) - \kappa_\ell(z)m_4(z)\,, \tag{3.19}$$

to further simplify the expression of $\mathcal{G}_j^{j_0}(z)$, $j \geq 1$, into:

$$\begin{aligned}\mathcal{G}_j^{j_0}(z) &= \kappa_r(z)^{j-1}\,\kappa_\ell^u(z)^{-j_0-1}\,\frac{2\,(m_1(z)m_4(z) - m_2(z)m_3(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)}\\ &= \kappa_r(z)^{j-1}\,\kappa_\ell^u(z)^{-j_0-1}\,\frac{2\,\det\mathbb{M}_{2,1}(z)\,\det\mathbb{M}_{1,0}(z)}{\alpha_\ell(1-\alpha_\ell)\Delta(z)}\\ &= \kappa_r(z)^{j-1}\,\kappa_\ell^u(z)^{-j_0-1}\,\frac{2\,\alpha_\ell(1+\alpha_\ell)\,(1+\alpha_m)}{\alpha_r^2(1-\alpha_r)(1-\alpha_m)(1-\alpha_\ell)\Delta(z)}\,.\end{aligned}$$

To obtain the expression (3.13) of $\mathcal{G}_j^{j_0}(z)$ for $j_0 \leq -1$ and $j \geq 1$, it only remains to use the relation $\kappa_\ell(z)\,\kappa_\ell^u(z) = -(1+\alpha_\ell)/(1-\alpha_\ell)$ (see (3.1a)), and the link between the determinant $\Delta(z)$ of $\mathbb{B}(z)$ and $\underline{\Delta}(z)$ (Lemma 3.3).

The only remaining task is to derive the expression (3.13) of $\mathcal{G}_j^{j_0}(z)$ for $j_0 \leq -1$ and $j \leq 0$. We go back to (3.18) and retain the second coordinate of the vector $G_j^{j_0}(z)$ for $j \leq 0$. We obtain:

$$\mathcal{G}_j^{j_0}(z) = \begin{cases} \kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1}\,\dfrac{2\,(-\kappa_r(z)(m_1(z)+\kappa_\ell^u(z)m_2(z)) + m_3(z)+\kappa_\ell^u(z)m_4(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)(\kappa_\ell^u(z)-\kappa_\ell(z))} \\ \qquad + \dfrac{2\,\kappa_\ell^u(z)^{j-j_0}}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j_0 < j \leq 0, \\ \kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1}\,\dfrac{2\,(-\kappa_r(z)(m_1(z)+\kappa_\ell^u(z)m_2(z)) + m_3(z)+\kappa_\ell^u(z)m_4(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)(\kappa_\ell^u(z)-\kappa_\ell(z))} \\ \qquad + \dfrac{2\,\kappa_\ell(z)^{j-j_0}}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j \leq j_0. \end{cases}$$

Recalling the relations (3.19), we get:

$$-\kappa_r(z)(m_1(z)+\kappa_\ell^u(z)m_2(z))+m_3(z)+\kappa_\ell^u(z)m_4(z)=(\kappa_\ell^u(z)-\kappa_\ell(z))(m_4(z)-\kappa_r(z)\,m_2(z))-\Delta(z)\,,$$

and this simplifies the expression of $\mathcal{G}_j^{j_0}(z)$ into:

$$\mathcal{G}_j^{j_0}(z)=\begin{cases}\kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1}\,\dfrac{2\,(m_4(z)-\kappa_r(z)m_2(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)} \\ \qquad +\dfrac{2(\kappa_\ell^u(z)^{j-j_0}-\kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1})}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j_0<j\leq 0, \\ \kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1}\,\dfrac{2\,(m_4(z)-\kappa_r(z)m_2(z))}{\alpha_\ell(1-\alpha_\ell)\Delta(z)} \\ \qquad +\dfrac{2(\kappa_\ell(z)^{j-j_0}-\kappa_\ell(z)^{j+1}\,\kappa_\ell^u(z)^{-j_0-1})}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j\leq j_0.\end{cases}$$

It turns out that we can slightly modify the above expression as follows. Using the expressions:

$$\Delta(z)=b_2(z)-\kappa_r(z)\,b_1(z)=\kappa_r(z)\,m_1(z)-m_3(z)+\kappa_\ell(z)(\kappa_r(z)\,m_2(z)-m_4(z))\,,$$
$$m_1(z)=\frac{\alpha_\ell(1+\alpha_\ell)}{\alpha_r(1-\alpha_m)}\,,\quad m_3(z)=-\frac{\alpha_\ell(1+\alpha_\ell)}{\alpha_r^2(1-\alpha_r)(1-\alpha_m)}(2(z-1)+\alpha_r(\alpha_r+\alpha_m))\,,$$

we notice that

$$\begin{aligned}\frac{1}{\kappa_\ell^u(z)}\,\frac{m_4(z)-\kappa_r(z)m_2(z)}{\alpha_\ell(1-\alpha_\ell)\Delta(z)}&=-\frac{1}{\alpha_\ell(1+\alpha_\ell)\Delta(z)}\,(-\Delta(z)+\kappa_r(z)m_1(z)-m_3(z))\\ &=\frac{1}{\alpha_\ell(1+\alpha_\ell)}+\frac{1}{\alpha_r(1-\alpha_m)\Delta(z)}\left(-\kappa_r(z)+\frac{2(1-z)-\alpha_r(\alpha_r+\alpha_m)}{\alpha_r(1-\alpha_r)}\right)\\ &=-\frac{1}{\alpha_\ell(1-\alpha_\ell)\kappa_\ell(z)\kappa_\ell^u(z)}+\frac{\alpha_r-\alpha_m-(1+\alpha_r)\,\kappa_r(z)^{-1}}{\alpha_r(1-\alpha_r)(1-\alpha_m)\Delta(z)}.\end{aligned}$$

We end up with the following expression

$$\mathcal{G}_j^{j_0}(z)=\begin{cases}\dfrac{2\,\kappa_\ell(z)^{j+1}\kappa_\ell^u(z)^{-j_0}}{\alpha_r(1-\alpha_r)(1-\alpha_m)\Delta(z)}\big(\alpha_r-\alpha_m-(1+\alpha_r)\,\kappa_r(z)^{-1}\big) \\ \qquad +\dfrac{2\big(\kappa_\ell(z)^{j-j_0}-\kappa_\ell(z)^{j}\,\kappa_\ell^u(z)^{-j_0}\big)}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j\leq j_0, \\ \dfrac{2\,\kappa_\ell(z)^{j+1}\kappa_\ell^u(z)^{-j_0}}{\alpha_r(1-\alpha_r)(1-\alpha_m)\Delta(z)}\big(\alpha_r-\alpha_m-(1+\alpha_r)\,\kappa_r(z)^{-1}\big) \\ \qquad +\dfrac{2\big(\kappa_\ell^u(z)^{j-j_0}-\kappa_\ell(z)^{j}\,\kappa_\ell^u(z)^{-j_0}\big)}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell^u(z)-\kappa_\ell(z))}, & j_0\leq j\leq 0.\end{cases}$$

We then use the expression of $\Delta(z)$ given in Lemma 3.3 and derive the expression (3.13) of $\mathcal{G}_j^0(z)$ given in Proposition 3.1 (for $j_0\leq -1$ and $j\leq 0$). The proof of Proposition 3.1 is therefore complete. □

The expression of the spatial Green's function given in Proposition 3.1 gives us in a straightforward way the following estimates away from the point 1. From inspection of the expressions (3.12) and (3.13), it is useful to introduce a tiny modification of the spatial Green's function that we define as follows. For any couple of integers $(j_0, j) \in \mathbb{Z}^2$, we define the following function:

$$\widetilde{\mathcal{G}}_j^{j_0}(z) := \begin{cases} \mathcal{G}_j^{j_0}(z) + \dfrac{2\,\kappa_r^u(z)^{j-j_0}}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}\,, & \text{if } 1 \le j \le j_0, \\[2ex] \mathcal{G}_j^{j_0}(z) + \dfrac{2\,\kappa_r(z)^{j-j_0}}{\alpha_r(1-\alpha_r)(\kappa_r^u(z)-\kappa_r(z))}\,, & \text{if } 1 \le j_0 < j, \\[2ex] \mathcal{G}_j^{j_0}(z) + \dfrac{2\,\kappa_\ell^u(z)^{j-j_0}}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}\,, & \text{if } j_0 \le j \le 0, \\[2ex] \mathcal{G}_j^{j_0}(z) + \dfrac{2\,\kappa_\ell(z)^{j-j_0}}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}\,, & \text{if } j < j_0 \le 0, \\[2ex] \mathcal{G}_j^{j_0}(z)\,, & \text{otherwise.} \end{cases} \tag{3.20}$$

Under the assumptions made in Proposition 3.1, Proposition 3.1 and Lemma 3.1 show that $\widetilde{\mathcal{G}}_j^{j_0}$ is well-defined on $\overline{\mathscr{U}} \setminus \{1\}$. Furthermore, this function is holomorphic on $\mathscr{U}$ and can be holomorphically extended in the neighborhood of any point of $\mathbb{S}^1 \setminus \{1\}$. The interest for defining this reduced function $\widetilde{\mathcal{G}}_j^{j_0}$ will be made clear in Chapter 4. Our result is the following.

Corollary 3.3. *Let the weak solution (2.2) satisfy the entropy inequalities (2.4). Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. Then for any $\varepsilon_\star > 0$, there exist constants $\eta_\star > 0$, $C > 0$ and $c > 0$ such that, if we define the set:*

$$\mathscr{Z}_{\varepsilon_\star,\eta_\star} := \{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-\eta_\star} \le |\zeta| \le 2\} \setminus \{\zeta = \mathrm{e}^\tau \in \mathbb{C} \,|\, \tau \in \mathbf{B}_{\varepsilon_\star}(0)\}\,, \tag{3.21}$$

then, for any couple $(j, j_0) \in \mathbb{Z}^2$, the function $\widetilde{\mathcal{G}}_j^{j_0}$ defined in (3.20) depends holomorphically on z on $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$ and it satisfies the uniform bound:

$$\forall\, z \in \mathscr{Z}_{\varepsilon_\star,\eta_\star}\,, \quad \forall\,(j_0, j) \in \mathbb{Z}^2\,, \quad \left|\widetilde{\mathcal{G}}_j^{j_0}(z)\right| \le C\,\exp(-c\,(|j| + |j_0|))\,.$$

The region $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$ is schematically depicted in Figure 3.2.

Proof of Corollary 3.3. The proof of Corollary 3.3 directly follows from the expressions (3.12) and (3.13) and the definition (3.20). Indeed, let $\varepsilon_\star > 0$ be given. Then the set:

$$\{\zeta \in \mathbb{C} \,|\, 1 \le |\zeta| \le 2\} \setminus \{\zeta = \mathrm{e}^\tau \in \mathbb{C} \,|\, \tau \in \mathbf{B}_{\varepsilon_\star}(0)\}\,,$$

is a compact subset of $\mathscr{O}$ and also of $\overline{\mathscr{U}} \setminus \{1\}$. Moreover, thanks to Assumption 1, we know that the Lopatinskii determinant $\underline{\Delta}$ does not vanish on that set. Lemma 3.1 also shows that the dispersion relation (3.1a), resp. (3.1b), has two distinct roots κ_ℓ and κ_ℓ^u, resp. κ_r and κ_r^u, for any z in that set. By using Lemma 3.1 (for the holomorphy properties of the roots of (3.1a) and (3.1b)), Lemma 3.2 (for the holomorphy properties of $\underline{\Delta}$) and Assumption 1, we can thus choose some $\eta_\star > 0$ such that:

- the set $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$ defined in (3.21) is a compact subset of $\mathscr{O}$ so that $\underline{\Delta}$ is holomorphic on $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$,

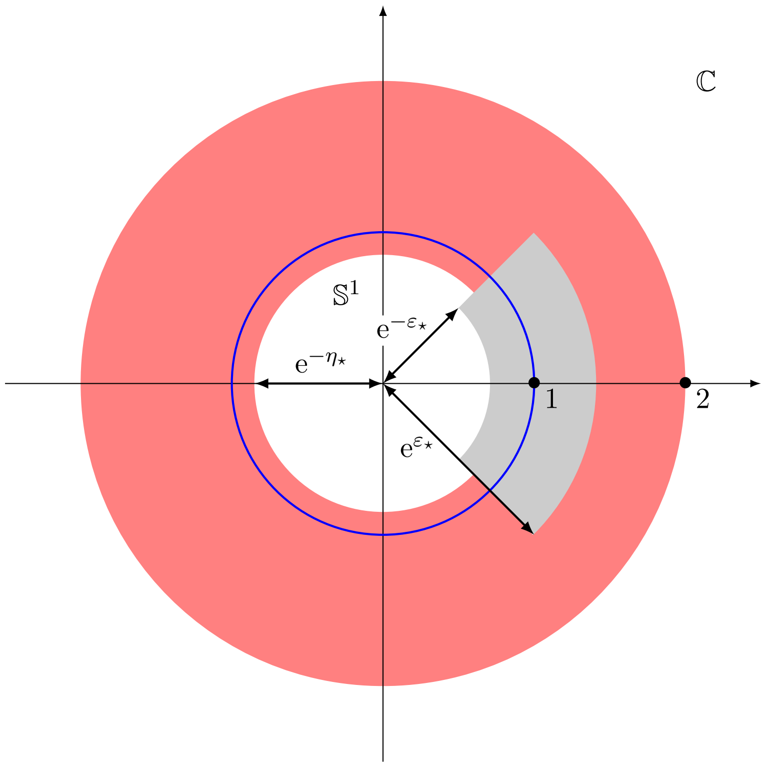

Figure 3.2: *The region* $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$ *(in red) of Corollary 3.3. In grey: the set* $\{\zeta = \mathrm{e}^\tau \in \mathbb{C} \,|\, \tau \in \mathbf{B}_{\varepsilon_\star}(0)\}$. *In blue: the unit circle* $\mathbb{S}^1$.

- $\kappa_\ell(z), \kappa_r^u(z)$ belong to $\mathscr{U}$, $\kappa_r(z), \kappa_\ell^u(z)$ belong to $\mathbb{D}$ for any $z \in \mathscr{Z}_{\varepsilon_\star,\eta_\star}$ and those four functions depend holomorphically on z on $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$,
- $\kappa_\ell(z)$ is different from $\kappa_\ell^u(z)$, resp. $\kappa_r(z)$ is different from $\kappa_r^u(z)$, for any $z \in \mathscr{Z}_{\varepsilon_\star,\eta_\star}$,
- $\underline{\Delta}$ is holomorphic and does not vanish on $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$.

All the above properties imply that for any couple $(j, j_0) \in \mathbb{Z}^2$, the function $\widetilde{\mathcal{G}}_j^{j_0}$ defined in (3.20) extends to a holomorphic function on $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$.

We now consider $j_0 \geq 1$ and look at the expression (3.12) for the spatial Green's function and the definition (3.20) for $\widetilde{\mathcal{G}}_j^{j_0}(z)$. Since $\mathscr{Z}_{\varepsilon_\star,\eta_\star}$ is compact, we can find some constants C such that for any $z \in \mathscr{Z}_{\varepsilon_\star,\eta_\star}$, there holds:

$$\left|\widetilde{\mathcal{G}}_j^{j_0}(z)\right| \leq \begin{cases} C\,|\kappa_r^u(z)|^{-j_0}\,|\kappa_\ell(z)|^j\,, & \text{if } j \leq 0, \\ \\ C\,|\kappa_r^u(z)|^{-j_0}\,|\kappa_r(z)|^j\,, & \text{if } j \geq 1. \end{cases}$$

It remains to use uniform lower or upper bounds:

$$|\kappa_r^u(z)| \geq \mathrm{e}^{-c}\,, \quad |\kappa_\ell(z)| \geq \mathrm{e}^{-c}\,, \quad |\kappa_r(z)| \leq \mathrm{e}^{-c}\,,$$

with a uniform constant $c > 0$, and the conclusion of Corollary 3.3 follows in the case $j_0 \geq 1$. The case $j_0 \leq 0$ follows from similar arguments by using the expression (3.13) for the spatial Green's function and the definition (3.20) for $\widetilde{\mathcal{G}}_j^{j_0}(z)$. □

3.3 Spectral stability. Proof of Theorem 2.3

This short paragraph is devoted to the proof of Theorem 2.3 (which is a more general version of Theorem 2.2) using all above ingredients, that is our analysis of the Lopatinskii determinant and the construction of the spatial Green's function for $z \in \overline{\mathscr{U}} \setminus \{1\}$.

We shall only give the proof of Theorem 2.3 and leave the analogous analysis for either convex or concave fluxes to the interested reader. Let us first quickly show that 1 is an eigenvalue of $\mathscr{L}$, which is reminiscent of the fact that the Lopatinskii determinant vanishes at 1 (Lemma 3.2), see [15, 5, 14, 28, 27]. The fact that 1 is an eigenvalue for $\mathscr{L}$ was already proven in [28, Theorem 2.3]. We just reproduce a proof here, with our notation, for the sake of completeness.

We consider the expressions (3.12) and (3.13) of the spatial Green's function. Since $\underline{\Delta}$ vanishes at 1, these expressions incorporate a (simple) pole at $z = 1$. We thus introduce the sequence $(\mathcal{H}_j)_{j \in \mathbb{Z}}$ that is defined by:

$$\mathcal{H}_j \, := \, \lim_{z \to 1} \, (z-1) \, \mathcal{G}_j^{j_0}(z) \, ,$$

and whose precise expression is given by:

$$\mathcal{H}_j \, = \, \begin{cases} -\dfrac{2\,(1-\alpha_m)}{\alpha_\ell \, \underline{\Delta}'(1)} \, \kappa_\ell(1)^j \, , & \text{if } j \le 0 \, , \\ \\ \dfrac{2\,(1+\alpha_m)}{\alpha_r \, \underline{\Delta}'(1)} \, \kappa_r(1)^{j-1} \, , & \text{if } j \ge 1 \, , \end{cases} \tag{3.22}$$

the expression being independent of $j_0 \in \mathbb{Z}$. The sequence given in (3.22) is nonzero and it belongs to any $\ell^q(\mathbb{Z};\mathbb{C})$ since it has exponential decay at infinity (recall that $\kappa_r(1)$ belongs to $\mathbb{D}$ and $\kappa_\ell(1)$ belongs to $\mathscr{U}$). It is also a mere algebra exercise to verify that the sequence given in (3.22) belongs to the kernel of the operator $\mathrm{Id} - \mathscr{L}$, as expected from the above formal analysis. This means that 1 is an eigenvalue for $\mathscr{L}$ in any $\ell^q(\mathbb{Z};\mathbb{C})$.

Let us now show that the set[9] $\overline{\mathscr{U}} \setminus \{1\}$ lies in the resolvent set for $\mathscr{L}$. We consider $\mathbf{h} \in \ell^q(\mathbb{Z};\mathbb{C})$ and we wish to construct a solution in $\ell^q(\mathbb{Z};\mathbb{C})$ to the resolvent equation:

$$(z \, \mathrm{Id} - \mathscr{L}) \, \mathbf{v}(z) \, = \, \mathbf{h} \, . \tag{3.23}$$

The solution will necessarily be unique because of Corollary 3.2. For $z \in \overline{\mathscr{U}} \setminus \{1\}$ and $j \in \mathbb{Z}$, we define:

$$v_j(z) \, := \, \sum_{j_0 \in \mathbb{Z}} \mathcal{G}_j^{j_0}(z) \, h_{j_0} \, ,$$

with $\mathcal{G}_j^{j_0}(z)$ given in Proposition 3.1. Let us first show that the sequence $\mathbf{v}(z) = (v_j(z))_{j \in \mathbb{Z}}$ thus defined belongs indeed to $\ell^q(\mathbb{Z};\mathbb{C})$. The arguments below are given for a fixed $z \in \overline{\mathscr{U}} \setminus \{1\}$ and the constants may depend on z. From the expressions (3.12), (3.13) and using Lemma 3.1 and Assumption 1, we obtain bounds of the form:

$$\forall \, (j_0, j) \in \mathbb{Z}^2 \, , \quad \left| \mathcal{G}_j^{j_0}(z) \right| \le C_z \, \exp(-c_z \, (|j| + |j_0|)) + C_z \, \exp(-c_z \, |j - j_0|) \, ,$$

[9] In the case of a convex or concave flux, one can show that the whole set $\mathscr{O}$ lies in the resolvent set of $\mathscr{L}$ since the Lopatinskii determinant $\underline{\Delta}$ does not vanish on $\mathscr{O}$ and Proposition 3.1 extends to any $z \in \mathscr{O}$ in that case.

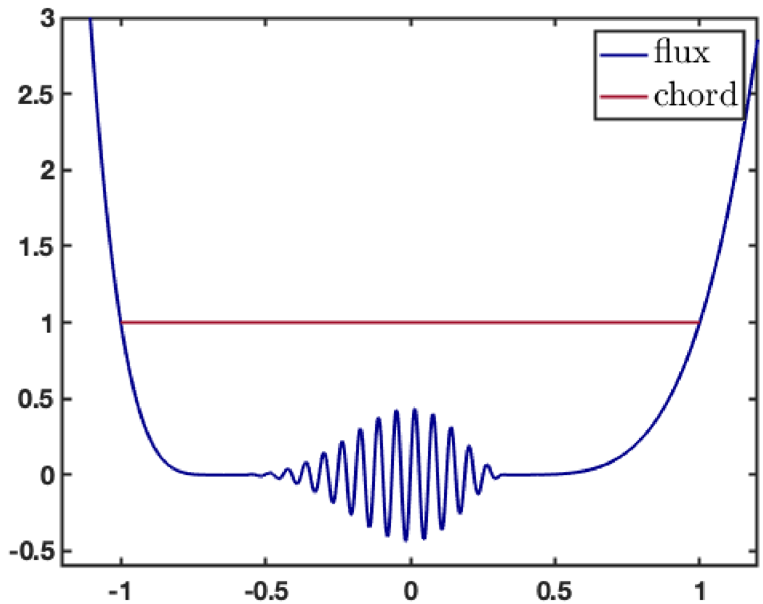

Figure 3.3: *An example of flux f that yields spectrally unstable shock profiles. The graph of the flux is depicted in blue and the chord between $u_r = -1$ and $u_\ell = +1$ is depicted in red. The Rankine-Hugoniot condition (2.3) and Oleinik's entropy condition are satisfied.*

with positive constants C_z, c_z that may depend on z but that do not depend on (j_0, j). Applying either the Hölder or the Young inequality, we obtain that the above defined sequence $\mathbf{v}(z) := (v_j(z))_{j\in\mathbb{Z}}$ belongs to $\ell^q(\mathbb{Z};\mathbb{C})$. It is then a mere exercise to verify that $\mathbf{v}(z)$ is a solution to the resolvent equation (3.23) (this is rather easy in this framework since $\mathscr{L}$ involves a finite stencil). We have thus shown that any point $z \in \overline{\mathscr{U}} \setminus \{1\}$ belongs to the resolvent set of $\mathscr{L}$, which completes the proof of Theorem 2.3.

3.4 Instability cases

This whole article will be devoted to *stable* discrete shock profiles, but let us just take a little time to discuss two *unstable* cases, just to show that spectral instabilities may occur. We go back to the expression (3.3) of the Lopatinskii determinant. As explained in Remark 3.2, in the symmetric case $f'(u_\ell) = -f'(u_r)$, the Lopatinskii determinant $\underline{\Delta}$ is independent of the mid-point derivative $f'((u_\ell + u_r)/2)$ and spectral stability[10] always holds.

Let us therefore assume from now on $f'(u_\ell) \neq -f'(u_r)$ so that one has $\alpha_\ell + \alpha_r \neq 0$. We first go back to the expression (3.4) of the derivative $\underline{\Delta}'(1)$. We observe from this expression that $\underline{\Delta}'(1)$ can be zero if α_m is given by:

$$\alpha_m = \frac{\alpha_r - \alpha_\ell + 2\,\alpha_\ell\,\alpha_r}{\alpha_\ell + \alpha_r}\,. \tag{3.24}$$

This gives for instance the value $\alpha_m = 13/3$ in the case $(\alpha_\ell, \alpha_r) = (1/3, -2/3)$. The case $\underline{\Delta}'(1) = 0$ is a weak form of instability that we do not study here but that can be achieved with a non-convex flux

[10]Spectral stability means here that Assumption 1 is satisfied.

function f. An example of such a function f is depicted in Figure 3.3 with the choice $u_r = -1$, $u_\ell = +1$ and $f(u_r) = f(u_\ell) = 1$ so that the Rankine-Hugoniot condition (2.3) is satisfied. The entropy inequalities (2.4) are also satisfied and we can choose the CFL parameter λ sufficiently small in such a way that the condition (2.12) is met. We can then tune the "small amplitude" oscillations near the origin in such a way that the derivative $f'(0)$ satisfies (3.24) (recall the relation $\alpha_m = \lambda\, f'(0)$, see (2.9)) and the graph of f between -1 and 1 lies below the horizontal chord of height 1. This means that not only the Lax shock entropy inequalities (2.4) are satisfied but also Oleinik's entropy condition which is stronger, see [12, 26].

A more severe instability scenario corresponds to finding a root of $\underline{\Delta}$ in the instability region $\mathscr{U}$. For concreteness, we still assume that the end points of the shock are $u_r = -1$, $u_\ell = +1$, that the flux f satisfies $f(u_r) = f(u_\ell) = 1$, $f'(u_r) = -2\, f'(u_\ell)$ and that the parameter λ has been chosen in such a way that $(\alpha_\ell, \alpha_r) = (1/3, -2/3)$. We then compute:

$$\kappa_\ell(2) \, = \, -5 \, - 3\sqrt{3}\,, \quad \kappa_r(2) \, = \, \frac{13 \, - 3\sqrt{21}}{10}\,.$$

We then see on the expression (3.3) that $\underline{\Delta}$ vanishes at $z = 2 \in \mathscr{U}$ provided that we have[11]:

$$\alpha_m \, \left(\frac{3}{2} + 2\sqrt{3} - \frac{\sqrt{21}}{2} \right) \, = \, \frac{7}{2} + 3 \, \left(\sqrt{3} + \sqrt{7} + \frac{\sqrt{21}}{2} \right) \, .$$

Once again, this can be achieved by tuning small amplitude oscillations near the origin to have the desired value for $f'(0)$, as shown in Figure 3.3. Spectral instabilities may therefore occur for stationary shock profiles of the Lax-Wendroff scheme even though Oleinik's entropy condition is satisfied.

3.5 Decomposing the spatial Green's function

The detailed expression of the spatial Green's function $(\mathcal{G}_j^{j_0}(z))_{j \in \mathbb{Z}}$ is given in Proposition 3.1. For later use, we need to decompose the expression of $\mathcal{G}_j^{j_0}(z)$ by isolating several parts in it and specifically its singular behavior near $z = 1$. A convenient way to do so is to introduce yet two other Green's functions which correspond to that of the Lax-Wendroff scheme for a constant coefficient transport operator on the whole real line. The chosen velocity will be either $f'(u_\ell)$ or $f'(u_r)$ depending on the sign of the initial position j_0. The choice of the velocity $f'(u_\ell)$ corresponds to the expression of the operator $\mathscr{L}$ in the region $\{j \le -1\}$, see (2.11), and the choice of the velocity $f'(u_r)$ corresponds to the expression of the operator $\mathscr{L}$ in the region $\{j \ge 2\}$, see again (2.11). We thus devote the following paragraph to recalling several facts on the Green's function for the Lax-Wendroff on the whole real line. Additional material in this case can be found in [10] and [9].

3.5.1 The free Green's function on the whole real line

We pause for a while and go back to the definition (2.11) of the linearized operator $\mathscr{L}$. In the regions $\{j \ge 2\}$ and $\{j \le -1\}$, the coefficients in the operator $\mathscr{L}$ are independent of the spatial index, meaning that $\mathscr{L}$ reduces to a convolution operator that corresponds to the linearization of the Lax-Wendroff scheme

[11] This gives the value $\alpha_m \simeq 8,79$.

at the constant state u_ℓ or u_r. We thus introduce the convolution operators $\mathscr{L}_\ell, \mathscr{L}_r$ that are defined on complex valued sequences $\mathbf{v} = (v_j)_{j \in \mathbb{Z}}$ defined on $\mathbb{Z}$ as follows:

$$\forall\, j \in \mathbb{Z}\,, \quad (\mathscr{L}_\ell\, \mathbf{v})_j \,:=\, v_j - \frac{\alpha_\ell}{2}\,(v_{j+1} - v_{j-1}) + \frac{\alpha_\ell^2}{2}\,(v_{j+1} - 2\,v_j + v_{j-1})\ , \tag{3.25a}$$

$$(\mathscr{L}_r\, \mathbf{v})_j \,:=\, v_j - \frac{\alpha_r}{2}\,(v_{j+1} - v_{j-1}) + \frac{\alpha_r^2}{2}\,(v_{j+1} - 2\,v_j + v_{j-1})\ . \tag{3.25b}$$

We recall that α_ℓ and α_r are defined in (2.9). The operators in (3.25) are nothing but the operators arising from the Lax-Wendroff scheme applied to the transport equation with velocity equal to either $f'(u_\ell)$ or $f'(u_r)$. Thanks to the Lévy-Wiener Theorem [22], the spectrum of the operators $\mathscr{L}_\ell$ and $\mathscr{L}_r$ on any space $\ell^q(\mathbb{Z};\mathbb{C})$ is completely known (see [30] for more on the spectral analysis of convolution operators). We have:

$$\sigma(\mathscr{L}_\ell) = \Big\{ 1 - 2\,\alpha_\ell^2\, \sin^2 \frac{\xi}{2} + \mathbf{i}\,\alpha_\ell\, \sin\xi \,|\, \xi \in \mathbb{R} \Big\}\,,$$
$$\sigma(\mathscr{L}_r) = \Big\{ 1 - 2\,\alpha_r^2\, \sin^2 \frac{\xi}{2} + \mathbf{i}\,\alpha_r\, \sin\xi \,|\, \xi \in \mathbb{R} \Big\}\,,$$

where the result is independent of $q \in [1,+\infty]$. For instance, the spectrum of $\mathscr{L}_\ell$ is represented as the black curve (an ellipse, actually) in Figure 3.1. Due to the restriction (2.12), the spectrum of both $\mathscr{L}_\ell$ and $\mathscr{L}_r$ is included in the closed unit disk $\overline{\mathbb{D}}$ with a single tangency point at 1 with the unit circle.

We can then proceed as in Section 3.2 above and introduce the spatial Green's function for either $\mathscr{L}_\ell$ or $\mathscr{L}_r$. Because of the spatial invariance in (3.25a) and (3.25b), it is sufficient to look at the case where the Dirac mass $\boldsymbol{\delta}_{j_0}$ is located at $j_0 = 0$. Hence, for z in the exterior $\mathscr{O}$ of the curve (2.13), we have that z lies in the resolvent set of both $\mathscr{L}_\ell$ and $\mathscr{L}_r$, and we can therefore introduce the solutions $\overline{\mathcal{G}}_\ell(z) = (\overline{\mathcal{G}}_{\ell,j}(z))_{j\in\mathbb{Z}}$ and $\overline{\mathcal{G}}_r(z) = (\overline{\mathcal{G}}_{r,j}(z))_{j\in\mathbb{Z}}$ to the resolvent problems:

$$(z\,\mathrm{Id} - \mathscr{L}_\ell)\,\overline{\mathcal{G}}_\ell(z) \,=\, \boldsymbol{\delta}_0\,, \quad (z\,\mathrm{Id} - \mathscr{L}_r)\,\overline{\mathcal{G}}_r(z) \,=\, \boldsymbol{\delta}_0\,. \tag{3.26}$$

The computations that lead to the precise expressions for $(\overline{\mathcal{G}}_{\ell,j}(z))_{j\in\mathbb{Z}}$ and $(\overline{\mathcal{G}}_{r,j}(z))_{j\in\mathbb{Z}}$ follow from the same methodology as in Sections 3.1 and 3.2 (we refer to [10] for an even more general analysis in the case of finite difference schemes with arbitrary, possibly infinite, stencils). The analysis is actually much simpler for pure convolution operators than what we did in Sections 3.1 and 3.2 since there is no Lopatinskii determinant involved. We thus feel free to state without proof the following result that is in the same vein as Proposition 3.1.

Proposition 3.2. *Under the condition (2.12) on λ, for any z in the exterior $\mathscr{O}$ of the spectral curve (2.13), there exist unique solutions $\overline{\mathcal{G}}_\ell(z) = (\overline{\mathcal{G}}_{\ell,j}(z))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C})$ and $\overline{\mathcal{G}}_r(z) = (\overline{\mathcal{G}}_{r,j}(z))_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{C})$ to the equations (3.26). These sequences are given by:*

$$\overline{\mathcal{G}}_{r,j}(z) \,=\, \begin{cases} \dfrac{-2\,\kappa_r^u(z)^j}{\alpha_r(1-\alpha_r)(\kappa_r^u(z) - \kappa_r(z))}, & \text{if } j \leq 0, \\[2ex] \dfrac{-2\,\kappa_r(z)^j}{\alpha_r(1-\alpha_r)(\kappa_r^u(z) - \kappa_r(z))}, & \text{if } j \geq 0, \end{cases} \tag{3.27}$$

and

$$\overline{\mathcal{G}}_{\ell,j}(z) \, = \begin{cases} \dfrac{-2\,\kappa_\ell(z)^j}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}, & \text{if } j \le 0, \\ \\ \dfrac{-2\,\kappa_\ell^u(z)^j}{\alpha_\ell(1-\alpha_\ell)(\kappa_\ell(z)-\kappa_\ell^u(z))}, & \text{if } j \ge 0. \end{cases} \tag{3.28}$$

The expressions in (3.28) and (3.27) can be recognized in (3.12) and (3.13). This is made more explicit in the following paragraph.

3.5.2 Decomposing the spatial Green's function

The decomposition proceeds as follows. From inspection of the expressions (3.12) and (3.13) and the result of Proposition 3.2, the definition (3.20) can be recast as follows:

$$\widetilde{\mathcal{G}}_j^{j_0}(z) \, = \begin{cases} \mathcal{G}_j^{j_0}(z) \, - \, \mathbb{1}_{j\ge 1}\,\overline{\mathcal{G}}_{r,j-j_0}(z)\,, & \text{if } j_0 \ge 1, \\ \mathcal{G}_j^{j_0}(z) \, - \, \mathbb{1}_{j\le 0}\,\overline{\mathcal{G}}_{\ell,j-j_0}(z)\,, & \text{if } j_0 \le 0, \end{cases} \tag{3.29}$$

where the notation $\mathbb{1}_{j\ge 1}$ is used to denote 1 if $j \ge 1$ and 0 otherwise (and similarly for $\mathbb{1}_{j\le 0}$). It is also useful to introduce the following two fourth degree polynomial functions φ_r and φ_ℓ:

$$\forall\,\tau \in \mathbb{C}\,, \quad \varphi_{r,\ell}(\tau) \, := \, -\,\frac{1}{\alpha_{r,\ell}}\,\tau \, + \, \frac{1-\alpha_{r,\ell}^2}{6\,\alpha_{r,\ell}^3}\,\tau^3 \, - \, \frac{1-\alpha_{r,\ell}^2}{8\,\alpha_{r,\ell}^3}\,\tau^4\,. \tag{3.30}$$

With such notation, our result is the following.

Proposition 3.3. *Let the weak solution (2.2) satisfy the entropy inequalities (2.4). Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. Then there exists $\varepsilon_0 > 0$, there exist constants $C > 0$ and $c > 0$, there exist two complex valued sequences $(\gamma_j^r)_{j\in\mathbb{Z}}$ and $(\gamma_j^\ell)_{j\in\mathbb{Z}}$, there exist two bounded holomorphic functions Ψ_r and Ψ_ℓ on the square $\mathbf{B}_{\varepsilon_0}(0)$, and there exist sequences $(\Phi_{r,j})_{j\in\mathbb{Z}}$, $(\Phi_{\ell,j})_{j\in\mathbb{Z}}$, $(\Theta_{r,j})_{j\in\mathbb{Z}}$, $(\Theta_{\ell,j})_{j\in\mathbb{Z}}$ and $(\Theta_{1,j})_{j\in\mathbb{Z}}$ of bounded holomorphic functions on the square $\mathbf{B}_{\varepsilon_0}(0)$ such that the following hold:*

- *the sequences $(\gamma_j^r)_{j\in\mathbb{Z}}$ and $(\gamma_j^\ell)_{j\in\mathbb{Z}}$ satisfy the estimates:*

$$\forall\,j \in \mathbb{Z}\,, \quad |\gamma_j^r| \, + \, |\gamma_j^\ell| \, \le \, C\,\exp(-c\,|j|)\,;$$

- *the sequences $(\Phi_{r,j})_{j\in\mathbb{Z}}$, $(\Phi_{\ell,j})_{j\in\mathbb{Z}}$, $(\Theta_{r,j})_{j\in\mathbb{Z}}$ and $(\Theta_{\ell,j})_{j\in\mathbb{Z}}$ satisfy the estimates:*

$$\forall\,j \in \mathbb{Z}\,, \quad \forall\,\tau \in \mathbf{B}_{\varepsilon_0}(0)\,, \quad |\Phi_{r,j}(\tau)| + |\Phi_{\ell,j}(\tau)| + |\Theta_{r,j}(\tau)| + |\Theta_{\ell,j}(\tau)| + |\Theta_{1,j}(\tau)| \, \le \, C\,\exp(-c\,|j|)\,;$$

- *for any couple of integers $(j_0, j) \in \mathbb{Z}^2$, the function:*

$$\tau \in \mathbf{B}_{\varepsilon_0}(0) \cap \Big\{\zeta \in \mathbb{C} \,|\, \mathrm{Re}\,\zeta > 0\Big\} \longmapsto \widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau)\,,$$

  whose expression is given in (3.29), has a meromorphic *extension to the square $\mathbf{B}_{\varepsilon_0}(0)$ with a first order pole at 0 only, and there holds:*

$$\mathrm{e}^\tau\,\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) \, = \begin{cases} \left(\dfrac{\mathcal{H}_j}{\tau} \, + \, \gamma_j^r \, + \, \tau\,\Phi_{r,j}(\tau)\right) \exp\big(-\,j_0\,\varphi_r(\tau) \, + \, j_0\,\tau^5\,\Psi_r(\tau)\big)\,, & \text{if } j_0 \ge 1, \\ \\ \left(\dfrac{\mathcal{H}_j}{\tau} \, + \, \gamma_j^\ell \, + \, \tau\,\Phi_{\ell,j}(\tau)\right) \exp\big(-\,j_0\,\varphi_\ell(\tau) \, + \, j_0\,\tau^5\,\Psi_\ell(\tau)\big)\,, & \text{if } j_0 \le 0, \end{cases} \tag{3.31}$$

for any $\tau \in \mathbf{B}_{\varepsilon_0}(0) \setminus \{0\}$, *where we recall that* $\mathcal{H}_j$ *is defined in* (3.22)*;*

- *for any couple of integers* $(j_0, j) \in \mathbb{Z}^2$, *the function:*

$$\tau \in \mathbf{B}_{\varepsilon_0}(0) \cap \Big\{\zeta \in \mathbb{C} \,|\, \mathrm{Re}\, \zeta > 0\Big\} \longmapsto \widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)$$

has a holomorphic *extension to the square* $\mathbf{B}_{\varepsilon_0}(0)$, *and there holds:*

$$\mathrm{e}^\tau \,\Big(\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)\Big) \,=\, \begin{cases} \left(\dfrac{\mathcal{H}_j}{\alpha_r} \,+\, \tau\, \Theta_{r,j}(\tau)\right)\, \exp\big(-\, j_0\, \varphi_r(\tau) \,+\, j_0\, \tau^5\, \Psi_r(\tau)\big)\,, & \text{if } j_0 \ge 2, \\ \dfrac{\mathcal{H}_j}{\alpha_r} \,+\, \gamma_j^r - \gamma_j^\ell + \tau\, \Theta_{1,j}(\tau)\,, & \text{if } j_0 = 1, \\ \left(\dfrac{\mathcal{H}_j}{\alpha_\ell} \,+\, \tau\, \Theta_{\ell,j}(\tau)\right)\, \exp\big(-\, j_0\, \varphi_\ell(\tau) \,+\, j_0\, \tau^5\, \Psi_\ell(\tau)\big)\,, & \text{if } j_0 \le 0, \end{cases} \tag{3.32}$$

for any $\tau \in \mathbf{B}_{\varepsilon_0}(0)$.

Proof. We give the proof in the case $j_0 \ge 1$ and construct all quantities associated with the "right" state u_r. The proof in the case $j_0 \le 0$ is entirely similar and is left to the interested reader. We thus always consider from on some $j_0 \ge 1$ and some arbitrary integer $j \in \mathbb{Z}$. We also define, for later use, the function:

$$z \longmapsto \Theta(z) \,:=\, \frac{\underline{\Delta}(z)}{z-1}\,,$$

where the Lopatinskii determinant $\underline{\Delta}$ is defined in (3.3). Thanks to Lemma 3.2 and Assumption 1, we know that Θ can be holomorphically extended to some set of the form $\{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-\delta_0} < |\zeta|\}$ for an appropriate $\delta_0 > 0$. This is because $\underline{\Delta}$ has a simple zero at 1. Furthermore, we know that, up to restricting δ_0, Θ does not vanish on the set $\{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-\delta_0} < |\zeta|\}$.

From the definition (3.20) and the expressions (3.12), (3.27), we get the factorization:

$$\forall\, z \in \mathscr{U}\,, \quad \widetilde{\mathcal{G}}_j^{j_0}(z) \,=\, \chi_j^r(z)\, \overline{\mathcal{G}}_{r,1-j_0}(z)\,,$$

where we have set

$$\chi_j^r(z) \,:=\, \begin{cases} \dfrac{\chi_r^-(z)}{z-1}\, \kappa_\ell(z)^j\,, & \text{if } j \le 0, \\ \left(\dfrac{\chi_r^+(z)}{z-1} \,-\, 1\right)\, \kappa_r(z)^{j-1}\,, & \text{if } j \ge 1, \end{cases} \tag{3.33}$$

with

$$\chi_r^+(z) := \frac{\big(\alpha_\ell - \alpha_m + (1-\alpha_\ell)\,\kappa_\ell(z)\big)\,(1-\alpha_r)\,(\kappa_r^u(z) - \kappa_r(z))}{\Theta(z)}\,,$$

$$\chi_r^-(z) := \frac{\alpha_r\,(1-\alpha_r)\,(1-\alpha_m)\,(\kappa_r^u(z) - \kappa_r(z))}{\alpha_\ell\,\Theta(z)}\,.$$

It appears from those expressions and from Lemma 3.1 that, up to restricting δ_0 again, both functions $\chi_r^\pm$ have a holomorphic extension to the set $\{\zeta \in \mathbb{C} \,|\, \mathrm{e}^{-\delta_0} < |\zeta|\}$ and there is no loss os generality in assuming that both κ_ℓ and κ_r are also holomorphic functions and do not vanish on that set (see Lemma 3.1). Moreover, we compute:

$$\chi_r^+(1) \,=\, -\frac{2\,(1+\alpha_m)}{\underline{\Delta}'(1)}\,,\quad \chi_r^-(1) \,=\, \frac{2\,\alpha_r\,(1-\alpha_m)}{\alpha_\ell\,\underline{\Delta}'(1)}\,.$$

For z in a neighborhood of 1, we may then write (see (3.33)):

$$\chi_j^r(z) \,=\, \frac{\xi_j^r}{z-1} \,+\, \Gamma_j^r \,+\, \Xi_j^r(z)\,,$$

where ξ_j^r and Γ_j^r are complex numbers defined by:

$$\xi_j^r \,:=\, \lim_{z\to 1}\,(z-1)\,\chi_j^r(z) \,=\, \begin{cases} \chi_r^-(1)\,\kappa_\ell(1)^j\,, & \text{if } j\le 0,\\ \chi_r^+(1)\,\kappa_r(1)^{j-1}\,, & \text{if } j\ge 1,\end{cases}$$

$$\Gamma_j^r \,:=\, \lim_{z\to 1}\,\left(\chi_j^r(z) \,-\, \frac{\xi_j^r}{z-1}\right)\,,$$

and the function $(z\mapsto \Xi_j^r(z))$ is defined by:

$$\Xi_j^r(z) \,:=\, \chi_j^r(z) \,-\, \frac{\xi_j^r}{z-1} \,-\, \Gamma_j^r\,,$$

so that Ξ_j^r has a holomorphic extension to the set $\{\zeta\in\mathbb{C}\,|\,\mathrm{e}^{-\delta_0}<|\zeta|\}$ and vanishes at 1. In other words, we have isolated the first order pole at 1 in χ_j^r. Moreover, it is not difficult to compare the expression for ξ_j^r and the defining equation (3.22) for $\mathcal{H}_j$ and to find the relation $\xi_j^r=-\alpha_r\,\mathcal{H}_j$.

We know from Lemma 3.1 and from Section 3.3 that $\kappa_\ell(1)$ belongs to $\mathscr{U}$ and $\kappa_r(1)$ belongs to $\mathbb{D}$. Hence we can infer from the above definitions the exponentially decaying bounds:

$$|\xi_j^r| \,+\, |\Gamma_j^r| \,\le\, C\,\mathrm{e}^{-c\,|j|}\,,$$

as well as the local bound in z close to 1:

$$|\Xi_j^r(z)| \,\le\, C\,|z-1|\,\mathrm{e}^{-c\,|j|}\,.$$

In the same way, we find from the expression (3.27) that for $j_0\ge 1$, the function $\overline{\mathcal{G}}_{r,1-j_0}$ has a holomorphic extension to the set $\{\zeta\in\mathbb{C}\,|\,\mathrm{e}^{-\delta_0}<|\zeta|\}$ (up to restricting δ_0 one more time). Since $\kappa_r^u(1)$ equals 1, for τ in a sufficiently small square $\mathbf{B}_{\varepsilon_0}(0)$ centered at the origin, we can write:

$$\kappa_r^u(\mathrm{e}^\tau) \,=\, \exp(\omega_r(\tau))\,,$$

where ω_r is holomorphic and bounded on $\mathbf{B}_{\varepsilon_0}(0)$. It is then a mere algebra exercise to infer from (3.1b) the Taylor expansion of ω_r at 0 and we get:

$$\omega_r(\tau) \,=\, \varphi_r(\tau) \,+\, \mathcal{O}(\tau^5)\,,$$

with the fourth degree polynomial φ_r defined in (3.30). We can thus write, for any $\tau\in\mathbf{B}_{\varepsilon_0}(0)$:

$$\mathrm{e}^\tau\,\overline{\mathcal{G}}_{r,1-j_0}(\mathrm{e}^\tau) \,=\, \left(-\frac{1}{\alpha_r} \,+\, \mu_r\,\tau \,+\, \tau^2\,\widetilde{\Phi}_r(\tau)\right)\,\exp\big(-\,j_0\,\varphi_r(\tau) \,+\, j_0\,\tau^5\,\Psi_r(\tau)\big)\,,$$

where $\widetilde{\Phi}_r$ and Ψ_r are two holomorphic bounded functions on $\mathbf{B}_{\varepsilon_0}(0)$ and μ_r is a complex number. It then remains to perform the Taylor expansion of $\chi_j^r(\mathrm{e}^\tau)$ at $\tau=0$ and to multiply with the above expansion of $\mathrm{e}^\tau\,\overline{\mathcal{G}}_{r,1-j_0}(\mathrm{e}^\tau)$ to get the result of Proposition 3.3. □

We now turn to the proof of our main estimates for the Green's function of the operator $\mathscr{L}$ (see Theorem 4.1 below). This will lead to time decay estimates as stated in Theorem 2.4.

Chapter 4

Linear stability

The goal of this chapter is to derive sharp bounds for the *temporal* Green's function, that is, for any given $j_0 \in \mathbb{Z}$, the solution $(\mathscr{G}^n(j,j_0))_{(n,j)\in\mathbb{N}\times\mathbb{Z}}$ to the recurrence relation:

$$\begin{cases} \mathscr{G}^{n+1}(j,j_0) \,=\, (\mathscr{L}\,\mathscr{G}^n(\cdot,j_0))_j, \quad (n,j)\in\mathbb{N}\times\mathbb{Z}\,, \\ \mathscr{G}^0(\cdot,j_0) \,=\, \boldsymbol{\delta}_{j_0}\,. \end{cases} \tag{4.1}$$

The relevance of this sequence is motivated by the fact that for any initial condition $\mathbf{h} \in \ell^q(\mathbb{Z};\mathbb{R})$, the solution to the recurrence relation:

$$\forall\, n\in\mathbb{N}\,,\quad \mathbf{v}^{n+1} \,=\, \mathscr{L}\,\mathbf{v}^n\,, \qquad \mathbf{v}^0 \,=\, \mathbf{h}\,,$$

can be decomposed into:

$$\forall\,(n,j)\in\mathbb{N}\times\mathbb{Z}\,,\quad v_j^n \,=\, \sum_{j_0\in\mathbb{Z}} \mathscr{G}^n(j,j_0)\,h_{j_0}\,,$$

and we expect that sharp bounds on $\mathscr{G}^n(j,j_0)$ will quantify the decay properties of the semigroup of operators $(\mathscr{L}^n)_{n\in\mathbb{N}}$. We first decompose the temporal Green's function by following the decomposition given in Proposition 3.3 for the (reduced) spatial Green's function. We then analyze each contribution in the decomposition and derive bounds that are meant to be as sharp as possible in order to obtain large time decaying bounds for the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$.

4.1 Preliminary facts

We always consider from now on that the weak solution (2.2) satisfies the entropy inequalities (2.4) and that the CFL parameter λ satisfies the stability condition (2.12). We also assume that Assumption 1 is satisfied so that the analysis of Chapter 3 can be used. The value of the temporal Green's function $\mathscr{G}^n(j,j_0)$ is given by the so-called functional calculus (see [8]):

$$\mathscr{G}^n(j,j_0) \,=\, \frac{1}{2\pi\mathbf{i}}\int_{\widetilde{\Gamma}} z^n\,\mathcal{G}_j^{j_0}(z)\,\mathrm{d}z, \tag{4.2}$$

where $\widetilde{\Gamma}$ is any contour that encompasses the spectrum of the operator $\mathscr{L}$. In view of Theorem 2.3, one can choose for instance $\widetilde{\Gamma} = (1+\delta)\,\mathbb{S}^1 = \{\zeta\in\mathbb{C}\,|\,|\zeta| = 1+\delta\}$ for any $\delta > 0$, since the spectrum of $\mathscr{L}$ is included in $\mathbb{D}\cup\{1\}$.

Two other key quantities of interest to us are the temporal Green's functions associated with the operators $\mathscr{L}_\ell$ and $\mathscr{L}_r$ that are defined in (3.25). The associated temporal Green's functions for these operators are defined as the solutions to the recurrences:

$$\begin{cases} \overline{\mathscr{G}}_\ell^{\,n+1} \, = \, \mathscr{L}_\ell \, \overline{\mathscr{G}}_\ell^{\,n} \,, \quad n \in \mathbb{N} \,, \\ \overline{\mathscr{G}}_\ell^{\,0} \, = \, \boldsymbol{\delta}_0 \,, \end{cases} \tag{4.3a}$$

$$\begin{cases} \overline{\mathscr{G}}_r^{\,n+1} \, = \, \mathscr{L}_r \, \overline{\mathscr{G}}_r^{\,n} \,, \quad n \in \mathbb{N} \,, \\ \overline{\mathscr{G}}_r^{\,0} \, = \, \boldsymbol{\delta}_0 \,, \end{cases} \tag{4.3b}$$

and the same functional calculus rules as before yield the expressions:

$$\overline{\mathscr{G}}_\ell^{\,n}(j) \, = \, \dfrac{1}{2\pi \mathbf{i}} \int_{\widetilde{\Gamma}} z^n \, \overline{\mathbb{G}}_{\ell,j}(z) \, \mathrm{d}z \,, \quad \overline{\mathscr{G}}_r^{\,n}(j) \, = \, \dfrac{1}{2\pi \mathbf{i}} \int_{\widetilde{\Gamma}} z^n \, \overline{\mathbb{G}}_{r,j}(z) \, \mathrm{d}z \,,$$

where $\widetilde{\Gamma}$ is once again any closed contour that encompasses the spectrum of both $\mathscr{L}_\ell$ and $\mathscr{L}_r$ in its interior. We can choose, for instance, $\widetilde{\Gamma} = (1+\delta) \, \mathbb{S}^1$ for any $\delta > 0$. A crucial observation for what follows is that the sequences $(\overline{\mathscr{G}}_\ell^{\,n}(j))_{(n,j) \in \mathbb{N} \times \mathbb{Z}}$ and $(\overline{\mathscr{G}}_r^{\,n}(j))_{(n,j) \in \mathbb{N} \times \mathbb{Z}}$ have been thoroughly studied in [9], though with a different point of view since the framework allows for the use of Fourier analysis. We shall feel free to use repeatedly several results from [9], which are themselves more accurate estimates for the free Green's functions (in the whole space) than previous bounds obtained in [16, 17]. The main results of [9] are gathered in Appendix A together with some supplementary material that is needed to carry out the analysis below.

We start our analysis with a first elementary observation. Since the Lax-Wendroff scheme as a finite stencil, we readily see from the recurrence equation (4.1) defining the Green's function that for all $n \in \mathbb{N}$ and $(j, j_0) \in \mathbb{Z}^2$:

$$|j - j_0| > n \, \Rightarrow \, \mathscr{G}^n(j, j_0) = 0 \,.$$

this fact is repeatedly used below in order to restrict the possible regimes for j, j_0, n. As a second crucial observation, we have the following Lemma.

Lemma 4.1. *Let $j_0 \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $j_0 \geq 1$ and $n \leq j_0 - 1$, then there holds:*

$$\forall \, j \in \mathbb{Z} \,, \quad \mathscr{G}^n(j, j_0) \, = \, \overline{\mathscr{G}}_r^{\,n}(j - j_0) \,.$$

If $j_0 \leq 0$ and $n \leq |j_0|$, then there holds:

$$\forall \, j \in \mathbb{Z} \,, \quad \mathscr{G}^n(j, j_0) \, = \, \overline{\mathscr{G}}_\ell^{\,n}(j - j_0) \,.$$

Proof. We give the proof in the case $j_0 \geq 1$ and leave the other situation to the interested reader.

The proof directly follows from the expression (2.11) of the operator $\mathscr{L}$ and from the definition (3.25b) of the convolution operator $\mathscr{L}_r$. Indeed, we easily see that if $\mathbf{w} = (w_j)_{j \in \mathbb{Z}}$ is a sequence that is supported in $\{j \in \mathbb{Z} \, | \, j \geq j_{\min}\}$ with $j_{\min} \geq 2$, then $\mathscr{L}\mathbf{w}$ equals $\mathscr{L}_r \mathbf{w}$ and both sequences are supported in $\{j \in \mathbb{Z} \, | \, j \geq j_{\min} - 1\}$.

For $j_0 = 1$, the proof of Lemma 4.1 is obvious. We thus assume $j_0 \geq 2$. Then we can prove by induction that for any $n = 0, \dots, j_0 - 2$, the two sequences $\mathscr{G}^n(\cdot, j_0)$ and $\overline{\mathscr{G}}_r^{\,n}(\cdot - j_0)$ are equal and they are supported in $\{j \in \mathbb{Z} \, | \, j \geq j_0 - n\}$. Applying one last time the above fact when n equals $j_0 - 2$, we see that $\mathscr{G}^{j_0 - 1}(\cdot, j_0)$ and $\overline{\mathscr{G}}_r^{\,j_0 - 1}(\cdot - j_0)$ are equal and they are supported in $\{j \in \mathbb{Z} \, | \, j \geq 1\}$. This is when we cannot use the above fact any longer. ☐

Finally, as a last preparatory step, we shall change variable in the integral representation (4.2) to write instead

$$\mathscr{G}^n(j,j_0) \;=\; \frac{1}{2\pi\mathbf{i}}\int_\Gamma \mathrm{e}^{n\,\tau}\,\mathcal{G}_j^{j_0}(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau,$$

with $\Gamma = \{\tau = \rho + \mathbf{i}\,\theta \mid \theta \in [-\pi,\pi]\}$, for any $\rho > 0$. This change of variable justifies the decomposition made in Proposition 3.3 (see (3.31)). We now introduce some further notations and clarify the bounds that we intend to prove on the Green's function of the operator $\mathscr{L}$.

4.2 Notation and bounds for the Green's function

We first introduce some constants:

$$c_{3,\ell} \;:=\; \frac{\alpha_\ell\,(1-\alpha_\ell^2)}{6} > 0\,, \qquad\qquad c_{4,\ell} \;:=\; \frac{\alpha_\ell^2\,(1-\alpha_\ell^2)}{8} \;>\; 0\,, \tag{4.4a}$$

$$c_{3,r} \;:=\; \frac{\alpha_r\,(1-\alpha_r^2)}{6} < 0\,, \qquad\qquad c_{4,r} \;:=\; \frac{\alpha_r^2\,(1-\alpha_r^2)}{8} \;>\; 0\,. \tag{4.4b}$$

We then define the following two functions[1] $\mathbf{A}_\ell$ and $\mathbf{A}_r$ on $\mathbb{R}\times\mathbb{R}^{+*}$ as follows. For $\eta > 0$, we set:

$$\forall\,(x,y)\in\mathbb{R}\times\mathbb{R}^{+*}\,,\quad \mathbf{A}_\ell(x,y) \;:=\; \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathrm{e}^{x\,(\eta+\mathbf{i}\,\theta)}\,\mathrm{e}^{-c_{3,\ell}\,y\,(\eta+\mathbf{i}\,\theta)^3}\,\mathrm{e}^{-c_{4,\ell}\,y\,(\eta+\mathbf{i}\,\theta)^4}\,\frac{\mathrm{d}\theta}{\eta+\mathbf{i}\,\theta}\,, \tag{4.5a}$$

$$\mathbf{A}_r(x,y) \;:=\; \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathrm{e}^{x\,(\eta+\mathbf{i}\,\theta)}\,\mathrm{e}^{-c_{3,r}\,y\,(\eta+\mathbf{i}\,\theta)^3}\,\mathrm{e}^{-c_{4,r}\,y\,(\eta+\mathbf{i}\,\theta)^4}\,\frac{\mathrm{d}\theta}{\eta+\mathbf{i}\,\theta}\,, \tag{4.5b}$$

where both definitions (4.5a) and (4.5b) make sense since $c_{4,\ell}$ and $c_{4,r}$ are positive. The definitions (4.5) are shown to be *independent* of $\eta > 0$ thanks to the Cauchy formula for holomorphic functions. Eventually, we introduce two other functions $\mathbf{M}_\ell$ and $\mathbf{M}_r$ on $\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}^{+*}$ as follows:

$$\forall\,(c,x,y)\in\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}^{+*}\,,\quad \mathbf{M}_\ell(c,x,y) \;:=\; \begin{cases} \dfrac{1}{y^{1/3}}\,\exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } x \geq 0, \\[2ex] \dfrac{1}{y^{1/3}}\,, & \text{if } -y^{1/3} \leq x \leq 0, \\[2ex] \dfrac{1}{|x|^{1/4}\,y^{1/4}}\,\exp\big(-c\,x^2/y\big)\,, & \text{if } x \leq -y^{1/3}, \end{cases} \tag{4.6a}$$

$$\mathbf{M}_r(c,x,y) \;:=\; \begin{cases} \dfrac{1}{y^{1/3}}\,\exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } x \leq 0, \\[2ex] \dfrac{1}{y^{1/3}}\,, & \text{if } 0 \leq x \leq y^{1/3}, \\[2ex] \dfrac{1}{|x|^{1/4}\,y^{1/4}}\,\exp\big(-c\,x^2/y\big)\,, & \text{if } y^{1/3} \leq x. \end{cases} \tag{4.6b}$$

These functions encode the bounds for the free Green's functions associated with the convolution operators $\mathscr{L}_\ell$ and $\mathscr{L}_r$, as recalled in Appendix A (see for instance Corollary A.1).

[1]The letter A refers to "activation" in analogy with [6].

A crucial property for what follows is that both $\mathbf{M}_\ell$ and $\mathbf{M}_r$ are non-increasing with respect to their first argument. When various positive constants $c_1, c_2, \dots$ appear, we can always use the largest function $\mathbf{M}_r(\min_i c_i, \cdot, \cdot)$ as an upper bound for all functions $\mathbf{M}_r(c_i, \cdot, \cdot)$. This will be used repeatedly in order to avoid using specific notations for the various small positive constants c that appear.

Our main result for the Green's function of the operator $\mathscr{L}$ reads as follows.

Theorem 4.1 (The Green's function of the linearized numerical scheme)**.** *Let the weak solution (2.2) satisfy the Rankine-Hugoniot condition (2.3) and the entropy inequalities (2.4). Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. Then there exist some positive constants C and c such that, for any $j_0 \leq 0$, there holds:*

$$\forall (n,j) \in \mathbb{N}^* \times \mathbb{Z}, \quad \left|\mathscr{G}^n(j,j_0) \, - \, \mathcal{H}_j \, \mathbf{A}_\ell(j_0 + n\,\alpha_\ell, n)\right| \leq C\,\mathbf{M}_\ell\,(c, j_0 - j + n\,\alpha_\ell, n)\;\mathbb{1}_{j\leq 0} + C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_\ell\,(c, j_0 + n\,\alpha_\ell, n) + C\,\mathrm{e}^{-c\,n}\,\mathrm{e}^{-c\,|j|}\,\mathrm{e}^{-c\,|j_0|}\,, \tag{4.7}$$

and for any $j_0 \geq 1$, there holds:

$$\forall (n,j) \in \mathbb{N}^* \times \mathbb{Z}, \quad \left|\mathscr{G}^n(j,j_0) \, - \, \mathcal{H}_j \, \mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)\right| \leq C\,\mathbf{M}_r\,(c, j - j_0 + n\,|\alpha_r|, n)\;\mathbb{1}_{j\geq 1} + C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r\,(c, -j_0 + n\,|\alpha_r|, n) + C\,\mathrm{e}^{-c\,n}\,\mathrm{e}^{-c\,|j|}\,\mathrm{e}^{-c\,|j_0|}\,, \tag{4.8}$$

where in both (4.7) and (4.8), $\mathcal{H}_j$ is defined in (3.22).

4.3 Decomposing the temporal Green's function

Throughout this section, we assume that $n \geq 1$ is an integer and $(j, j_0) \in \mathbb{Z}^2$ satisfy $|j - j_0| \leq n$. We will mainly focus on the case $j_0 \geq 1$, and present at the end of the section the corresponding results for the case $j_0 \leq 0$. From Lemma 4.1, we shall further assume that $1 \leq j_0 \leq n$ since for $n \leq j_0 - 1$ we have already obtained an explicit expression for the temporal Green's function in terms of the free Green's function associated to $\mathscr{L}_r$ (so the final derivation of a bound for the Green's function will be simpler).

The starting point of our analysis is to exploit our expression for the spatial Green's function to get

$$\mathscr{G}^n(j,j_0) \, = \, \overline{\mathscr{G}}_r^n(j - j_0)\,\mathbb{1}_{j\geq 1} \, + \, \frac{1}{2\pi\mathbf{i}} \int_\Gamma \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau\,, \tag{4.9}$$

and define

$$\widetilde{\mathscr{G}}^n(j,j_0) \, := \, \frac{1}{2\pi\mathbf{i}} \int_\Gamma \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau\,.$$

At this stage, Γ denotes the segment $\{\tau = \rho + \mathbf{i}\,\theta \mid \theta \in [-\pi, \pi]\}$ for any $\rho > 0$, but we shall be allowed to deform Γ thanks to Cauchy's formula.

Let $\varepsilon_0 > 0$ be given by Proposition 3.3 and let $C_0 > 0$ be such for all $\tau \in \mathbf{B}_{\varepsilon_0}(0)$ one has the uniform bound:

$$\left|\tau^5\,\Psi_r(\tau)\right| \, \leq \, C_0\,\left(|\mathrm{Re}(\tau)|^5 + |\mathrm{Im}(\tau)|^5\right)\,. \tag{4.10}$$

Then, let $\varepsilon_* \in (0, \varepsilon_0)$ be such that for all $\varepsilon \in (0, \varepsilon_*)$ the following conditions are satisfied:

$$0 < \varepsilon < \min\left[\frac{1 - \alpha_r^2}{16\,\alpha_r^2\,C_0}\,, \left(\frac{1}{3}\right)^{1/5}\right]\,, \tag{4.11}$$

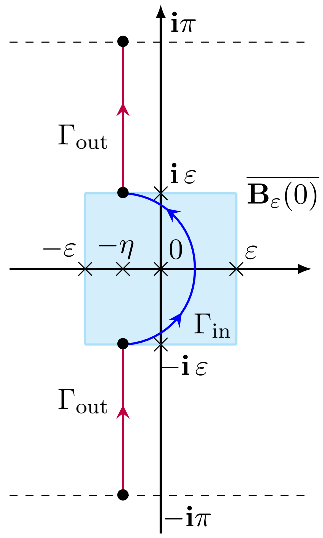

Figure 4.1: *Schematic illustration of the contour* Γ *and its decomposition into* Γ_{out} *(in red) and* Γ_{in} *(in blue). The contour* Γ_{in} *can be any path joining* $-\eta - \mathbf{i}\,\varepsilon$ *to* $-\eta + \mathbf{i}\,\varepsilon$*, which remains within* $\overline{\mathbf{B}_\varepsilon(0)}$ *and passes to the right of the origin. The black bullets represent the end points of the contours.*

$$\frac{4\,|\alpha_r|}{1-\alpha_r^2}\left[\max\left(\frac{|\alpha_r|}{2}, 1-\alpha_r^2\right) + |\alpha_r| C_0\right]\varepsilon + \varepsilon^3 + \frac{3}{2}\,\varepsilon^8 + \frac{1}{3}\,\varepsilon^{11} + \frac{4\,\alpha_r^2}{1-\alpha_r^2}\,C_0\,\varepsilon^{21} < \frac{1}{8}\,, \tag{4.12}$$

$$\frac{1-\alpha_r^2}{4|\alpha_r|}\left(1 + \frac{3}{2}\varepsilon^5\right)\varepsilon^2 + \frac{\alpha_r^2}{2}\,C_0\,\varepsilon^{20} < \frac{|\alpha_r|}{4}\,. \tag{4.13}$$

Finally, once for all, we fix $\varepsilon \in (0, \varepsilon_*)$, and we let $\eta_\varepsilon > 0$ be provided by Corollary 3.3 associated to this $\varepsilon > 0$ and we also set $0 < \eta < \min(\eta_\varepsilon, \varepsilon^5)$.

We can now proceed by choosing an appropriate contour Γ in our integral defining $\widetilde{\mathscr{G}}^n(j, j_0)$. We would like to choose the segment $\{\tau = -\eta + \mathbf{i}\,\theta \mid \theta \in [-\pi, \pi]\}$ but this is not possible right away because of the pole at the origin, so we shall make a detour on the right of the origin. We thus decompose Γ into two pieces Γ_{out} and Γ_{in} where

$$\Gamma_{\text{out}} := \{-\eta + \mathbf{i}\,\theta \mid \varepsilon \le |\theta| \le \pi\}\ ,$$

and Γ_{in} is any path joining $-\eta - \mathbf{i}\,\varepsilon$ to $-\eta + \mathbf{i}\,\varepsilon$, which remains within $\overline{\mathbf{B}_\varepsilon(0)}$ and passes to the right of the origin. These contours are depicted in Figure 4.1. From Corollary 3.3 in Chapter 3, we infer that

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{out}}} \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-\eta\,n}\,\mathrm{e}^{-c\,(|j|+|j_0|)}\,. \tag{4.14}$$

Since $\Gamma_{\text{in}} \subset \overline{\mathbf{B}_\varepsilon(0)} \subset \mathbf{B}_{\varepsilon_0}(0)$, we can use Proposition 3.3 and the expression (3.31) to decompose the remaining integral as

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau = \widetilde{\mathscr{G}}^n_{1,r}(j,j_0) + \widetilde{\mathscr{G}}^n_{2,r}(j,j_0) + \mathscr{R}^n_{1,r}(j,j_0)\,,$$

where we have set

$$\widetilde{\mathscr{G}}^n_{1,r}(j,j_0) := \mathcal{H}_j \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right) \frac{\mathrm{d}\tau}{\tau}\,,$$

$$\widetilde{\mathscr{G}}^n_{2,r}(j,j_0) := \gamma^r_j \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right) \mathrm{d}\tau\,, \tag{4.15}$$

and

$$\mathscr{R}^n_{1,r}(j,j_0) := \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right) \tau\,\Phi_{r,j}(\tau)\,\mathrm{d}\tau\,. \tag{4.16}$$

Thank's to Cauchy formula for holomorphic functions, it is important to remark that the above integrals do not depend on $\Gamma_{\rm in}$, and we shall later on in this section choose specific contours $\Gamma_{\rm in}$ depending on various regimes between n and j_0. In fact, for $\widetilde{\mathscr{G}}^n_{2,r}(j,j_0)$ and $\mathscr{R}^n_{1,r}(j,j_0)$, we can select any contour $\Gamma_{\rm in}$ that joins $-\eta - \mathbf{i}\,\varepsilon$ to $-\eta + \mathbf{i}\,\varepsilon$ and which remains within $\overline{\mathbf{B}_\varepsilon(0)}$ since both integrand are holomorphic functions in $\overline{\mathbf{B}_\varepsilon(0)}$, while for $\widetilde{\mathscr{G}}^n_{1,r}(j,j_0)$ we need to keep the constraint that the contour $\Gamma_{\rm in}$ passes to the right of the origin because of the pole at the origin of the integrand.

Our next task is to extract the leading order term of $\widetilde{\mathscr{G}}^n_{1,r}(j,j_0)$. For that, we may simply further decompose

$$\widetilde{\mathscr{G}}^n_{1,r}(j,j_0) = \mathcal{H}_j\,\mathscr{A}^n_r(j_0) + \widetilde{\mathscr{G}}^n_{3,r}(j,j_0) + \mathscr{R}^n_{2,r}(j,j_0),$$

where we have set

$$\mathscr{A}^n_r(j_0) := \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\,, \tag{4.17}$$

$$\widetilde{\mathscr{G}}^n_{3,r}(j,j_0) := \mathcal{H}_j \times j_0 \times \Psi_r(0) \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\tau^4\,\mathrm{d}\tau\,, \tag{4.18}$$

and

$$\mathscr{R}^n_{2,r}(j,j_0) := \mathcal{H}_j \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \left(\frac{\exp\left(j_0\,\tau^5\,\Psi_r(\tau)\right) - 1 - j_0\,\tau^5\,\Psi_r(0)}{\tau}\right) \mathrm{d}\tau\,. \tag{4.19}$$

For convenience, we further define $\mathscr{B}^n_r(j,j_0) := \widetilde{\mathscr{G}}^n_{2,r}(j,j_0) + \widetilde{\mathscr{G}}^n_{3,r}(j,j_0)$ (with the definitions (4.15) and (4.18)) and $\mathscr{R}^n_r(j,j_0) = \mathscr{R}^n_{1,r}(j,j_0) + \mathscr{R}^n_{2,r}(j,j_0)$, such that we have obtained the intermediate decomposition

$$\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}^{j_0}_j(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau = \mathcal{H}_j\,\mathscr{A}^n_r(j_0) + \mathscr{B}^n_r(j,j_0) + \mathscr{R}^n_r(j,j_0)\,. \tag{4.20}$$

For future reference, we gather the previous definitions of $\mathscr{B}^n_r(j,j_0)$ and $\mathscr{R}^n_r(j,j_0)$:

$$\begin{aligned}
\mathscr{B}^n_r(j,j_0) =& \gamma^r_j \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right) \mathrm{d}\tau \\
&+ \Psi_r(0) \times \mathcal{H}_j \times j_0 \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\tau^4 \mathrm{d}\tau\,,
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\mathscr{R}^n_r(j,j_0) =& \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right) \tau\,\Phi_{r,j}(\tau)\,\mathrm{d}\tau \\
&+ \mathcal{H}_j \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \left(\frac{\exp\left(j_0\,\tau^5\,\Psi_r(\tau)\right) - 1 - j_0\,\tau^5\,\Psi_r(0)}{\tau}\right) \mathrm{d}\tau\,.
\end{aligned} \tag{4.22}$$

We shall now study each term appearing in (4.20) separately.

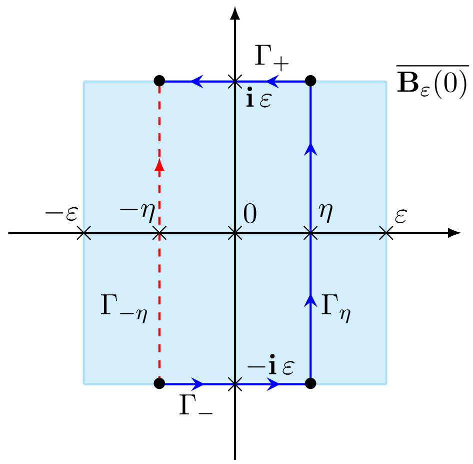

Figure 4.2: *In blue: the contour $\Gamma_{\rm in}$ used in the regime $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$ in Lemma 4.2 and its decomposition into three parts (Γ_-, Γ_η and Γ_+). In red dash: the contour $\Gamma_{-\eta}$. The black bullets represent the end points of the contours.*

4.3.1 Estimates of the leading order term $\mathscr{A}_r^n(j_0)$

In this subsection, we obtain a sharp estimate on the term $\mathscr{A}_r^n(j_0)$ in (4.17) and prove that it behaves like an activation function. This clarifies the behavior of the first term in the right-hand side of the decomposition (4.20). We shall consider here and in all the remainder of this Section, two different regimes for n and j_0:

(i) the main regime : $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$;

(ii) the tail : $1 \leq j_0 < \frac{n\,|\alpha_r|}{2}$.

The main result of this subsection is the following Lemma.

Lemma 4.2. *Let $n \geq 1$ and let $1 \leq j_0 \leq n$. Let $\mathscr{A}_r^n(j_0)$ be defined in (4.17). Then there exist some positive constants C and c that are uniform with respect to n and j_0 such that with the functions $\mathbf{A}_r$ defined in (4.5b) and $\mathbf{M}_r$ defined in (4.6b), there holds:*

$$\left|\mathscr{A}_r^n(j_0)\, -\, \mathbf{A}_r\left(-j_0+n\,|\alpha_r|,n\right)\right| \leq C\,\mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\, +\, C\,\mathrm{e}^{-c\,n-c\,j_0}\,. \tag{4.23}$$

Proof. **Case (i).** We assume that j_0 and n satisfy $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$. The contour $\Gamma_{\rm in}$ is decomposed into three parts as depicted in Figure 4.2. From the definition (4.17), we thus have:

$$\mathscr{A}_r^n(j_0) \,=\, \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_-\cup\Gamma_+} \mathrm{e}^{n\,\tau-j_0\,\varphi_r(\tau)}\frac{\mathrm{d}\tau}{\tau} + \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\eta} \mathrm{e}^{n\,\tau-j_0\,\varphi_r(\tau)}\frac{\mathrm{d}\tau}{\tau},$$

where $\Gamma_\eta = \{\eta+\mathbf{i}\,\theta \mid |\theta| \leq \varepsilon\}$ and $\Gamma_\pm$ are the two horizontal paths joining $-\eta \pm \mathbf{i}\,\varepsilon$ to $\eta \pm \mathbf{i}\,\varepsilon$. Upon denoting

$$\omega := \frac{j_0 - n\,|\alpha_r|}{n} \in \left[-\frac{|\alpha_r|}{2}, 1-|\alpha_r|\right] \quad \text{and} \quad \zeta := \frac{j_0}{n}\,\frac{1-\alpha_r^2}{2\,\alpha_r^2} \in \left[\frac{1-\alpha_r^2}{4\,|\alpha_r|}, \frac{1-\alpha_r^2}{2\,\alpha_r^2}\right]\,, \tag{4.24}$$

we get

$$n\,\tau - j_0\,\varphi_r(\tau) \;=\; \frac{n}{|\alpha_r|}\,\left[-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right]\,. \tag{4.25}$$

Let now $\tau \in \Gamma_-$ so that τ reads $\tau = t - \mathbf{i}\,\varepsilon$ with $t \in [-\eta, \eta]$. We compute

$$\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) \;=\; -\omega\,t - \zeta\,\varepsilon^2\,t + \frac{\zeta}{3}\,t^3 + \frac{3\,\zeta}{2}\,\varepsilon^2\,t^2 - \frac{\zeta}{4}\,t^4 - \frac{\zeta}{4}\,\varepsilon^4\,.$$

Recalling that $\eta \le \varepsilon^5$ by assumption, we obtain that

$$\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) \;\le\; -\frac{\zeta}{4}\,\varepsilon^4 + |\omega|\,\varepsilon^5 + \zeta\,\varepsilon^7 + \frac{3\,\zeta}{2}\,\varepsilon^{12} + \frac{\zeta}{3}\,\varepsilon^{15} \;\le\; -\frac{\zeta}{8}\,\varepsilon^4\,, \tag{4.26}$$

thanks to our smallness assumption (4.12) on ε. Thus, we get the estimate

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_-} e^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\right| \le C\,\mathrm{e}^{-n\,\frac{1-\alpha_r^2}{32\alpha_r^2}\,\varepsilon^4} \;=\; C\,\mathrm{e}^{-c\,n}\,.$$

There is of course a similar estimate on Γ_+. Since n dominates j_0, one deduces that

$$\left|\mathscr{A}_r^n(j_0) - \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\eta} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\right| \;\le\; C\,\mathrm{e}^{-c\,n - c\,j_0}\,,$$

for constants $C, c > 0$ that are independent of both n and j_0.

Next, we complete the remaining integral along Γ_η as follows:

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\eta} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau} \;=\; \frac{1}{2\pi\mathbf{i}}\int_{\eta+\mathbf{i}\mathbb{R}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau} - \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\eta^c} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\,,$$

where $\Gamma_\eta^c := \{\eta + \mathbf{i}\,\theta \mid |\theta| > \varepsilon\}$. The reader can verify indeed that the integral along the line $\eta + \mathbf{i}\mathbb{R}$ converges. This is due to the form of the function φ_r and the fact that j_0 is positive. We now estimate the integral along Γ_η^c and show that it is a remainder term just like several other contributions that we have already estimated.

We keep the notation (4.24) and use the relation (4.25). For $\tau \in \Gamma_\eta^c$, we write $\tau = \eta + \mathbf{i}\,\theta$ with $|\theta| > \varepsilon$, so that we have

$$\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) \;=\; -\omega\,\eta + \frac{\zeta}{3}\,\eta^3 - \frac{\zeta}{4}\,\eta^4 - \frac{\zeta}{4}\,\theta^4 - \zeta\,\eta\,\left(1 - \frac{3}{2}\eta\right)\,\theta^2\,.$$

Since $0 < \eta \le \varepsilon^5$ and ε is assumed to be small enough to satisfy (4.11) and (4.12), we have that

$$\begin{aligned}\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) \;&\le\; -\omega\,\eta + \frac{\zeta}{3}\,\eta^3 - \frac{\zeta}{4}\,\theta^4 - \frac{\zeta\,\eta}{2}\,\theta^2 \\ &\le -\frac{\zeta}{4}\,\varepsilon^4 + |\omega|\,\varepsilon^5 + \frac{\zeta}{3}\,\varepsilon^{15} - \frac{\zeta\,\eta}{2}\,\theta^2 \le -\frac{\zeta}{8}\varepsilon^4 - \frac{\zeta\,\eta}{2}\,\theta^2\,,\end{aligned}$$

and as a consequence, we obtain

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\eta^c} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\right| \;\le\; C\,\mathrm{e}^{-n\,\frac{1-\alpha_r^2}{32\alpha_r^2}\,\varepsilon^4}\;\int_\varepsilon^{+\infty} \mathrm{e}^{-n\,\frac{\zeta\eta}{2}\,\theta^2}\mathrm{d}\theta \;\le\; C\,\mathrm{e}^{-c\,n} \;\le\; C\,\mathrm{e}^{-c\,n - c\,j_0}\,,$$

where we have used once again that n dominates j_0. As a partial summary, we have obtained that

$$\left| \mathscr{A}_r^n(j_0) - \frac{1}{2\pi \mathbf{i}} \int_{\eta+\mathbf{i}\mathbb{R}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \frac{\mathrm{d}\tau}{\tau} \right| \le C\,\mathrm{e}^{-c\,n - c\,j_0}\,,$$

for positive constants C and c that are independent of n and j_0 and for integers that satisfy $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$.

Cauchy's formula shows that the value of the integral:

$$\frac{1}{2\pi \mathbf{i}} \int_{\eta+\mathbf{i}\mathbb{R}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \frac{\mathrm{d}\tau}{\tau}$$

does not depend on $\eta > 0$ so choosing the parametrization $\tau = |\alpha_r|\,(\eta + \mathbf{i}\,\theta)$ in the integral, we get:

$$\frac{1}{2\pi \mathbf{i}} \int_{\eta+\mathbf{i}\mathbb{R}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \frac{\mathrm{d}\tau}{\tau} = \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathrm{e}^{-(j_0+n\,\alpha_r)\,(\eta+\mathbf{i}\,\theta)}\,\mathrm{e}^{j_0\,\frac{1-\alpha_r^2}{6}\,(\eta+\mathbf{i}\,\theta)^3}\,\mathrm{e}^{j_0\,\alpha_r\,\frac{1-\alpha_r^2}{8}\,(\eta+\mathbf{i}\,\theta)^4}\,\frac{\mathrm{d}\theta}{\eta+\mathbf{i}\,\theta}\,.$$

Recalling the definitions (4.4b) and (4.5b), we may write the above integral as

$$\frac{1}{2\pi \mathbf{i}} \int_{\eta+\mathbf{i}\mathbb{R}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \frac{\mathrm{d}\tau}{\tau} = \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right)\,,$$

so that we have obtained the estimate:

$$\left| \mathscr{A}_r^n(j_0) - \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) \right| \le C\,\mathrm{e}^{-c\,n - c\,j_0}\,.$$

With the definition (4.6b), Corollary A.7 in Appendix A can be rewritten as:

$$\left| \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) - \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, n\right) \right| \le C\,\mathbf{M}_r(c, -j_0 + n\,|\alpha_r|, n)\,,$$

for suitable constants C and c, since we have assumed $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$. We can combine the previous two estimates and obtain the estimate (4.23) that we were aiming at. This completes the analysis of case (i).

Case (ii). For $1 \le j_0 < \frac{n\,|\alpha_r|}{2}$, we rather use the residue theorem. In other words, as suggested in Figure 4.2, we close the contour Γ_{in} by using the segment $\Gamma_{-\eta}$ and we thus get:

$$\mathscr{A}_r^n(j_0) = 1 + \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_{-\eta}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau)} \frac{\mathrm{d}\tau}{\tau},$$

with $\Gamma_{-\eta} := \{-\eta + \mathbf{i}\,\theta \mid |\theta| \le \varepsilon\}$. Along $\Gamma_{-\eta}$, for each $\tau = -\eta + \mathbf{i}\theta$ with $|\theta| \le \varepsilon$, we keep the notation (4.24) and compute (recall (4.25)):

$$\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) = \omega\,\eta - \frac{\zeta}{3}\,\eta^3 - \frac{\zeta}{4}\,\eta^4 - \frac{\zeta}{4}\,\theta^4 + \zeta\,\eta\left(1 + \frac{3}{2}\,\eta\right)\theta^2\,,$$

where ω and ζ are defined in (4.24) and now satisfy

$$\omega \in \left(-|\alpha_r|, -\frac{|\alpha_r|}{2}\right), \quad \text{and} \quad \zeta \in \left(0, \frac{1-\alpha_r^2}{4|\alpha_r|}\right)\,.$$

As a consequence, we have

$$\mathrm{Re}\left(-\omega\,\tau+\frac{\zeta}{3}\,\tau^3-\frac{\zeta}{4}\,\tau^4\right)\,\leq\,\eta\,\left(-\frac{|\alpha_r|}{2}+\zeta\,\left(1+\frac{3}{2}\,\varepsilon^5\right)\,\varepsilon^2\right)\,\leq\,-\frac{|\alpha_r|}{4}\,\eta\,,$$

thanks to our smallness assumption (4.13) on ε. Thus, we get

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{-\eta}}\mathrm{e}^{n\,\tau-j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\right|\,\leq\,C\,\mathrm{e}^{-c\,n-c\,j_0}\,,$$

which implies that

$$|\mathscr{A}_r^n(j_0)-1|\,\leq\,C\,\mathrm{e}^{-c\,n-c\,j_0}\,,$$

for positive constants C and c that are independent of n and j_0, and $1\leq j_0<\frac{n\,|\alpha_r|}{2}$.

Using now Corollary A.6 in Appendix A, we have the estimate:

$$|1-\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\leq\,C\,\mathrm{e}^{-c\,n}\leq\,C\,\mathrm{e}^{-c\,n-c\,j_0}\,,$$

where the second inequality comes from the fact that n dominates j_0. Adding the two previous estimates, we get

$$|\mathscr{A}_r^n(j_0)-\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\leq\,C\,\mathrm{e}^{-c\,n-c\,j_0}\,.$$

This completes the analysis of case (ii). □

4.3.2 Estimates of the next order term $\mathscr{B}_r^n(j,j_0)$

We now focus our attention to the next term $\mathscr{B}_r^n(j,j_0)$ appearing in the decomposition (4.20). Our main result is the following.

Lemma 4.3. *Let $n\geq 1$ and let $1\leq j_0\leq n$. Let $\mathscr{B}_r^n(j,j_0)$ be defined in (4.21). Then there exist some positive constants C and c that are uniform with respect to n, $j\in\mathbb{Z}$ and j_0 such that with the function $\mathbf{M}_r$ defined in (4.6b), there holds:*

$$\left|\mathscr{B}_r^n(j,j_0)\right|\leq C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,+\,C\,\mathrm{e}^{-c\,n-c\,|j|-c\,j_0}\,.\tag{4.27}$$

Proof. Once again, we shall consider two different regimes identified in the previous section, namely:

(i) $\frac{n\,|\alpha_r|}{2}\leq j_0\leq n$;

(ii) $1\leq j_0<\frac{n\,|\alpha_r|}{2}$.

From its definition, $\mathscr{B}_r^n(j,j_0)$ splits into two parts:

$$\mathscr{B}_r^n(j,j_0)\,=\,\widetilde{\mathscr{G}}_{2,r}^n(j,j_0)\,+\,\widetilde{\mathscr{G}}_{3,r}^n(j,j_0),$$

with quantities $\widetilde{\mathscr{G}}_{2,r}^n(j,j_0)$ and $\widetilde{\mathscr{G}}_{3,r}^n(j,j_0)$ defined in (4.15) and (4.18). We shall estimate separately each of theses two terms below for either case (i) or case (ii).

Case (i). We shall first focus on the term $\widetilde{\mathscr{G}}^n_{2,r}(j,j_0)$ whose expression is given (see (4.15)) by:

$$\widetilde{\mathscr{G}}^n_{2,r}(j,j_0) \,=\, \gamma_j^r \,\times\, \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\rm in}} \exp\big(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\big)\,\mathrm{d}\tau\,,$$

where, at first, $\Gamma_{\rm in}$ is a contour that joins $-\eta-\mathbf{i}\,\varepsilon$ to $-\eta+\mathbf{i}\,\varepsilon$ and that passes to the right of the origin (as depicted in Figure 4.1). Since we now integrate a holomorphic function (there is no longer a pole at the origin !), we shall feel free to deform $\Gamma_{\rm in}$ and choose any contour that joins $-\eta-\mathbf{i}\,\varepsilon$ to $-\eta+\mathbf{i}\,\varepsilon$ and remains within the closed square $\overline{\mathbf{B}_\varepsilon(0)}$ on which Ψ_r is a holomorphic function. We shall only focus on the integral that depends on j_0 and n since we already know that the factor γ_j^r satisfies an exponential estimate (see Proposition 3.3).

For $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$, we may take $\Gamma_{\rm in}$ as the union of the following paths:

$$\Gamma_{\rm in} \,=\, \Gamma_- \,\cup\, \Gamma_+ \,\cup\, \Gamma_0\,,$$

where $\Gamma_0 := \{\mathbf{i}\,\theta \mid |\theta| \le \varepsilon\}$ and $\Gamma_\pm$ are horizontal paths joining $-\eta\pm\mathbf{i}\,\varepsilon$ to $\pm\mathbf{i}\,\varepsilon$ (this is rather similar to what is depicted in Figure 4.2 except that we have shifted the right segment to the abscissa 0 instead of $+\eta$). Upon writing as usual now (see (4.24) and (4.25)):

$$n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau) = \frac{n}{|\alpha_r|}\,\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4 + \frac{j_0}{n}\,|\alpha_r|\,\tau^5\,\Psi_r(\tau)\right)\,,$$

and noticing that for any $\tau \in \Gamma_\pm \subset \mathbf{B}_{\varepsilon_0}(0)$ there holds (see (4.10)):

$$|\tau^5\,\Psi_r(\tau)| \,\le\, C_0\,(\varepsilon^5\,+\,\eta^5)\,,$$

we have for each $\tau \in \Gamma_\pm$:

$$\begin{aligned}\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4 + \frac{j_0}{n}\,|\alpha_r|\,\tau^5\,\Psi_r(\tau)\right) \,&\le\, -\frac{\zeta}{4}\,\varepsilon^4 + |\omega|\,\eta + \zeta\,\eta\,\varepsilon^2 + \frac{3\,\zeta}{2}\,\eta^2\,\varepsilon^2 + |\alpha_r|\,C_0\,(\varepsilon^5+\eta^5)\\ &\le\, -\frac{\zeta}{4}\,\varepsilon^4 + (|\omega| + |\alpha_r|\,C_0)\,\varepsilon^5 + \zeta\,\varepsilon^7 + \frac{3\,\zeta}{2}\,\varepsilon^{12} + |\alpha_r|\,C_0\,\varepsilon^{25}\\ &\le\, -\frac{\zeta}{8}\,\varepsilon^4\,,\end{aligned}$$

thanks to condition (4.12) on ε. As a consequence, we have

$$\left|\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_\pm} \exp\big(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\big)\,\mathrm{d}\tau\right| \,\le\, C\,\mathrm{e}^{-c\,n} \,\le\, C\,\mathrm{e}^{-c\,n-c\,j_0}\,,$$

since n dominates j_0 in this regime.

For the remaining integral along Γ_0, we use the parametrization $\tau = \mathbf{i}\,|\alpha_r|\,\theta$ and we thus get the expression:

$$\begin{aligned}&\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_0} \exp\big(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\big)\,\mathrm{d}\tau\\ &\quad= \frac{|\alpha_r|}{2\pi} \int_{-\frac{\varepsilon}{|\alpha_r|}}^{\frac{\varepsilon}{|\alpha_r|}} \exp\left(\mathbf{i}\,(-j_0 + n\,|\alpha_r|)\,\theta + \mathbf{i}\,\frac{j_0}{|\alpha_r|}\,c_{3,r}\,\theta^3 - \frac{j_0}{|\alpha_r|}\,c_{4,r}\,\theta^4 + j_0\,\mathbf{i}\,\theta^5\,|\alpha_r|^5\,\Psi_r(\mathbf{i}\,|\alpha_r|\,\theta)\right)\,\mathrm{d}\theta\,.\end{aligned}$$

We can now apply Theorem A.2 from Appendix A, and obtain that there exists some small enough $\underline{\varepsilon} > 0$ with $\underline{\varepsilon} \leq \varepsilon / |\alpha_r|$ some constants C and c such that for any $j_0, n \in \mathbb{N}^*$ with $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$, there holds[2]:

$$\left| \int_{-\underline{\varepsilon}}^{\underline{\varepsilon}} \exp\left(\mathbf{i}\,(-j_0 + n\,|\alpha_r|)\,\theta + \mathbf{i}\,\frac{j_0}{|\alpha_r|}\,c_{3,r}\,\theta^3 - \frac{j_0}{|\alpha_r|}\,c_{4,r}\,\theta^4 + j_0\,\mathbf{i}\,\theta^5\,|\alpha_r|^5\,\Psi_r(\mathbf{i}\,|\alpha_r|\,\theta) \right) \mathrm{d}\theta \right| \\ \leq C\,\mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|} \right).$$

Since we are in the regime $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$, we deduce that we always have an upper estimate

$$\mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|} \right) \leq C\,\mathbf{M}_r(c, -j_0 + n\,|\alpha_r|, n)\,,$$

so that the latter estimate also reads:

$$\left| \int_{-\underline{\varepsilon}}^{\underline{\varepsilon}} \exp\left(\mathbf{i}\,(-j_0 + n\,|\alpha_r|)\,\theta + \mathbf{i}\,\frac{j_0}{|\alpha_r|}\,c_{3,r}\,\theta^3 - \frac{j_0}{|\alpha_r|}\,c_{4,r}\,\theta^4 + j_0\,\mathbf{i}\,\theta^5\,|\alpha_r|^5\,\Psi_r(\mathbf{i}\,|\alpha_r|\,\theta) \right) \mathrm{d}\theta \right| \\ \leq C\,\mathbf{M}_r(c, -j_0 + n\,|\alpha_r|, n)\,.$$

The only remaining task is to control the integral with respect to θ on the two intervals $[-\varepsilon/|\alpha_r|, -\underline{\varepsilon}]$ and $[\underline{\varepsilon}, \varepsilon/|\alpha_r|]$. This is entirely similar to what we have already done on the segments $\Gamma_\pm$ so we feel free to skip the details. The final estimate on these segments is of exponential type with respect to both n and j_0 (that are of comparable sizes in case (i)). Combining all above estimates on the segments $\Gamma_\pm$ and on Γ_0, we have obtained the following estimate for $\widetilde{\mathscr{G}}_{2,r}^n(j, j_0)$:

$$\left| \widetilde{\mathscr{G}}_{2,r}^n(j, j_0) \right| \leq C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r(c, -j_0 + n\,|\alpha_r|, n) + C\,\mathrm{e}^{-c\,n - c\,|j| - c\,j_0}\,,$$

for suitable constants C and c and $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$ (the integer $j \in \mathbb{Z}$ is arbitrary).

We finally turn our attention to the next term $\widetilde{\mathscr{G}}_{3,r}^n(j, j_0)$ that enters the definition of $\mathscr{B}_r^n(j, j_0)$. We recall that $\widetilde{\mathscr{G}}_{3,r}^n(j, j_0)$ is defined as follows:

$$\widetilde{\mathscr{G}}_{3,r}^n(j, j_0) = \mathcal{H}_j \times \Psi_r(0) \times j_0 \times \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\mathrm{in}}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) \right) \tau^4 \mathrm{d}\tau\,,$$

and we shall mainly focus our efforts on the above integral on the right-hand side since we already know that $\mathcal{H}_j$ decays exponentially with respect to j, and $\Psi_r(0)$ is just a constant factor. We still focus here on case (i), that is on the regime $\frac{n\,|\alpha_r|}{2} \leq j_0 \leq n$. It is important to observe that we need to absorb the factor j_0 that is unbounded.

Since we integrate a holomorphic function, we may use Cauchy's formula and deform the contour Γ_{in}. We therefore choose Γ_{in} to be the union of the paths $\Gamma_0 := \{\mathbf{i}\,\theta \mid |\theta| \leq \varepsilon\}$ and $\Gamma_\pm$ which are horizontal paths joining $-\eta \pm \mathbf{i}\,\varepsilon$ to $\pm\mathbf{i}\,\varepsilon$, just as we did above for the analysis of the term $\widetilde{\mathscr{G}}_{2,r}^n(j, j_0)$. For the integrals

[2] With the notation of Theorem A.2, we use $x := -j_0 + n\,|\alpha_r|$ and $y := j_0/|\alpha_r|$. In case (i), we are in a regime where $|x|/y$ is bounded and y is bounded from below so that we can tune the constant $\underline{\mathbf{C}}$.

along $\Gamma_\pm$, we may use again the estimate (4.26) and combine with a uniform bound for the factor τ^4. Since n and j_0 are comparable in this first regime, we readily obtain the estimate:

$$\left| j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_\pm} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau \right| \le C\,j_0\,\mathrm{e}^{-c\,n - c\,j_0} \le C\,\mathrm{e}^{-c\,n - c\,j_0}\,.$$

For the remaining integral along Γ_0, we add and subtract to get:

$$\begin{aligned} j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_0} &\exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau \\ &= j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\mathbf{i}\,\mathbb{R}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau - j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_0^c} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau\,, \end{aligned}$$

with rather obvious notation. For $\tau = \mathbf{i}\,\theta \in \Gamma_0^c$, we have (see the definition (3.30)):

$$\mathrm{Re}\,\Big(n\,\tau - j_0\,\varphi_r(\tau)\Big) = -c\,j_0\,\theta^4\,,$$

for some constant $c > 0$, so we have the straightforward estimate:

$$\left| j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\Gamma_0^c} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau \right| \le C\,j_0 \int_\varepsilon^{+\infty} \theta^4\,\mathrm{e}^{-c\,j_0\,\theta^4}\,\mathrm{d}\theta \le C\,j_0\,\mathrm{e}^{-c\,j_0} \le C\,\mathrm{e}^{-c\,j_0}\,.$$

Since j_0 and n are comparable in case (i), we can collect all the above estimates and already obtain the estimate:

$$\left| \widetilde{\mathscr{G}}_{3,r}^n(j,j_0) - \mathcal{H}_j \times \Psi_r(0) \times j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\mathbf{i}\,\mathbb{R}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau \right| \le C\,\mathrm{e}^{-c\,n - c\,|j| - c\,j_0}\,,$$

for suitable uniform constants C and c, and integers n, j_0, j that satisfy $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$ and $j \in \mathbb{Z}$.

It thus only remains to estimate the integral over the imaginary axis for which we refer to the definition (A.49) in Appendix A of the function $\mathbf{G}_4$ (the constants c_3 and c_4 should be taken to be $c_{3,r}$ and $c_{4,r}$ since we deal here with the right state of the shock). Using the parametrization $\tau = \mathbf{i}\,|\alpha_r|\,\theta$ in the integral, we get the relation:

$$j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\mathbf{i}\,\mathbb{R}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau = |\alpha_r|^5\,j_0\,\mathbf{G}_4\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right)\,,$$

and we can then use the estimates provided by Theorem A.4 to get[3]:

$$\left| j_0 \times \frac{1}{2\pi \mathbf{i}} \int_{\mathbf{i}\,\mathbb{R}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau \right| \le C\,\mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) \le C\,\mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, n\right)\,,$$

since we have $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$. Collecting all the above estimates for the various contributions in the decomposition of $\widetilde{\mathscr{G}}_{3,r}^n(j,j_0)$, we end up with the estimate:

$$\left| \widetilde{\mathscr{G}}_{3,r}^n(j,j_0) \right| \le C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r(c, -j_0 + n\,|\alpha_r|, n) + C\,\mathrm{e}^{-c\,n - c\,|j| - c\,j_0}\,,$$

for $\frac{n\,|\alpha_r|}{2} \le j_0 \le n$, $j \in \mathbb{Z}$, and for suitable constants C and c that are uniform with respect to n, j and j_0. This completes the analysis of case (i) by combining with the estimate for $\widetilde{\mathscr{G}}_{2,r}^n(j,j_0)$.

[3]The fact that we consider the function $\mathbf{G}_4$ is crucial here since 4 is the first index where the gain in the estimates of Theorem A.4 is sufficient to absorb the factor j_0.

Case (ii). In the case $1 \leq j_0 \leq \frac{n\,|\alpha_r|}{2}$, we simply take $\Gamma_{in} = \Gamma_{-\eta}$ (see Figure 4.2) for evaluating the integrals arising in the definition of both $\widetilde{\mathscr{G}}^n_{2,r}(j,j_0)$ and $\widetilde{\mathscr{G}}^n_{3,r}(j,j_0)$. This is legitimate because we integrate holomorphic functions on the closed square $\overline{\mathbf{B}_\varepsilon(0)}$ so we can apply Cauchy's formula. Reproducing similar computations as in the previous subsection, we get that for each $\tau = -\eta + \mathbf{i}\,\theta \in \Gamma_{-\eta}$ (with therefore $|\theta| \leq \varepsilon$), there holds:

$$\begin{aligned}\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4 + \frac{j_0}{n}\,|\alpha_r|\,\tau^5\,\Psi_r(\tau)\right) &\leq \eta\,\left(-\frac{|\alpha_r|}{2} + \zeta\,\left(1 + \frac{3}{2}\,\varepsilon^5\right)\,\varepsilon^2 + \frac{\alpha_r^2}{2}\,C_0\,\varepsilon^{20}\right)\\ &\quad - \theta^4\,\left(\frac{\zeta}{4} - |\alpha_r|\,C_0\,|\theta|\right)\\ &\leq -\frac{|\alpha_r|}{4}\,\eta\,,\end{aligned}$$

and similarly:

$$\mathrm{Re}\left(-\omega\,\tau + \frac{\zeta}{3}\,\tau^3 - \frac{\zeta}{4}\,\tau^4\right) \leq -\frac{|\alpha_r|}{4}\,\eta\,,$$

where once again we have used conditions (4.11) and (4.12) on ε and our choice for η. By applying the triangle inequality (with a uniform bound for $\tau \in \Gamma_{-\eta}$), we deduce that

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{-\eta}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\,\mathrm{d}\tau\right| \leq C\,\mathrm{e}^{-c\,n} \leq C\,\mathrm{e}^{-c\,n - c\,j_0}\,,$$

and

$$\left|j_0\,\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{-\eta}} \exp\left(n\,\tau - j_0\,\varphi_r(\tau)\right)\,\tau^4\,\mathrm{d}\tau\right| \leq C\,j_0\,\mathrm{e}^{-c\,n - c\,j_0} \leq C\,\mathrm{e}^{-c\,n - c\,j_0}\,,$$

where we use once again the fact that n dominates j_0. As a consequence, using the exponential estimate of γ^r_j from Proposition 3.3 and the fact that $\mathcal{H}_j$ is also exponentially decaying, we obtain the uniform exponential estimate:

$$|\mathscr{B}^n_r(j,j_0)|\;C\,\mathrm{e}^{-c\,n - c\,|j| - c\,j_0}\,,$$

for $1 \leq j_0 \leq \frac{n\,|\alpha_r|}{2}$ and $j \in \mathbb{Z}$. This completes the analysis of case (ii). □

4.3.3 Estimates of the remainder term $\mathscr{R}^n_r(j,j_0)$

In this section, we shall prove some estimates on the remainder term $\mathscr{R}^n_r(j,j_0)$ whose expression is gathered in (4.22). We recall that $\mathscr{R}^n_r(j,j_0)$ is decomposed into $\mathscr{R}^n_r(j,j_0) = \mathscr{R}^n_{1,r}(j,j_0) + \mathscr{R}^n_{2,r}(j,j_0)$ with $\mathscr{R}^n_{1,r}(j,j_0)$ and $\mathscr{R}^n_{2,r}(j,j_0)$ defined in (4.16) and (4.19). In both (4.16) and (4.19), the path Γ_{in} is any path joining $-\eta - \mathbf{i}\,\varepsilon$ to $-\eta + \mathbf{i}\,\varepsilon$ which remains in $\overline{\mathbf{B}_\varepsilon(0)}$ since the integrand is holomorphic with respect to τ on that set. We further recall that the sequence $(\mathcal{H}_j)_{j\in\mathbb{Z}}$ in (3.22) is exponentially decreasing and the sequence of holomorphic functions $(\Phi_{r,j})_{j\in\mathbb{Z}}$ satisfies the exponential bound stated in Proposition 3.3, uniformly with respect to $\tau \in \mathbf{B}_{\varepsilon_0}(0)$. Restricting to the smaller set $\overline{\mathbf{B}_\varepsilon(0)} \subset \mathbf{B}_{\varepsilon_0}(0)$, we thus have

$$|\mathcal{H}_j| + |\Phi_{r,j}(\tau)| \leq C\,\mathrm{e}^{-c\,|j|}\,,$$

for all $j \in \mathbb{Z}$ and $\tau \in \overline{\mathbf{B}_\varepsilon(0)}$ with uniform constants $C, c > 0$. Our main result in this section is the following.

Proposition 4.1. *There exist $C, c > 0$ such that for all $n \in \mathbb{N}^*$ and $(j, j_0) \in \mathbb{Z}^2$ such that $1 \leq j_0 \leq n$, one has:*

$$|\mathscr{R}_r^n(j, j_0)| \leq C \begin{cases} \dfrac{1}{n^{1/4}} \dfrac{e^{-c|j|}}{n^{1/3}} \exp\left(-c\left(\dfrac{j_0 - n|\alpha_r|}{n^{1/3}}\right)^{3/2}\right), & \text{if } j_0 - n|\alpha_r| \geq 0, \\ \dfrac{e^{-c|j|}}{n^{7/12}}, & \text{if } -n^{1/3}|\alpha_r| \leq j_0 - n|\alpha_r| \leq 0, \\ \dfrac{1}{n^{1/8}} \dfrac{e^{-c|j|}}{n^{1/2}} \exp\left(-c\left(\dfrac{|j_0 - n|\alpha_r||}{n^{1/2}}\right)^2\right), & \text{if } j_0 - n|\alpha_r| \leq -n^{1/3}|\alpha_r|. \end{cases}$$

In particular, we have:

$$|\mathscr{R}_r^n(j, j_0)| \leq C\, \mathrm{e}^{-c|j|}\, \mathbf{M}_r(c, -j_0 + |\alpha_r|\, n, n),$$

with the function $\mathbf{M}_r$ defined in (4.6b).

Proof. We will mainly focus on the remainder term $\mathscr{R}_{1,r}^n(j, j_0)$ and briefly explain how to recover similar estimates for the second term $\mathscr{R}_{2,r}^n(j, j_0)$ in the last part of this section. We shall decompose the proof into several steps, which corresponds to different regimes for ω, which we recall is defined as

$$\omega := \frac{j_0 - n|\alpha_r|}{n} \in (-|\alpha_r|, 1 - |\alpha_r|), \quad \text{when} \quad 1 \leq j_0 \leq n.$$

More precisely, we define the following three regimes:

$$\text{(I) } |\omega| \leq n^{-2/3}|\alpha_r|, \quad \text{(II) } n^{-2/3}|\alpha_r| \leq \omega \leq 1 - |\alpha_r|, \quad \text{(III) } -|\alpha_r| \leq \omega \leq -n^{-2/3}|\alpha_r|.$$

Case (I) – Uniform bound. For $|\omega| \leq n^{-2/3}|\alpha_r|$, we provide a uniform bound for $\mathscr{R}_r^n(j, j_0)$ using classical results from oscillatory integrals (see Proposition A.1 in Appendix A for similar arguments).

Lemma 4.4. *There exist constants $C, c > 0$ such that for all $n \geq 1$, $1 \leq j_0 \leq n$ with $|j - j_0| \leq n$ one has*

$$\left|\mathscr{R}_{1,r}^n(j, j_0)\right| \leq \frac{C}{n^{7/12}} e^{-c|j|},$$

for $|\omega| \leq n^{-2/3}|\alpha_r|$.

Proof. Let us first observe that for $|\omega| \leq n^{-2/3}|\alpha_r|$, there holds:

$$\left|\frac{j_0}{n|\alpha_r|} - 1\right| \leq n^{-2/3},$$

so for $n \geq 2$, we get a uniform bound from below $j_0/n \geq c > 0$. The case $n = 1$ should be dealt with separately but it is far easier since we have $j_0 = 1$ and $j \in \{0, 1, 2\}$ so we only deal with finitely many integrals then. We shall therefore assume $n \geq 2$ from now on and use that the quantity ζ defined in (4.24) is uniformly bounded from below by a positive constant.

We take Γ_{in} as the union of the following paths:

$$\Gamma_{\text{in}} = \Gamma_- \cup \Gamma_+ \cup \Gamma_0,$$

where $\Gamma_0 = \{\mathbf{i}\,\theta \mid |\theta| \leq \varepsilon\}$ and $\Gamma_\pm$ are horizontal paths joining $-\eta \pm \mathbf{i}\,\varepsilon$ to $\pm\mathbf{i}\,\varepsilon$. Upon writing as usual now

$$n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau) = \frac{n}{|\alpha_r|}\left(-\omega\tau + \frac{\zeta}{3}\tau^3 - \frac{\zeta}{4}\tau^4 + \frac{j_0}{n}|\alpha_r|\tau^5\Psi_r(\tau)\right),$$

we have already proved that for each $\tau \in \Gamma_\pm$, there holds:

$$\mathrm{Re}\left(-\omega\tau + \frac{\zeta}{3}\tau^3 - \frac{\zeta}{4}\tau^4 + \frac{j_0}{n}|\alpha_r|\tau^5\Psi_r(\tau)\right) \leq -\frac{\zeta}{8}\varepsilon^8,$$

thanks to the smallness condition (4.12) on ε. As a consequence, we get

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\pm}\exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\,\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\,\mathrm{e}^{-c\,|j|-c\,j_0}\,.$$

But since $|\omega| \leq n^{-2/3}|\alpha_r|$, we readily get $\mathrm{e}^{-j_0} \leq C\mathrm{e}^{-c\,n}$, from which we deduce that

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\pm}\exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\,\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\,\frac{\mathrm{e}^{-c\,|j|}}{n^{7/12}}\,.$$

Along the remaining integral on the vertical segment Γ_0, we have

$$\begin{aligned}\frac{1}{2\pi\mathbf{i}}&\int_{\Gamma_0}\exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\,\Phi_{r,j}(\tau)\mathrm{d}\tau \\ &= \frac{1}{2\pi}\int_{-\varepsilon}^{\varepsilon}\exp\left(\frac{n}{|\alpha_r|}\left[-\omega\mathbf{i}\,\theta - \frac{\zeta}{3}\mathbf{i}\,\theta^3 - \frac{\zeta}{4}\theta^4 + \frac{j_0}{n}|\alpha_r|\mathbf{i}\,\theta^5\Psi_r(\mathbf{i}\,\theta)\right]\right)\theta\,\Phi_{r,j}(\mathbf{i}\,\theta)\,\mathrm{d}\theta\,.\end{aligned}$$

We introduce two functions (that depend on (j, j_0, n)):

$$h(\theta) := \exp\left(-n\mathbf{i}\left(\frac{\omega}{|\alpha_r|}\theta + \frac{\zeta}{3|\alpha_r|}\theta^3\right)\right),\quad g(\theta) := \exp\left(-n\frac{\zeta}{4|\alpha_r|}\theta^4 + n\mathbf{i}\frac{2\zeta\alpha_r^2}{1-\alpha_r^2}\theta^5\Psi_r(\mathbf{i}\theta)\right)\theta\,\Phi_{r,j}(\mathbf{i}\,\theta).$$

Using [24, Lemma 3.1], we have the estimate

$$\left|\int_{-\varepsilon}^{\varepsilon}h(\theta)\,g(\theta)\,\mathrm{d}\theta\right| \leq \left(\sup_{x\in[-\varepsilon,\varepsilon]}\left|\int_{-\varepsilon}^{x}h(\theta)\mathrm{d}\theta\right|\right)\left(\|g\|_{L^\infty([-\varepsilon,\varepsilon])} + \|g'\|_{L^1([-\varepsilon,\varepsilon])}\right).\tag{4.28}$$

By an application of the van der Corput Lemma, there exists a constant $C > 0$, independent of ω and n, such that[4]

$$\forall x \in [-\varepsilon, \varepsilon],\quad \left|\int_{-\varepsilon}^{x}h(\theta)\mathrm{d}\theta\right| \leq \frac{C}{n^{1/3}}.$$

Furthermore, with our choice (4.11) of $\varepsilon > 0$, we have

$$\forall\theta \in [-\varepsilon, \varepsilon],\quad |g(\theta)| \leq C\,|\theta|\,\mathrm{e}^{-n\frac{\zeta}{8|\alpha_r|}\theta^4}\mathrm{e}^{-c\,|j|}\,.$$

Differentiating the expression for $g(\theta)$, we also get the bound

$$\forall\theta \in [-\varepsilon, \varepsilon],\quad \left|g'(\theta)\right| \leq C\,\left(1 + n|\theta|^4\right)\,\mathrm{e}^{-n\frac{\zeta}{8|\alpha_r|}\theta^4}\,\mathrm{e}^{-c\,|j|},$$

[4] This holds because the parameter ζ is uniformly bounded from below in the considered regime.

such that we get that

$$\|g'\|_{L^1([-\varepsilon,\varepsilon])} \leq \frac{C}{n^{1/4}} e^{-c\,|j|},$$

since ζ is uniformly positive in the considered regime. Using estimate (4.28), we arrive at the final bound

$$\left|\mathscr{R}^n_{1,r}(j,j_0)\right| \leq \frac{C}{n^{1/3+1/4}} \mathrm{e}^{-c\,|j|},$$

for some constants $C, c > 0$ independent of n, j_0 and j. This concludes the proof of the lemma. $\square$

Case (II) − **Fast decaying tail.** We now turn our attention to the second regime (II) where $n^{-2/3}|\alpha_r| < \omega \leq 1 - |\alpha_r|$ implying that necessarily $j_0 - n\,|\alpha_r| > 0$ where we expect to observe a fast decaying bound for $\mathscr{R}^n_{1,r}(j,j_0)$.

Lemma 4.5. *There exist constants $C, c > 0$ such that for all $n \geq 1$, $1 \leq j_0 \leq n$ with $|j - j_0| \leq n$ one has*

$$\left|\mathscr{R}^n_{1,r}(j,j_0)\right| \leq C\,\frac{\mathrm{e}^{-c\,|j|}}{n^{2/3}} \left(\frac{j_0 - n\,|\alpha_r|}{n^{1/3}}\right)^{-1/2} \exp\left(-c\left(\frac{j_0 - n\,|\alpha_r|}{n^{1/3}}\right)^{3/2}\right),$$

as long as $n^{-2/3}|\alpha_r| \leq \omega \leq 1 - |\alpha_r|$.

Proof. We first note that when $\omega > 0$, one has $\frac{1-\alpha_r^2}{2|\alpha_r|} < \zeta \leq \frac{1-\alpha_r^2}{2\alpha_r^2}$ so ζ is uniformly positive. With the constant $C_0 > 0$ in (4.10) (associated to ε_0 given by Proposition 3.3), we choose $\omega_\varepsilon \in (0, 1 - |\alpha_r|)$ small enough such that the following inequalities are satisfied

$$\omega_\varepsilon \leq \frac{1-\alpha_r^2}{2|\alpha_r|}\epsilon^2, \quad \sqrt{\omega_\varepsilon} \leq \frac{1}{8|\alpha_r|C_0}\left(\frac{1-\alpha_r^2}{2|\alpha_r|}\right)^{3/2}, \quad \sqrt{\omega_\varepsilon} \leq \frac{1}{3}\left(\frac{1-\alpha_r^2}{2|\alpha_r|}\right)^{1/2}. \tag{4.29}$$

Next, we fix $n^{-2/3}|\alpha_r| \leq \omega \leq \omega_\varepsilon$. We introduce a family of parametrized curves (see[5] Figure 4.3 for an illustration) indexed by ω as follows

$$\Gamma_- := \{t - \mathbf{i}\,\varepsilon \mid t \in [-\eta, 0]\}, \quad \Gamma_+ := \{-t + \mathbf{i}\,\varepsilon \mid t \in [0, \eta]\}, \quad \Gamma_\omega := \left\{\sqrt{\frac{\omega}{\zeta}} + \mathbf{i}\,\theta \mid \theta \in [-\varepsilon, \varepsilon]\right\},$$

together with

$$\Gamma^\omega_> := \left\{t - \mathbf{i}\,\varepsilon \mid t \in \left[0, \sqrt{\frac{\omega}{\zeta}}\right]\right\}, \quad \Gamma^\omega_< := \left\{\sqrt{\frac{\omega}{\zeta}} - t + \mathbf{i}\,\varepsilon \mid t \in \left[0, \sqrt{\frac{\omega}{\zeta}}\right]\right\}.$$

Along $\Gamma_\pm$, we have already proved that

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\pm} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\,\Phi_{r,j}(\tau)\,\mathrm{d}\tau\right| \leq C\,\mathrm{e}^{-c\,|j| - c\,j_0}.$$

Next, we remark that since $0 < \omega \leq 1 - |\alpha_r|$ we deduce that

$$-1 \leq -\frac{1}{\sqrt{1-|\alpha_r|}}\left(\frac{j_0 - n\,|\alpha_r|}{n}\right)^{1/2},$$

[5]The reader may compare our choice with the one made in [9] that corresponds to the parametrization of the Green's function for the Cauchy problem. See also Appendix A. The difference here is that we rather parametrize all curves in terms of the time frequency rather than with respect to the space frequency.

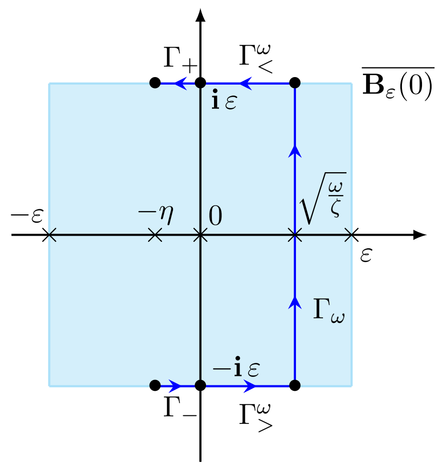

Figure 4.3: *In blue: the contour Γ_{in} and its decomposition into five parts (Γ_-, $\Gamma^\omega_>$, Γ_ω, $\Gamma^\omega_<$ and Γ_+) in the regime where $n^{-2/3}|\alpha_r| \le \omega \le 1-|\alpha_r|$. The black bullets represent the end points of the contours.*

and thus

$$-j_0 = -n\,|\alpha_r| - (j_0 - n\,|\alpha_r|) \le -n\,|\alpha_r| - \frac{1}{\sqrt{1-|\alpha_r|}}\left(\frac{j_0 - n\,|\alpha_r|}{n^{1/3}}\right)^{3/2}.$$

As a consequence, we have obtained

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\pm} \exp\left(n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\,\Phi_{r,j}(\tau)\,\mathrm{d}\tau\right| \le C\,\mathrm{e}^{-c\,n - c\,|j|}\exp\left(-c\left(\frac{j_0 - n\,|\alpha_r|}{n^{1/3}}\right)^{3/2}\right),$$

which can be subsumed into our desired bound.

Next, we handle the contributions along $\Gamma^\omega_>$ and $\Gamma^\omega_<$. For example, in the former case, upon denoting $\Lambda_r(\tau) := n\,\tau - j_0\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)$, we have that for each $\tau = t - \mathbf{i}\epsilon \in \Gamma^\omega_>$ with $t \in \left[0, \sqrt{\frac{\omega}{\zeta}}\right]$,

$$\begin{aligned}
\mathrm{Re}\left(\Lambda_r\left(t - \mathbf{i}\,\varepsilon\right)\right) &\le \frac{n}{|\alpha_r|}\left[-\omega t + \frac{\zeta}{3}\left(t^3 - 3\varepsilon^2 t\right) - \frac{\zeta}{4}\left(t^4 + \varepsilon^4 - 6\varepsilon^2 t^2\right) + |\alpha_r| C_0 (t^5 + \varepsilon^5)\right] \\
&\le \frac{n}{|\alpha_r|}\left[-t\left(\frac{2}{3}\omega + \zeta\varepsilon^2\left(1 - \frac{3}{2}\sqrt{\frac{\omega}{\zeta}}\right)\right) - t^4\left(\frac{\zeta}{4} - |\alpha_r| C_0 t\right) - \varepsilon^4\left(\frac{\zeta}{4} - |\alpha_r| C_0 \varepsilon\right)\right] \\
&\le \frac{n}{|\alpha_r|}\left[-t\left(\frac{2}{3}\omega + \frac{\zeta}{2}\varepsilon^2\right) - \frac{\zeta}{8}\varepsilon^4\right],
\end{aligned}$$

since from (4.11) one has $8|\alpha_r| C_0 \varepsilon < \zeta$ and from (4.29) one has $\sqrt{\omega} < \sqrt{\zeta}/3$ and $8|\alpha_r| C_0 \sqrt{\omega} \le \zeta^{3/2}$. Hence,

$$\begin{aligned}
\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\omega_>} \exp\left(n\,\tau - j_0\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)\right)\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| &\le C e^{-n\frac{\zeta\varepsilon^4}{8|\alpha_r|} - c\,|j|}\int_0^{\sqrt{\frac{\omega}{\zeta}}} \mathrm{e}^{-nt\frac{\zeta}{2|\alpha_r|}\varepsilon^2}\mathrm{d}t \\
&\le C \mathrm{e}^{-nc - c\,|j|}\exp\left(-c\left(\frac{j_0 - n\,|\alpha_r|}{n^{1/3}}\right)^{3/2}\right).
\end{aligned}$$

We finally turn our attention to the last integral along Γ_ω. We note that in that case, for each $\theta\in[-\varepsilon,\varepsilon]$, one has

$$\begin{aligned}\mathrm{Re}\left(\Lambda_r\left(\sqrt{\frac{\omega}{\zeta}}+\mathbf{i}\theta\right)\right) &\leq \frac{n}{|\alpha_r|}\left(-\frac{2}{3\sqrt{\zeta}}\omega^{3/2}-\frac{\omega^2}{\zeta^2}\left(\frac{\zeta}{4}-|\alpha_r|C_0\sqrt{\frac{\omega}{\zeta}}\right)-\sqrt{\omega}\left(\sqrt{\zeta}-\frac{3}{2}\sqrt{\omega}\right)\theta^2\right)\\ &\quad -\frac{n}{|\alpha_r|}\left(\frac{\zeta}{4}-|\alpha_r|C_0\varepsilon\right)\theta^4\\ &\leq \frac{n}{|\alpha_r|}\left(-\frac{2}{3\sqrt{\zeta}}\omega^{3/2}-\frac{3\sqrt{\zeta}}{4}\sqrt{\omega}\theta^2\right),\end{aligned}$$

thanks to our assumption (4.29). As a consequence, we get

$$\begin{aligned}\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\omega}\mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| &\leq C\mathrm{e}^{-\frac{2n}{3|\alpha_r|\sqrt{\zeta}}\omega^{3/2}-c|j|}\int_{-\varepsilon}^{\varepsilon}\left(\sqrt{\omega}+|\theta|\right)\mathrm{e}^{-\frac{3\sqrt{\zeta}}{4|\alpha_r|}n\sqrt{\omega}\theta^2}\mathrm{d}\theta\\ &\leq C\mathrm{e}^{-\frac{2n}{3|\alpha_r|\sqrt{\zeta}}\omega^{3/2}-c|j|}\left(\frac{\omega^{1/4}}{n^{1/2}}+\frac{1}{n\omega^{1/2}}\right)\\ &\leq C\frac{e^{-c|j|}}{n^{2/3}}\left(\frac{|1-j_0|-n|\alpha_r|}{n^{1/3}}\right)^{-1/2}\exp\left(-c\left(\frac{j_0-n|\alpha_r|}{n^{1/3}}\right)^{3/2}\right).\end{aligned}$$

We now move to the case where $\omega_\varepsilon\leq\omega\leq 1-|\alpha_r|$. We follow the same strategy and use the contours $\Gamma_\pm$, $\Gamma_{\omega_\varepsilon}$ and $\Gamma_{\lessgtr}^{\omega_\varepsilon}$ independently of ω. We readily notice that with our careful choice of ω_ε all the previous computations remain valid and thus we also have in that case

$$\begin{aligned}\left|\mathscr{R}_{1,r}^n(j,j_0)\right| &\leq \left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_\pm}\mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right|+\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_>^{\omega_\varepsilon}\cup\Gamma_<^{\omega_\varepsilon}}\mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right|\\ &\quad+\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\omega_\varepsilon}}\mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right|\\ &\leq C\left[e^{-c\,j_0-c|j|}+e^{-cn-c|j|}+\mathrm{e}^{-\frac{n}{|\alpha_r|}\left(\omega-\frac{\omega_\varepsilon}{\zeta}\right)\sqrt{\frac{\omega_\varepsilon}{\zeta}}-c|j|}\left(\frac{\omega_\varepsilon^{1/4}}{n^{1/2}}+\frac{1}{n\omega_\varepsilon^{1/2}}\right)\right]\\ &\leq C\frac{\mathrm{e}^{-c|j|}}{n^{2/3}}\left(\frac{j_0-n|\alpha_r|}{n^{1/3}}\right)^{-1/2}\exp\left(-c\left(\frac{j_0-n|\alpha_r|}{n^{1/3}}\right)^{3/2}\right),\end{aligned}$$

since $\omega_\varepsilon\leq\omega\leq 1-|\alpha_r|$. This concludes the proof of the lemma. □

Case (III) **– Oscillatory tail.** We finally move to the last regime (III) where $-|\alpha_r|<\omega\leq -n^{-2/3}|\alpha_r|$. The analysis is further split into two parts.

Lemma 4.6. *There exist constants $\omega_*>0$ and $C,c>0$ such that for each $1\leq j_0\leq n$ and $|j-j_0|\leq n$, on has*

$$\left|\mathscr{R}_{1,r}^n(j,j_0)\right|\leq C\,\frac{\mathrm{e}^{-c|j|}}{n^{5/8}}\left(\frac{1}{n^{1/8}}\left(\frac{|j_0-n|\alpha_r||}{n^{1/2}}\right)^{-1/2}+\left(\frac{|j_0-n|\alpha_r||}{n^{1/2}}\right)^{1/4}\right)\mathrm{e}^{-c\left(\frac{|j_0-n|\alpha_r||}{n^{1/2}}\right)^2},$$

as long as $-\omega_*<\omega\leq -n^{-2/3}|\alpha_r|$.

Proof. We let $\omega_* \in (0, |\alpha_r|/3)$ be fixed as follows:

$$\omega_* \leq \frac{1-\alpha_r^2}{3|\alpha_r|}\left(-1+\sqrt{1+4\varepsilon}\right)^2, \quad |\alpha_r| C_0 \sqrt{\omega_*} \leq \frac{1}{8}\left(\frac{1-\alpha_r^2}{3|\alpha_r|}\right)^{3/2}, \quad \sqrt{\omega_*} \leq \frac{1}{3}\left(\frac{1-\alpha_r^2}{3|\alpha_r|}\right)^{1/2}. \tag{4.30}$$

As a consequence, for each $|\omega| \in [n^{-2/3}|\alpha_r|, \omega_*]$, one has

$$\frac{1-\alpha_r^2}{3|\alpha_r|} = \frac{2|\alpha_r|}{3}\frac{1-\alpha_r^2}{2\alpha_r^2} < (|\alpha_r| - \omega_*)\frac{1-\alpha_r^2}{2\alpha_r^2} \leq \frac{j_0}{n}\frac{1-\alpha_r^2}{2\alpha_r^2} = \zeta.$$

Defining:

$$\chi_\omega := \frac{|\omega|}{2\zeta} + \sqrt{\frac{|\omega|}{\zeta}} > 0,$$

the above conditions imply in particular that

$$\chi_\omega \leq \frac{\varepsilon}{4}, \quad |\alpha_r| C_0 \sqrt{\frac{|\omega|}{\zeta}} \leq \frac{\zeta}{8}, \quad \sqrt{\frac{|\omega|}{\zeta}} \leq \frac{1}{3}, \qquad \text{with } |\omega| \in [n^{-2/3}, \omega_*].$$

We then introduce the following contours which are illustrated in Figure 4.4 (compare again with the choice made in [9] that is entirely similar):

- two horizontal contours $\Gamma_\pm$ defined as

$$\Gamma_- = \{t - \mathbf{i}\,\varepsilon \mid t \in [-\eta, 0]\}, \quad \Gamma_+ = \{-t + \mathbf{i}\,\varepsilon \mid t \in [0, \eta]\};$$

- two horizontal contours $\Gamma^\omega_{\lessgtr}$ defined as

$$\Gamma^\omega_> = \left\{t - \mathbf{i}\,\varepsilon \mid t \in \left[0, \sqrt{\frac{|\omega|}{\zeta}}\right]\right\}, \quad \Gamma^\omega_< = \left\{\sqrt{\frac{|\omega|}{\zeta}} - t + \mathbf{i}\,\varepsilon \mid t \in \left[0, \sqrt{\frac{|\omega|}{\zeta}}\right]\right\};$$

- two vertical contours $\Gamma^v_\pm$ defined as

$$\Gamma^v_- = \left\{\sqrt{\frac{|\omega|}{\zeta}} + \mathbf{i}\,\theta \mid \theta \in \left[-\varepsilon, -\chi_\omega - \sqrt{\frac{|\omega|}{\zeta}}\right]\right\}, \quad \Gamma^v_+ = \left\{\sqrt{\frac{|\omega|}{\zeta}} + \mathbf{i}\,\theta \mid \theta \in \left[\chi_\omega + \sqrt{\frac{|\omega|}{\zeta}}, \varepsilon\right]\right\};$$

- two oblique contours $\Gamma^o_\pm$

$$\Gamma^o_- = \left\{-\frac{|\omega|}{2\zeta} - \mathbf{i}\sqrt{\frac{|\omega|}{\zeta}} + t e^{3\mathbf{i}\pi/4} \mid t \in \left[-\sqrt{2}\chi_\omega, \sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}\right]\right\},$$

$$\Gamma^o_+ = \left\{-\frac{|\omega|}{2\zeta} + \mathbf{i}\sqrt{\frac{|\omega|}{\zeta}} + t e^{\mathbf{i}\pi/4} \mid t \in \left[-\sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}, \sqrt{2}\chi_\omega\right]\right\}.$$

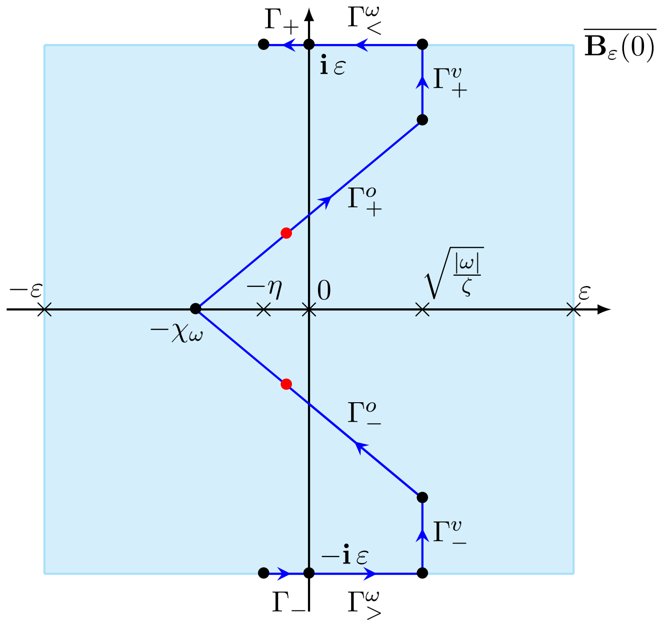

Figure 4.4: *In blue: the contour* Γ_{in} *within* $\overline{\mathbf{B}_\varepsilon(0)}$ *and its decomposition into eight parts (*Γ_-, $\Gamma^\omega_>$, Γ^v_-, Γ^0_-, Γ^0_+, Γ^+_v, $\Gamma^\omega_<$ *and* Γ_+*) in the regime where* $-\omega_* < \omega \leq -n^{-2/3}|\alpha_r|$*. The two red dots correspond to the approximate saddles of the phase* $|\omega|\tau + \frac{\zeta}{3}\tau^3 - \frac{\zeta}{4}\tau^4$ *in the complex plane and the black bullets represent the end points of the contours.*

Using Cauchy's formula, we write

$$\begin{aligned}\mathscr{R}^n_r(j,j_0) = &\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_-\cup\Gamma_+} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau + \frac{1}{2\pi\mathbf{i}}\int_{\Gamma^\omega_>\cup\Gamma^\omega_<} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau,\\ &+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^v_-\cup\Gamma^v_+} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau + \frac{1}{2\pi\mathbf{i}}\int_{\Gamma^o_-\cup\Gamma^o_+} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau.\end{aligned}$$

As in the preceding case, we have that the uniform bound

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_-\cup\Gamma_+} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\,\mathrm{e}^{-c\,j_0-c\,|j|},$$

and remark that since $|\omega|\in[n^{-2/3}|\alpha_r|,\omega_*]$, an exponential bound in j_0 leads to an exponential bound in both j_0 and n.

Bounds along $\Gamma^\omega_\lessgtr$. For each $\tau = t-\mathbf{i}\,\varepsilon\in\Gamma^\omega_>$ with $t\in\left[0,\sqrt{\frac{|\omega|}{\zeta}}\right]$,

$$\begin{aligned}\mathrm{Re}\left(\Lambda_r\left(t-\mathbf{i}\,\varepsilon\right)\right) &\leq \frac{n}{|\alpha_r|}\left[|\omega|t+\frac{\zeta}{3}\left(t^3-3\varepsilon^2t\right)-\frac{\zeta}{4}\left(t^4+\varepsilon^4-6\varepsilon^2t^2\right)+|\alpha_r|C_0(t^5+\varepsilon^5)\right]\\ &\leq \frac{n}{|\alpha_r|}\left[t\left(\frac{4}{3}|\omega|-\zeta\varepsilon^2\left(1-\frac{3}{2}\sqrt{\frac{|\omega|}{\zeta}}\right)\right)-t^4\left(\frac{\zeta}{4}-|\alpha_r|C_0t\right)-\varepsilon^4\left(\frac{\zeta}{4}-|\alpha_r|C_0\varepsilon\right)\right]\\ &\leq \frac{n}{|\alpha_r|}\left[t\left(\frac{4}{3}|\omega|-\frac{\zeta}{2}\varepsilon^2\right)-\frac{\zeta}{8}\varepsilon^4\right] \leq -\frac{n}{|\alpha_r|}\left[t\frac{\zeta}{6}\varepsilon^2+\frac{\zeta}{8}\varepsilon^4\right].\end{aligned}$$

Thus, we obtain

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{\omega}_{>}} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\mathrm{e}^{-n\frac{\zeta\varepsilon^4}{8|\alpha_r|}-c|j|}\int_0^{\sqrt{\frac{|\omega|}{\zeta}}} e^{-n\frac{\zeta}{6|\alpha_r|}t}\mathrm{d}t \leq C\mathrm{e}^{-cn-c|j|}\,.$$

By symmetry, a similar estimate holds along $\Gamma^{\omega}_{<}$. As for the bounds along $\Gamma_{\pm}$, we remark that since $|\omega|\in[n^{-2/3}|\alpha_r|,\omega_*]$, an exponential bound in n leads to an exponential bound in both j_0 and n.

Bounds along $\Gamma^{v}_{\pm}$. For each $\theta\in\left[-\varepsilon,-\chi_\omega-\sqrt{\frac{|\omega|}{\zeta}}\right]$, we have that

$$\mathrm{Re}\left(\Lambda_r\left(\sqrt{\frac{|\omega|}{\zeta}}+\mathbf{i}\theta\right)\right) \leq \frac{n}{|\alpha_r|}\left(\frac{4}{3\sqrt{\zeta}}|\omega|^{3/2}-\sqrt{|\omega|}\left(\sqrt{\zeta}-\frac{3}{2}\sqrt{|\omega|}\right)\theta^2-\left(\frac{\zeta}{4}-|\alpha_r|C_0\right)\theta^4\right)$$
$$-\frac{n}{|\alpha_r|}\frac{|\omega|^2}{\zeta^2}\left(\frac{\zeta}{4}-|\alpha_r|C_0\sqrt{\frac{|\omega|}{\zeta}}\right).$$

Thanks to our careful choice for ε and ω_*, we get that

$$\mathrm{Re}\left(\Lambda_r\left(\sqrt{\frac{|\omega|}{\zeta}}+\mathbf{i}\theta\right)\right) \leq \frac{n}{|\alpha_r|}\left(\frac{4}{3\sqrt{\zeta}}|\omega|^{3/2}-\frac{\sqrt{\zeta}}{2}\sqrt{|\omega|}\theta^2\right)-\frac{n|\omega|^2}{8|\alpha_r|\zeta},\quad \theta\in\left[-\varepsilon,-\chi_\omega-\sqrt{\frac{|\omega|}{\zeta}}\right].$$

As a consequence, we obtain

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{v}_{-}} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\,\mathrm{e}^{\frac{4}{3\sqrt{\zeta}|\alpha_r|}n|\omega|^{3/2}-\frac{n|\omega|^2}{8|\alpha_r|\zeta}-c|j|}\int_{-\varepsilon}^{-\chi_\omega-\sqrt{\frac{|\omega|}{\zeta}}}(\sqrt{|\omega|}+|\theta|)\mathrm{e}^{-\frac{\sqrt{\zeta}}{2|\alpha_r|}\sqrt{|\omega|}n\theta^2}\mathrm{d}\theta\,.$$

The last integral on the right-hand side should be estimated carefully since there is an exponentially growing factor in front of the integral. We use Lemma A.2 in Appendix A and obtain the bound:

$$\int_{-\varepsilon}^{-\chi_\omega-\sqrt{\frac{|\omega|}{\zeta}}}\mathrm{e}^{-\frac{\sqrt{\zeta}}{2|\alpha_r|}\sqrt{|\omega|}n\theta^2}\,\mathrm{d}\theta \leq \int_{2\sqrt{\frac{|\omega|}{\zeta}}}^{+\infty}\mathrm{e}^{-\frac{\sqrt{\zeta}}{2|\alpha_r|}\sqrt{|\omega|}n\theta^2}\,\mathrm{d}\theta \leq \frac{C}{n\,|\omega|}\,\mathrm{e}^{-\frac{2}{\sqrt{\zeta}|\alpha_r|}n|\omega|^{3/2}}\,,$$

and the analogous bound:

$$\int_{-\varepsilon}^{-\chi_\omega-\sqrt{\frac{|\omega|}{\zeta}}}|\theta|\,\mathrm{e}^{-\frac{\sqrt{\zeta}}{2|\alpha_r|}\sqrt{|\omega|}n\theta^2}\,\mathrm{d}\theta \leq \int_{2\sqrt{\frac{|\omega|}{\zeta}}}^{+\infty}\theta\,\mathrm{e}^{-\frac{\sqrt{\zeta}}{2|\alpha_r|}\sqrt{|\omega|}n\theta^2}\,\mathrm{d}\theta \leq \frac{C}{n\sqrt{|\omega|}}\,\mathrm{e}^{-\frac{2}{\sqrt{\zeta}|\alpha_r|}n|\omega|^{3/2}}\,,$$

This leads to the final estimate:

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^{v}_{-}} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\,\frac{\mathrm{e}^{-\frac{2}{3\sqrt{\zeta}|\alpha_r|}n|\omega|^{3/2}-\frac{n|\omega|^2}{8|\alpha_r|\zeta}-c|j|}}{n\sqrt{|\omega|}}\,,$$

since we have $|\omega|\leq\omega_*$. A similar estimate holds along Γ^{v}_{+}.

Bounds along $\Gamma^o_\pm$. We finally handle the contributions along the oblique contours $\Gamma^o_\pm$. We focus first on Γ^o_-. For each $\tau \in \Gamma^o_-$, we have the parametrization

$$\tau(t) = -\frac{|\omega|}{2\zeta} - \mathbf{i}\sqrt{\frac{|\omega|}{\zeta}} + te^{3\mathbf{i}\pi/4}, \quad t \in \left[-\sqrt{2}\chi_\omega, \sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}\right],$$

and we compute that

$$|\omega|\tau(t) + \frac{\zeta}{3}\tau(t)^3 - \frac{\zeta}{4}\tau(t)^4 = \sum_{k=0}^{4} p_k(|\omega|)t^k,$$

where each p_k depends on $|\omega|$ only and is a complex valued function whose expansions as $|\omega| \to 0$ is given by

$$\begin{aligned}
\mathrm{Re}(p_0(|\omega|)) &= -\frac{1}{4\zeta}|\omega|^2 + \mathcal{O}(|\omega|^3), \quad \mathrm{Im}(p_0(|\omega|)) = -\frac{2}{3\sqrt{\zeta}}|\omega|^{3/2} + \mathcal{O}(|\omega|^{5/2}), \quad p_1(|\omega|) = \mathcal{O}(|\omega|^2),\\
\mathrm{Re}(p_2(|\omega|)) &= -\sqrt{\zeta|\omega|} + \mathcal{O}(|\omega|^{3/2}), \quad \mathrm{Im}(p_2(|\omega|)) = \mathcal{O}(|\omega|),\\
p_3(|\omega|) &= \frac{\zeta}{3}e^{\mathbf{i}\pi/4} + \mathcal{O}(|\omega|^{1/2}), \quad p_4(|\omega|) = \frac{\zeta}{4}.
\end{aligned}$$

As a consequence, we get that for each $t \in \left[-\sqrt{2}\chi_\omega, \sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}\right]$

$$\begin{aligned}
\sum_{k=1}^{4} \mathrm{Re}(p_k(|\omega|))t^k = -\sqrt{\zeta|\omega|}t^2 + \frac{\zeta}{3\sqrt{2}}t^3 + \underbrace{\mathrm{Re}(p_1(|\omega|))}_{\mathcal{O}(|\omega|^2)}t + \underbrace{\left(\mathrm{Re}(p_2(|\omega|)) + \sqrt{\zeta|\omega|}\right)}_{\mathcal{O}(|\omega|^{3/2})}t^2\\
+ \underbrace{\left(\mathrm{Re}(p_3(|\omega|)) - \frac{\zeta}{3\sqrt{2}}\right)}_{\mathcal{O}(|\omega|^{1/2})}t^3 + \frac{\zeta}{4}t^4,
\end{aligned}$$

together with

$$\mathrm{Re}(p_0(|\omega|)) + \frac{j_0}{n}|\alpha_r|\mathrm{Re}\left(\tau(t)^5\Psi_r(\tau(t))\right) \leq -\frac{1}{4\zeta}|\omega|^2 + \underbrace{\left(\mathrm{Re}(p_0(|\omega|)) + \frac{1}{4\zeta}|\omega|^2\right)}_{\mathcal{O}(|\omega|^3)} + |\alpha_r|C_0|\tau(t)|^5.$$

Upon decreasing further ω_* if necessary, we get the existence of a constant $c > 0$ independent of $|\omega|$ and n, such that

$$\sum_{k=1}^{4} \mathrm{Re}(p_k(|\omega|))t^k \leq -c|\alpha_r|\sqrt{|\omega|}t^2, \quad \mathrm{Re}(p_0(|\omega|)) + \frac{j_0}{n}|\alpha_r|\mathrm{Re}\left(\tau(t)^5\Psi_r(\tau(t))\right) \leq -c|\alpha_r||\omega|^2,$$

for all $t \in \left[-\sqrt{2}\chi_\omega, \sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}\right]$. As a consequence, we have

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma^o_-} \mathrm{e}^{\Lambda_r(\tau)}\tau\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\sqrt{|\omega|}\mathrm{e}^{-cn|\omega|^2 - c|j|}\int_{-\sqrt{2}\chi_\omega}^{\sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}} \mathrm{e}^{-cn\sqrt{|\omega|}t^2}\mathrm{d}t.$$

Next, we remark that

$$\int_{-\sqrt{2}\chi_\omega}^{\sqrt{2}\sqrt{\frac{|\omega|}{\zeta}}} \mathrm{e}^{-cn\sqrt{|\omega|}t^2}\mathrm{d}t \leq \frac{C}{n^{1/2}|\omega|^{1/4}}, \quad n^{-2/3}|\alpha_r| \leq |\omega| \leq \omega_*,$$

such that

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_-^o} \mathrm{e}^{\Lambda_r(\tau)}\tau^5\Phi_{r,j}(\tau)\mathrm{d}\tau\right| \leq C\frac{|\omega|^{1/4}}{n^{1/2}}\mathrm{e}^{-cn|\omega|^2-c|j|}, \quad n^{-2/3}|\alpha_r| \leq |\omega| \leq \omega_*.$$

And a similar estimate holds along Γ_+^o.

Conclusion. In summary, combining all the bounds for the eight segments and retaining only the "worst" contributions, we have obtained the estimate

$$\left|\mathscr{R}_{1,r}^n(j,j_0)\right| \leq C\left(\frac{1}{n\sqrt{|\omega|}}+\frac{|\omega|^{1/4}}{n^{1/2}}\right)\mathrm{e}^{-cn|\omega|^2-c|j|}, \quad n^{-2/3}|\alpha_r| \leq |\omega| \leq \omega_*.$$

We may rewrite the above estimate as

$$\left|\mathscr{R}_{1,r}^n(j,j_0)\right| \leq C\frac{\mathrm{e}^{-c|j|}}{n^{5/8}}\left(\frac{1}{n^{1/8}}\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^{-1/2}+\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^{1/4}\right)e^{-c\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^2},$$

valid in the range $n^{-2/3}|\alpha_r| \leq |\omega| \leq \omega_*$. □

We should now deal with the final regime $\omega \leq -\omega_*$, with $\omega_* > 0$ being given by Lemma 4.6.

Lemma 4.7. *Let $\omega_* > 0$ be given by Lemma 4.6. Then there exist constants $C, c > 0$ such that for each $1 \leq j_0 \leq n$ and $|j-j_0| \leq n$, there holds:*

$$\left|\mathscr{R}_{1,r}^n(j,j_0)\right| \leq C\mathrm{e}^{-cn-c\,j_0-c\,|j|},$$

as long as $-|\alpha_r| < \omega \leq -\omega_*$.

Proof. We simply take the vertical contour $\Gamma_{\text{in}} = \Gamma_{-\eta}$, and recall that, thanks to conditions (4.11) and (4.12) on ε, we have

$$\mathrm{Re}\left(\Lambda_r(-\eta+\mathbf{i}\,\theta)\right) \leq -\frac{|\alpha_r|}{4}\eta, \quad \theta \in [-\varepsilon,\varepsilon].$$

We readily obtain the desired bound since $-|\alpha_r| < \omega \leq -\omega_*$. □

Combining Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.7, we have proved Proposition 4.1 for $\mathscr{R}_{1,r}^n(j,j_0)$. Indeed, in the range $n^{-2/3}|\alpha_r| \leq |\omega| \leq \omega_*$, we can further bound

$$\frac{1}{n^{1/8}}\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^{-1/2} \leq n^{1/12-1/8} \leq 1,$$

and

$$\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^{1/4}\mathrm{e}^{-c\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^2} \leq \tilde{C}\,e^{-\tilde{c}\left(\frac{|j_0-n\,|\alpha_r||}{n^{1/2}}\right)^2},$$

with a smaller constant $\tilde{c} > 0$ and a suitable constant $\tilde{C}$.

Regarding the second remainder term $\mathscr{R}^n_{2,r}(j,j_0)$, we first observe that for each $\tau \in \overline{\mathbf{B}_\varepsilon(0)}$ one has

$$\frac{\exp\left(j_0\,\tau^5\,\Psi_r(\tau)\right) - 1 - j_0\tau^5\Psi_r(0)}{\tau} = \frac{\exp\left(j_0\,\tau^5\,\Psi_r(\tau)\right) - 1 - j_0\tau^5\Psi_r(\tau)}{\tau} + j_0\tau^4\left(\Psi_r(\tau) - \Psi_r(0)\right),$$

such that

$$\left|\frac{\exp\left(j_0\,\tau^5\,\Psi_r(\tau)\right) - 1 - j_0\tau^5\Psi_r(0)}{\tau}\right| \leq C\left[\left(j_0|\tau|^4\right) + j_0|\tau|^4\right]|\tau|\,.$$

All the previous lemmas can be easily adapted to obtain similar estimates for $\mathscr{R}^n_{2,r}(j,j_0)$ as the ones we derived for $\mathscr{R}^n_{1,r}(j,j_0)$. For example, regarding the uniform bound in Lemma 4.4, the map $g(\theta)$ now reads

$$g(\theta) := \mathrm{e}^{-n\frac{\zeta}{4}\theta^4}\left(\frac{\exp\left(j_0\mathbf{i}\theta^5\Psi_r(\mathbf{i}\theta)\right) - 1 - j_0\mathbf{i}\theta^5\Psi_r(0)}{\theta}\right), \quad \theta \in [-\varepsilon,\varepsilon],$$

and we note that

$$\left|g'(\theta)\right| \leq C\left(1 + n\,\theta^4 + (n\,\theta^4)^2\right)\mathrm{e}^{-cn\theta^4},$$

since $1 \leq j_0 \leq n$, and we observe that $\|g'\|_{L^1([-\varepsilon,\varepsilon])} \leq Cn^{-1/4}$. As a consequence, we naturally retrieve the estimate of Lemma 4.4 for $\mathscr{R}^n_{2,r}(j,j_0)$. We let the other cases to the interested reader. □

4.3.4 Final decomposition of the temporal Green's function. Proof of Theorem 4.1

Let us recall that, for $j_0 \geq 1$, we have first decomposed (see (4.9)):

$$\mathscr{G}^n(j,j_0) = \overline{\mathscr{G}}^n_r(j-j_0)\,\mathbb{1}_{j\geq 1} + \widetilde{\mathscr{G}}^n(j,j_0)\,,$$

and the reduced Green's function $\widetilde{\mathscr{G}}^n(j,j_0)$ has been decomposed into:

$$\widetilde{\mathscr{G}}^n(j,j_0) = \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\mathrm{out}}} \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}^{j_0}_j(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau + \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\mathrm{in}}} \mathrm{e}^{n\,\tau}\,\widetilde{\mathcal{G}}^{j_0}_j(\mathrm{e}^\tau)\,\mathrm{e}^\tau\,\mathrm{d}\tau\,,$$

with suitable contours Γ_{out} and Γ_{in}. The part on Γ_{out} satisfies the exponential bound (4.14) and the part on Γ_{in} has been further decomposed in (4.20).

Coming back to our decomposition (4.20) of part of the (reduced) temporal Green's function $\widetilde{\mathscr{G}}^n(j,j_0)$, and recalling the initial decomposition (4.9), we can gather Lemma 4.2 (together with the exponential decay of $\mathcal{H}_j$), Lemma 4.3 and Proposition 4.1 to obtain for $j \in \mathbb{Z}$ and $1 \leq j_0 \leq n$:

$$\begin{aligned}|\mathscr{G}^n(j,j_0) - \mathcal{H}_j\,\mathbf{A}_r(-j_0 + n\,|\alpha_r|,n)| &\leq |\overline{\mathscr{G}}^n_r(j-j_0)|\,\mathbb{1}_{j\geq 1} + |\widetilde{\mathscr{G}}^n(j,j_0) - \mathcal{H}_j\,\mathbf{A}_r(-j_0 + n\,|\alpha_r|,n)| \\ &\leq C\,\mathbf{M}_r\left(c, j - j_0 + n\,|\alpha_r|, n\right)\,\mathbb{1}_{j\geq 1} \\ &\quad + C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, n\right) + C\,\mathrm{e}^{-c\,n}\,\mathrm{e}^{-c\,|j|}\,\mathrm{e}^{-c\,|j_0|}\,,\end{aligned}$$

where we have used Corollary A.1 in Appendix A to derive the bound of the free Green's function $\overline{\mathscr{G}}_r$ (and the definition (4.6b) of the function $\mathbf{M}_r$). This shows the validity of the bound (4.8) in Theorem 4.1 for the case $1 \leq j_0 \leq n$.

Let us finally consider the regime $1 \leq n \leq j_0 - 1$ and show that the estimate (4.8) in Theorem 4.1 is still valid. We apply Lemma 4.1 and obtain:

$$\begin{aligned} |\mathscr{G}^n(j,j_0) - \mathcal{H}_j \, \mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)| &= \left|\overline{\mathscr{G}}_r^n(j-j_0) - \mathcal{H}_j \, \mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)\right| \\ &\leq \left|\overline{\mathscr{G}}_r^n(j-j_0)\right| + |\mathcal{H}_j| \, |\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)| \\ &\leq C\,\mathbf{M}_r(c, j - j_0 + n\,|\alpha_r|, n) \, + \, C\,\mathrm{e}^{-c\,|j|} \, |\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)| \, , \end{aligned}$$

where we have again used Corollary A.1 in Appendix A to derive the bound of the free Green's function $\overline{\mathscr{G}}_r$. It only remains to show that the final term on the right-hand side is exponentially small with respect to both j_0 and n and the proof will be complete. We apply Corollary A.6 in Appendix A to estimate the term with the function $\mathbf{A}_r$ on the right-hand side in the considered regime $1 \leq n \leq j_0 - 1$. We get:

$$|\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)| \, \leq \, C \, \exp\left(-c\,\frac{|j_0 - n\,|\alpha_r||^{4/3}}{n^{1/3}}\right) \, \leq \, C\,\mathrm{e}^{-c\,j_0} \, ,$$

where the final estimate comes from the fact that we now consider the regime $1 \leq n \leq j_0 - 1$. Since now j_0 dominates n, we get the exponential estimate:

$$|\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)| \, \leq \, C\,\mathrm{e}^{-c\,n}\,\mathrm{e}^{-c\,j_0} \, ,$$

and we have thus proved the validity of the bound (4.8) in Theorem 4.1 for the case $1 \leq n \leq j_0 - 1$.

The proof of Theorem 4.1 is entirely similar in the case $j_0 \leq 0$ except that all involved functions are now associated with the left state u_ℓ of the shock rather than with u_r. We leave the details to the interested reader.

4.4 Derivative of the temporal Green's functions

As it will be made clear in the last chapter of this article, it will also be necessary to obtain large time decaying bounds for the family of operators $(\mathscr{L}^n(\mathrm{Id} - \mathbf{S}))_{n \in \mathbb{N}}$ where $\mathbf{S} : \ell^q(\mathbb{Z};\mathbb{R}) \to \ell^q(\mathbb{Z};\mathbb{R})$ is the shift operator defined as $(\mathbf{S}\mathbf{h})_j := h_{j+1}$ for all $j \in \mathbb{Z}$ for any sequence $\mathbf{h} = (h_j)_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z};\mathbb{R})$. By definition of the operator $\mathscr{L}^n$, for any $\mathbf{h} \in \ell^q(\mathbb{Z};\mathbb{R})$, we have the decomposition

$$\forall\,(n,j) \in \mathbb{N} \times \mathbb{Z} \, , \quad (\mathscr{L}^n(\mathrm{Id} - \mathbf{S})\mathbf{h})_j = \sum_{j_0 \in \mathbb{Z}} (\mathscr{G}^n(j,j_0) - \mathscr{G}^n(j,j_0-1))\,h_{j_0} \, .$$

This motivates the definition of the following quantity

$$\forall\,(n,j,j_0) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \, , \quad \mathscr{D}^n(j,j_0) := \mathscr{G}^n(j,j_0) - \mathscr{G}^n(j,j_0-1). \tag{4.31}$$

The above quantity is thus a discrete spatial derivative of $\mathscr{G}^n(j,j_0)$ with respect to its second argument, and we shall refer to it simply as the derivative of the temporal Green's function. We will now follow the same strategy as presented in the previous sections of this chapter. That is, we shall decompose the derivative of the temporal Green's function into several contributions and derive bounds for each such contributions which are meant to be sufficiently sharp in order to obtain large time decaying bounds for the family of operators $(\mathscr{L}^n(\mathrm{Id} - \mathbf{S}))_{n \in \mathbb{N}}$.

Before stating our main result, we recall that the temporal Green's function satisfies $\mathscr{G}^n(j,j_0)=0$ whenever $|j-j_0|>n$, since the Lax-Wendroff scheme has a finite stencil. As an elementary consequence, we have that for all $n\in\mathbb{N}$ and $(j,j_0)\in\mathbb{Z}^2$:

$$j-j_0>n \;\text{ or }\; j-j_0<-n-1 \;\Rightarrow\; \mathscr{D}^n(j,j_0)=0.$$

As a consequence, throughout this section, we shall only consider $(n,j,j_0)\in\mathbb{N}\times\mathbb{Z}\times\mathbb{Z}$ that satisfies $-n-1\le j-j_0\le n$. Furthermore, a direct application of Lemma 4.1 gives the following result.

Lemma 4.8. *Let $j_0\in\mathbb{Z}$ and $n\in\mathbb{N}$. If $j_0\ge 2$ and $n\le j_0-2$, then there holds:*

$$\forall\, j\in\mathbb{Z}\,,\quad \mathscr{D}^n(j,j_0)\;=\;\overline{\mathscr{G}}_r^n(j-j_0)\;-\;\overline{\mathscr{G}}_r^n(j-j_0+1)\,.$$

If $j_0\le 0$ and $n\le|j_0|$, then there holds:

$$\forall\, j\in\mathbb{Z}\,,\quad \mathscr{D}^n(j,j_0)\;=\;\overline{\mathscr{G}}_\ell^n(j-j_0)-\overline{\mathscr{G}}_\ell^n(j-j_0+1)\,.$$

Finally, we introduce two functions $\mathbf{K}_\ell$ and $\mathbf{K}_r$ on $\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}^{+*}$ as follows:

$$\mathbf{K}_\ell(c,x,y)\;:=\;\begin{cases}\dfrac{1}{y^{7/12}}\,\exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } x\ge 0,\\[2ex] \dfrac{1}{y^{7/12}}\,, & \text{if } -y^{1/3}\le x\le 0,\\[2ex] \dfrac{1}{y^{5/8}}\,\exp\big(-c\,x^2/y\big)\,, & \text{if } x\le -y^{1/3},\end{cases}\tag{4.32a}$$

$$\mathbf{K}_r(c,x,y)\;:=\;\begin{cases}\dfrac{1}{y^{7/12}}\,\exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } x\le 0,\\[2ex] \dfrac{1}{y^{7/12}}\,, & \text{if } 0\le x\le y^{1/3},\\[2ex] \dfrac{1}{y^{5/8}}\,\exp\big(-c\,x^2/y\big)\,, & \text{if } y^{1/3}\le x,\end{cases}\tag{4.32b}$$

for all $(c,x,y)\in\mathbb{R}^{+*}\times\mathbb{R}\times\mathbb{R}^{+*}$. Once again, we crucially note that both $\mathbf{K}_\ell$ and $\mathbf{K}_r$ are non-increasing with respect to their first argument.

We can now state our main result regarding the derivative of the Green's function.

Theorem 4.2 (Pointwise bounds on the derivative of the Green's function)**.** *Let the weak solution (2.2) satisfy the Rankine-Hugoniot condition (2.3) and the entropy inequalities (2.4). Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. Then there exist some positive constants C and c such that for each $n\ge 1$ and $(j,j_0)\in\mathbb{Z}^2$ with $-n-1\le j-j_0\le n$, the derivative $\mathscr{D}^n(j,j_0)$ of the Green's function enjoys the following pointwise bounds:*

- *for any $j_0\le 0$:*

$$|\mathscr{D}^n(j,j_0)|\le C\,\mathbf{K}_\ell(c,j_0-j+n\,\alpha_\ell,n)\mathbb{1}_{j\le 0}+C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_\ell\,(c,j_0+\alpha_\ell n,n)+C\,\mathrm{e}^{-cn-c|j-j_0|}\,;\tag{4.33}$$

- *and for any $j_0 \geq 1$:*

$$|\mathscr{D}^n(j,j_0)| \leq C\,\mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)\mathbb{1}_{j\geq 1} + C\,\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r\left(c, -j_0+|\alpha_r|n, n\right) + C\,\mathrm{e}^{-cn-c|j-j_0|}. \quad (4.34)$$

Proof. Throughout the proof we assume that $n \geq 1$ and $(j,j_0) \in \mathbb{Z}^2$ with $-n-1 \leq j-j_0 \leq n$ and we only consider $j_0 \geq 1$ since the analysis for $j_0 \leq 0$ follows similar lines.

We first focus on the regime $2 \leq j_0 \leq n+1$. Using the expression for $\mathscr{G}^n(j,j_0)$, we may instead write

$$\mathscr{D}^n(j,j_0) = \mathbb{1}_{j\geq 1}\left[\underbrace{\overline{\mathscr{G}}_r^n(j-j_0) - \overline{\mathscr{G}}_r^n(j-j_0+1)}_{:=\mathscr{K}_r^n(j-j_0)}\right] + \underbrace{\frac{1}{2\pi\mathbf{i}}\int_\Gamma \mathrm{e}^{n\,\tau}\left[\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\mathrm{d}\tau}_{:=\widetilde{\mathscr{D}}^n(j,j_0)}. \quad (4.35)$$

We first handle the second term $\widetilde{\mathscr{D}}^n(j,j_0)$. We let $\varepsilon \in (0,\varepsilon_*)$ be fixed as in previous section, that is satisfying conditions (4.11)-(4.12)-(4.13), and $0 < \eta < \min(\eta_\varepsilon, \varepsilon^5)$. From Proposition 3.3, since the map $\tau \mapsto \widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)$ has a holomorphic extension to $\mathbf{B}_{\varepsilon_0}(0)$, we can decompose the contour Γ into $\Gamma_{\text{out}} = \{-\eta + \mathbf{i}\theta \mid \varepsilon \leq \theta \leq \pi\}$ and Γ_{in}, where Γ_{in} can be any path joining $-\eta - \mathbf{i}\,\varepsilon$ to $-\eta + \mathbf{i}\,\varepsilon$ which remains in $\mathbf{B}_\varepsilon(0)$ (see Figure 4.1). As a consequence, we have

$$\left|\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{out}}} \mathrm{e}^{n\,\tau}\left[\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\mathrm{d}\tau\right| \leq C\mathrm{e}^{-\eta n}e^{-c\,|j|-c|j_0|},$$

together with

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau}\left[\widetilde{\mathcal{G}}_j^{j_0}(\mathrm{e}^\tau) - \widetilde{\mathcal{G}}_j^{j_0-1}(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\mathrm{d}\tau = \frac{\mathcal{H}_j}{\alpha_r} \times \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)}\mathrm{d}\tau$$
$$+ \frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau - j_0\,\varphi_r(\tau) + j_0\,\tau^5\,\Psi_r(\tau)}\tau\,\Theta_{r,j}(\tau)\mathrm{d}\tau,$$

where the sequence $(\mathcal{H}_j)_{j\in\mathbb{Z}}$ is defined in (3.22) and the sequence $(\Theta_{r,j})_{j\in\mathbb{Z}}$ of bounded of holomorphic functions is given by Proposition 3.3. The first integral is similar to $\widetilde{\mathscr{G}}_{2,r}^n(j,j_0)$ (with definition (4.15)) and enjoys a similar bound, while the second integral is similar to $\widetilde{\mathscr{R}}_{1,r}^n(j,j_0)$ (with definition (4.16)) and enjoys a similar bound. As a consequence, for $2 \leq j_0 \leq n+1$, we have the estimate

$$\left|\widetilde{\mathscr{D}}^n(j,j_0)\right| \leq C\,\mathrm{e}^{-c\,|j|}\mathbf{M}_r\left(c, -j_0+|\alpha_r|n, n\right) + C\,\mathrm{e}^{-cn-c\,j_0-c\,|j|}, \quad (4.36)$$

for some $C > 0$ and $c > 0$ which are independent of n, j and j_0.

Let us focus now on the first term $\mathscr{K}_r^n(j-j_0)$ of (4.35) which can also be written as

$$\mathscr{K}_r^n(j-j_0) = \frac{1}{2\pi\mathbf{i}}\int_\Gamma \mathrm{e}^{n\,\tau}\left[\overline{\mathcal{G}}_{r,j-j_0}(\mathrm{e}^\tau) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\mathrm{d}\tau.$$

From the expression (3.27) of Proposition 3.2 of the free spatial Green's function there exists $\kappa_{\mathscr{O}} > 0$ depending on the set $\mathscr{O}$ defined in Chapter 3, such that for $j_0 \leq j$ one has

$$\left|\left[\overline{\mathcal{G}}_{r,j-j_0}(\mathrm{e}^\tau) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\right| \leq C\,\mathrm{e}^{-c|j-j_0|},$$

for all $\tau \in \mathbb{C}$ with $\mathrm{Re}(\tau) \geq -\kappa_{\mathscr{O}}$. As a consequence, upon taking $\Gamma = \{-\eta + \mathbf{i}\theta \mid -\pi \leq \theta \leq \pi\}$ with $0 < \eta < \kappa_{\mathscr{O}}$, we readily get in that case that

$$|\mathscr{K}_r^n(j-j_0)| \leq C\,\mathrm{e}^{-\eta n - c|j-j_0|}.$$

On the other hand, for $1 \leq j \leq j_0 \leq n+1$, one has from the expression (3.27) that for all $\tau \in \overline{\mathbf{B}_{\varepsilon_0}(0)}$

$$\left[\overline{\mathcal{G}}_{r,j-j_0}(e^{\tau}) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau} = \tau \Xi_r(\tau) \exp\left((j-j_0)\varphi_r(\tau) - (j-j_0)\tau^5 \Psi_r(\tau)\right),$$

for some bounded holomorphic function Ξ_r, with the same ε_0 as the one given by Proposition 3.3. We can then once again let $\varepsilon \in (0, \varepsilon_*)$ be fixed as in the previous section and set $0 < \eta < \min(\eta_\varepsilon, \varepsilon^5)$. Using Corollary 3.3, we also have

$$\left|\left[\overline{\mathcal{G}}_{r,j-j_0}(e^{\tau}) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau}\right| \leq C\,\mathrm{e}^{-c|j-j_0|},$$

for all $\tau = -\eta + \mathbf{i}\theta$ with $\varepsilon \leq \theta \leq \pi$. As a consequence, with our now usual notation:

$$\Gamma_{\text{out}} = \{-\eta + \mathbf{i}\theta \mid \varepsilon \leq \theta \leq \pi\},$$

we get

$$\left|\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\text{out}}} \mathrm{e}^{n\,\tau} \left[\overline{\mathcal{G}}_{r,j-j_0}(e^{\tau}) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau}\right| \mathrm{d}\tau \leq C\,\mathrm{e}^{-\eta n - c|j-j_0|}.$$

On the other hand, if Γ_{in} denotes any path joining $-\eta - \mathbf{i}\,\varepsilon$ to $-\eta + \mathbf{i}\,\varepsilon$ which remains in $\overline{\mathbf{B}_{\varepsilon}(0)}$, one has

$$\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau} \left[\overline{\mathcal{G}}_{r,j-j_0}(e^{\tau}) - \overline{\mathcal{G}}_{r,j-j_0-1}(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau} \mathrm{d}\tau = \frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau + (j-j_0)\varphi_r(\tau) - (j-j_0)\tau^5 \Psi_r(\tau)} \tau \Xi_r(\tau) \mathrm{d}\tau.$$

Then applying Lemma 4.4, Lemma 4.5, Lemma 4.6 and Lemma 4.7 with $\Phi_{r,j}$ replaced by $\Xi_r(\tau)$ and $-j_0$ by $j - j_0$, we readily obtain that

$$\left|\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\text{in}}} \mathrm{e}^{n\,\tau} \left[\overline{\mathcal{G}}_{r,j-j_0}(e^{\tau}) - \overline{\mathcal{G}}_{r,j-j_0+1}(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau} \mathrm{d}\tau\right| \leq C\,\mathbf{K}_r(c, j - j_0 + n\,|\alpha_r|, n),$$

where the definition of $\mathbf{K}_r$ is given in (4.32b). Combining the above bound with the estimate (4.36) proves the bound (4.34) of $\mathscr{D}^n(j, j_0)$ in the range $2 \leq j_0 \leq n+1$.

For $j_0 \geq 2$ and $n \leq j_0 - 2$, we use Lemma 4.8 which gives that

$$\forall\, j \in \mathbb{Z}, \quad \mathscr{D}^n(j, j_0) \;=\; \overline{\mathscr{G}}^n_r(j - j_0) \;-\; \overline{\mathscr{G}}^n_r(j - j_0 + 1).$$

We can then proceed along similar lines as above, and get that

$$|\mathscr{D}^n(j, j_0)| \leq C\,\mathbf{K}_r(c, j - j_0 + n\,|\alpha_r|, n) + C\,\mathrm{e}^{-\eta n - c|j-j_0|}.$$

Let us now turn to the case $j_0 = 1$. Using the expressions of $\mathscr{G}^n(j, 1)$ and $\mathscr{G}^n(j, 0)$, we have

$$\mathscr{D}^n(j, 1) = \mathbb{1}_{j\geq 1} \overline{\mathscr{G}}^n_r(j-1) - \mathbb{1}_{j\leq 0} \overline{\mathscr{G}}^n_\ell(j) + \underbrace{\frac{1}{2\pi\mathbf{i}} \int_{\Gamma} \mathrm{e}^{n\,\tau} \left[\widetilde{\mathcal{G}}^1_j(e^{\tau}) - \widetilde{\mathcal{G}}^0_j(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau} \mathrm{d}\tau}_{:=\widetilde{\mathscr{D}}^n(j,1)}.$$

First, inspecting the contribution stemming from $\widetilde{\mathscr{D}}^n(j, 1)$ and decomposing Γ with Γ_{out} and Γ_{in}, we observe that from (3.32) of Proposition 3.3, we have:

$$\left|\frac{1}{2\pi\mathbf{i}} \int_{\Gamma_{\text{out}}} \mathrm{e}^{n\,\tau} \left[\widetilde{\mathcal{G}}^1_j(e^{\tau}) - \widetilde{\mathcal{G}}^0_j(\mathrm{e}^{\tau})\right] \mathrm{e}^{\tau} \mathrm{d}\tau\right| \leq C\mathrm{e}^{-\eta n} e^{-c\,|j|},$$

together with

$$\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\rm in}} \mathrm{e}^{n\,\tau}\left[\widetilde{\mathcal{G}}^1_j(e^\tau)-\widetilde{\mathcal{G}}^0_j(\mathrm{e}^\tau)\right]\mathrm{e}^\tau\mathrm{d}\tau = \left(\frac{\mathcal{H}_j}{\alpha_r}+\gamma_j^r-\gamma_j^\ell\right)\times\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\rm in}}\mathrm{e}^{n\,\tau}\mathrm{d}\tau+\frac{1}{2\pi\mathbf{i}}\int_{\Gamma_{\rm in}}\mathrm{e}^{n\,\tau}\tau\,\Theta_{1,j}(\tau)\mathrm{d}\tau\,.$$

As a consequence, upon taking $\Gamma_{\rm in}=\{-\eta+\mathbf{i}\theta\mid|\theta|\le\varepsilon\}$, we get

$$\left|\widetilde{\mathscr{D}}^n(j,1)\right|\le C\mathrm{e}^{-cn-c\,|j|}.$$

Upon noticing that $\mathbf{M}_r\,(c,-1+|\alpha_r|n,n)=\mathcal{O}\left(\mathrm{e}^{-c\,n}\right)$, estimate (4.36) also holds true for $j_0=1$. Coming back to the first two terms in the expression of $\mathscr{D}^n(j,1)$, we simply note, using Corollary A.1, that

$$\forall j\ge1\,,\quad \left|\overline{\mathscr{G}}^n_r(j-1)\right|\le C\mathrm{e}^{-c\,n},\ \text{and}\quad \forall j\le0\,,\quad\left|\overline{\mathscr{G}}^n_\ell(j)\right|\le C\mathrm{e}^{-c\,n}.$$

But since one has $-n\le j\le n+1$, the above exponential bound in n leads to an exponential bound in both j and n:

$$\left|\mathbb{1}_{j\ge1}\overline{\mathscr{G}}^n_r(j-1)-\mathbb{1}_{j\le0}\,\overline{\mathscr{G}}^n_\ell(j)\right|\le C\,\mathrm{e}^{-cn-c\,|j|}.$$

To conclude, we note that $\mathbf{K}_r(c,j-1+n\,|\alpha_r|,n)=\mathcal{O}\left(\mathrm{e}^{-cn-c\,|j|}\right)$ which implies that estimate (4.34) also holds true for $j_0=1$. ☐

4.5 Linear estimates

In this final section, we derive large time decaying bounds for the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$ and the family of operators $(\mathscr{L}^n(\mathrm{Id}-\mathbf{S}))_{n\in\mathbb{N}}$ acting on algebraically weighted spaces which are crucial for our forthcoming nonlinear stability analysis.

We first recall some notation for the polynomially weighted spaces of sequences. Given a real number $\gamma\ge0$ and $q\in[1,+\infty]$, if we define the weight sequence $\boldsymbol{\omega}_\gamma:=(1+|j|^\gamma)_{j\in\mathbb{Z}}$, we recall our definition of algebraically weighted ℓ^q spaces:

$$\ell^q_\gamma(\mathbb{Z};\mathbb{R})=\left\{\mathbf{h}\in\ell^q(\mathbb{Z};\mathbb{R})\mid\boldsymbol{\omega}_\gamma\mathbf{h}\in\ell^q(\mathbb{Z};\mathbb{R})\right\},$$

where $\boldsymbol{\omega}_\gamma\mathbf{h}$ stands for the sequence $((1+|j|^\gamma)\,h_j)_{j\in\mathbb{Z}}$. For any sequence $\mathbf{h}\in\ell^q_\gamma(\mathbb{Z};\mathbb{R})$, the norm of $\mathbf{h}$ is defined as $\|\mathbf{h}\|_{\ell^q_\gamma}:=\|\boldsymbol{\omega}_\gamma\mathbf{h}\|_{\ell^q}$.

The main result of this Chapter reads as follows (the reader will observe that this statement is a refinement of Theorem 2.4 that was made more simple for the reader's convenience).

Theorem 4.3. *Let the weak solution (2.2) satisfy the Rankine-Hugoniot relation (2.3) and the entropy inequalities (2.4). Let the parameter λ satisfy the CFL condition (2.12) and let Assumption 1 be satisfied. For any $\gamma_2\ge\gamma_1\ge0$, there exists $C_{\mathscr{L}}(\gamma_1,\gamma_2)>0$ such that we have the following estimates on the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$:*

$$\forall n\in\mathbb{N}\,,\quad\|\mathscr{L}^n\,\mathbf{h}\|_{\ell^1_{\gamma_1}}\le\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(1+n)^{\gamma_2-\gamma_1-1/8}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,\qquad \textit{for } \mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})\ \textit{with}\ \sum_{j\in\mathbb{Z}}h_j=0\,,\tag{4.37a}$$

$$\forall n\in\mathbb{N}\,,\quad\|\mathscr{L}^n\,\mathbf{h}\|_{\ell^\infty_{\gamma_1}}\le\frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(1+n)^{\gamma_2-\gamma_1+\min(1/3,\gamma_1)}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,\qquad \textit{for } \mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})\ \textit{with}\ \sum_{j\in\mathbb{Z}}h_j=0\,,\tag{4.37b}$$

and the following estimates on the family of operators $(\mathscr{L}^n(\mathrm{Id}-\mathbf{S}))_{n\in\mathbb{N}}$:

$$\forall n \in \mathbb{N}\,, \quad \|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^1_{\gamma_1}} \le \frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(1+n)^{\gamma_2-\gamma_1+1/8}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,, \qquad \textit{for } \mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})\,, \tag{4.38a}$$

$$\forall n \in \mathbb{N}\,, \quad \|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^\infty_{\gamma_1}} \le \frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(1+n)^{\gamma_2-\gamma_1+1/3+\min(1/4,\gamma_1)}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,, \qquad \textit{for } \mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})\,, \tag{4.38b}$$

$$\forall n \in \mathbb{N}\,, \quad \|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^\infty_{\gamma_1}} \le \frac{C_{\mathscr{L}}(\gamma_1,\gamma_2)}{(1+n)^{\gamma_2-\gamma_1+\min(1/8,\gamma_1-1/8)}}\,\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}}\,, \qquad \textit{for } \mathbf{h}\in\ell^\infty_{\gamma_2}(\mathbb{Z};\mathbb{R})\,. \tag{4.38c}$$

4.5.1 Proof of the estimates (4.37a) and (4.37b) on the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$

Before proceeding with the proof of the estimates (4.37a) and (4.37b) on the semigroup $(\mathscr{L}^n)_{n\in\mathbb{N}}$ of Theorem 4.3, let us comment on the strategy that we shall follow. For $\gamma\ge 0$ and $\mathbf{h}\in\ell^q_\gamma(\mathbb{Z};\mathbb{R})$ we recall that the action of the semigroup $\mathscr{L}^n$ on $\mathbf{h}$ is given for each $n\in\mathbb{N}$ by

$$\forall j\in\mathbb{Z}\,,\quad (\mathscr{L}^n\mathbf{h})_j = \sum_{j_0\in\mathbb{Z}} \mathscr{G}^n(j,j_0)h_{j_0}\,,$$

where $\mathscr{G}^n(\cdot,j_0)$ is the temporal Green's function solution of (4.1). The bounds (4.37a)-(4.37b) are trivial for $n=0$ since we have $\gamma_1\le\gamma_2$ and therefore $\|\mathbf{h}\|_{\ell^1_{\gamma_1}}\le\|\mathbf{h}\|_{\ell^1_{\gamma_2}}$ so we assume from now on $n\in\mathbb{N}^*$. Motivated by the estimates obtained on the temporal Green's function in Theorem 4.1 (for $n\in\mathbb{N}^*$), we introduce a family of operators $(\mathscr{L}^n_{\mathrm{act}})_{n\in\mathbb{N}^*}$ acting on a given sequence $\mathbf{h}\in\ell^q_\gamma(\mathbb{Z};\mathbb{R})$ as follows:

$$\begin{aligned}\forall (n,j)\in\mathbb{N}^*\times\mathbb{Z}\,,\quad (\mathscr{L}^n_{\mathrm{act}}\,\mathbf{h})_j := \mathcal{H}_j \sum_{j_0\ge 1} \mathbb{1}_{|j-j_0|\le n}\,\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0}\\ +\mathcal{H}_j\sum_{j_0\le 0}\mathbb{1}_{|j-j_0|\le n}\,\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0}\,.\end{aligned}$$

It is important to observe that $\mathscr{L}^n_{\mathrm{act}}$ does not stand for the n-th power of $\mathscr{L}^1_{\mathrm{act}}$, while $\mathscr{L}^n$ is the n-th power of $\mathscr{L}$. However, we hope that this will not create any confusion for the reader and we keep the notation as such to bear in mind that all operators are considered at the discrete time n.

We recall that the Green's function satisfies $\mathscr{G}^n(j,j_0)=0$ for $|j-j_0|>n$. With the previous definition for $\mathscr{L}^n_{\mathrm{act}}$, for all $(n,j)\in\mathbb{N}^*\times\mathbb{Z}$, one can therefore decompose

$$\begin{aligned}(\mathscr{L}^n\mathbf{h})_j = (\mathscr{L}^n_{\mathrm{act}}\mathbf{h})_j &+ \sum_{j_0\ge1}\left(\mathscr{G}^n(j,j_0)-\mathbb{1}_{|j-j_0|\le n}\,\mathcal{H}_j\,\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\right)\,h_{j_0}\\ &+\sum_{j_0\le 0}\left(\mathscr{G}^n(j,j_0)-\mathbb{1}_{|j-j_0|\le n}\,\mathcal{H}_j\,\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\right)\,h_{j_0}\,.\end{aligned}\tag{4.39}$$

We now introduce some notation. For $(n,j)\in\mathbb{N}^*\times\mathbb{Z}$ and $\mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, we define:

$$\begin{aligned}\mathfrak{S}_{r,1}(n,j,\mathbf{h}) &:= \mathbb{1}_{j\ge1}\sum_{j_0\ge1}\mathbb{1}_{|j-j_0|\le n}\,\mathbf{M}_r\left(c,j-j_0+n\,|\alpha_r|,n\right)\,|h_{j_0}|\,,\\ \mathfrak{S}_{\ell,1}(n,j,\mathbf{h}) &:= \mathbb{1}_{j\le0}\sum_{j_0\le0}\mathbb{1}_{|j-j_0|\le n}\,\mathbf{M}_\ell\left(c,j_0-j+n\,\alpha_\ell,n\right)\,|h_{j_0}|\,,\end{aligned}$$

and

$$\begin{aligned}
\mathfrak{S}_{r,2}(n,j,\mathbf{h}) &:= \mathrm{e}^{-c\,|j|} \sum_{j_0 \geq 1} \mathbf{M}_r\left(c, -j_0 + n\,|\alpha_r|, n\right) |h_{j_0}| \,, \\
\mathfrak{S}_{\ell,2}(n,j,\mathbf{h}) &:= \mathrm{e}^{-c\,|j|} \sum_{j_0 \leq 0} \mathbf{M}_\ell\left(c, j_0 + n\,\alpha_\ell, n\right) |h_{j_0}| \,,
\end{aligned}$$

where we recall that the functions $\mathbf{M}_\ell$ and $\mathbf{M}_r$ are defined in (4.6). We also define:

$$\mathfrak{S}_{\exp}(n,j,\mathbf{h}) := \mathrm{e}^{-c\,n-c\,|j|} \sum_{j_0 \in \mathbb{Z}} \mathrm{e}^{-c\,|j_0|} \, |h_{j_0}| \,.$$

Using the estimates derived in Theorem 4.1 and applying the triangle inequality, we remark that for some suitable positive constants C and c that do not depend on n, j nor $\mathbf{h}$, we have

$$\left| \sum_{j_0 \geq 1} \left(\mathscr{G}^n(j,j_0) - \mathbb{1}_{|j-j_0| \leq n} \, \mathcal{H}_j \, \mathbf{A}_r(-j_0 + n\,|\alpha_r|, n) \right) h_{j_0} \right| \\
\leq C\,\mathfrak{S}_{r,1}(n,j,\mathbf{h}) + C\,\mathfrak{S}_{r,2}(n,j,\mathbf{h}) + C\,\mathfrak{S}_{\exp}(n,j,\mathbf{h}) \,,$$

together with

$$\left| \sum_{j_0 \leq 0} \left(\mathscr{G}^n(j,j_0) - \mathbb{1}_{|j-j_0| \leq n} \, \mathcal{H}_j \, \mathbf{A}_\ell(j_0 + n\,\alpha_\ell, n) \right) h_{j_0} \right| \\
\leq C\,\mathfrak{S}_{\ell,1}(n,j,\mathbf{h}) + C\,\mathfrak{S}_{\ell,2}(n,j,\mathbf{h}) + C\,\mathfrak{S}_{\exp}(n,j,\mathbf{h}) \,.$$

Combining those two inequalities with the triangle inequality in (4.39), we have for all $(n,j) \in \mathbb{N}^* \times \mathbb{Z}$:

$$\left| (\mathscr{L}^n \mathbf{h})_j \right| \leq \left| (\mathscr{L}^n_{\mathrm{act}} \mathbf{h})_j \right| + C \left(\mathfrak{S}_{r,1}(n,j,\mathbf{h}) + \mathfrak{S}_{\ell,1}(n,j,\mathbf{h}) + \mathfrak{S}_{r,2}(n,j,\mathbf{h}) + \mathfrak{S}_{\ell,2}(n,j,\mathbf{h}) + \mathfrak{S}_{\exp}(n,j,\mathbf{h}) \right) . \tag{4.40}$$

In order to prove the estimates (4.37a) and (4.37b), one simply needs to prove estimates for each of the terms appearing in the right-hand side of the above inequality. From now on, we shall proceed term by term.

We start by estimating $\left(\mathfrak{S}_{r,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}}$ and $\left(\mathfrak{S}_{\ell,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}}$.

Lemma 4.9. *For any $\gamma_2 \geq \gamma_1 \geq 0$, there exists $C > 0$, such that for all $n \geq 1$ and $\mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, one has:*

$$\begin{aligned}
\left\| \left(\mathfrak{S}_{r,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} + \left\| \left(\mathfrak{S}_{\ell,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2-\gamma_1-1/8}} \|\mathbf{h}\|_{\ell^1_{\gamma_2}} \,, \\
\left\| \left(\mathfrak{S}_{r,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} + \left\| \left(\mathfrak{S}_{\ell,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2-\gamma_1+1/3}} \|\mathbf{h}\|_{\ell^1_{\gamma_2}} \,.
\end{aligned}$$

Proof. We only prove the estimates for $\left(\mathfrak{S}_{r,1}(n,j,\mathbf{h})\right)_{j \in \mathbb{Z}}$, the other case is handled similarly. For $\mathbf{h} \in$

$\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$ and $n \in \mathbb{N}^*$, we have that

$$\begin{aligned}\left\|(\mathfrak{S}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} &= \sum_{j\in\mathbb{Z}} (1+|j|^{\gamma_1})\,\mathbb{1}_{j\geq 1}\left(\sum_{j_0\geq 1}\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\;|h_{j_0}|\right)\\ &= \underbrace{\sum_{j\geq 1}(1+|j|^{\gamma_1})\left(\sum_{j_0=1}^{j}\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)}_{=:\mathscr{I}_1^n}\\ &\quad+\underbrace{\sum_{j\geq 1}(1+|j|^{\gamma_1})\left(\sum_{j_0\geq j+1}\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\;|h_{j_0}|\right)}_{=:\mathscr{I}_2^n}.\end{aligned}$$

From the definition (4.6b) of $\mathbf{M}_r$, we note that for $1 \leq j_0 \leq j$ with $|j-j_0| \leq n$, one has

$$\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n) \leq C\mathrm{e}^{-c\,n}\,.$$

Furthermore, Peetre's inequality implies that for $|j-j_0| \leq n$, there holds:

$$(1+|j|^{\gamma_1}) \;\leq\; C\,(1+|j_0|^{\gamma_1})\,(1+n^{\gamma_1})\,.$$

We thus obtain the following estimate for the first contribution $\mathscr{I}_1^n$:

$$\begin{aligned}\mathscr{I}_1^n &\leq\; C\,(1+n^{\gamma_1})\,\mathrm{e}^{-c\,n}\sum_{j\geq 1}\sum_{j_0=1}^{j}\mathbb{1}_{|j-j_0|\leq n}\,(1+|j_0|)^{\gamma_1}\,|h_{j_0}|\\ &=\; C\,(1+n^{\gamma_1})\,\mathrm{e}^{-c\,n}\sum_{j_0\geq 1}\left(\sum_{j\geq j_0}\mathbb{1}_{|j-j_0|\leq n}\right)\,(1+|j_0|^{\gamma_1})\,|h_{j_0}|\\ &\leq\; C\,n\,(1+n^{\gamma_1})\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_1}} \;\leq\; C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.\end{aligned}$$

On the other hand, for the second contribution $\mathscr{I}_2^n$, we decompose the sum into two parts:

$$\begin{aligned}\mathscr{I}_2^n = &\sum_{j\geq 1}(1+|j|^{\gamma_1})\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0<\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\;|h_{j_0}|\right)\\ &+\sum_{j\geq 1}(1+|j|^{\gamma_1})\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right).\end{aligned}$$

In both sums, since we have $1 \leq j \leq j_0$, we can always use the inequality:

$$(1+|j|^{\gamma_1}) \;\leq\; (1+|j_0|^{\gamma_1})\,.$$

Furthermore, we notice once again that for $j \geq 1$ and $j_0 \geq j+1$ with $1 \leq j_0 < \frac{n\,|\alpha_r|}{2}$, one has an exponential bound:

$$\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n) \;\leq\; C\,\mathrm{e}^{-c\,n}\,.$$

As a consequence, one gets

$$\begin{aligned}\sum_{j\geq 1}(1+|j|^{\gamma_1})&\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0<\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&\leq\,C\,\mathrm{e}^{-c\,n}\sum_{j\geq 1}\sum_{j_0\geq j+1}\mathbb{1}_{j_0<\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,(1+|j_0|^{\gamma_1})\,|h_{j_0}|\\&\leq\,C\,\mathrm{e}^{-c\,n}\sum_{j_0\in\mathbb{Z}}\left(\sum_{j=1}^{j_0}\mathbb{1}_{|j-j_0|\leq n}\right)\,(1+|j_0|^{\gamma_1})\,|h_{j_0}|\,\leq\,C\,n\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_1}}\,\leq\,C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.\end{aligned}$$

For the remaining contribution, we have

$$\begin{aligned}\sum_{j\geq 1}(1+|j|^{\gamma_1})&\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&\leq\,\sum_{j_0\geq 1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\,\left(\sum_{j=1}^{j_0}\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\right)\,(1+|j_0|^{\gamma_1})\,|h_{j_0}|\\&\leq\,C\,\|\mathbf{M}_r\,(c,\cdot+n\,|\alpha_r|,n)\,\|_{\ell^1(\mathbb{Z})}\,\sum_{j_0\geq 1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\,\frac{(1+|j_0|^{\gamma_2})}{(1+|j_0|^{\gamma_2-\gamma_1})}|h_{j_0}|\\&\leq\,\frac{C}{n^{\gamma_2-\gamma_1}}\,\|\mathbf{M}_r\,(c,\cdot+n\,|\alpha_r|,n)\,\|_{\ell^1(\mathbb{Z})}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.\end{aligned}$$

In order to conclude, we just use the following estimate that can be obtained from the definition (4.6b) and a mere comparison between a series and an integral:

$$\forall\,n\in\mathbb{N}^*\,,\quad\left\|(\mathbf{M}_r\,(c,j+n\,|\alpha_r|,n))_{j\in\mathbb{Z}}\right\|_{\ell^1(\mathbb{Z})}\,\leq\,C\,n^{1/8}\,,\tag{4.41}$$

where, of course, all constants are uniform with respect to $n\in\mathbb{N}^*$. The proof of the $\ell^1_{\gamma_1}$ estimate for $\mathfrak{S}_{r,1}(n,\cdot,\mathbf{h})$ is now complete.

We now turn our attention to the second estimate in $\ell^\infty_{\gamma_1}$. By definition of the norm in $\ell^\infty_{\gamma_1}$, we have

$$\begin{aligned}\left\|(\mathfrak{S}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\,&=\,\sup_{j\in\mathbb{Z}}\,(1+|j|^{\gamma_1})\,\mathbb{1}_{j\geq 1}\,\left(\sum_{j_0\geq 1}\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&\leq\,\sup_{j\geq 1}\,(1+|j|^{\gamma_1})\,\left(\sum_{j_0=1}^{j}\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&\quad+\sup_{j\geq 1}\,(1+|j|^{\gamma_1})\,\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0<\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&\quad+\sup_{j\geq 1}\,(1+|j|^{\gamma_1})\,\left(\sum_{j_0\geq j+1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\,\mathbb{1}_{|j-j_0|\leq n}\,\mathbf{M}_r\,(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\&=:\,\mathscr{I}_1^n+\mathscr{I}_2^n+\mathscr{I}_3^n\,.\end{aligned}$$

Performing similar computations and estimates as above, one gets first:

$$\mathscr{J}_1^n + \mathscr{J}_2^n \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}},$$

which is an even better estimate than the one we are aiming at. On the other hand, we have

$$\begin{aligned}\mathscr{J}_3^n &\leq C\,\|\mathbf{M}_r\,(c,\cdot+n\,|\alpha_r|,n)\,\|_{\ell^\infty(\mathbb{Z})}\sum_{j_0\geq\frac{n\,|\alpha_r|}{2}}(1+|j_0|^{\gamma_1})\,|h_{j_0}|\\ &\leq \frac{C}{n^{\gamma_2-\gamma_1}}\,\|\mathbf{M}_r\,(c,\cdot+n\,|\alpha_r|,n)\,\|_{\ell^\infty(\mathbb{Z})}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}},\end{aligned}$$

and it simply remains to use the property:

$$\left\|(\mathbf{M}_r\,(c,j+n\,|\alpha_r|,n))_{j\in\mathbb{Z}}\right\|_{\ell^\infty(\mathbb{Z})} \leq \frac{1}{n^{1/3}}, \tag{4.42}$$

that can be easily deduced from the definition (4.6b). This concludes the proof of Lemma 4.9. □

Next, we estimate $(\mathfrak{S}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}$ and $(\mathfrak{S}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}$.

Lemma 4.10. *For any $\gamma_2\geq\gamma_1\geq 0$, there exists a constant $C>0$ such that for all $n\geq 1$ and $\mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, one has:*

$$\left\|(\mathfrak{S}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} + \left\|(\mathfrak{S}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} \leq \frac{C}{n^{\gamma_2+1/3}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}},$$

and therefore:

$$\left\|(\mathfrak{S}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} + \left\|(\mathfrak{S}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} \leq \frac{C}{n^{\gamma_2+1/3}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}.$$

Proof. For $\mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, we have that

$$\begin{aligned}\left\|(\mathfrak{S}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} &= \left(\sum_{j\in\mathbb{Z}}(1+|j|^{\gamma_1})\,\mathrm{e}^{-c\,|j|}\right)\left(\sum_{j_0\geq 1}\mathbf{M}_r\,(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\right)\\ &\leq C\sum_{j_0=1}^{\frac{n\,|\alpha_r|}{2}}\mathbf{M}_r\,(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}| + C\sum_{j_0\geq\frac{n\,|\alpha_r|}{2}}\mathbf{M}_r\,(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|.\end{aligned}$$

We then either use the exponential bound given by the definition (4.6b) or the global bound (4.42), which yields:

$$\|\mathfrak{S}_{r,2}(n,\cdot,\mathbf{h})\|_{\ell^1_{\gamma_1}} \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1} + \frac{C}{n^{1/3}}\sum_{j_0\geq\frac{n\,|\alpha_r|}{2}}\frac{(1+|j_0|^{\gamma_2})}{n^{\gamma_2}}\,|h_{j_0}| \leq \frac{C}{n^{\gamma_2+1/3}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}.$$

The $\ell^\infty_{\gamma_1}$ estimate is straightforward by just observing that the $\ell^\infty_{\gamma_1}$ norm is always smaller than the $\ell^1_{\gamma_1}$ norm. The proof of Lemma 4.10 is thus complete. □

We now handle the most delicate estimates on the family of operators $(\mathscr{L}^n_{\mathrm{act}})_{n\in\mathbb{N}^*}$.

Lemma 4.11. *For any $\gamma_2 \geq \gamma_1 \geq 0$, there exists a constant $C > 0$ such that the family of operators $(\mathscr{L}^n_{\mathrm{act}})_{n\in\mathbb{N}^*}$ satisfies the following estimate:*

$$\forall n \in \mathbb{N}^*\,, \quad \|\mathscr{L}^n_{\mathrm{act}}\,\mathbf{h}\|_{\ell^1_{\gamma_1}} \,\leq\, \frac{C}{n^{\gamma_2}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,, \qquad \textit{for any } \mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R}) \textit{ with } \sum_{j\in\mathbb{Z}} h_j = 0\,,$$

and therefore:

$$\forall n \in \mathbb{N}^*\,, \quad \|\mathscr{L}^n_{\mathrm{act}}\,\mathbf{h}\|_{\ell^\infty_{\gamma_1}} \,\leq\, \frac{C}{n^{\gamma_2}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,, \qquad \textit{for any } \mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R}) \textit{ with } \sum_{j\in\mathbb{Z}} h_j = 0\,.$$

Proof. We consider first $\mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$ without making any assumption on its mass. The first step of the proof consists in writing:

$$\begin{aligned}
&\sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|\leq n}\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0} + \sum_{j_0\leq 0} \mathbb{1}_{|j-j_0|\leq n}\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0} \\
&\qquad = \sum_{j_0\geq 1} \mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0} + \sum_{j_0\leq 0} \mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0} \\
&\qquad\quad - \sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|>n}\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0} - \sum_{j_0\leq 0} \mathbb{1}_{|j-j_0|>n}\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0}\,,
\end{aligned} \tag{4.43}$$

where the two infinite sums on the right-hand side converge since $\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)$ is uniformly bounded (see Corollary A.6 in Appendix A) and $\mathbf{h}$ is integrable. Since $|\alpha_r| \in (0,1)$, we can take $\beta \in (|\alpha_r|,1)$. As a consequence, we can use the exponential decay of $(\mathcal{H}_j)_{j\in\mathbb{Z}}$ and further decompose:

$$\begin{aligned}
\left|\mathcal{H}_j \sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|>n}\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)h_{j_0}\right| &\leq C \sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\,|h_{j_0}|\,\,\mathrm{e}^{-c\,|j|} \\
&= C \sum_{j_0=1}^{n\beta} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\,|h_{j_0}|\,\,\mathrm{e}^{-c\,|j|} \\
&\quad + C \sum_{j_0>n\beta} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\,|h_{j_0}|\,\,\mathrm{e}^{-c\,|j|}\,.
\end{aligned}$$

In the first sum, since $n < |j-j_0|$ and $1 \leq j_0 \leq n\beta$, we get that $(1-\beta)n < |j|$, and thus

$$\begin{aligned}
\sum_{j_0=1}^{n\beta} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\,|h_{j_0}|\,\,\mathrm{e}^{-c\,|j|} &\leq \sum_{j_0=1}^{n\beta} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\,\,|h_{j_0}|\,\,\mathrm{e}^{-\frac{c}{2}|j|}\mathrm{e}^{-\frac{c(1-\beta)}{2}n} \\
&\leq C\,n\,\mathrm{e}^{-\frac{c}{2}(1-\beta)n}\,\mathrm{e}^{-\frac{c}{2}|j|}\,\|\mathbf{h}\|_{\ell^\infty(\mathbb{Z})}\,,
\end{aligned}$$

where we have used the property from Corollary A.6 that the quantity $|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|$ is uniformly bounded and we have also used the fact that the sum with respect to j_0 gathers at most n terms (since $\beta \geq 1$). We now consider the sum with respect to $j_0 > n\beta$. Still using Corollary A.6 in Appendix A, but this time for $j_0 > n\beta$ with $\beta \in (|\alpha_r|,1)$, we have that

$$|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)| \leq C\,\exp\left(-c\,\frac{|j_0-n\,|\alpha_r||^{4/3}}{n^{1/3}}\right)\,,$$

and thus

$$\sum_{j_0>n\beta} |\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)| \leq C\,\mathrm{e}^{-c\,n}\,.$$

As a consequence, we have

$$\sum_{j_0>n\beta} \mathbb{1}_{|j-j_0|>n}\,|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)|\;|h_{j_0}|\;\mathrm{e}^{-c\,|j|} \leq C\,\mathrm{e}^{-c\,n-c\,|j|}\,\|\mathbf{h}\|_{\ell^\infty(\mathbb{Z})}\,.$$

Summing up, we have proved the following estimate

$$\sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|\mathcal{H}_j\sum_{j_0\geq 1}\mathbb{1}_{|j-j_0|>n}\,\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0}\right| \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^\infty(\mathbb{Z})} \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,$$

and similarly, we also have

$$\sum_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\left|\mathcal{H}_j\sum_{j_0\leq 0}\mathbb{1}_{|j-j_0|>n}\,\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0}\right| \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.$$

We now go back to the decomposition (4.43) and consider the two series in the right-hand side. We assume from now on that the sequence $\mathbf{h}$ has zero mass, that is:

$$\sum_{j_0\in\mathbb{Z}} h_{j_0} \;=\; 0\,.$$

We can therefore write:

$$\begin{aligned}\sum_{j_0\geq 1}\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)\,h_{j_0}+\sum_{j_0\leq 0}\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)\,h_{j_0}&\\ =\sum_{j_0\geq 1}\left(\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)-1\right)\,h_{j_0}&+\sum_{j_0\leq 0}\left(\mathbf{A}_\ell(j_0+n\,\alpha_\ell,n)-1\right)\,h_{j_0}\,,\end{aligned}$$

and we shall study each sum separately. For the first contribution, we further split the sum into two parts:

$$\begin{aligned}\sum_{j_0\geq 1}\left(\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)-1\right)\,h_{j_0}=&\sum_{j_0=1}^{\frac{n\,|\alpha_r|}{2}}\left(\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)-1\right)\,h_{j_0}\\&+\sum_{j_0>\frac{n\,|\alpha_r|}{2}}\left(\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)-1\right)\,h_{j_0}\,.\end{aligned}$$

Once again, we will rely on Corollary A.6 in Appendix A. On the one hand, we use the fact for $1\leq j_0\leq \frac{n\,|\alpha_r|}{2}$, one has an exponential bound:

$$|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)-1|\;\leq\;C\,\mathrm{e}^{-c\,n}\,,$$

and on the other hand, that the factor $\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)$ is uniformly bounded with respect to $j_0 \in \mathbb{Z}$ and $n \in \mathbb{N}^*$. As a consequence, we get

$$\left| \sum_{j_0 \geq 1} \left(\mathbf{A}_r(-j_0 + n\,|\alpha_r|, n) - 1 \right) h_{j_0} \right| \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1} + C \sum_{j_0 > \frac{n\,|\alpha_r|}{2}} |h_{j_0}|$$

$$\leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1} + C \sum_{j_0 > \frac{n\,|\alpha_r|}{2}} \frac{1 + |j_0|^{\gamma_2}}{1 + |(n\,|\alpha_r|/2)|^{\gamma_2}} |h_{j_0}| \leq \frac{C}{n^{\gamma_2}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.$$

And similarly, we also have

$$\left| \sum_{j_0 \leq 0} \left(\mathbf{A}_\ell(j_0 + n\,\alpha_\ell, n) - 1 \right) h_{j_0} \right| \leq \frac{C}{n^{\gamma_2}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.$$

Summing up, we have obtained

$$\sum_{j \in \mathbb{Z}} (1 + |j|)^{\gamma_1} \left| \mathcal{H}_j \left(\sum_{j_0 \geq 1} \mathbf{A}_r(-j_0 + n\,|\alpha_r|, n)\,h_{j_0} + \sum_{j_0 \leq 0} \mathbf{A}_\ell(j_0 + n\,\alpha_\ell, n)\,h_{j_0} \right) \right| \leq \frac{C}{n^{\gamma_2}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.$$

Going back to the decomposition (4.43), this concludes the proof of Lemma 4.11. ☐

The very last contribution to handle is the one coming from the exponential terms $(\mathfrak{S}_{\exp}(n, j, \mathbf{h}))_{j \in \mathbb{Z}}$, and we have the following result whose proof is trivial.

Lemma 4.12. *For any $\gamma_2 \geq \gamma_1 \geq 0$, there exists a constant $C > 0$ such that for all $n \geq 1$ and $\mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, one has*

$$\|(\mathfrak{S}_{\exp}(n, j, \mathbf{h}))_{j \in \mathbb{Z}}\|_{\ell^1_{\gamma_1}} \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,$$

and therefore:

$$\|(\mathfrak{S}_{\exp}(n, j, \mathbf{h}))_{j \in \mathbb{Z}}\|_{\ell^\infty_{\gamma_1}} \leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,.$$

Going back to the inequality (4.40), the proof of the estimates (4.37a) and (4.37b) on the semigroup $(\mathscr{L}^n)_{n \in \mathbb{N}}$ of Theorem 4.3 is then a direct consequence of Lemma 4.9, Lemma 4.10, Lemma 4.11 and Lemma 4.12. Indeed, for proving (4.37a), we consider $\mathbf{h} \in \ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$ with mass zero and combine those four preliminary results to obtain:

$$\|\mathscr{L}^n\,\mathbf{h}\|_{\ell^1_{\gamma_1}} \leq C \left(\frac{1}{n^{\gamma_2}} + \frac{1}{n^{\gamma_2 - \gamma_1 - 1/8}} + \frac{1}{n^{\gamma_2 + 1/3}} + \mathrm{e}^{-c\,n} \right) \|\mathbf{h}\|_{\ell^1_{\gamma_2}} \leq \frac{C}{n^{\gamma_2 - \gamma_1 - 1/8}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,$$

and the proof of (4.37b) is entirely similar.

4.5.2 Proof of the estimates (4.38a), (4.38b) and (4.38c) on the family $(\mathscr{L}^n(\mathrm{Id} - \mathbf{S}))_{n \in \mathbb{N}}$

The starting point of the proof of the estimates (4.38a), (4.38b) and (4.38c) of Theorem 4.3 follows similar lines as in the previous subsection for proving (4.37a) and (4.37b) on the semigroup $(\mathscr{L}^n)_{n \in \mathbb{N}}$. For $\gamma \geq 0$

and $\mathbf{h} \in \ell^q_\gamma(\mathbb{Z};\mathbb{R})$ ($q=1$ or $q=+\infty$) we recall that the action of the family of operators $(\mathscr{L}^n(\mathrm{Id}-\mathbf{S}))_{n\in\mathbb{N}}$ on $\mathbf{h}$ is given for each $n \in \mathbb{N}$ by:

$$\forall\,(n,j) \in \mathbb{N}\times\mathbb{Z}\,,\quad (\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h})_j = \sum_{j_0\in\mathbb{Z}} \left(\mathscr{G}^n(j,j_0)-\mathscr{G}^n(j,j_0-1)\right) h_{j_0} = \sum_{j_0\in\mathbb{Z}} \mathscr{D}^n(j,j_0)\, h_{j_0}\,,$$

by definition (4.31) of $\mathscr{D}^n(j,j_0)$. Restricting to $n\geq 1$ (the case $n=0$ is trivial) and using the estimates obtained on the derivative of the temporal Green's function in Theorem 4.2, we obtain that

$$\begin{aligned}\left|(\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h})_j\right| \leq\, & C\sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|\leq n+1}\left[\mathbb{1}_{j\geq 1}\mathbf{K}_r(c,j-j_0+n\,|\alpha_r|,n)+\mathrm{e}^{-c\,|j|}\,\mathbf{M}_r(c,-j_0+|\alpha_r|n,n)\right]\,|h_{j_0}| \\ & +C\sum_{j_0\leq 0} \mathbb{1}_{|j-j_0|\leq n+1}\left[\mathbb{1}_{j\leq 0}\,\mathbf{K}_\ell(c,j_0-j+n\,\alpha_\ell,n)+\mathrm{e}^{-c\,|j|}\,\mathbf{M}_\ell(c,j_0+n\,\alpha_\ell,n)\right]\,|h_{j_0}| \\ & +C\,\mathrm{e}^{-c\,n}\sum_{j_0\in\mathbb{Z}} \mathbb{1}_{|j-j_0|\leq n+1}\,\mathrm{e}^{-c\,|j-j_0|}\,|h_{j_0}|\,.\end{aligned}$$

This motivates the definition of the following quantities for $(n,j)\in\mathbb{N}^*\times\mathbb{Z}$:

$$\begin{aligned}\mathfrak{D}_{r,1}(n,j,\mathbf{h}) &:= \mathbb{1}_{j\geq 1}\sum_{j_0\geq 1}\mathbb{1}_{|j-j_0|\leq n+1}\,\mathbf{K}_r(c,j-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\,,\\ \mathfrak{D}_{\ell,1}(n,j,\mathbf{h}) &:= \mathbb{1}_{j\leq 0}\sum_{j_0\leq 0}\mathbb{1}_{|j-j_0|\leq n+1}\,\mathbf{K}_\ell(c,j_0-j+n\,\alpha_\ell,n)\,|h_{j_0}|\,,\end{aligned}$$

and

$$\begin{aligned}\mathfrak{D}_{r,2}(n,j,\mathbf{h}) &:= \mathrm{e}^{-c\,|j|}\sum_{j_0\geq 1}\mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}|\,,\\ \mathfrak{D}_{\ell,2}(n,j,\mathbf{h}) &:= \mathrm{e}^{-c\,|j|}\sum_{j_0\leq 0}\mathbf{M}_\ell(c,j_0+n\,\alpha_\ell,n)\,|h_{j_0}|\,,\end{aligned}$$

together with:

$$\mathfrak{D}_{\exp}(n,j,\mathbf{h}) := \mathrm{e}^{-c\,n}\sum_{j_0\in\mathbb{Z}}\mathbb{1}_{|j-j_0|\leq n+1}\,\mathrm{e}^{-c|j-j_0|}\,|h_{j_0}|\,.$$

With these notations at hand, we readily obtain that for all $(n,j)\in\mathbb{N}^*\times\mathbb{Z}$, we have:

$$\left|(\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\,\mathbf{h})_j\right| \leq C\,\left(\mathfrak{D}_{r,1}(n,j,\mathbf{h})+\mathfrak{D}_{\ell,1}(n,j,\mathbf{h})+\mathfrak{D}_{r,2}(n,j,\mathbf{h})+\mathfrak{D}_{\ell,2}(n,j,\mathbf{h})+\mathfrak{D}_{\exp}(n,j,\mathbf{h})\right)\,.$$

We will mainly focus our efforts on the terms $\mathfrak{D}_{r,1}(n,j,\mathbf{h})$, $\mathfrak{D}_{\ell,1}(n,j,\mathbf{h})$ and $\mathfrak{D}_{\exp}(n,j,\mathbf{h})$. Indeed, we note that $\mathfrak{D}_{r,2}(n,j,\mathbf{h})$ and $\mathfrak{D}_{\ell,2}(n,j,\mathbf{h})$ are identical to $\mathfrak{S}_{r,2}(n,j,\mathbf{h})$ and $\mathfrak{S}_{\ell,2}(n,j,\mathbf{h})$ in the previous subsection and thus enjoy the same estimates as the ones derived in Lemma 4.10. We shall need however an additional estimate to the one proved in Lemma 4.10 but the proof of it will be very similar to what has already been done. Let us start indeed with the estimates for $\mathfrak{D}_{r,2}(n,j,\mathbf{h})$ and $\mathfrak{D}_{\ell,2}(n,j,\mathbf{h})$.

Lemma 4.13. *For any $\gamma_2\geq\gamma_1\geq 0$, there exists a constant $C>0$ such that for all $n\geq 1$ and $\mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R})$, one has*

$$\begin{aligned}\left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2+1/3}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\,,\\ \left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2+1/3}}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\end{aligned}$$

and for $\mathbf{h} \in \ell^\infty_{\gamma_2}(\mathbb{Z};\mathbb{R})$*, one has*

$$\left\| (\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} + \left\| (\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} \leq \frac{C}{n^{\gamma_2-1/8}} \|\mathbf{h}\|_{\ell^\infty_{\gamma_2}} .$$

Proof. The first two estimates in Lemma 4.13 have already been proved in Lemma 4.10 so we switch directly to the last estimate where the novelty is that now the sequence $\mathbf{h}$ is assumed to belong to the larger space $\ell^\infty_{\gamma_2}(\mathbb{Z};\mathbb{R})$. We thus consider $n \in \mathbb{N}^*$ and $\mathbf{h} \in \ell^\infty_{\gamma_2}(\mathbb{Z};\mathbb{R})$. From the definition of $\mathfrak{D}_{r,2}(n,j,\mathbf{h})$, we have

$$\begin{aligned}\left\| (\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} &= \left(\sup_{j\in\mathbb{Z}} (1+|j|^{\gamma_1})\, \mathrm{e}^{-c|j|} \right) \left(\sum_{j_0\geq 1} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}| \right) \\ &\leq C \sum_{j_0=1}^{\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}| + C \sum_{j_0\geq\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}| \\ &\leq C\,\|\mathbf{h}\|_{\ell^\infty} \sum_{j_0=1}^{\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n) + C \sum_{j_0\geq\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0}| .\end{aligned}$$

By summing the definition (4.6b) and comparing the corresponding series with an integral, we get the exponential estimate:

$$\sum_{j_0=1}^{\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n) \leq C\,\mathrm{e}^{-c\,n} .$$

For the second sum, we have:

$$\begin{aligned}\sum_{j_0\geq\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,|h_{j_0} &\leq C \sum_{j_0\geq\frac{n\,|\alpha_r|}{2}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n)\,\frac{(1+|j_0|^{\gamma_2})}{n^{\gamma_2}}\,|h_{j_0}| \\ &\quad + \frac{C}{n^{\gamma_2}} \|\mathbf{h}\|_{\ell^\infty_{\gamma_2}} \sum_{j_0\in\mathbb{Z}} \mathbf{M}_r(c,-j_0+n\,|\alpha_r|,n) ,\end{aligned}$$

and we simply use (4.41) to conclude. ☐

Now, we handle the contributions from $\mathfrak{D}_{r,1}(n,j,\mathbf{h})$ and $\mathfrak{D}_{\ell,1}(n,j,\mathbf{h})$.

Lemma 4.14. *For any* $\gamma_2 \geq \gamma_1 \geq 0$*, there exists* $C>0$*, such that for all* $n\geq 1$*, one has*

$$\begin{aligned}\left\| (\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} + \left\| (\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2-\gamma_1+1/8}} \|\mathbf{h}\|_{\ell^1_{\gamma_2}} , \text{ for } \mathbf{h}\in\ell^1_{\gamma_2} , \\ \left\| (\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} + \left\| (\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2-\gamma_1+7/12}} \|\mathbf{h}\|_{\ell^1_{\gamma_2}} , \text{ for } \mathbf{h}\in\ell^1_{\gamma_2} , \\ \left\| (\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} + \left\| (\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} &\leq \frac{C}{n^{\gamma_2-\gamma_1+1/8}} \|\mathbf{h}\|_{\ell^\infty_{\gamma_2}} , \text{ for } \mathbf{h}\in\ell^\infty_{\gamma_2} .\end{aligned}$$

Proof. For $h \in \ell^1_{\gamma_2}$, we have that

$$\begin{aligned}
\left\| (\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^1_{\gamma_1}} &= \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&= \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0=1}^{j} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\quad + \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0<\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\quad + \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0\geq\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}|.
\end{aligned}$$

Now from the definition (4.32b) of $\mathbf{K}_r$, we infer the following facts

- for $1 \leq j_0 \leq j$ with $|j-j_0| \leq n+1$, one has $\mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n) \leq C\,\mathrm{e}^{-c\,n}$,
- for $j \geq 1$ and $j_0 \geq j+1$ with $1 \leq j_0 < \frac{n\,|\alpha_r|}{2}$, one has $\mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n) \leq C\,\mathrm{e}^{-c\,n}$.

As a consequence, one readily derives, as in the previous cases, that

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0=1}^{j} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\qquad + \sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0<\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\qquad \leq C\mathrm{e}^{-c\,n} \|\mathbf{h}\|_{\ell^1_{\gamma_2}}.
\end{aligned}$$

For the last contribution, inverting the sums gives

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0\geq\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\qquad \leq C \left\| (\mathbf{K}_r(j+n\,|\alpha_r|, n))_{j\in\mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})} \sum_{j_0\geq 1} \mathbb{1}_{j_0\geq\frac{n|\alpha_r|}{2}} (1+|j_0|)^{\gamma_1} |h_{j_0}| \leq \frac{C}{n^{\gamma_2-\gamma_1+1/8}} \|\mathbf{h}\|_{\ell^1_{\gamma_2}},
\end{aligned}$$

since $\left\| (\mathbf{K}_r(j+n\,|\alpha_r|, n))_{j\in\mathbb{Z}} \right\|_{\ell^1(\mathbb{Z})} \leq C/n^{1/8}$.

Finally, the proof of the second estimate is similar to the one for the estimate of $(\mathfrak{S}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}$ and $(\mathfrak{S}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}$ in Lemma 4.9, this time using that $\left\| (\mathbf{K}_r(j+n\,|\alpha_r|, n))_{j\in\mathbb{Z}} \right\|_{\ell^\infty(\mathbb{Z})} \leq C/n^{7/12}$.

For the last estimate, we first note that, for $\mathbf{h} \in \ell^\infty_{\gamma_2}$, we have that

$$\begin{aligned}
\left\| (\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}} \right\|_{\ell^\infty_{\gamma_1}} &= \sup_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq 1} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&= \sup_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0=1}^{j} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\quad + \sup_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0<\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}| \\
&\quad + \sup_{j\in\mathbb{Z}} (1+|j|)^{\gamma_1} \mathbb{1}_{j\geq 1} \sum_{j_0\geq j+1} \mathbb{1}_{j_0\geq\frac{n|\alpha_r|}{2}} \mathbb{1}_{|j-j_0|\leq n+1} \mathbf{K}_r(c, j-j_0+n\,|\alpha_r|, n)|h_{j_0}|.
\end{aligned}$$

The first two terms contribute to

$$\sup_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\mathbb{1}_{j\geq 1}\sum_{j_0=1}^{j}\mathbb{1}_{|j-j_0|\leq n+1}\mathbf{K}_r(c,j-j_0+n\,|\alpha_r|,n)|h_{j_0}|$$
$$+\sup_{j\in\mathbb{Z}}(1+|j|)^{\gamma_1}\mathbb{1}_{j\geq 1}\sum_{j_0\geq j+1}\mathbb{1}_{j_0<\frac{n\,|\alpha_r|}{2}}\mathbb{1}_{|j-j_0|\leq n+1}\mathbf{K}_r(c,j-j_0+n\,|\alpha_r|,n)|h_{j_0}|$$
$$\leq C\mathrm{e}^{-c\,n}\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}},$$

while for the third term, we note that for each $j\geq 1$

$$(1+|j|)^{\gamma_1}\sum_{j_0\geq j+1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}\mathbb{1}_{|j-j_0|\leq n+1}\mathbf{K}_r(c,j-j_0+n\,|\alpha_r|,n)|h_{j_0}|$$
$$\leq C\sum_{j_0\geq 1}\mathbb{1}_{j_0\geq\frac{n\,|\alpha_r|}{2}}|\mathscr{K}_r^n(c,j-j_0)|\frac{(1+|j_0|)^{\gamma_2}}{(1+|j_0|)^{\gamma_2-\gamma_1}}|h_{j_0}|$$
$$\leq\frac{C}{n^{\gamma_2-\gamma_1+1/8}}\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}},$$

since $\|\left(\mathbf{K}_r(j+n\,|\alpha_r|,n)\right)_{j\in\mathbb{Z}}\|_{\ell^1(\mathbb{Z})}\leq C/n^{1/8}$. □

The very last contribution to handle is the one coming from the exponential terms $(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}$. And we have the following result whose proof is trivial and let to the interested reader.

Lemma 4.15. *For any $\gamma_2\geq\gamma_1\geq 0$, there exists a constant $C>0$ such that for all $n\geq 1$, one has:*

$$\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\|_{\ell^1_{\gamma_1}}\leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^1_{\gamma_2}},\qquad\text{for any }\mathbf{h}\in\ell^1_{\gamma_2}(\mathbb{Z};\mathbb{R}),$$
$$\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\|_{\ell^\infty_{\gamma_1}}\leq C\,\mathrm{e}^{-c\,n}\,\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}},\qquad\text{for any }\mathbf{h}\in\ell^\infty_{\gamma_2}(\mathbb{Z};\mathbb{R}).$$

We can now conclude the proof of Theorem 4.3. For the first estimate (4.38a), we use the estimates provided by Lemma 4.14, Lemma 4.13 and Lemma 4.15. We thus have

$$\|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^1_{\gamma_1}}\leq C\left(\left\|(\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\right)$$
$$+C\left(\left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\right)$$
$$\leq C\left(\frac{1}{n^{\gamma_2-\gamma_1+1/8}}+\frac{1}{n^{\gamma_2+1/3}}+\mathrm{e}^{-c\,n}\right)\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\leq\frac{C}{n^{\gamma_2-\gamma_1+1/8}}\|\mathbf{h}\|_{\ell^1_{\gamma_2}}.$$

For the second estimate (4.38b), we have

$$
\begin{aligned}
\|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^\infty_{\gamma_1}} &\leq C\left(\left\|(\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\right)\\
&\quad +C\left(\left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\right)\\
&\leq C\left(\left\|(\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\right)\\
&\quad +C\left(\left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}+\left\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^1_{\gamma_1}}\right)\\
&\leq C\left(\frac{1}{n^{\gamma_2-\gamma_1+7/12}}+\frac{1}{n^{\gamma_2+1/3}}+\mathrm{e}^{-c\,n}\right)\|\mathbf{h}\|_{\ell^1_{\gamma_2}}\leq\frac{C}{n^{\gamma_2-\gamma_1+1/3+\min(1/4,\gamma_1)}}\|\mathbf{h}\|_{\ell^1_{\gamma_2}}.
\end{aligned}
$$

Finally, for the third estimate (4.38c), we use once more the estimates provided by Lemma 4.14, Lemma 4.13 and Lemma 4.15 to get:

$$
\begin{aligned}
\|\mathscr{L}^n(\mathrm{Id}-\mathbf{S})\mathbf{h}\|_{\ell^\infty_{\gamma_1}} &\leq C\left(\left\|(\mathfrak{D}_{r,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,1}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\right)\\
&\quad +C\left(\left\|(\mathfrak{D}_{r,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\ell,2}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}+\left\|(\mathfrak{D}_{\exp}(n,j,\mathbf{h}))_{j\in\mathbb{Z}}\right\|_{\ell^\infty_{\gamma_1}}\right)\\
&\leq C\left(\frac{1}{n^{\gamma_2-\gamma_1+1/8}}+\frac{1}{n^{\gamma_2-1/8}}+\mathrm{e}^{-c\,n}\right)\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}}\leq\frac{C}{n^{\gamma_2-\gamma_1+\min(1/8,\gamma_1-1/8)}}\|\mathbf{h}\|_{\ell^\infty_{\gamma_2}}.
\end{aligned}
$$

This completes the proof of Theorem 4.3.

Chapter 5

Nonlinear orbital stability

In this last chapter, we finally prove the nonlinear orbital stability of the family of stationary discrete profiles $\left\{\mathbf{v}^\theta, \theta \in (-\underline{\theta}, \underline{\theta})\right\}$ given by Theorem 2.1 in some well-chosen algebraically weighted ℓ^p spaces. More precisely, given an initial sequence $\mathbf{h} = (h_j)_{j\in\mathbb{Z}}$ in $\ell^1(\mathbb{Z};\mathbb{R})$ satisfying the mass condition

$$\sum_{j\in\mathbb{Z}} h_j = \theta\,, \quad \theta \in (-\underline{\theta}, \underline{\theta}),$$

we let $\mathbf{u}^n = (u_j^n)_{j\in\mathbb{Z},n\in\mathbb{N}}$ denote the corresponding solution to the Lax-Wendroff scheme (2.5)-(2.6) starting from the initial condition $\mathbf{u}^0 = \overline{\mathbf{u}} + \mathbf{h}$, where we recall that $\overline{\mathbf{u}} = \mathbf{v}^0$ is the element of the family $\left\{\mathbf{v}^\theta, \theta \in (-\underline{\theta}, \underline{\theta})\right\}$ given by the explicit expression (2.7). By mass conservation, we automatically have:

$$\forall\, n \in \mathbb{N}\,, \quad \sum_{j\in\mathbb{Z}} u_j^n - \overline{u}_j \;=\; \sum_{j\in\mathbb{Z}} u_j^0 - \overline{u}_j \;=\; \sum_{j\in\mathbb{Z}} h_j \;=\; \theta\,.$$

As a consequence, for each $n \in \mathbb{N}$, it is natural to define the *perturbation* $\mathbf{p}^n = (p_j^n)_{j\in\mathbb{Z}}$ as

$$\mathbf{p}^n \;:=\; \mathbf{u}^n - \mathbf{v}^\theta\,,$$

where in particular the mass conservation property of $\mathbf{u}^n$ implies that for any $n \in \mathbb{N}$, the sequence $\mathbf{p}^n$ has zero mass:

$$\forall\, n \in \mathbb{N}\,, \quad \sum_{j\in\mathbb{Z}} p_j^n \;=\; 0\,.$$

From the recurrence formula (2.5) for $\mathbf{u}^n = \mathbf{v}^\theta + \mathbf{p}^n$, we deduce that the sequence $\mathbf{p}^n$ satisfies the following equation:

$$\forall\, n \in \mathbb{N}\,, \quad \mathbf{p}^{n+1} = \mathcal{L}^\theta\,\mathbf{p}^n + (\mathrm{Id} - \mathbf{S})\,\mathscr{N}^\theta(\mathbf{p}^n)\,, \tag{5.1}$$

where the bounded linear operator $\mathcal{L}^\theta : \ell^q(\mathbb{Z};\mathbb{C}) \mapsto \ell^q(\mathbb{Z};\mathbb{C})$ is defined as the linearization of the Lax-Wendroff scheme around the stationary discrete shock profile $\mathbf{v}^\theta$, that is:

$$\begin{aligned}\forall\, j \in \mathbb{Z}\,, \quad \left(\mathcal{L}^\theta\,\mathbf{p}\right)_j \;:=\; p_j &- \lambda\left[\partial_u\mathscr{F}_\lambda(v_j^\theta, v_{j+1}^\theta)\,p_j + \partial_v\mathscr{F}_\lambda(v_j^\theta, v_{j+1}^\theta)\,p_{j+1}\right] \\ &+ \lambda\left[\partial_u\mathscr{F}_\lambda(v_{j-1}^\theta, v_j^\theta)\,p_{j-1} + \partial_v\mathscr{F}_\lambda(v_{j-1}^\theta, v_j^\theta)\,p_j\right]\,,\end{aligned} \tag{5.2}$$

and the nonlinear operator $\mathscr{N}^\theta$ is defined by:

$$\begin{aligned}\forall\, j \in \mathbb{Z}\,, \quad \left(\mathscr{N}^\theta(\mathbf{p})\right)_j := \lambda\, \mathscr{F}_\lambda(v^\theta_{j-1}+p_{j-1}, v^\theta_j + p_j) - \lambda\, \mathscr{F}_\lambda(v^\theta_{j-1}, v^\theta_j) \\ - \lambda\, \left[\partial_u \mathscr{F}_\lambda(v^\theta_{j-1}, v^\theta_j)\, p_{j-1} + \partial_v \mathscr{F}_\lambda(v^\theta_{j-1}, v^\theta_j)\, p_j\right]\,.\end{aligned} \tag{5.3}$$

We also recall that $\mathbf{S}$ is the shift operator defined as $(\mathbf{S}\,\mathbf{p})_j := p_{j+1}$ for all $j \in \mathbb{Z}$. Actually, we shall rewrite the recurrence relation in time (5.1) more conveniently as

$$\forall\, n \in \mathbb{N}\,, \quad \mathbf{p}^{n+1} = \mathscr{L}\,\mathbf{p}^n + \left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p}^n + (\mathrm{Id} - \mathbf{S})\,\mathscr{N}^\theta(\mathbf{p}^n)\,,$$

where the operator $\mathscr{L} = \mathcal{L}^0$, given by (2.11), is the linearization of (2.5)-(2.6) around the discrete shock profile $\overline{\mathbf{u}} = \mathbf{v}^0$ given by (2.7). Finally, an application of Duhamel's formula gives the final expression for the perturbation $\mathbf{p}^n$ at any time $n \in \mathbb{N}$:

$$\forall\, n \in \mathbb{N}\,, \quad \mathbf{p}^n = \mathscr{L}^n\,\mathbf{p}^0 + \sum_{m=0}^{n-1} \mathscr{L}^{n-1-m}\,\left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p}^m + \sum_{m=0}^{n-1} \mathscr{L}^{n-1-m}\,(\mathrm{Id} - \mathbf{S})\,\mathscr{N}^\theta(\mathbf{p}^m)\,. \tag{5.4}$$

As already emphasized in Chapter 2, our nonlinear orbital stability result (that is, Theorem 2.5) holds true in algebraically weighted spaces. Recalling our definition from the previous chapter, for $\gamma \in \mathbb{R}^+$, we define the weight $\boldsymbol{\omega}_\gamma = (1 + |j|^\gamma)_{j\in\mathbb{Z}}$. Then, for $p \in [1, +\infty]$, we introduce the weighted space:

$$\ell^p_\gamma(\mathbb{Z};\mathbb{R}) \;=\; \{\mathbf{h} \in \ell^p(\mathbb{Z};\mathbb{R}) \mid \boldsymbol{\omega}_\gamma \mathbf{h} \in \ell^p(\mathbb{Z};\mathbb{R})\}\,,$$

with $\boldsymbol{\omega}_\gamma \mathbf{h} = ((1 + |j|^\gamma)\, h_j)_{j\in\mathbb{Z}}$ and for $\mathbf{h} \in \ell^p_\gamma$, we denote $\|\mathbf{h}\|_{\ell^p_\gamma} = \|\boldsymbol{\omega}_\gamma \mathbf{h}\|_{\ell^p}$ the norm of $\mathbf{h}$ in the space ℓ^p_γ. Our strategy will be to bound each term appearing in the right-hand side of (5.4) using the semi-group estimates obtained for $(\mathscr{L}^n)_{n\in\mathbb{N}}$ and $(\mathscr{L}^n(\mathrm{Id} - \mathbf{S}))_{n\in\mathbb{N}}$ in Theorem 4.3. This is exactly the same strategy as in [7] except that the exponents in Theorem 4.3 are different so the whole process should be carried out in order to determine the correct constraints for β and σ in Theorem 2.5.

We shall need three technical lemmas which we now state. The first lemma provides semi-group estimates for the family of operators $(\mathscr{L}^n\left(\mathcal{L}^\theta - \mathscr{L}\right))_{n\in\mathbb{N}}$.

Lemma 5.1. *For any $(\nu_1, \nu_2) \in [0, +\infty)^2$, there exists a constant $C_{\mathcal{L}}(\nu_1, \nu_2) > 0$ such that for all sequence $\mathbf{p} \in \ell^\infty(\mathbb{Z};\mathbb{R})$ one has $\left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p} \in \cap_{\gamma\geq 0}\ell^1_\gamma$ and:*

$$\forall\, n \in \mathbb{N}\,, \quad \left\|\mathscr{L}^n\left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p}\right\|_{\ell^1_{\nu_1}} \;\leq\; \frac{C_{\mathcal{L}}(\nu_1,\nu_2)}{(n+1)^{\nu_2}}\, |\theta|\, \|\mathbf{p}\|_{\ell^\infty}\,,$$

for all $\theta \in (-\underline{\theta}, \underline{\theta})$.

The proof of the above lemma can be found in [7, Proposition 2], and it uses crucially the exponential localisation of the family of discrete shock profiles $\mathbf{v}^\theta$ as stated in Theorem 2.1 and the key remark that for a bounded sequence $\mathbf{p} \in \ell^\infty(\mathbb{Z};\mathbb{R})$ one has

$$\sum_{j\in\mathbb{Z}} \left(\left(\mathcal{L}^\theta - \mathscr{L}\right)\mathbf{p}\right)_j = 0,$$

such that one can rely on estimate (4.37a) of Theorem 4.3 with $(\gamma_1, \gamma_2) = \left(\nu_1, \nu_1 + \nu_2 + \frac{1}{8}\right)$.

The second lemma establishes algebraically weighted bounds for the nonlinear operator $\mathscr{N}^\theta$, and a proof is given in [7, Lemma 3.1].

Lemma 5.2. *Let* $\left(\mathscr{N}^\theta\right)_{\theta\in(-\underline{\theta},\underline{\theta})}$ *be the family of nonlinear operators defined by* (5.3)*. Let* $\gamma_1 \geq 0$ *and* $\rho > 0$ *be given. There exists a constant* $C_{\mathscr{N}}(\gamma_1,\rho) > 0$ *such that the following holds.*

(i) If $\mathbf{p} \in \ell^1_{\gamma_1}(\mathbb{Z};\mathbb{R})$ *with* $\|\mathbf{p}\|_{\ell^\infty} \leq \rho$*, then for all* $\theta \in (-\underline{\theta},\underline{\theta})$*, one has* $\mathscr{N}^\theta(\mathbf{p}) \in \ell^1_{2\gamma_1}(\mathbb{Z};\mathbb{R})$ *and*

$$\forall\,\theta \in (-\underline{\theta},\underline{\theta})\,, \quad \left\|\mathscr{N}^\theta(\mathbf{p})\right\|_{\ell^1_{2\gamma_1}} \leq C_{\mathscr{N}}(\gamma_1,\rho)\,\|\mathbf{p}\|_{\ell^1_{\gamma_1}}\,\|\mathbf{p}\|_{\ell^\infty_{\gamma_1}}\,.$$

(ii) If $\mathbf{p} \in \ell^\infty_{\gamma_1}(\mathbb{Z};\mathbb{R})$ *with* $\|\mathbf{p}\|_{\ell^\infty} \leq \rho$*, then for all* $\theta \in (-\underline{\theta},\underline{\theta})$ *one has* $\mathscr{N}^\theta(\mathbf{p}) \in \ell^\infty_{2\gamma_1}(\mathbb{Z};\mathbb{R})$ *and*

$$\forall\,\theta \in (-\underline{\theta},\underline{\theta})\,, \quad \left\|\mathscr{N}^\theta(\mathbf{p})\right\|_{\ell^\infty_{2\gamma_1}} \leq C_{\mathscr{N}}(\gamma_1,\rho)\,\|\mathbf{p}\|^2_{\ell^\infty_{\gamma_1}}\,.$$

Finally, we shall also need the following estimates whose proof can be found in [7, Lemma 3.2]. These are discrete versions of [31, Lemma 2.3]. Here and in all this Chapter, we let $\lfloor x\rfloor$ denote the integer part of a real number x.

Lemma 5.3. *Let* a*,* b *and* c *be positive real numbers. Then there exists a constant* $\boldsymbol{C}(a,b,c) > 0$ *such that*

$$\forall\,n \in \mathbb{N}\,, \quad \sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor} \frac{1}{(1+m-n)^a}\,\frac{1}{(1+m)^b} \leq \frac{\boldsymbol{C}(a,b,c)}{(2+n)^c}\,,$$

whenever $0 < a-c$ *if* $b = 1$*, or* $1-b \leq a-c$ *if* $b \in [0,1)$ *or* $0 \leq a-c$ *if* $b > 1$*, together with*

$$\forall\,n \in \mathbb{N}\,, \quad \sum_{m=\lfloor\frac{n+1}{2}\rfloor+1}^{n} \frac{1}{(1+m-n)^a}\,\frac{1}{(1+m)^b} \leq \frac{\boldsymbol{C}(a,b,c)}{(2+n)^c}\,,$$

whenever $0 < b-c$ *if* $a = 1$*, or* $1-a \leq b-c$ *if* $a \in [0,1)$ *or* $0 \leq b-c$ *if* $a > 1$*.*

5.1 Fixing the constants

The first step towards the proof of Theorem 2.5 is to fix the two constants $C_0 > 0$ and $\epsilon > 0$ that appear in its statement. As a consequence, once for all we fix $\sigma+\beta \geq \frac{5}{12}$ together with $0 \leq \sigma < \beta + \frac{1}{8}$, and we set $\gamma := \sigma+\beta+\frac{1}{8}$. We also fix a positive constant[1] $\varrho > 0$.

Next, using Theorem 2.1(iii), we get the existence of a constant $C_m(\gamma) > 0$ such that

$$\forall\,\theta \in (-\underline{\theta},\underline{\theta})\,, \quad \|\mathbf{v}^\theta - \overline{\mathbf{u}}\|_{\ell^1_\gamma} \leq C_m(\gamma)\,|\theta|\,.$$

We shall denote by $C_{\mathscr{L}}(\beta,\gamma) > 0$ the constant from Theorem 4.3 with $(\gamma_1,\gamma_2) = (\beta,\gamma)$. We can thus set

$$C_0 := (1 + C_{\mathscr{L}}(\beta,\gamma))\,(1 + C_m(\gamma))\,. \tag{5.5}$$

For any real number x, we use the standard notation $x_+ := \max(x,0)$. With the above parameter σ, we introduce the number:

$$\underline{\nu}_2 := \frac{7}{12} + \left(\sigma - \frac{7}{12}\right)_+\,. \tag{5.6}$$

[1] In case the flux f of the conservation law or the numerical flux $\mathscr{F}_\lambda$ would be defined on an open set and not on the whole space ($\mathbb{R}$ or $\mathbb{R}^2$), this parameter ϱ would help controlling the ℓ^∞ norm of the numerical solution so that the numerical solution would be defined for all times.

Next, we introduce the following two constants

$$C_1 := 2\, C_0\, C_{\mathcal{L}}(\beta, \underline{\nu}_2)\, \boldsymbol{C}\left(\underline{\nu}_2, \sigma + \frac{11}{24}, \sigma\right) + 2\, C_0^2\, C_{\mathscr{L}}(\beta, 2\beta)\, C_{\mathscr{N}}(\beta, \varrho)\, \boldsymbol{C}\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{24}, \sigma\right),$$

$$\begin{aligned} C_2 := {} & 2\, C_0\, C_{\mathcal{L}}\left(\beta, \sigma + \frac{25}{24}\right) \boldsymbol{C}\left(\sigma + \frac{25}{24}, \sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right) \\ & + C_0^2\, C_{\mathscr{L}}(\beta, 2\beta)\, C_{\mathscr{N}}(\beta, \varrho)\, \boldsymbol{C}\left(\beta + \frac{7}{12}, 2\sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right) \\ & + C_0^2\, C_{\mathscr{L}}(\beta, 2\beta)\, C_{\mathscr{N}}(\beta, \varrho)\, \boldsymbol{C}\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{12}, \sigma + \frac{11}{24}\right), \end{aligned}$$

where the constants $C_{\mathcal{L}}(\beta, \underline{\nu}_2)$ and $C_{\mathcal{L}}\left(\beta, \sigma + \frac{25}{24}\right)$ are given by Lemma 5.1 with $(\nu_1, \nu_2) = (\beta, \underline{\nu}_2)$ and $(\nu_1, \nu_2) = \left(\beta, \sigma + \frac{25}{24}\right)$, $C_{\mathscr{L}}(\beta, 2\beta)$ is given by Theorem 4.3 with $(\gamma_1, \gamma_2) = (\beta, 2\beta)$, $C_{\mathscr{N}}(\beta, \varrho)$ is given by Lemma 5.2 with $(\gamma_1, \rho) = (\beta, \varrho)$, and the four constants $\boldsymbol{C}\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{24}, \sigma\right)$, $\boldsymbol{C}\left(\sigma + \frac{25}{24}, \sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right)$, $\boldsymbol{C}\left(\beta + \frac{7}{12}, 2\sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right)$ and $\boldsymbol{C}\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{12}, \sigma + \frac{11}{24}\right)$ are given by Lemma 5.3 with either one of the triplets:

$$\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{24}, \sigma\right), \left(\sigma + \frac{25}{24}, \sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right), \left(\beta + \frac{7}{12}, 2\sigma + \frac{11}{24}, \sigma + \frac{11}{24}\right), \left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{12}, \sigma + \frac{11}{24}\right).$$

At last, we choose $\epsilon > 0$ small enough such that

$$0 < \epsilon < \min\left(\underline{\theta}, \frac{\varrho}{C_0}, \frac{1 + C_m(\gamma)}{C_1}, \frac{1 + C_m(\gamma)}{C_2}\right). \tag{5.7}$$

5.2 Proof of Theorem 2.5

We now complete the proof of Theorem 2.5. We consider an initial perturbation $\mathbf{h} \in \ell^1_\gamma(\mathbb{Z}; \mathbb{R})$, with γ previously defined, and we assume that $\mathbf{h}$ is small enough so that

$$\|\mathbf{h}\|_{\ell^1_\gamma} < \epsilon\,.$$

We then define the excess mass:

$$\theta := \sum_{j \in \mathbb{Z}} h_j\,.$$

We readily remark that necessarily

$$|\theta| = \left|\sum_{j \in \mathbb{Z}} h_j\right| \leq \|\mathbf{h}\|_{\ell^1} \leq \|\mathbf{h}\|_{\ell^1_\gamma} < \epsilon < \underline{\theta}\,,$$

the last inequality coming from our choice (5.7) for ϵ. We can therefore use below the discrete shock profile $\mathbf{v}^\theta$ associated with the excess mass θ. We define the initial condition $\mathbf{u}^0 := \overline{\mathbf{u}} + \mathbf{h}$ and since the numerical flux $\mathscr{F}_\lambda \in \mathscr{C}^\infty(\mathbb{R}^2; \mathbb{R})$ is globally defined, we directly obtain the existence of $(\mathbf{u}^n)_{n \in \mathbb{N}}$ solution of (2.5)-(2.6) starting from this $\mathbf{u}^0$. There is no need here to control the ℓ^∞ norm of the numerical solution at each time step in order to remain within the set where the numerical flux is defined. However, the ℓ^∞

control will be crucial in order to obtain uniform bounds for the quadratic remainder term $\mathscr{N}^\theta$. Thus, we can introduce the sequence of perturbations $(\mathbf{p}^n)_{n\in\mathbb{N}}$ defined as

$$\mathbf{p}^n := \mathbf{u}^n - \mathbf{v}^\theta \,,$$

which is given by (5.4), that is

$$\forall\, n \in \mathbb{N}\,, \quad \mathbf{p}^n = \mathscr{L}^n\,\mathbf{p}^0 + \sum_{m=0}^{n-1} \mathscr{L}^{n-1-m} \left(\mathcal{L}^\theta - \mathscr{L}\right) \mathbf{p}^m + \sum_{m=0}^{n-1} \mathscr{L}^{n-1-m} \left(\mathrm{Id} - \mathbf{S}\right) \mathscr{N}^\theta(\mathbf{p}^m)\,.$$

We shall now prove by induction that for all $n \in \mathbb{N}$ one has $\mathbf{p}^n \in \ell^1_\beta(\mathbb{Z};\mathbb{R})$ together with the bounds:

$$\forall\, m = 0, \dots, n\,, \quad \|\mathbf{p}^m\|_{\ell^1_\beta} \le \frac{C_0}{(1+m)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,, \quad \|\mathbf{p}^m\|_{\ell^\infty_\beta} \le \frac{C_0}{(1+m)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,, \quad \text{and} \quad \|\mathbf{p}^m\|_{\ell^\infty} \le \varrho\,. \tag{5.8}$$

5.2.1 Initialization step

At $n = 0$, we have by definition that

$$\mathbf{p}^0 = \mathbf{u}^0 - \mathbf{v}^\theta = \overline{\mathbf{u}} - \mathbf{v}^\theta + \mathbf{h}\,.$$

As a consequence, using $0 \le \beta \le \gamma$ and obvious inequalities between norms, we have the estimates:

$$\|\mathbf{p}^0\|_{\ell^\infty} \le \|\mathbf{p}^0\|_{\ell^\infty_\beta} \le \|\mathbf{p}^0\|_{\ell^1_\beta} \le \|\mathbf{p}^0\|_{\ell^1_\gamma} \le \left\|\overline{\mathbf{u}} - \mathbf{v}^\theta\right\|_{\ell^1_\gamma} + \|\mathbf{h}\|_{\ell^1_\gamma} \le (1 + C_m(\gamma))\|\mathbf{h}\|_{\ell^1_\gamma} \le C_0\|\mathbf{h}\|_{\ell^1_\gamma} \le C_0\epsilon \le \varrho\,. \tag{5.9}$$

This means that the induction assumption (5.8) is satisfied for $n = 0$ (the above chain of inequalities encompasses the three estimates of (5.8)). We finally also recall that $\mathbf{p}^0$ has zero mass:

$$\sum_{j\in\mathbb{Z}} p_j^0 = 0\,,$$

and this property will be automatically propagated at later times since we consider a conservative scheme.

5.2.2 Induction step

We assume that (5.8) is satisfied up to some integer $n \in \mathbb{N}$ and we now show that it propagates to the integer $n+1$. Using Duhamel's formula at $n+1$, we have the following expression for $\mathbf{p}^{n+1}$:

$$\mathbf{p}^{n+1} = \mathscr{L}^{n+1}\,\mathbf{p}^0 + \sum_{m=0}^{n} \mathscr{L}^{n-m} \left(\mathcal{L}^\theta - \mathscr{L}\right) \mathbf{p}^m + \sum_{m=0}^{n} \mathscr{L}^{n-m} \left(\mathrm{Id} - \mathbf{S}\right) \mathscr{N}^\theta(\mathbf{p}^m)\,.$$

We shall now bound each term separately.

The ℓ^1_β estimate. Since the initial perturbation $\mathbf{p}^0 \in \ell^1_\gamma$ is of zero mass, we directly have by Theorem 4.3:

$$\left\|\mathscr{L}^{n+1}\,\mathbf{p}^0\right\|_{\ell^1_\beta} \le \frac{C_{\mathscr{L}}(\beta,\gamma)}{(2+n)^\sigma}\,\|\mathbf{p}^0\|_{\ell^1_\gamma} \le \frac{C_{\mathscr{L}}(\beta,\gamma)(1+C_m(\gamma))}{(2+n)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,, \tag{5.10}$$

where we have used the following consequence of (5.9):

$$\|\mathbf{p}^0\|_{\ell^1_\gamma} \;\le\; (1+C_m(\gamma))\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.$$

Regarding the second term, we use Lemma 5.1 with $\nu_1 = \beta$ and $\underline{\nu}_2$ defined in (5.6), the inequality $|\theta| < \epsilon$ and the induction assumption (5.8) to obtain:

$$\begin{aligned}
\left\|\mathscr{L}^{n-m}\left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p}^m\right\|_{\ell^1_\beta} \;&\le\; \frac{C_{\mathcal{L}}(\beta,\underline{\nu}_2)}{(n-m+1)^{\underline{\nu}_2}}\,|\theta|\,\|\mathbf{p}^m\|_{\ell^\infty} \\
&\le\; \frac{C_{\mathcal{L}}(\beta,\underline{\nu}_2)}{(n-m+1)^{\underline{\nu}_2}}\,|\theta|\,\|\mathbf{p}^m\|_{\ell^\infty_\beta} \\
&\le\; \frac{\epsilon\,C_0\,C_{\mathcal{L}}(\beta,\underline{\nu}_2)}{(n-m+1)^{\underline{\nu}_2}\,(1+m)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.
\end{aligned}$$

Finally, using Lemma 5.3 with $a = \underline{\nu}_2$, $b = \sigma + \frac{11}{24}$ and $c = \sigma$ (the reader can easily verify that we are always in a position to apply the two inequalities provided by Lemma 5.3 with our choice (5.6) of $\underline{\nu}_2$), we directly obtain that

$$\begin{aligned}
\left\|\sum_{m=0}^{n}\mathscr{L}^{n-m}\left(\mathcal{L}^\theta - \mathscr{L}\right)\,\mathbf{p}^m\right\|_{\ell^1_\beta} &\le \sum_{m=0}^{n}\frac{\epsilon\,C_0\,C_{\mathcal{L}}(\beta,\underline{\nu}_2)}{(n-m+1)^{\underline{\nu}_2}\,(1+m)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma} \\
&\le \frac{2\,\epsilon\,C_0\,C_{\mathcal{L}}(\beta,\underline{\nu}_2)\,\boldsymbol{C}\left(\underline{\nu}_2,\sigma+\dfrac{11}{24},\sigma\right)}{(2+n)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.
\end{aligned} \tag{5.11}$$

For the third term that incorporates the nonlinear contributions, we shall use the estimate (4.38a) of Theorem 4.3 and Lemma 5.2 to derive that

$$\begin{aligned}
\left\|\sum_{m=0}^{n}\mathscr{L}^{n-m}\,(\mathrm{Id}-\mathbf{S})\,\mathscr{N}^\theta(\mathbf{p}^m)\right\|_{\ell^1_\beta} &\le \sum_{m=0}^{n}\frac{C_{\mathscr{L}}(\beta,2\beta)}{(1+n-m)^{\beta+\frac{1}{8}}}\,\left\|\mathscr{N}^\theta(\mathbf{p}^m)\right\|_{\ell^1_{2\beta}} \\
&\le \sum_{m=0}^{n}\frac{C_{\mathscr{L}}(\beta,2\beta)\,C_{\mathscr{N}}(\beta,\varrho)}{(1+n-m)^{\beta+\frac{1}{8}}}\,\|\mathbf{p}^m\|_{\ell^1_\beta}\,\|\mathbf{p}^m\|_{\ell^\infty_\beta} \\
&\le \sum_{m=0}^{n}\frac{C_0^2\,C_{\mathscr{L}}(\beta,2\beta)\,C_{\mathscr{N}}(\beta,\varrho)}{(1+n-m)^{\beta+\frac{1}{8}}(1+m)^{2\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|^2_{\ell^1_\gamma} \\
&\le \frac{2\,\epsilon\,C_0^2\,C_{\mathscr{L}}(\beta,2\beta)\,C_{\mathscr{N}}(\beta,\varrho)\,\boldsymbol{C}\left(\beta+\frac{1}{8},2\sigma+\frac{11}{24},\sigma\right)}{(2+n)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.
\end{aligned}$$

For the last inequality, we have applied Lemma 5.3 with $a = \beta + \frac{1}{8}$, $b = 2\sigma + \frac{11}{24}$ and $c = \sigma$. We can readily verify that

$$0 < \beta + \frac{1}{8} - \sigma = a - c\,,\quad 0 < \sigma + \frac{11}{24} = b - c\,,\quad \text{and}\quad b - c + a - 1 = \beta + \sigma + \frac{1}{8} + \frac{11}{24} - 1 = \sigma + \beta - \frac{5}{12} \ge 0\,,$$

thanks to our assumptions on $\beta+\sigma\geq\frac{5}{12}$ and $0\leq\sigma<\beta+\frac{1}{8}$.

As a consequence (recalling the definition of the constant C_1), we have obtained that

$$\|\mathbf{p}^{n+1}\|_{\ell^1_\beta}\leq\frac{1}{(2+n)^\sigma}\Big(C_{\mathscr{L}}(\beta,\gamma)\,(1+C_m(\gamma))+\epsilon\,C_1\Big)\,\|\mathbf{h}\|_{\ell^1_\gamma}\leq\frac{C_0}{(2+n)^\sigma}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,,$$

thanks to our choice of C_0 and the restrictions on ϵ.

The ℓ^∞_β estimate. Using once more that the initial perturbation $\mathbf{p}^0\in\ell^1_\gamma$ is of zero mass, we directly find, using the semi-group estimate of $(\mathscr{L}^n)_{n\in\mathbb{N}}$ in ℓ^∞_β that

$$\left\|\mathscr{L}^{n+1}\,\mathbf{p}^0\right\|_{\ell^\infty_\beta}\leq\frac{C_{\mathscr{L}}(\beta,\gamma)}{(2+n)^{\sigma+\frac{11}{24}}}\,\|\mathbf{p}^0\|_{\ell^1_\gamma}\leq\frac{C_{\mathscr{L}}(\beta,\gamma)\,(1+C_m(\gamma))}{(2+n)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,,$$

where we have used one more time the following consequence of (5.9):

$$\|\mathbf{p}^0\|_{\ell^1_\gamma}\leq(1+C_m(\gamma))\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.$$

Regarding the second term, we use Lemma 5.1 with $\nu_1=\beta$ and $\nu_2=\sigma+\frac{25}{24}$ to obtain

$$\begin{aligned}\left\|\mathscr{L}^{n-m}\left(\mathcal{L}^\theta-\mathscr{L}\right)\mathbf{p}^m\right\|_{\ell^\infty_\beta}\leq\left\|\mathscr{L}^{n-m}\left(\mathcal{L}^\theta-\mathscr{L}\right)\mathbf{p}^m\right\|_{\ell^1_\beta}&\leq\frac{C_{\mathcal{L}}\left(\beta,\sigma+\frac{25}{24}\right)}{(n-m+1)^{\sigma+\frac{25}{24}}}\,|\theta|\,\|\mathbf{p}^m\|_{\ell^\infty}\\&\leq\frac{\epsilon\,C_0\,C_{\mathcal{L}}\left(\beta,\sigma+\frac{25}{24}\right)}{(n-m+1)^{\sigma+\frac{25}{24}}\,(1+m)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.\end{aligned}$$

Next, using Lemma 5.3 with $a=\sigma+\frac{25}{24}>1$ and $b=c=\sigma+\frac{11}{24}$, noticing that $c<a$, we directly obtain that

$$\begin{aligned}\left\|\sum_{m=0}^{n}\mathscr{L}^{n-m}\left(\mathcal{L}^\theta-\mathscr{L}\right)\mathbf{p}^m\right\|_{\ell^\infty_\beta}&\leq\sum_{m=0}^{n}\frac{\epsilon\,C_0\,C_{\mathcal{L}}\left(\beta,\sigma+\frac{25}{24}\right)}{(n-m+1)^{\sigma+\frac{25}{24}}(1+m)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\\&\leq\frac{2\,\epsilon\,C_0\,C_{\mathcal{L}}\left(\beta,\sigma+\frac{25}{24}\right)\,\boldsymbol{C}\left(\sigma+\frac{25}{24},\sigma+\frac{11}{24},\sigma+\frac{11}{24}\right)}{(2+n)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.\end{aligned}$$

For the third term, we shall use the estimate (4.38b) of Theorem 4.3 and Lemma 5.2 to derive that

$$\begin{aligned}\left\|\sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}\mathscr{L}^{n-m}\,(\mathrm{Id}-\mathbf{S})\,\mathscr{N}^\theta(\mathbf{p}^m)\right\|_{\ell^\infty_\beta}&\leq\sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}\frac{C_{\mathscr{L}}(\beta,2\beta)}{(1+n-m)^{\beta+\frac{1}{3}+\min\left(\frac{1}{4},\beta\right)}}\left\|\mathscr{N}^\theta(\mathbf{p}^m)\right\|_{\ell^1_{2\beta}}\\&\leq\sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}\frac{C_{\mathscr{L}}(\beta,2\beta)C_{\mathscr{N}}(\beta,\varrho)}{(1+n-m)^{\beta+\frac{7}{12}}}\,\|\mathbf{p}^m\|_{\ell^1_\beta}\,\|\mathbf{p}^m\|_{\ell^\infty_\beta}\\&\leq\sum_{m=0}^{\lfloor\frac{n+1}{2}\rfloor}\frac{C_0^2\,C_{\mathscr{L}}(\beta,2\beta)\,C_{\mathscr{N}}(\beta,\varrho)}{(1+n-m)^{\beta+\frac{7}{12}}(1+m)^{2\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|^2_{\ell^1_\gamma}\\&\leq\frac{\epsilon\,C_0^2\,C_{\mathscr{L}}(\beta,2\beta)\,C_{\mathscr{N}}(\beta,\varrho)\,\boldsymbol{C}\left(\beta+\frac{7}{12},2\sigma+\frac{11}{24},\sigma+\frac{11}{24}\right)}{(2+n)^{\sigma+\frac{11}{24}}}\,\|\mathbf{h}\|_{\ell^1_\gamma}\,.\end{aligned}$$

For the last inequality, we have used the first inequality of Lemma 5.3 with $a = \beta + \frac{7}{12}$, $b = 2\sigma + \frac{11}{24}$ and $\sigma + \frac{11}{24}$, noticing that

$$a - c = \beta + \frac{1}{8} - \sigma > 0, \quad \text{and} \quad a - c + b - 1 = \sigma + \beta - \frac{5}{12} \geq 0,$$

since $0 \leq \sigma < \beta + \frac{1}{8}$ and $\beta + \sigma \geq \frac{5}{12}$.

For the remaining part of the sum, we use estimate (4.38c) of Theorem 4.3 and Lemma 5.2 to obtain

$$\begin{aligned}
\left\| \sum_{m=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \mathscr{L}^{n-m} \left(\mathrm{Id} - \mathbf{S}\right) \mathscr{N}^{\theta}(\mathbf{p}^m) \right\|_{\ell^\infty_\beta} &\leq \sum_{m=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \frac{C_{\mathscr{L}}(\beta, 2\beta)}{(1+n-m)^{\beta + \min\left(\frac{1}{8}, \beta - \frac{1}{8}\right)}} \left\| \mathscr{N}^{\theta}(\mathbf{p}^m) \right\|_{\ell^\infty_{2\beta}} \\
&\leq \sum_{m=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \frac{C_{\mathscr{L}}(\beta, 2\beta) C_{\mathscr{N}}(\beta, \varrho)}{(1+n-m)^{\beta + \frac{1}{8}}} \left\| \mathbf{p}^m \right\|^2_{\ell^\infty_\beta} \\
&\leq \sum_{m=\lfloor \frac{n+1}{2} \rfloor + 1}^{n} \frac{C_{\mathscr{L}}(\beta, 2\beta) C_{\mathscr{N}}(\beta, \varrho) C_0^2}{(1+n-m)^{\beta + \frac{1}{8}} (1+m)^{2\sigma + \frac{11}{12}}} \left\| \mathbf{h} \right\|^2_{\ell^1_\gamma} \\
&\leq \frac{\epsilon\, C_0^2\, C_{\mathscr{L}}(\beta, 2\beta)\, C_{\mathscr{N}}(\beta, \varrho)\, \boldsymbol{C}\left(\beta + \frac{1}{8}, 2\sigma + \frac{11}{12}, \sigma + \frac{11}{24}\right)}{(2+n)^{\sigma + \frac{11}{24}}} \left\| \mathbf{h} \right\|_{\ell^1_\gamma}.
\end{aligned}$$

For the last inequality, we have used the second inequality of Lemma 5.3 with $a = \beta + \frac{1}{8}$, $b = 2\sigma + \frac{11}{12}$ and $\sigma + \frac{11}{24}$, noticing that

$$b - c > 0, \quad \text{and} \quad a - c + b - 1 = \sigma + \beta - \frac{5}{12} \geq 0.$$

As a consequence (recalling our definition for the constant C_2), we have obtained that

$$\left\| \mathbf{p}^{n+1} \right\|_{\ell^\infty_\beta} \leq \frac{1}{(2+n)^{\sigma + \frac{11}{24}}} \left(C_{\mathscr{L}}(\beta, \gamma) \left(1 + C_m(\gamma)\right) + \epsilon\, C_2 \right) \left\| \mathbf{h} \right\|_{\ell^1_\gamma} \leq \frac{C_0}{(2+n)^{\sigma + \frac{11}{24}}} \left\| \mathbf{h} \right\|_{\ell^1_\gamma}.$$

From there, we also deduce that

$$\left\| \mathbf{p}^{n+1} \right\|_{\ell^\infty} \leq \left\| \mathbf{p}^{n+1} \right\|_{\ell^\infty_\beta} \leq C_0\, \epsilon \leq \varrho\,.$$

This concludes the proof of Theorem 2.5.

Appendix A

The exact and approximate Green's functions for the Cauchy problem

In this appendix, we study the Lax-Wendroff scheme when applied to the linear transport equation:

$$\partial_t v \,+\, a\,\partial_x v \,=\, 0\,,$$

with $a \neq 0$ and the equation is considered on the whole real line $\mathbb{R}$. As we have already seen earlier in this article, in the linear case, the Lax-Wendroff scheme can be recast under the form:

$$\forall\, n \in \mathbb{N}\,, \quad v^{n+1} \,=\, \overline{\mathscr{L}}\, v^n\,, \tag{A.1}$$

where for any integer $n \in \mathbb{N}$, v^n denotes the sequence $(v_j^n)_{j\in\mathbb{Z}}$, and $\overline{\mathscr{L}}$ is the discrete convolution operator defined on any real or complex valued sequence $v = (v_j)_{j\in\mathbb{Z}}$ by:

$$\forall\, j \in \mathbb{Z}\,, \quad (\overline{\mathscr{L}}\, v)_j \,:=\, v_j - \frac{\alpha}{2}\,(v_{j+1} - v_{j-1}) + \frac{\alpha^2}{2}\,(v_{j+1} - 2v_j + v_{j-1})\;,$$

where, as in the core of this article, α is a short notation for $\lambda\, a$, $\lambda > 0$ denoting the fixed ratio $\Delta t/\Delta x$ between the time and space steps. The constant α thus has the sign of the transport velocity a. In what follows, we shall apply the results below to either $\alpha = \alpha_\ell \in (0,1)$ or $\alpha = \alpha_r \in (-1,0)$, so that the above operator $\overline{\mathscr{L}}$ corresponds to either one of the operators $\mathscr{L}_\ell$ or $\mathscr{L}_r$ defined in (3.25).

In this appendix, we recall or prove several bounds on the *exact* and *approximate* Green's functions for the Lax-Wendroff scheme. The approximate Green's function is defined below in (A.8) and is meant to reproduce the leading qualitative and quantitative features of the exact Green's function of (A.1). The study of the Green's function of (A.1) was the purpose of the article [9] by one of the authors. We shall recall below the main conclusions of [9] since they are useful for our purpose here. Unsurprisingly, many arguments below for studying the approximate Green's function follow what has been already done in [9] but there are also several new regimes that need to be considered and that did not appear in [9]. The bounds that we prove below (see in particular Theorem A.3) are used in this article to study the so-called activation function $\mathbf{A}$ in our decomposition of the Green's function for the linearized operator $\mathscr{L}$ in (2.11) (see Theorem 4.1) and various bounds for some remainder terms. This appendix can also be seen as a main building block for a complete justification of the local limit theorem for finite difference approximations of the transport equation that exhibit *dispersion* and *dissipation* (we refer to [23] and

[24, 25] for a presentation and some recent advances on the local limit theorem and its connection to probability theory). The local limit theorem in the non-dispersive (or rather parabolic) case is justified in [11] and the complete justification of the local limit theorem in the dispersive case is a work in progress. We refer to [24] for a justification of the leading order term in the local limit theorem for sequences that exhibit dispersion and dissipation as we consider here.

A.1 The exact and approximate Green's functions. Main results

As we have recalled above, the Lax-Wendroff scheme for the transport equation on the whole real line reads:

$$v_j^{n+1} = v_j^n - \frac{\alpha}{2}\,(v_{j+1}^n - v_{j+1}^n) + \frac{\alpha^2}{2}\,(v_{j+1}^n - 2\,v_j^n + v_{j+1}^n)\,, \tag{A.2}$$

with $\alpha := \lambda\, a$. The *exact* Green's function for (A.2) corresponds to the initial condition defined by:

$$\forall\, j \in \mathbb{Z}\,, \quad \overline{\mathscr{G}}_j^0 := \begin{cases} 1\,, & \text{if } j = 0, \\ 0\,, & \text{otherwise,} \end{cases}$$

which leads to the solution $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ for (A.2). This sequence $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ is studied in [9], see also [16, 17]. Its main features are recalled below.

We restrict from now on to the case $\alpha \in (-1,1)\setminus\{0\}$ in order to stick to the two particular situations we are interested in, that is, either $\alpha = \alpha_\ell \in (0,1)$ or $\alpha = \alpha_r \in (-1,0)$. We compute the so-called amplification factor for (A.2) and obtain (see for instance [9]):

$$\forall\, \theta \in \mathbb{R}\,, \quad \widehat{F}_{\text{LW}}(\theta) = 1 - 2\,\alpha^2 \sin^2\frac{\theta}{2} + \mathbf{i}\,\alpha\,\sin\theta\,.$$

As already reported in [9], the expansion of the amplification factor $\widehat{F}_{\text{LW}}$ near the frequency 0 reads:

$$\widehat{F}_{\text{LW}}(\theta) = \exp\left(\mathbf{i}\,\alpha\,\theta - \mathbf{i}\,\frac{\alpha\,(1-\alpha^2)}{6}\,\theta^3 - \frac{\alpha^2\,(1-\alpha^2)}{8}\,\theta^4 + \mathcal{O}(\theta^5)\right)\,, \tag{A.3}$$

and we have furthermore the dissipation property:

$$\forall\, \theta \in [-\pi,\pi]\setminus\{0\}\,, \quad |\widehat{F}_{\text{LW}}(\theta)| < 1\,.$$

Since $\widehat{F}_{\text{LW}}$ is a trigonometric polynomial, it extends as a holomorphic function with respect to θ on the whole complex plane $\mathbb{C}$. For later use, we introduce the coefficients:

$$c_3 := \frac{\alpha\,(1-\alpha^2)}{6} \neq 0\,, \quad c_4 := \frac{\alpha^2\,(1-\alpha^2)}{8} > 0\,, \tag{A.4}$$

in such a way that (A.3) equivalently reads:

$$\widehat{F}_{\text{LW}}(\theta) = \exp\Big(\mathbf{i}\,\alpha\,\theta - \mathbf{i}\,c_3\,\theta^3 - c_4\,\theta^4 + \mathcal{O}(\theta^5)\Big)\,,$$

as θ tends to 0. The main result proved in [9] can be formulated as follows. It uses the notation c_3 and c_4 for the coefficients in (A.4) that arise in the Taylor's expansion of the amplification factor $\widehat{F}_{\text{LW}}$ at 0.

Theorem A.1. *Assume that the constant* c_3 *in* (A.4) *is positive, that is* $\alpha \in (0,1)$*. Then there exist two constants* $C > 0$ *and* $c > 0$ *such that the Green's function* $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ *for* (A.2) *satisfies the uniform bounds:*

$$\forall\, n \in \mathbb{N}^*\,, \quad \forall\, j \in \mathbb{Z}\,, \quad \big|\,\overline{\mathscr{G}}_j^n\,\big| \,\le\, \frac{C}{n^{1/3}}\,\exp\left(-\,c\,\left(\frac{j\,-\,\alpha\,n}{n^{1/3}}\right)^{3/2}\right)\,, \quad \text{if } j\,-\,\alpha\,n\,\ge\,0\,, \tag{A.5}$$

and:

$$\forall\, n \in \mathbb{N}^*\,, \quad \forall\, j \in \mathbb{Z}\,,$$
$$\big|\,\overline{\mathscr{G}}_j^n\,-\,2\,\mathrm{Re}\,\mathfrak{g}_j^n\,\big| \,\le\, \frac{C}{n^{1/3}}\,\exp\left(-\,c\,\left(\frac{|j\,-\,\alpha\,n|}{n^{1/3}}\right)^{3/2}\right)\,+\,\frac{C}{n^{1/2}}\,\exp\left(-\,c\,\frac{(j\,-\,\alpha\,n)^2}{n}\right)\,,$$
$$\text{if } j\,-\,\alpha\,n\,<\,0\,, \tag{A.6}$$

where $\mathfrak{g}_j^n$ *is defined for* $n \in \mathbb{N}^*$ *and* $j \in \mathbb{Z}$ *as:*

$$\forall\,(n,j) \in \mathbb{N}^* \times \mathbb{Z}\,, \quad \mathfrak{g}_j^n \,:=\, \frac{1}{2\,\pi}\,\exp\left(-\,\frac{c_4\,(j-\alpha\,n)^2}{9\,c_3^2\,n}\right)\,\exp\left(\mathbf{i}\,\frac{2\,|j-\alpha\,n|^{3/2}}{3\,\sqrt{3\,|c_3|\,n}}\,-\,\mathbf{i}\,\frac{\pi}{4}\right)$$
$$\times\int_{-\sqrt{\frac{2\,|j-\alpha\,n|}{3\,|c_3|\,n}}}^{\sqrt{\frac{2\,|j-\alpha\,n|}{3\,|c_3|\,n}}} \mathrm{e}^{-\,\sqrt{3\,|c_3|\,|j-\alpha\,n|\,n}\,u^2}\,\mathrm{e}^{c_3\,n\,\mathrm{e}^{-\mathbf{i}\pi/4}\,u^3}\,\mathrm{d}u\,. \tag{A.7}$$

If c_3 *is negative, the bounds depending on the sign of* $j\,-\,\alpha\,n$ *should be switched.*

In order to be absolutely complete, it is important to note that Theorem A.1 above is not exactly the statement that is proved in [9]. This is because, in the interval between the completion of [9] and the present work, an error was found in [9], which made us modify the statement and the proof of the main result in [9]. The new statement, that is correct, and whose corollaries follow exactly as in [9], is Theorem A.1 above. The error occurred in the proof of the estimate (A.6) and in the definition (A.7) of the approximation $\mathfrak{g}_j^n$ that incorporates the damped oscillations of the Green's function. Rather than reproducing the whole proof of Theorem A.1 (only a tiny part of the proof needs to be corrected), we shall rather give the proof of Theorem A.3 below for the *approximate* Green's function since the analysis of the approximate Green's function needs to incorporate new regimes that were not considered in [9]. The proof of the above estimate (A.6) corresponds to the bound (A.12) below for the approximate Green's function. The reader will most certainly experiment no difficulty to adapt the arguments below to derive the statement in (A.6), (A.7).

Let us recall an immediate consequence of Theorem A.1, see [9] for an even more precise statement.

Corollary A.1. *Let the constant* c_3 *in* (A.4) *be positive, that is,* $\alpha \in (0,1)$*. Then there exist two constants* $C > 0$ *and* $c > 0$ *such that the Green's function* $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ *for* (A.2) *satisfies:*

$$\forall\, n \in \mathbb{N}^*\,, \quad \big|\overline{\mathscr{G}}_j^n\big| \,\le\, C\,\begin{cases} \dfrac{1}{n^{1/3}}\,\exp\big(-\,c\,|j-\alpha\,n|^{3/2}/n^{1/2}\big)\,, & \text{if } j-\alpha\,n \ge 0, \\[2ex] \dfrac{1}{n^{1/3}}\,, & \text{if } -n^{1/3} \le j-\alpha\,n \le 0, \\[2ex] \dfrac{1}{|j-\alpha\,n|^{1/4}\,n^{1/4}}\,\exp\big(-\,c\,|j-\alpha\,n|^2/n\big)\,, & \text{if } j-\alpha\,n \le -n^{1/3}, \end{cases}$$

together with the ℓ^1 estimate:

$$\forall\, n \in \mathbb{N}\,, \quad \sum_{j \in \mathbb{Z}} \big| \overline{\mathscr{G}}_j^n \big| \,\le\, C\,(1+n)^{1/8}\,.$$

If c_3 is negative, that is, $\alpha \in (-1,0)$, the pointwise estimates for the Green's function $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ read:

$$\forall\, n \in \mathbb{N}^*\,, \quad \big| \overline{\mathscr{G}}_j^n \big| \,\le\, C \begin{cases} \dfrac{1}{n^{1/3}}\, \exp\big(-c\,|j-\alpha\,n|^{3/2}/n^{1/2}\big)\,, & \text{if } j-\alpha\,n \le 0, \\ \dfrac{1}{n^{1/3}}\,, & \text{if } 0 \le j-\alpha\,n \le n^{1/3}, \\ \dfrac{1}{|j-\alpha\,n|^{1/4}\,n^{1/4}}\, \exp\big(-c\,|j-\alpha\,n|^2/n\big)\,, & \text{if } j-\alpha\,n \ge n^{1/3}, \end{cases}$$

and the ℓ^1 estimate is unchanged.

The analysis in Chapter 4 will use a very slight variation on Theorem A.1 which we state here for convenience. Actually, the statement in Theorem A.2 below is the core of the proof of Theorem A.1 in [9] even though this result was not stated in such generality in [9]. Once again, the proof of Theorem A.2 below follows from the exact same arguments as those we develop below in the proofs of Theorem A.3 and Theorem A.4. The absorption of the remainder term $\theta^5\,\Psi(\theta)$ in the integral is made by choosing δ small enough, just like we did in the proof of Proposition 4.1 in Chapter 4. We therefore feel free to skip the proof of Theorem A.2.

Theorem A.2. *Let $\tilde{c}_3$ and $\tilde{c}_4$ be two positive real numbers. Let Ψ denote a holomorphic function on some neighborhood of 0 in $\mathbb{C}$. Let $\underline{\mathbf{C}}$ be a positive real number. Then there exists some positive real number $\delta_0 > 0$ such that the following property holds: for any $\delta \in (0,\delta_0]$, there exist two constants $C>0$ and $c>0$ such that there holds:*

$$\forall\,(x,y) \in \mathbb{R} \times [\underline{\mathbf{C}}^{-1},+\infty)\,, \quad \left| \int_{-\delta}^{\delta} \mathrm{e}^{\mathbf{i}\,x\,\theta + \mathbf{i}\,\tilde{c}_3\,y\,\theta^3 - \tilde{c}_4\,y\,\theta^4 + y\,\theta^5\,\Psi(\theta)}\, \mathrm{d}\theta \right|$$

$$\le\, C \begin{cases} \dfrac{1}{y^{1/3}}\, \exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } 0 \le x \le \underline{\mathbf{C}}\,y, \\ \dfrac{1}{y^{1/3}}\,, & \text{if } -y^{1/3} \le x \le 0, \\ \dfrac{1}{|x|^{1/4}\,y^{1/4}}\, \exp\big(-c\,|x|^2/y\big)\,, & \text{if } -\underline{\mathbf{C}}\,y \le x \le -y^{1/3}. \end{cases}$$

If now $\tilde{c}_3$ is negative ($\tilde{c}_4$ being kept positive), the result still holds but with estimates that now read:

$$\forall\,(x,y)\in\mathbb{R}\times[\underline{\mathbf{C}}^{-1},+\infty)\,,\quad \left|\int_{-\delta}^{\delta}\mathrm{e}^{\mathbf{i}\,x\,\theta+\mathbf{i}\,\tilde{c}_3\,y\,\theta^3-\tilde{c}_4\,y\,\theta^4+y\,\theta^5\,\Psi(\theta)}\,\mathrm{d}\theta\right|$$
$$\leq C\begin{cases}\dfrac{1}{y^{1/3}}\,\exp\big(-c\,|x|^{3/2}/y^{1/2}\big)\,, & \text{if } -\underline{\mathbf{C}}\,y\leq x\leq 0,\\[2ex] \dfrac{1}{y^{1/3}}\,, & \text{if } 0\leq x\leq y^{1/3},\\[2ex] \dfrac{1}{|x|^{1/4}\,y^{1/4}}\,\exp\big(-c\,|x|^2/y\big)\,, & \text{if } y^{1/3}\leq x\leq\underline{\mathbf{C}}\,y.\end{cases}$$

We now turn to the approximate Green's function for (A.2). Recalling that we have the following formula for the exact Green's function:

$$\forall\,(n,j)\in\mathbb{N}\times\mathbb{Z}\,,\quad \overline{\mathscr{G}}_j^n\;=\;\frac{1}{2\,\pi}\int_{-\pi}^{\pi}\mathrm{e}^{-\mathbf{i}\,j\,\theta}\,\widehat{F}_{\mathrm{LW}}(\theta)^n\,\mathrm{d}\theta\;=\;\frac{1}{2\,\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\mathbf{i}\,j\,\theta}\,\widehat{F}_{\mathrm{LW}}(-\theta)^n\,\mathrm{d}\theta\,,$$

the expansion (A.3) and the dissipation property suggests the introduction of the approximate Green's function $\mathbb{G}_j^n$ defined by:

$$\forall\,(n,j)\in\mathbb{N}^*\times\mathbb{Z}\,,\quad \mathbb{G}_j^n\;:=\;\frac{1}{2\,\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathbf{i}\,(j-\alpha\,n)\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,n\,\theta^3}\,\mathrm{e}^{-c_4\,n\,\theta^4}\,\mathrm{d}\theta\,. \tag{A.8}$$

The relevance of $\mathbb{G}_j^n$ for analyzing the behavior of the exact Green's function for (A.2) is illustrated by numerous simulations in [3]. A rigorous justification (by means of sharp analytical bounds) of these numerical observations is, to some extent, the purpose of the so-called local limit theorem and is the content of a future work by the authors. We show below that the bounds for $\mathbb{G}_j^n$ are "consistent" with those given in Theorem A.1 for the exact Green's function $\overline{\mathscr{G}}_j^n$. Let us quickly observe that the above definition (A.8) only makes sense for $n\in\mathbb{N}^*$ so that the integral is absolutely convergent. We have replaced the compact integration interval $[-\pi,\pi]$ by the whole real line $\mathbb{R}$ for convenience since high frequencies will not modify much the behavior of $\mathbb{G}_j^n$.

For technical reasons that will be made more clear below, it is useful to "extend" the approximate Green's function to a continuous setting and we therefore introduce the function of two variables $\mathbf{G}$ that is defined by:

$$\forall\,(x,y)\in\mathbb{R}\times\mathbb{R}^{+*}\,,\quad \mathbf{G}(x,y)\;:=\;\frac{1}{2\,\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,. \tag{A.9}$$

From the definition (A.8), we directly get the relation $\mathbb{G}_j^n=\mathbf{G}(j-\alpha\,n,n)$ so that any detailed information or bounds on $\mathbf{G}$ will give us information or bounds on $\mathbb{G}_j^n$.

We thus consider from now on the function $\mathbf{G}$ defined in (A.9). Our main result, that is Theorem A.3 below, makes use of an auxiliary function that, to some extent, captures the oscillating behavior of the function $\mathbf{G}$ on the side where x has the opposite sign of c_3. This auxiliary function $\mathfrak{g}$ is defined as follows:

$$\forall\,(x,y)\in\mathbb{R}\times\mathbb{R}^{+*}\,,\quad \mathfrak{g}(x,y)\;:=\;\frac{1}{2\,\pi}\,\exp\left(-\frac{c_4\,x^2}{9\,c_3^2\,y}\right)\,\exp\left(\mathbf{i}\,\frac{2\,|x|^{3/2}}{3\,\sqrt{3\,|c_3|\,y}}\,-\,\mathbf{i}\,\frac{\pi}{4}\right)$$
$$\times\int_{-\sqrt{\frac{2\,|x|}{3\,|c_3|\,y}}}^{\sqrt{\frac{2\,|x|}{3\,|c_3|\,y}}}\mathrm{e}^{-\,\sqrt{3\,|c_3|\,|x|\,y}\,t^2}\,\mathrm{e}^{c_3\,y\,\mathrm{e}^{-\mathbf{i}\pi/4}\,t^3}\,\mathrm{d}t\,.\qquad\text{(A.10)}$$

Comparing with (A.7), we see that (A.7) corresponds to $x=j-\alpha\,n$ and $y=n$ in (A.10). This explains why the proof of the above estimate (A.6) is entirely similar to the proof of (A.12) below. Our main result is then the following.

Theorem A.3. *Let us assume that the coefficient c_3 in (A.4) is positive, that is $\alpha\in(0,1)$. Let $y_{\min}>0$ and let $\underline{\mathbf{c}}>0$ be given. Then there exist some constants*[1] *$C>0$ and $c>0$ such that, for any $(x,y)\in\mathbb{R}\times[y_{\min},+\infty)$, there holds:*

$$|\mathbf{G}(x,y)|\;\le\;\begin{cases}\dfrac{C}{y^{1/4}}\,\exp(-c\,x^{4/3}/y^{1/3})\,, & \text{if } x\ge\underline{\mathbf{c}}\,y,\\ \dfrac{C}{y^{1/3}}\,\exp(-c\,x^{3/2}/y^{1/2})\,, & \text{if } 0\le x\le\underline{\mathbf{c}}\,y,\\ \dfrac{C}{y^{1/3}}\,, & \text{if } -y^{1/3}\le x\le 0,\\ \dfrac{C}{y^{1/4}}\,\exp(-c\,|x|^{4/3}/y^{1/3})\,, & \text{if } x\le-\underline{\mathbf{c}}\,y,\end{cases}\qquad\text{(A.11)}$$

and:

$$|\mathbf{G}(x,y)\,-\,2\,\mathrm{Re}\;\mathfrak{g}(x,y)|\;\le\;\frac{C}{y^{1/3}}\,\exp\left(-c\,\frac{|x|^{3/2}}{y^{1/2}}\right)\,+\,\frac{C}{y^{1/2}}\,\exp\left(-c\,\frac{x^2}{y}\right)\,,\qquad\text{(A.12)}$$

if $-\underline{\mathbf{c}}\,y\le x\le -y^{1/3}$.

In particular, there exist another constant $\widetilde{C}\ge C$ and another constant $\tilde{c}\in(0,c]$ such that for any $(x,y)\in\mathbb{R}\times[y_{\min},+\infty)$, there holds:

$$\left|\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;\begin{cases}\widetilde{C}\,\exp(-\tilde{c}\,|x|^{4/3}/y^{1/3})\,, & \text{if } x\le-\underline{\mathbf{c}}\,y,\\ \widetilde{C}\,, & \text{if } -\underline{\mathbf{c}}\,y\le x\le\underline{\mathbf{c}}\,y.\end{cases}\qquad\text{(A.13)}$$

and:

$$\left|1-\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;\widetilde{C}\,\exp(-\tilde{c}\,x^{4/3}/y^{1/3})\,,\quad\text{if } x\ge\underline{\mathbf{c}}\,y.\qquad\text{(A.14)}$$

Let us note that the case where c_3 is negative (that is, where α is negative) simply corresponds to changing the sign of x since the definition (A.9) gives:

$$\mathbf{G}(-x,y)\;:=\;\frac{1}{2\,\pi}\int_{\mathbb{R}}\mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,(-c_3)\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

We shall thus feel free to use without proof the analog of Theorem A.3 in the case where α is negative. (This is precisely what we shall do in Chapter 4 where one of the relevant values for α is $\alpha_r\in(-1,0)$.)

[1]These constants depend only on $y_{\min}$, $\underline{\mathbf{c}}$, c_3 and c_4, or equivalently on $y_{\min}$, $\underline{\mathbf{c}}$ and α.

As follows from [29], see also [16, 17], it is already known that the Green's function $(\overline{\mathscr{G}}_j^n)_{(j,n)\in\mathbb{Z}\times\mathbb{N}}$ of the Lax-Wendroff scheme (A.2) is not uniformly bounded in $\ell^1(\mathbb{Z})$ (where "uniformly" refers to uniformity with respect to $n \in \mathbb{N}$). This is a characteristic feature of numerical schemes that exhibit *dispersion*. Since the function $\mathbf{G}$ is meant to reproduce the leading behavior of the Green's function, there is no hope to show that $\mathbf{G}(\cdot, y)$ belongs to L^1_x uniformly with respect to $y > 0$ (or, say, $y \geq y_{\min} > 0$ in order to get rid of potential trouble when y becomes arbitrarily small). However, a crucial point in our analysis of the linearized operator $\mathscr{L}$ around our reference discrete shock profile is to obtain a uniform bound for the activation function $\mathbf{A}$. As we shall see later on, this activation function corresponds to the primitive function of $\mathbf{G}$ with respect to its first variable. Deriving a bound for this primitive function *can not* follow from the triangle inequality. We thus need to capture the leading oscillating behavior of $\mathbf{G}$ in order to show that its primitive (with respect to its first variable) is uniformly bounded with respect to *both* variables. This is the meaning of the estimates (A.13) and (A.14) and this is where the estimate (A.12) is crucial.

The proof of Theorem A.3 is split in several paragraphs in order to emphasize the distinctions between the fast decaying side of the Green's function and its oscillating side (as for the Airy function). Most of what follows is an adaptation of the results in [9] even though several regimes did not appear in [9] because the exact Green's function was considered there and this one has compact support for any n. Furthermore, it appears that an error occurred in [9] when introducing the analog of the above function $\mathfrak{g}$ and in proving and error bound between the Green's function and its (presumably) leading behavior. This appendix is therefore the opportunity to correct this error and to generalize some of the results in [9] to a continuous setting. In the final paragraph of this appendix, we connect the result of Theorem A.3 with the analysis of the activation function $\mathbf{A}_r$ (or $\mathbf{A}_\ell$), see Corollary A.6, and we explain why the bounds in Theorem A.3 give exactly all that is needed in the analysis of Sections 4.3 and 4.5.

A.2 The uniform bound

It is sometimes useful to consider $y > 0$ in (A.9) and later restrict to $y \geq y_{\min}$ as in Theorem A.3. This is mostly what we shall do in what follows, where the final restriction $y \geq y_{\min}$ will be used to derive bounds as claimed in Theorem A.3. We first derive a uniform bound for $\mathbf{G}(x, y)$ with respect to $x \in \mathbb{R}$.

Proposition A.1. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *with a nonzero constant* c_3. *Then there exists a constant* $C > 0$ *such that for any* $y > 0$, *there holds:*

$$\sup_{x\in\mathbb{R}} |\mathbf{G}(x,y)| \leq \frac{C}{y^{1/3}}\,.$$

Proof. The proof of Proposition A.1 is a mere adaptation of the proof of [9, Proposition 2.3]. We recall the details for the sake of completeness. From [24, Lemma 3.1] and the so-called van der Corput Lemma, we have:

$$\left| \int_a^b \mathrm{e}^{\mathbf{i}\, f(\theta)}\, g(\theta)\, \mathrm{d}\theta \right| \leq \frac{C_0}{\Big(\min_{\theta\in[a,b]} |\, f^{(3)}(\theta)\, | \Big)^{1/3}} \Big(\|\, g\,\|_{L^\infty([a,b])} + \|\, g'\,\|_{L^1([a,b])} \Big)\,, \tag{A.15}$$

as long as f is real valued and the minimum of $|f^{(3)}|$ on the interval $[a, b]$ is positive. The crucial observation here is that the constant C_0 in (A.15) *does not* depend on a nor b (nor f and g, of course). The function g could be complex valued.

We apply the above inequality to $f(\theta) := x\,\theta + c_3\,y\,\theta^3$ and $g(\theta) := \exp(-c_4\,y\,\theta^4)$, so that both norms $\|\,g\,\|_{L^\infty([a,b])}$ and $\|\,g'\,\|_{L^1([a,b])}$ can be bounded uniformly with respect to a, b and y. We thus obtain the bound:

$$\left|\int_a^b \mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\right| \le \frac{C}{y^{1/3}}\,,$$

and it remains to pass to the limit in both a and b to conclude, the constant C being independent of the interval $[a, b]$. ☐

A.3 The fast decaying side

We assume from now on that the coefficient c_3 in (A.4) is positive, the case where this coefficient is negative being obtained by switching the sign of x. Now that the sign of c_3 is fixed, we first deal with the case $x \ge 0$.

Proposition A.2. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let the constant* c_3 *be positive. Then there exist some constants* $\mathbf{c}_\sharp > 0$*,* $C > 0$ *and* $c > 0$ *such that for any* $(x, y) \in \mathbb{R} \times \mathbb{R}^{+*}$*, there holds:*

$$|\mathbf{G}(x,y)| \le \begin{cases} \dfrac{C}{y^{1/4}\,\max(1, x^{1/4})}\,\exp\left(-c\,\dfrac{x^{3/2}}{y^{1/2}}\right), & \text{if } 0 \le x \le \mathbf{c}_\sharp\,y,\\[2ex] \dfrac{C}{y^{1/2}}\,\exp\left(-c\,\dfrac{x^{4/3}}{y^{1/3}}\right), & \text{if } x \ge \mathbf{c}_\sharp\,y. \end{cases}$$

Proof. Integrating the holomorphic function:

$$z \in \mathbb{C} \longmapsto \mathrm{e}^{\mathbf{i}\,x\,z}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,z^3}\,\mathrm{e}^{-c_4\,y\,z^4}$$

over a large rectangle and passing to the limit, we easily see that the real line $\mathbb{R}$ over which we integrated it to obtain the defining equation (A.9) can be switched to any line $\mathbf{i}\mu + \mathbb{R}$ with $\mu \in \mathbb{R}$. In other words, the Cauchy formula yields:

$$\forall\,\mu \in \mathbb{R}\,,\quad \forall\,(x,y) \in \mathbb{R} \times \mathbb{R}^{+*}\,,\quad \mathbf{G}(x,y) = \frac{1}{2\,\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathbf{i}\,x\,(\mathbf{i}\mu+\theta)}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,(\mathbf{i}\mu+\theta)^3}\,\mathrm{e}^{-c_4\,y\,(\mathbf{i}\mu+\theta)^4}\,\mathrm{d}\theta\,. \tag{A.16}$$

Expanding all quantities within the integral and applying the triangle inequality, we obtain the bound:

$$\forall\,\mu \in \mathbb{R}\,,\quad \forall\,(x,y) \in \mathbb{R} \times \mathbb{R}^{+*}\,,\quad |\mathbf{G}(x,y)| \le \frac{\mathrm{e}^{-x\,\mu+c_3\,y\,\mu^3-c_4\,y\,\mu^4}}{2\,\pi}\int_{\mathbb{R}} \mathrm{e}^{-3\,y\,\mu\,(c_3-2\,c_4\,\mu)\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,. \tag{A.17}$$

This bound is rather crude but it allows to deal with many regimes in (x, y) by optimizing with respect to the free parameter $\mu \in \mathbb{R}$.

Let us now consider $x \ge 0$ and let us choose $\mu = \mu_0 := (x/(3\,c_3\,y))^{1/2} \ge 0$. We use the bound (A.17) and use the obvious inequality $\exp(-c_4\,y\,\mu_0^4) \le 1$ to obtain:

$$|\mathbf{G}(x,y)| \le \frac{1}{2\,\pi}\,\exp\left(-\frac{2\,x^{3/2}}{3\,\sqrt{3\,c_3}\,y^{1/2}}\right)\int_{\mathbb{R}} \mathrm{e}^{-3\,y\,\mu_0\,(c_3-2\,c_4\,\mu_0)\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

We define once and for all the positive constant $\mathbf{c}_\sharp := 3\,c_3^3/(16\,c_4^2)$. If we consider the regime $0 \le x \le \mathbf{c}_\sharp\, y$, then we have $2\,c_4\,\mu_0 \le c_3/2$, so that we get:

$$|\mathbf{G}(x,y)| \;\le\; \frac{1}{2\,\pi}\,\exp\left(-\frac{2\,x^{3/2}}{3\,\sqrt{3\,c_3}\,y^{1/2}}\right)\int_{\mathbb{R}} \mathrm{e}^{-(3/2)\,c_3\,y\,\mu_0\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

Since x is nonnegative (and therefore μ_0 is nonnegative too), we can either use one exponential term or the other within the integral to derive a bound for this integral. In the case $x=0$, only the exponential term $\exp(-c_4\,y\,\theta^4)$ makes the integral converge. Recalling the value $\mu_0 = (x/(3\,c_3\,y))^{1/2}$, we thus get:

$$\int_{\mathbb{R}} \mathrm{e}^{-(3/2)\,c_3\,y\,\mu_0\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \le \frac{C}{y^{1/4}\,\max(1,x^{1/4})}\,,$$

with a constant C that is independent of y and $x \in [0,\mathbf{c}_\sharp\,y]$. This yields our first bound (see the statement of Proposition A.2):

$$\forall\, x \in [0,\mathbf{c}_\sharp\,y]\,,\quad |\mathbf{G}(x,y)| \;\le\; \frac{C}{y^{1/4}\,\max(1,x^{1/4})}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,. \tag{A.18}$$

We assume from now on $x \ge \mathbf{c}_\sharp\,y > 0$ with the above (already fixed) positive constant $\mathbf{c}_\sharp = 3\,c_3^3/(16\,c_4^2)$. We go back to (A.17) and restrict from now on to positive values of the parameter μ. We use Young's inequality:

$$6\,c_4\,y\,\mu^2\,\theta^2 \;\le\; c_4\,y\,\theta^4 \;+\; 9\,c_4\,y\,\mu^4\,,$$

to derive the bound:

$$\forall\,\mu > 0\,,\quad |\mathbf{G}(x,y)| \;\le\; \frac{C}{\sqrt{y\,\mu}}\,\exp\big(-x\,\mu + c_3\,y\,\mu^3 + 8\,c_4\,y\,\mu^4\big)\,, \tag{A.19}$$

where the constant C is independent of x, y and μ. Our final choice for μ will depend on x and y so it is crucial to get constants that do not depend on μ all along.

The function :

$$f\;:\;\mu \in \mathbb{R}^+ \longmapsto -x\,\mu + c_3\,y\,\mu^3 + 8\,c_4\,y\,\mu^4\,, \tag{A.20}$$

is smooth and strictly convex. Since $f(0)=0$, $f'(0)<0$ and f tends to $+\infty$ at $+\infty$, it appears that f attains its unique global minimum on $\mathbb{R}^+$ at some $\underline{\mu} > 0$ that is characterized by $f'(\underline{\mu}) = 0$. Multiplying the relation $f'(\underline{\mu}) = 0$ by $\underline{\mu}$, we obtain:

$$f(\underline{\mu}) \;=\; -2\,c_3\,y\,\underline{\mu}^3 - 24\,c_4\,y\,\underline{\mu}^4 \;\le\; -24\,c_4\,y\,\underline{\mu}^4\,.$$

Since we have $x/y \ge \mathbf{c}_\sharp$, it is not hard to show that we can choose some small positive constant $\mathbf{c}_\natural > 0$, whose choice only depends on c_3, c_4 $\mathbf{c}_\sharp$, and such that:

$$f'\left(\mathbf{c}_\natural\,\frac{x^{1/3}}{y^{1/3}}\right) < 0\,.$$

From the strict convexity of f, this means that we have $\underline{\mu} \ge \mathbf{c}_\natural\,(x/y)^{1/3}$ for some constant $\mathbf{c}_\natural$ that only depends on c_3 and c_4 (we recall that $\mathbf{c}_\sharp$ only depends on c_3 and c_4).

Going back to (A.19) and using $\underline{\mu} \geq \mathbf{c}_\natural\,(x/y)^{1/3}$, we thus obtain the bound:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{\sqrt{y\,\underline{\mu}}}\,\exp\big(f(\underline{\mu})\big) \,\leq\, \frac{C}{\sqrt{y\,\underline{\mu}}}\,\exp\big(-24\,c_4\,y\,\underline{\mu}^4\big) \,\leq\, \frac{C}{y^{1/3}\,x^{1/6}}\,\exp\left(-c\,\frac{x^{4/3}}{y^{1/3}}\right)\,.$$

We simplify a little bit more this last bound by using again the inequality $x \geq \mathbf{c}_\sharp\,y$ and this yields the estimate that we announced in Proposition A.2, namely:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/2}}\,\exp\left(-c\,\frac{x^{4/3}}{y^{1/3}}\right)\,.$$

The proof of Proposition A.2 is now complete. □

Let us observe that in Proposition A.2, the parameter y can take arbitrarily small positive values. We now restrict to $y \geq y_{\min} > 0$ in order to prove the result of Theorem A.3. This will be done in two steps.

Corollary A.2. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let* $y_{\min} > 0$ *be given. Then there exist some constants* $\mathbf{c}_\sharp > 0$, $C > 0$ *and* $c > 0$ *such that for any* $(x,y) \in \mathbb{R} \times [y_{\min}, +\infty)$, *there holds:*

$$|\mathbf{G}(x,y)| \,\leq\, \begin{cases} \dfrac{C}{y^{1/3}}\,\exp\left(-c\,\dfrac{x^{3/2}}{y^{1/2}}\right)\,, & \text{if } 0 \leq x \leq \mathbf{c}_\sharp\,y, \\ \dfrac{C}{y^{1/4}}\,\exp\left(-c\,\dfrac{x^{4/3}}{y^{1/3}}\right)\,, & \text{if } x \geq \mathbf{c}_\sharp\,y. \end{cases}$$

Proof. We fix the constant $\mathbf{c}_\sharp > 0$ as the one given in Proposition A.2. Let us first assume $x \geq \mathbf{c}_\sharp\,y$ so that Proposition A.2 gives:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/2}}\,\exp\left(-c\,\frac{x^{4/3}}{y^{1/3}}\right)\,.$$

We then use the bound $y^{1/2} \geq y_{\min}^{1/4}\,y^{1/4}$ to conclude. We can now assume $0 \leq x \leq \mathbf{c}_\sharp\,y$ so that Proposition A.2 gives:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/4}\,\max(1,x^{1/4})}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,.$$

In particular, we have $\max(1,x^{1/4}) \geq x^{1/4}$ so if $x \geq y^{1/3} > 0$, we get:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/4}\,x^{1/4}}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right) \,\leq\, \frac{C}{y^{1/3}}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,. \tag{A.21}$$

It remains to examine the case $0 \leq x \leq \min(y^{1/3}, \mathbf{c}_\sharp\,y)$. Proposition A.1 gives the bound:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/3}} \,\leq\, \frac{C\,\mathrm{e}^c}{y^{1/3}}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,,$$

with the same constant $c > 0$ as in (A.21). This gives the expected bound for $|\mathbf{G}(x,y)|$ in the regime $0 \leq x \leq \mathbf{c}_\sharp\,y$. The proof of Corollary A.2 is complete. □

It appears that allowing us to choose the value of the constant $\mathbf{c}_\sharp$ that separates the two regimes in Corollary A.2 will be convenient in the analysis (while in Corollary A.2 the constant $\mathbf{c}_\sharp$ is given and is therefore not free to be fixed). We thus prove the following result which is a slight adaptation of Corollary A.2.

Corollary A.3. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let* $y_{\min} > 0$ *be given. Then for any constant* $\underline{\mathbf{c}} > 0$*, there exist some constants* $C > 0$ *and* $c > 0$ *such that for any* $(x, y) \in \mathbb{R} \times [y_{\min}, +\infty)$*, there holds:*

$$|\mathbf{G}(x,y)| \le \begin{cases} \dfrac{C}{y^{1/3}} \exp\left(-c\,\dfrac{x^{3/2}}{y^{1/2}}\right) , & \text{if } 0 \le x \le \underline{\mathbf{c}}\, y, \\ \dfrac{C}{y^{1/4}} \exp\left(-c\,\dfrac{x^{4/3}}{y^{1/3}}\right) , & \text{if } x \ge \underline{\mathbf{c}}\, y. \end{cases}$$

Proof. We let $\underline{\mathbf{c}} > 0$ be given and assume that there holds $\underline{\mathbf{c}} \ge \mathbf{c}_\sharp$ where $\mathbf{c}_\sharp$ is given in Corollary A.2. (The case $\underline{\mathbf{c}} \le \mathbf{c}_\sharp$ is dealt with similarly as below.) Assuming first $x \ge \underline{\mathbf{c}}\, y$, we have $x \ge \mathbf{c}_\sharp\, y$ and we can therefore use Corollary A.2 and obtain the desired bound for $|\mathbf{G}(x,y)|$. Corollary A.2 also gives the desired bound in the regime $0 \le x \le \mathbf{c}_\sharp\, y$ and it therefore only remains to consider the regime $\mathbf{c}_\sharp\, y \le x \le \underline{\mathbf{c}}\, y$. From Corollary A.2, we already have the bound:

$$|\mathbf{G}(x,y)| \le \frac{C_1}{y^{1/4}} \exp\left(-c_1\,\frac{x^{4/3}}{y^{1/3}}\right) ,$$

for some constants C_1 and c_1 that are independent of x and y. Since we have $\mathbf{c}_\sharp\, y \le x$, we can first use a slight part of the exponential term to improve the power of the algebraic factor:

$$|\mathbf{G}(x,y)| \le \frac{C_1}{y^{1/4}} \exp(-c_1\,\mathbf{c}_\sharp^{4/3}\, y) \le \frac{C_2}{y^{1/3}} \exp\left(-\frac{c_1\,\mathbf{c}_\sharp^{4/3}}{2}\, y\right) ,$$

and we now wish to show that the right-hand side of the latter inequality is less than:

$$\frac{C}{y^{1/3}} \exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)$$

for some appropriate constants C and c (recalling that x belongs to the interval $[\mathbf{c}_\sharp\, y, \underline{\mathbf{c}}\, y]$). We fix $c_2 := c_1\,\mathbf{c}_\sharp^{4/3}/(2\,\underline{\mathbf{c}}^{3/2})$ so that we have (we use $x \le \underline{\mathbf{c}}\, y$):

$$\exp\left(-\frac{c_1\,\mathbf{c}_\sharp^{4/3}}{2}\, y\right) \le \exp\left(-c_2\,\frac{x^{3/2}}{y^{1/2}}\right) ,$$

and we therefore obtain:

$$|\mathbf{G}(x,y)| \le \frac{C_2}{y^{1/3}} \exp\left(-c_2\,\frac{x^{3/2}}{y^{1/2}}\right) ,$$

as expected. The proof of Corollary A.3 is complete. □

A.4 The oscillating side

We still assume that the coefficient c_3 in (A.4) is positive but we now consider the case where x is negative. This corresponds to the oscillating side of the Green's function. The analysis is split in four regimes. The case $|x| \leq y^{1/3}$ is dealt with straightforwardly with the uniform bound of Proposition A.1. The second (and most difficult) regime corresponds to $x \leq -y^{1/3}$ and $|x|/y$ is sufficiently small. The third regime corresponds to the case where $|x|/y$ is sufficiently large and the fourth and last regime corresponds to the case where $|x|$ and y are of comparable sizes. In the end, we collect all bounds and prove the result announced in Theorem A.3. We start with the most difficult case for which $|x|/y$ is small.

Proposition A.3. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let the function* $\mathfrak{g}$ *be defined in* (A.10). *Then there exist some constants* $\mathbf{c}_\flat > 0$, $C > 0$ *and* $c > 0$ *such that for any* $(x,y) \in \mathbb{R} \times \mathbb{R}^{+*}$, *there holds:*

$$|\mathbf{G}(x,y) \, - \, 2\,\mathrm{Re}\,\mathfrak{g}(x,y)| \, \leq \, \frac{C}{y^{1/3}\,\min(1,y^{1/3})} \, \exp\left(-c\,\frac{|x|^{3/2}}{y^{1/2}}\right) \, + \, \frac{C}{y^{1/2}} \, \exp\left(-c\,\frac{x^2}{y}\right) \, ,$$

if $-\mathbf{c}_\flat\, y \leq x \leq -y^{1/3}$.

Proof. We always consider $x < 0$ and introduce the notation $\omega := |x|/y$. We follow the analysis in [9] with slight modifications since this is precisely at this stage that [9] has some incorrect argument. Rather than integrating the function:

$$z \in \mathbb{C} \longmapsto \mathrm{e}^{\mathbf{i}\,x\,z}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,z^3}\,\mathrm{e}^{-c_4\,y\,z^4}$$

on a horizontal line, we apply once again the Cauchy formula and use the contour depicted in Figure A.1, that is:

- A horizontal half-line from $-\infty + \mathbf{i}\,\sqrt{\omega/(3\,c_3)}$ to $-2\sqrt{\omega/(3\,c_3)} - (2\,c_4/(9\,c_3^2))\omega + \mathbf{i}\sqrt{\omega/(3\,c_3)}$,
- A segment (with slope $-\pi/4$) from the point $-2\sqrt{\omega/(3\,c_3)} - (2\,c_4/(9\,c_3^2))\omega + \mathbf{i}\sqrt{\omega/(3\,c_3)}$ to the point $-\mathbf{i}(\sqrt{\omega/(3\,c_3)} + (2\,c_4/(9\,c_3^2))\omega)$,
- A segment (with slope $\pi/4$) from the point $-\mathbf{i}(\sqrt{\omega/(3\,c_3)} + (2\,c_4/(9\,c_3^2))\omega)$ to the point $2\sqrt{\omega/(3\,c_3)} + (2\,c_4/(9\,c_3^2))\omega + \mathbf{i}\sqrt{\omega/(3\,c_3)}$,
- A horizontal half-line from $2\sqrt{\omega/(3\,c_3)} + (2\,c_4/(9\,c_3^2))\omega + \mathbf{i}\sqrt{\omega/(3\,c_3)}$ to $+\infty + \mathbf{i}\sqrt{\omega/(3\,c_3)}$.

The corresponding contributions $\varepsilon_1(x,y)$, $\mathscr{H}_\flat(x,y)$, $\mathscr{H}_\sharp(x,y)$ and $\varepsilon_2(x,y)$ are reported in Figure A.1 and their exact expressions are given below.

In this way, we obtain the decomposition:

$$\mathbf{G}(x,y) \, = \, \varepsilon_1(x,y) \, + \, \varepsilon_2(x,y) \, + \, \mathscr{H}_\flat(x,y) \, + \, \mathscr{H}_\sharp(x,y) \, , \tag{A.22}$$

where the four contributions $\varepsilon_1(x,y)$, $\varepsilon_2(x,y)$, $\mathscr{H}_\flat(x,y)$ and $\mathscr{H}_\sharp(x,y)$ have the following expressions:

$$\varepsilon_1(x,y) \, = \, \frac{\mathrm{e}^{-x\,\mu+c_3\,y\,\mu^3-c_4\,y\,\mu^4}}{2\,\pi} \int_{-\infty}^{-\Xi(\omega)} \mathrm{e}^{\mathbf{i}\,x\,\theta+\mathbf{i}\,c_3\,y\,\theta^3-\mathbf{i}\,3\,c_3\,y\,\mu^2\,\theta+\mathbf{i}\,4\,c_4\,y\,\mu^3\,\theta-\mathbf{i}\,4\,c_4\,y\,\mu\,\theta^3} \\ \times \mathrm{e}^{-3\,c_3\,y\,\mu\,\theta^2+6\,c_4\,y\,\mu^2\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \, ,$$

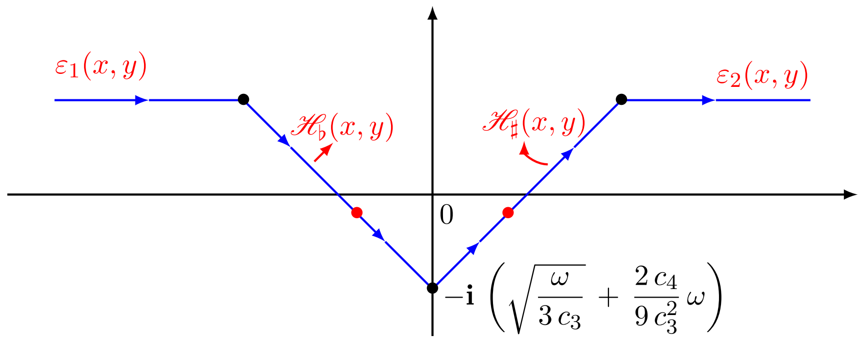

Figure A.1: *The integration contour in the case $x \le -y^{1/3}$ and $|x|/y$ small. The two red bullets represent the approximate saddle points of the phase $-\mathbf{i}\,\omega\,z + \mathbf{i}\,c_3\,z^3 - c_4\,z^4$ and the black bullets represent the end points of the contours along which we compute the contributions $\varepsilon_1(x,y)$, $\varepsilon_2(x,y)$, $\mathscr{H}_\flat(x,y)$, $\mathscr{H}_\sharp(x,y)$.*

where the upper bound $\Xi(\omega)$ and the parameter μ in the expression of $\varepsilon_1(x,y)$ are defined as:

$$\Xi(\omega) \,:=\, 2\sqrt{\frac{\omega}{3\,c_3}} + \frac{2\,c_4}{9\,c_3^2}\,\omega\,, \quad \mu \,:=\, \sqrt{\frac{\omega}{3\,c_3}}\,. \tag{A.23}$$

The contribution $\varepsilon_2(x,y)$ has the following expression (with the same definition for $\Xi(\omega)$ and μ):

$$\varepsilon_2(x,y) \,=\, \frac{\mathrm{e}^{-x\,\mu + c_3\,y\,\mu^3 - c_4\,y\,\mu^4}}{2\,\pi} \int_{\Xi(\omega)}^{+\infty} \mathrm{e}^{\mathbf{i}\,x\,\theta + \mathbf{i}\,c_3\,y\,\theta^3 - \mathbf{i}\,3\,c_3\,y\,\mu^2\,\theta + \mathbf{i}\,4\,c_4\,y\,\mu^3\,\theta - \mathbf{i}\,4\,c_4\,y\,\mu\,\theta^3} \\ \times \mathrm{e}^{-3\,c_3\,y\,\mu\,\theta^2 + 6\,c_4\,y\,\mu^2\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \,=\, \overline{\varepsilon_1(x,y)}\,. \tag{A.24}$$

The contribution $\mathscr{H}_\flat(x,y)$ is given by:

$$\mathscr{H}_\flat(x,y) \,=\, \frac{\mathrm{e}^{-\mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \mathrm{e}^{-\mathbf{i}\,\omega\,y\,\Theta(t) + \mathbf{i}\,c_3\,y\,\Theta(t)^3 - c_4\,y\,\Theta(t)^4}\,\mathrm{d}t\,, \tag{A.25}$$

with:

$$\forall\, t \in \left[-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega, \sqrt{\frac{2\,\omega}{3\,c_3}}\right], \quad \Theta(t) \,:=\, -\sqrt{\frac{\omega}{3\,c_3}} - \frac{2\,c_4}{9\,c_3^2}\,\omega + t\,\mathrm{e}^{-\mathbf{i}\,\pi/4}\,, \tag{A.26}$$

which corresponds to the parametrization of the first slanted segment in Figure A.1 (the one with slope $-\pi/4$). The parametrization of the second segment is entirely similar and we obtain the expression:

$$\mathscr{H}_\sharp(x,y) \,=\, \overline{\mathscr{H}_\flat(x,y)}\,.$$

Let us start with the estimate of $\varepsilon_2(x,y)$ whose expression is given in (A.24). By using the expression for the parameter μ and recalling that we have $x = -\omega\,y$, we get:

$$\varepsilon_2(x,y) \,=\, \frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}} - \frac{c_4}{9\,c_3^2}\,\omega^2\,y\right) \int_{\Xi(\omega)}^{+\infty} \mathrm{e}^{\mathbf{i}\,\cdots} \times \mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,\theta^2 + \frac{2\,c_4}{c_3}\,\omega\,y\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,,$$

where the three dots within the integral stand for a *real* quantity whose precise expression is useless since we are going to use the triangle inequality so that the modulus of this exponential term will be bounded by 1. Indeed, applying the triangle inequality yields the bound:

$$\begin{aligned}|\varepsilon_2(x,y)| &\leq \frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}-\frac{c_4}{9\,c_3^2}\,\omega^2\,y\right)\int_{\Xi(\omega)}^{+\infty}\mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,\theta^2+\frac{2\,c_4}{c_3}\,\omega\,y\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\\ &\leq \frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\int_{\Xi(\omega)}^{+\infty}\mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,\theta^2+\frac{2\,c_4}{c_3}\,\omega\,y\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,.\end{aligned}$$

We first restrict $\omega=|x|/y$ by imposing:

$$\omega\,\leq\,\frac{3\,c_3^3}{16\,c_4^2}\,,\tag{A.27}$$

so that we have:

$$\frac{2\,c_4}{c_3}\,\omega\,\leq\,\frac{1}{2}\,\sqrt{3\,c_3\,\omega}\,.$$

We thus get:

$$\begin{aligned}|\varepsilon_2(x,y)|\,&\leq\,\frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\int_{\Xi(\omega)}^{+\infty}\mathrm{e}^{-\frac{\sqrt{3\,c_3\,\omega}}{2}\,y\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\\ &\leq\,\frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\int_{\Xi(\omega)}^{+\infty}\mathrm{e}^{-\frac{\sqrt{3\,c_3\,\omega}}{2}\,y\,\theta^2}\,\mathrm{d}\theta\\ &\leq\,\frac{1}{2\,\pi}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\int_{2\,\sqrt{\frac{\omega}{3\,c_3}}}^{+\infty}\mathrm{e}^{-\frac{\sqrt{3\,c_3\,\omega}}{2}\,y\,\theta^2}\,\mathrm{d}\theta\,,\end{aligned}$$

where the final inequality comes from the definition of $\Xi(\omega)$. We then use the inequality:

$$\forall\,a>0\,,\quad\forall\,X>0\,,\quad\int_X^{+\infty}\mathrm{e}^{-a\,\theta^2}\,\mathrm{d}\theta\,\leq\,\frac{1}{2\,a\,X}\,\mathrm{e}^{-a\,X^2}\,,$$

to obtain (for a suitable constant C):

$$|\varepsilon_2(x,y)|\,\leq\,\frac{C}{\omega\,y}\,\exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\,\exp\left(-\frac{2\,\omega^{3/2}\,y}{\sqrt{3\,c_3}}\right)\,=\,\frac{C}{\omega\,y}\,\exp\left(-\frac{2\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\,,$$

and this gives, going back to x and y, the estimate:

$$|\varepsilon_2(x,y)|\,\leq\,\frac{C}{|x|}\,\exp\left(-c\,\frac{|x|^{3/2}}{y^{1/2}}\right)\,.$$

Restricting to the regime $x\leq -y^{1/3}$, we obtain the final estimate:

$$|\varepsilon_2(x,y)|\,\leq\,\frac{C}{y^{1/3}}\,\exp\left(-c\,\frac{|x|^{3/2}}{y^{1/2}}\right)\,,\tag{A.28}$$

as long as we have $x \leq -y^{1/3}$ and $|x|/y \leq 3\,c_3^3/(16\,c_4^2)$. Of course a similar estimate holds for $\varepsilon_1(x,y)$ since it is the complex conjugate of $\varepsilon_2(x,y)$.

We now turn to the contribution $\mathscr{H}_\flat(x,y)$ in (A.25). With the definition (A.26) for $\Theta(t)$, we expand:

$$-\mathbf{i}\,\omega\,y\,\Theta(t) + \mathbf{i}\,c_3\,y\,\Theta(t)^3 - c_4\,y\,\Theta(t)^4 \;=\; y\sum_{k=0}^{4} p_k(\omega)\,t^k\,,$$

where the quantities $p_0(\omega),\dots,p_4(\omega)$ have the following behavior[2] as $\omega > 0$ tends to zero:

$$\operatorname{Re}\,p_0(\omega) \;=\; -\,\frac{c_4}{9\,c_3^2}\,\omega^2 \;+\; O(\omega^3)\,, \tag{A.29a}$$
$$\operatorname{Im}\,p_0(\omega) \;=\; \frac{2}{3\sqrt{3\,c_3}}\,\omega^{3/2} \;+\; O(\omega^{5/2})\,, \tag{A.29b}$$
$$p_1(\omega) \;=\; O(\omega^2)\,, \tag{A.29c}$$
$$\operatorname{Re}\,p_2(\omega) \;=\; -\,\sqrt{3\,c_3\,\omega} \;+\; O(\omega^{3/2})\,, \tag{A.29d}$$
$$\operatorname{Im}\,p_2(\omega) \;=\; O(\omega)\,, \tag{A.29e}$$
$$p_3(\omega) \;=\; c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4} \;+\; O(\omega^{\,1/2})\,, \tag{A.29f}$$
$$p_4(\omega) \;=\; c_4\,. \tag{A.29g}$$

We thus have:

$$\mathscr{H}_\flat(x,y) \;=\; \frac{\mathrm{e}^{p_0(\omega)\,y-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(\sum_{k=1}^{4} p_k(\omega)\,y\,t^k\Big)\,\mathrm{d}t\,.$$

We now use the inequality:

$$\forall\, z\in\mathbb{C}\,,\quad |\mathrm{e}^z - 1| \;\leq\; |z|\,\mathrm{e}^{|z|}\,, \tag{A.30}$$

with the specific value:

$$z \;:=\; y\,\Big\{p_1(\omega)\,t \;+\; (p_2(\omega)+\sqrt{3\,c_3\,\omega})\,t^2 \;+\; (p_3(\omega) - c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4})\,t^3 \;+\; p_4(\omega)\,t^4\Big\}\,.$$

On the considered integration interval for t, we thus have (see (A.29c), (A.29d), (A.29e), (A.29f), (A.29g)):

$$|z| \;\leq\; C\,y\,\big(\omega^{5/2} \;+\; \omega\,t^2\big)\,,$$

with a constant C that is independent of y, ω and t (for the relevant values of t). This gives the estimate:

$$\begin{aligned}\Bigg|\mathscr{H}_\flat(x,y) \;-\; \frac{\mathrm{e}^{p_0(\omega)\,y-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2 + \mathbf{i}\,c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\Bigg| \\ \leq C\,y\,\mathrm{e}^{\operatorname{Re}\,p_0(\omega)\,y + C\,\omega^{5/2}\,y}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \big(\omega^{5/2} \;+\; \omega\,t^2\big)\,\mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,t^2 + \frac{c_3}{\sqrt{2}}\,y\,t^3 + C\,\omega\,y\,t^2}\,\mathrm{d}t\,.\end{aligned}$$

[2] Here we see that our choice of contour only uses approximate saddle points because $p_1(\omega)$ is not zero. This does not matter so much since $p_1(\omega)$ will be small enough to produce error terms that can be controlled in our analysis.

We first use (A.29a) to estimate the real part of $p_0(\omega)$ and choose ω small enough so that we have:

$$\mathrm{Re}\; p_0(\omega) + C\,\omega^{5/2} \;\leq\; -c\,\omega^2\,,$$

for a suitable constant $c > 0$ that does not depend on ω. We thus get the estimate:

$$\left|\mathscr{H}_\flat(x,y) - \frac{\mathrm{e}^{p_0(\omega)\,y - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(- \sqrt{3\,c_3\,\omega}\,y\,t^2 + \mathbf{i}\,c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right|$$
$$\leq C\,y\,\mathrm{e}^{-c\,\omega^2\,y} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \big(\omega^{5/2} \;+\; \omega\,t^2\big)\,\mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,t^2 + \frac{c_3}{\sqrt{2}}\,y\,t^3 + C\,\omega\,y\,t^2}\,\mathrm{d}t\,.$$

As in [9], let us now observe that when t is nonpositive in the integral, we have $c_3\,t^3 \leq 0$ (recall that c_3 is positive), and when t is positive, on the considered integration interval, we have:

$$\frac{c_3}{\sqrt{2}}\,t^3 \;\leq\; \sqrt{\frac{c_3\,\omega}{3}}\,t^2\,.$$

If ω is chosen sufficiently small (and this choice is, of course, independent of y), we thus get:

$$\mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,t^2 + \frac{c_3}{\sqrt{2}}\,y\,t^3 + C\,\omega\,y\,t^2} \;\leq\; \mathrm{e}^{-c\,\sqrt{\omega}\,y\,t^2}\,,$$

for the relevant values of t. This gives the estimate:

$$\left|\mathscr{H}_\flat(x,y) - \frac{\mathrm{e}^{p_0(\omega)\,y - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(- \sqrt{3\,c_3\,\omega}\,y\,t^2 + \mathbf{i}\,c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right|$$
$$\leq C\,y\,\mathrm{e}^{-c\,\omega^2\,y} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \big(\omega^{5/2} \;+\; \omega\,t^2\big)\,\mathrm{e}^{-c\,\sqrt{\omega}\,y\,t^2}\,\mathrm{d}t\,.$$

Integrating now with respect to t (over $\mathbb{R}$ rather than over the above compact interval), we thus have:

$$\left|\mathscr{H}_\flat(x,y) - \frac{\mathrm{e}^{p_0(\omega)\,y - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(- \sqrt{3\,c_3\,\omega}\,y\,t^2 + \mathbf{i}\,c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right|$$
$$\leq C\,y^{1/2}\,\omega^{9/4}\,\mathrm{e}^{-c\,\omega^2\,y} \;+\; \frac{C}{\sqrt{y}}\,\omega^{1/4}\,\mathrm{e}^{-c\,\omega^2\,y} \;\leq\; \frac{C}{\sqrt{y}}\,(1+\omega^2\,y)\,\mathrm{e}^{-c\,\omega^2\,y}\,.$$

Up to diminishing the constant $c > 0$ and increasing the constant C, we thus get the final estimate (recall $\omega = |x|/y$):

$$\left|\mathscr{H}_\flat(x,y) - \frac{\mathrm{e}^{p_0(\omega)\,y - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(- \sqrt{3\,c_3\,\omega}\,y\,t^2 + \mathbf{i}\,c_3\,\mathrm{e}^{-\,\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right|$$
$$\leq \frac{C}{y^{1/2}}\,\exp\left(-c\,\frac{x^2}{y}\right)\,. \qquad \text{(A.31)}$$

We now use the inequality (A.30) one more time with the value:

$$z := y\left\{p_0(\omega) + \frac{c_4}{9\,c_3^2}\,\omega^2 - \mathbf{i}\,\frac{2}{3\sqrt{3\,c_3}}\,\omega^{3/2}\right\},$$

which gives (use (A.29a) and (A.29b)):

$$\left|\mathrm{e}^{p_0(\omega)\,y} - \mathrm{e}^{-\frac{c_4}{9\,c_3^2}\,\omega^2\,y}\,\mathrm{e}^{\mathbf{i}\,\frac{2}{3\sqrt{3\,c_3}}\,\omega^{3/2}\,y}\right| \leq C\,\omega^{5/2}\,y\,\mathrm{e}^{C\,\omega^{5/2}\,y}\,.$$

We can then follow the same kind of estimate as just above and derive the estimate (combining with (A.31)):

$$\left|\mathscr{H}_\flat(x,y) - \frac{\mathrm{e}^{-\frac{c_4\,x^2}{9\,c_3^2\,y}}\,\mathrm{e}^{\mathbf{i}\,\frac{2\,|x|^{3/2}}{3\sqrt{3\,c_3}\,y^{1/2}}-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2+\mathbf{i}\,c_3\,\mathrm{e}^{-\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right| \leq \frac{C}{y^{1/2}}\,\exp\left(-c\,\frac{x^2}{y}\right)\,, \quad \text{(A.32)}$$

provided that ω is sufficiently small.

If we compare with the definition (A.10) of $\mathfrak{g}$, we see that we have almost made the quantity $\mathfrak{g}(x,y)$ appear, except for the fact that the integration interval is not exactly the good one since it is not symmetric. This final estimate is not more difficult than the previous ones. Namely, we easily estimate:

$$\left|\frac{\mathrm{e}^{-\frac{c_4\,x^2}{9\,c_3^2\,y}}\,\mathrm{e}^{\mathbf{i}\,\frac{2\,|x|^{3/2}}{3\sqrt{3\,c_3}\,y^{1/2}}-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2+\mathbf{i}\,c_3\,\mathrm{e}^{-\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right| \leq C\,\mathrm{e}^{-c\,\omega^2\,y}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2+\frac{c_3}{\sqrt{2}}\,y\,t^3\Big)\,\mathrm{d}t\,.$$

Since t is negative on the considered interval, we have:

$$\left|\frac{\mathrm{e}^{-\frac{c_4\,x^2}{9\,c_3^2\,y}}\,\mathrm{e}^{\mathbf{i}\,\frac{2\,|x|^{3/2}}{3\sqrt{3\,c_3}\,y^{1/2}}-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2+\mathbf{i}\,c_3\,\mathrm{e}^{-\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right| \leq C\,\mathrm{e}^{-c\,\omega^2\,y}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2\Big)\,\mathrm{d}t\,,$$

and we can then bound the final integral by the maximum of the integrated function times the length of the interval. This gives the estimate:

$$\left|\frac{\mathrm{e}^{-\frac{c_4\,x^2}{9\,c_3^2\,y}}\,\mathrm{e}^{\mathbf{i}\,\frac{2\,|x|^{3/2}}{3\sqrt{3\,c_3}\,y^{1/2}}-\mathbf{i}\,\pi/4}}{2\,\pi}\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}}\exp\Big(-\sqrt{3\,c_3\,\omega}\,y\,t^2+\mathbf{i}\,c_3\,\mathrm{e}^{-\mathbf{i}\,\pi/4}\,y\,t^3\Big)\,\mathrm{d}t\right| \leq C\,\omega\,\mathrm{e}^{-c\,\omega^2\,y}\,\mathrm{e}^{-c\,\omega^{3/2}\,y} \leq \frac{C}{\omega^{1/2}\,y}\,(\omega^{3/2}\,y)\,\mathrm{e}^{-c\,\omega^{3/2}\,y}\,.$$

We can then again decrease the constant c and increase C to get:

$$\left| \frac{\mathrm{e}^{-\frac{c_4\, x^2}{9\, c_3^2\, y}}\, \mathrm{e}^{\mathbf{i}\, \frac{2\,|x|^{3/2}}{3\,\sqrt{3\,c_3}\, y^{1/2}} - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(-\sqrt{3\,c_3\,\omega}\, y\, t^2 + \mathbf{i}\, c_3\, \mathrm{e}^{-\mathbf{i}\,\pi/4}\, y\, t^3 \Big)\, \mathrm{d}t \right| \\ \leq \frac{C}{\omega^{1/2}\, y}\, \mathrm{e}^{-c\,\omega^{3/2}\, y}\,.$$

Since we restrict to the case $|x| \geq y^{1/3}$, we have $\omega^{1/2} \geq y^{-1/3}$ and this gives:

$$\left| \frac{\mathrm{e}^{-\frac{c_4\, x^2}{9\, c_3^2\, y}}\, \mathrm{e}^{\mathbf{i}\, \frac{2\,|x|^{3/2}}{3\,\sqrt{3\,c_3}\, y^{1/2}} - \mathbf{i}\,\pi/4}}{2\,\pi} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}} - \frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{-\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(-\sqrt{3\,c_3\,\omega}\, y\, t^2 + \mathbf{i}\, c_3\, \mathrm{e}^{-\mathbf{i}\,\pi/4}\, y\, t^3 \Big)\, \mathrm{d}t \right| \\ \leq \frac{C}{y^{2/3}}\, \exp\left(-c\, \frac{|x|^{3/2}}{y^{1/2}} \right)\,. \quad \text{(A.33)}$$

We now combine (A.32) and (A.33) and recall the definition (A.10) to get:

$$|\mathscr{H}_\flat(x,y)\; -\; \mathfrak{g}(x,y)| \;\leq\; \frac{C}{y^{2/3}}\, \exp\left(-c\, \frac{|x|^{3/2}}{y^{1/2}} \right) + \frac{C}{y^{1/2}}\, \exp\left(-c\, \frac{x^2}{y} \right)\,.$$

Taking twice the real part and recalling that the remaining contribution $\mathscr{H}_\sharp(x,y)$ is the complex conjugate of $\mathscr{H}_\flat(x,y)$, we get:

$$|\mathscr{H}_\flat(x,y)\; +\; \mathscr{H}_\sharp(x,y)\; -\; 2\,\mathrm{Re}\; \mathfrak{g}(x,y)| \;\leq\; \frac{C}{y^{2/3}}\, \exp\left(-c\, \frac{|x|^{3/2}}{y^{1/2}} \right) + \frac{C}{y^{1/2}}\, \exp\left(-c\, \frac{x^2}{y} \right)\,.$$

It remains[3] to combine this estimate with (A.28) and the corresponding one for $\varepsilon_1(x,y)$ to obtain the estimate of Proposition A.3 (recall the decomposition (A.22)). We have not kept track all along of the smallness requirement on $\omega = |x|/y$ but the final estimate holds as long as ω is small enough, and this smallness condition is independent of y, which corresponds to the statement of Proposition A.3 for the condition on x: $x \geq -\mathbf{c}_\flat\, y$ for some $\mathbf{c}_\flat > 0$. □

We now deal with the regime where x is negative and $|x|/y$ is large enough. As in Proposition A.2, this is a regime that had not been considered in [9].

Proposition A.4. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let* $\mathbf{c}_\flat > 0$ *be the constant given in Proposition A.3. Then there exist some constants* $\mathbf{C}_\flat > \mathbf{c}_\flat$, $C > 0$ *and* $c > 0$ *such that for any* $(x,y) \in \mathbb{R} \times \mathbb{R}^{+*}$ *with* $x \leq -\mathbf{C}_\flat\, y$, *there holds:*

$$|\mathbf{G}(x,y)| \;\leq\; \frac{C}{y^{1/4}}\, \exp\left(-c\, \frac{|x|^{4/3}}{y^{1/3}} \right)\,.$$

[3]The error in [9] occurred here since the author had tried to get rid of the term t^3 in the integral by applying similar arguments but an error occurred when estimating the corresponding integrals. Hence the final expressions for the analog of the function $\mathfrak{g}$ and the error bound are not correct.

Proof. We go back to the bound (A.17) and only consider from now on *negative* values for both x and μ. We use Young's inequality:

$$6\,c_4\,y\,\mu^2\,\theta^2 \;\leq\; \frac{c_4}{2}\,y\,\theta^4 \;+\; 18\,c_4\,y\,\mu^4\,,$$

as well as the obvious inequality $\exp(c_3\,y\,\mu^3) \leq 1$ to obtain:

$$\forall\,\mu \leq 0\,,\quad |\mathbf{G}(x,y)| \;\leq\; \frac{\mathrm{e}^{-x\,\mu+17\,c_4\,y\,\mu^4}}{2\,\pi}\;\int_{\mathbb{R}} \mathrm{e}^{3\,c_3\,y\,|\mu|\,\theta^2}\,\mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

We choose the specific value $\mu = \mu_0 := -(|x|/(68\,c_4\,y))^{1/3} < 0$ so as to minimize the exponential factor before the integral. We thus get a bound[4]:

$$|\mathbf{G}(x,y)| \;\leq\; C\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\;\int_0^{+\infty} \mathrm{e}^{3\,c_3\,y\,|\mu_0|\,\theta^2}\,\mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

The integral is estimated by cutting at $\theta_0 > 0$ such that $3\,c_3\,|\mu_0| = (c_4/4)\,\theta_0^2$, so that we have:

$$\int_{\theta_0}^{+\infty} \mathrm{e}^{3\,c_3\,y\,|\mu_0|\,\theta^2}\,\mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta \;=\; \int_{\theta_0}^{+\infty} \mathrm{e}^{\frac{c_4}{4}\,y\,\theta_0^2\,\theta^2}\,\mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta \;\leq\; \int_{\theta_0}^{+\infty} \mathrm{e}^{-\frac{c_4}{4}\,y\,\theta^4}\,\mathrm{d}\theta \;\leq\; \frac{C}{y^{1/4}}\,,$$

and we also have:

$$\int_0^{\theta_0} \mathrm{e}^{3\,c_3\,y\,|\mu_0|\,\theta^2}\,\mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta \;\leq\; \int_0^{\theta_0} \mathrm{e}^{3\,c_3\,y\,|\mu_0|\,\theta^2}\,\mathrm{d}\theta \;\leq\; \theta_0\,\mathrm{e}^{3\,c_3\,y\,|\mu_0|\,\theta_0^2} \;\leq\; C\,|\mu_0|^{1/2}\,\mathrm{e}^{C\,y\,\mu_0^2}\,.$$

Going back to the definition of μ_0 in terms of x and y, we thus get the estimate:

$$|\mathbf{G}(x,y)| \;\leq\; \frac{C}{y^{1/4}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right) \;+\; C\,\left(\frac{|x|}{y}\right)^{1/6}\,\exp(C\,y^{1/3}\,|x|^{2/3})\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\,,$$

where all constants C and c are independent of $x < 0$ and $y > 0$. It is now not difficult to show that if the constant $\mathbf{C}_\flat$ is chosen large enough and is we restrict to the regime $x \leq -\mathbf{C}_\flat\,y$, then the decaying exponential term $\exp(-c\,|x|^{4/3}/y^{1/3})$ can absorb the large term $\exp(C\,y^{1/3}\,|x|^{2/3})$. There is of course no loss of generality in assuming that $\mathbf{C}_\flat$ is chosen larger than the constant $\mathbf{c}_\flat$ given in Proposition A.3. Now that $\mathbf{C}_\flat$ is fixed and that we restrict to the regime $x \leq -\mathbf{C}_\flat\,y$, we get the estimate:

$$\begin{aligned}|\mathbf{G}(x,y)| \;&\leq\; \frac{C}{y^{1/4}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right) + \frac{C}{y^{1/8}}\,\left(\frac{|x|}{y^{1/4}}\right)^{1/6}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\\ &\leq\; \frac{C}{y^{1/4}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right) + \frac{C}{y^{1/8}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\,,\end{aligned}$$

with (new) constants C and c that are still independent of $y > 0$ and $x \leq -\mathbf{C}_\flat\,y$. We can still spare half of the last decaying exponential term on the right-hand side to gain a decaying factor $\exp(-c\,y)$ and then use a crude bound:

$$\frac{1}{y^{1/8}}\,\exp(-c\,y) \;\leq\; \frac{C}{y^{1/4}}\,,$$

[4] The integral is reduced to the domain $\mathbb{R}^+$ by an obvious change of variable, which only modifies the constant in the inequality by a factor 2.

so that, eventually, we get the claimed bound:

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{y^{1/4}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\,,$$

for $x \leq -\mathbf{C}_\flat\, y$. The proof of Proposition A.4 is complete. □

We now turn to the final, intermediate regime where x is negative and $|x|$ and y are comparable.

Proposition A.5. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let the constant* $\mathbf{c}_\flat$ *be the one given by Proposition A.3 and let the constant* $\mathbf{C}_\flat$ *be the one given by Proposition A.4. Then there exist some constants* $C > 0$ *and* $c > 0$ *such that for any* $(x,y) \in \mathbb{R} \times \mathbb{R}^{+*}$ *with* $-\mathbf{C}_\flat\, y \leq x \leq -\mathbf{c}_\flat\, y$, *there holds:*

$$|\mathbf{G}(x,y)| \,\leq\, \frac{C}{\min(1,y^{1/4})}\,\exp(-c\,y)\,.$$

Proof. We go back to the definition (A.9) for $\mathbf{G}(x,y)$ but we now deform the integration axis $\mathbb{R}$ as depicted on Figure A.2. Namely, we introduce some parameter $\delta > 0$ to be fixed later on and first consider the contribution:

$$\varepsilon(x,y) \,:=\, \frac{1}{2\,\pi}\int_{-\infty}^{-\delta} \mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \,+\, \frac{1}{2\,\pi}\int_{\delta}^{+\infty} \mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,. \tag{A.34}$$

The integral $\mathscr{H}_1(x,y)$ corresponds to the integral along the segment from $-\delta$ to $-\mathbf{i}\,\delta$, namely:

$$\mathscr{H}_1(x,y) \,:=\, \frac{\delta\,(1-\mathbf{i})}{2\,\pi}\int_0^1 \mathrm{e}^{\mathbf{i}\,x\,(-\delta\,(1-t)\,-\,\mathbf{i}\,\delta\,t)}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,(-\delta\,(1-t)\,-\,\mathbf{i}\,\delta\,t)^3}\,\mathrm{e}^{-\,c_4\,y\,(-\delta\,(1-t)\,-\,\mathbf{i}\,\delta\,t)^4}\,\mathrm{d}t\,, \tag{A.35}$$

while $\mathscr{H}_2(x,y)$ corresponds to the integral along the segment from $-\mathbf{i}\,\delta$ to δ, namely:

$$\mathscr{H}_2(x,y) \,:=\, \frac{\delta\,(1+\mathbf{i})}{2\,\pi}\int_0^1 \mathrm{e}^{\mathbf{i}\,x\,(\delta\,t\,-\,\mathbf{i}\,\delta\,(1-t))}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,(\delta\,t\,-\,\mathbf{i}\,\delta\,(1-t))^3}\,\mathrm{e}^{-\,c_4\,y\,(\delta\,t\,-\,\mathbf{i}\,\delta\,(1-t))^4}\,\mathrm{d}t\,. \tag{A.36}$$

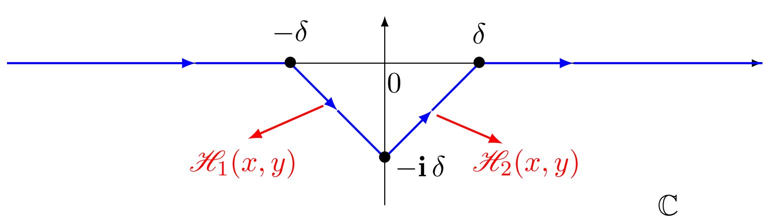

Figure A.2: *The integration contour in the case* $-\mathbf{C}_\flat\, y \leq x \leq -\mathbf{c}_\flat\, y$.

Cauchy's formula allows us to decompose $\mathbf{G}(x,y)$ as:

$$\mathbf{G}(x,y) \,=\, \mathscr{H}_1(x,y) \,+\, \mathscr{H}_2(x,y) \,+\, \varepsilon(x,y)\,,$$

and we now estimate each contribution in $\mathbf{G}(x,y)$ one by one. We first estimate the integral $\mathscr{H}_1(x,y)$ defined in (A.35) and explain how the parameter $\delta > 0$ is fixed. We restrict from on to the regime $-\mathbf{C}_\flat\, y \leq x \leq -\mathbf{c}_\flat\, y$ where the constant $\mathbf{c}_\flat$, resp. $\mathbf{C}_\flat$, is given by Proposition A.3, resp. Proposition A.4.

This implies in particular that x is negative. We expand the polynomial quantities within the exponential terms in (A.35) and use the triangle inequality to obtain:

$$|\mathscr{H}_1(x,y)| \,\leq\, C\,\delta \int_0^1 \exp\Big(-\,|x|\,\delta\,t \,-\, c_3\,y\,\delta^3\,t^3 \,+\, 3\,c_3\,y\,\delta^3\,(1-t)^2\,t\Big) \\ \times\,\exp\Big(-\,c_4\,y\,\delta^4\,(1-t)^4 \,+\, 6\,c_4\,y\,\delta^4\,(1-t)^2\,t^2 \,-\, c_4\,y\,\delta^4\,t^4\Big)\,\mathrm{d}t\,.$$

Estimating exponentials of negative terms by 1, we have:

$$|\mathscr{H}_1(x,y)| \,\leq\, C\,\delta \int_0^1 \exp\Big(-\,|x|\,\delta\,t \,+\, 3\,c_3\,\delta^3\,y\,t \,+\, 6\,c_4\,\delta^4\,y\,t^2 \,-\, c_4\,\delta^4\,y\,(1-t)^4\Big)\,\mathrm{d}t\,.$$

We fix the constant $\delta > 0$ by imposing:

$$3\,c_3\,\delta^2 \,\leq\, \frac{\mathbf{c}_\flat}{3}\,,\quad \text{and}\quad 6\,c_4\,\delta^3 \,\leq\, \frac{\mathbf{c}_\flat}{3}\,. \tag{A.37}$$

Since δ is fixed and only depends on already fixed parameters, we allow constants below to depend on δ. We use the restrictions (A.37) in the previous estimate for $\mathscr{H}_1(x,y)$ and use furthermore the inequality $|x| \geq \mathbf{c}_\flat\,y$ to obtain:

$$|\mathscr{H}_1(x,y)| \,\leq\, C \int_0^1 \exp\Big(-\,y\,\Big(\frac{\mathbf{c}_\flat}{3}\,\delta\,t \,+\, c_4\,\delta^4\,(1-t)^4\Big)\Big)\,\mathrm{d}t\,.$$

We now observe that (A.37) implies that the function:

$$t \in [0,1] \longmapsto \frac{\mathbf{c}_\flat}{3}\,\delta\,t \,+\, c_4\,\delta^4\,(1-t)^4\,,$$

is increasing. We thus have the exponentially decaying bound:

$$|\mathscr{H}_1(x,y)| \,\leq\, C\,\exp(-\,c_4\,\delta^4\,y) \,=\, C\,\exp(-\,c\,y)\,.$$

By a simple change of variable $t \to 1-t$, we easily find that the integral $\mathscr{H}_2(x,y)$ in (A.36) equals the complex conjuugate of $\mathscr{H}_1(x,y)$ so the previous estimate for $\mathscr{H}_1(x,y)$ also applies to $\mathscr{H}_2(x,y)$.

We end the argument with the last remaining term $\varepsilon(x,y)$ defined in (A.34). We recall that $\delta > 0$ has been fixed in the analysis of the contributions $\mathscr{H}_1(x,y)$ and $\mathscr{H}_2(x,y)$. By applying the triangle inequality, we have:

$$|\varepsilon(x,y)| \,\leq\, \frac{1}{2\,\pi} \int_{-\infty}^{-\delta} \mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \,+\, \frac{1}{2\,\pi} \int_{\delta}^{+\infty} \mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta \,=\, \frac{1}{\pi} \int_{\delta}^{+\infty} \mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,.$$

We thus get the estimate:

$$|\varepsilon(x,y)| \,\leq\, \frac{\mathrm{e}^{-\frac{c_4}{2}\,\delta^4\,y}}{\pi} \int_{\delta}^{+\infty} \mathrm{e}^{-\frac{c_4}{2}\,y\,\theta^4}\,\mathrm{d}\theta \,\leq\, \frac{C}{y^{1/4}}\,\exp(-\,c\,y)\,.$$

Going back to the decomposition of $\mathbf{G}(x,y)$, we collect the estimates of $\mathscr{H}_1(x,y)$, $\mathscr{H}_2(x,y)$ and $\varepsilon(x,y)$ to get:

$$|\mathbf{G}(x,y)| \,\leq\, C\,\exp(-\,c\,y) \,+\, \frac{C}{y^{1/4}}\,\exp(-\,c\,y)\,.$$

The conclusion of Proposition A.5 follows. □

We first collect the results of Proposition A.4 and Proposition A.5 to obtain the following unified estimate, the proof of which is left to the interested reader. The constant $\mathbf{c}_\flat$ in Corollary A.4 below is, of course, the same as the one given by Proposition A.3.

Corollary A.4. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let* $y_{\min} > 0$ *be given. Then there exist some constants* $\mathbf{c}_\flat > 0$, $C > 0$ *and* $c > 0$ *such that for any* $(x, y) \in \mathbb{R} \times [y_{\min}, +\infty)$, *there holds:*

$$|\mathbf{G}(x,y)| \le \frac{C}{y^{1/4}} \exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right),$$

as long as x *satisfies* $x \le -\mathbf{c}_\flat\, y$.

The final Corollary is precisely what we are aiming at, namely at estimates that are analogous to those of Proposition A.3 and Corollary A.4 but with now a transition at $-\underline{\mathbf{c}}\, y$ with an arbitrary constant $\underline{\mathbf{c}}$. The result is the following.

Corollary A.5. *Let the function* $\mathbf{G}$ *be defined in* (A.9) *and let* c_3 *be positive. Let the function* $\mathfrak{g}$ *be defined in* (A.10). *Let also* $y_{\min} > 0$ *be given. Then for any constant* $\underline{\mathbf{c}} > 0$, *there exist constants* $C > 0$ *and* $c > 0$ *such that for any* $(x, y) \in \mathbb{R} \times [y_{\min}, +\infty)$, *there holds:*

$$|\mathbf{G}(x,y) - 2\,\mathrm{Re}\ \mathfrak{g}(x,y)| \le \frac{C}{y^{1/3}} \exp\left(-c\,\frac{|x|^{3/2}}{y^{1/2}}\right) + \frac{C}{y^{1/2}} \exp\left(-c\,\frac{x^2}{y}\right), \tag{A.38}$$

if $-\underline{\mathbf{c}}\, y \le x \le -y^{1/3}$, *and there holds:*

$$|\mathbf{G}(x,y)| \le \frac{C}{y^{1/4}} \exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right), \tag{A.39}$$

if $x \le -\underline{\mathbf{c}}\, y$.

Proof. The argument is mostly similar to that we used in the proof of Corollary A.3 except that we now need to control also $\mathfrak{g}(x, y)$ in the region where $|x|$ and y are comparable. Let us for instance deal with the case $\underline{\mathbf{c}} \ge \mathbf{c}_\flat$ where $\mathbf{c}_\flat > 0$ is the constant given in Proposition A.3. If $x \le -\underline{\mathbf{c}}\, y$, we have $x \le -\mathbf{c}_\flat\, y$ and the desired estimate (A.39) for $|\mathbf{G}(x, y)|$ is given by Corollary A.4. Moreover, if x satisfies $-\mathbf{c}_\flat\, y \le x \le -y^{1/3}$, the estimate (A.38) is given by Proposition A.3 and by using $y \ge y_{\min} > 0$ in order to estimate from below the quantity $\min(1, y^{1/3})$.

It thus remains to control the left hand side of (A.38) in the case where x is negative and $|x|/y \in [\mathbf{c}_\flat, \underline{\mathbf{c}}]$, which we assume from now on. We have:

$$\begin{aligned}|\mathbf{G}(x,y) - 2\,\mathrm{Re}\ \mathfrak{g}(x,y)| &\le |\mathbf{G}(x,y)| + 2\,|\mathfrak{g}(x,y)| \\ &\le \frac{C}{y^{1/4}} \exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right) + 2\,|\mathfrak{g}(x,y)|,\end{aligned}$$

where the second inequality follows from Corollary A.4 since we have $x \le -\mathbf{c}_\flat\, y$. The quantity $\mathfrak{g}(x, y)$ is estimated directly from the definition (A.10). Applying the triangle inequality, we have:

$$|\mathfrak{g}(x,y)| \le C \exp\left(-c\,\frac{x^2}{y}\right) \int_{-\sqrt{\frac{2\,|x|}{3\,c_3\,y}}}^{\sqrt{\frac{2\,|x|}{3\,c_3\,y}}} \mathrm{e}^{-\sqrt{3\,c_3\,|x|\,y}\,t^2 + \frac{c_3}{\sqrt{2}}\,y\,t^3}\,\mathrm{d}t,$$

and we have already seen that on the considered integration interval, there holds:

$$\mathrm{e}^{-\sqrt{3\,c_3\,|x|\,y}\,t^2+\frac{c_3}{\sqrt{2}}\,y\,t^3}\;\le\;\mathrm{e}^{-c\,\sqrt{|x|\,y}\,t^2}$$

for some appropriate numerical constant $c>0$. Since we consider the regime $|x|/y\in[\mathbf{c}_\flat,\underline{\mathbf{c}}]$, we thus have:

$$|\mathfrak{g}(x,y)|\;\le\;C\,\exp\left(-c\,\frac{x^2}{y}\right)\,\int_{\mathbb{R}}\mathrm{e}^{-c\,y_{\min}\,t^2}\,\mathrm{d}t\;\le\;C\,\exp\left(-c\,y\right)\,,$$

which yields:

$$|\mathbf{G}(x,y)\;-\;2\,\mathrm{Re}\;\mathfrak{g}(x,y)|\;\le\;C\,\exp\left(-c\,y\right)\,.$$

As we have already seen several times before, there is no difficulty at this stage to derive the estimate (A.38) for the regime $|x|/y\in[\mathbf{c}_\flat,\underline{\mathbf{c}}]$ and $y\ge y_{\min}$. This completes the proof of Corollary A.5. □

A.5 Proof of Theorem A.3

The combination of Proposition A.1, Corollary A.3 and Corollary A.5 already gives the estimates (A.11) and (A.12) of the function $\mathbf{G}$ as given in Theorem A.3. It therefore only remains to prove the estimates (A.13) and (A.14) for the primitive function of $\mathbf{G}$ with respect to its first variable. We still assume $c_3>0$ and start with the first estimate in (A.13). We consider a fixed positive constant $\underline{\mathbf{c}}$ and let $x\le-\underline{\mathbf{c}}\,y$. We then use the estimate in the fourth case of (A.11) to get:

$$\left|\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;\int_{-\infty}^{x}|\mathbf{G}(\xi,y)|\,\mathrm{d}\xi\;\le\;\frac{C}{y^{1/4}}\,\int_{-\infty}^{x}\exp\left(-c\,\frac{|\xi|^{4/3}}{y^{1/3}}\right)\,\mathrm{d}\xi\,.$$

We then perform a change of variable to obtain:

$$\left|\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;C\,\int_{-\infty}^{x/y^{1/4}}\exp(-c\,|\eta|^{4/3})\,\mathrm{d}\eta\;=\;C\,\int_{-\infty}^{x/y^{1/4}}\frac{(4/3)\,c\,|\eta|^{1/3}}{(4/3)\,c\,|\eta|^{1/3}}\,\exp(-c\,|\eta|^{4/3})\,\mathrm{d}\eta\,.$$

Since we assume $x\le-\underline{\mathbf{c}}\,y$, we have $|\eta|^{1/3}\ge\underline{\mathbf{c}}^{1/3}\,y^{1/4}$ in the last integral on the right-hand side, and this gives:

$$\left|\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;\frac{C}{y^{1/4}}\,\int_{-\infty}^{x/y^{1/4}}(4/3)\,c\,|\eta|^{1/3}\,\exp(-c\,|\eta|^{4/3})\,\mathrm{d}\eta\,.$$

We thus end up with the estimate:

$$\left|\int_{-\infty}^{x}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;\frac{C}{y^{1/4}}\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\;\le\;C\,\exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right)\,,$$

for $x\le-\underline{\mathbf{c}}\,y$ and $y\ge y_{\min}$, an estimate from which the first half of (A.13) follows directly. Note in particular that we have the uniform estimate:

$$\left|\int_{-\infty}^{-\underline{\mathbf{c}}\,y}\mathbf{G}(\xi,y)\,\mathrm{d}\xi\right|\;\le\;C\,,\tag{A.40}$$

for any $y\ge y_{\min}$.

From the definition (A.9), we know that for any $y > 0$, $\mathbf{G}(\cdot, y)$ is the inverse Fourier transform of the Schwartz class function:

$$\theta \in \mathbb{R} \longmapsto \mathrm{e}^{\mathbf{i}\, c_3\, y\, \theta^3}\, \mathrm{e}^{-c_4\, y\, \theta^4}\,.$$

In particular, this means that the (partial) Fourier transform of $\mathbf{G}$ with respect to its first variable is given by:

$$\mathscr{F}_x(\mathbf{G})(\theta, y) \;=\; \mathrm{e}^{\mathbf{i}\, c_3\, y\, \theta^3}\, \mathrm{e}^{-c_4\, y\, \theta^4}\,.$$

Evaluating at $\theta = 0$, we get the relation:

$$\int_{\mathbb{R}} \mathbf{G}(x, y)\, \mathrm{d}x \;=\; 1\,,$$

so that proving the bound (A.14) amounts to showing the bound:

$$\left| \int_x^{+\infty} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \right| \;\leq\; \widetilde{C}\, \exp(-\tilde{c}\, y)\,, \quad \text{if } x \geq \underline{\mathbf{c}}\, y.$$

The proof of this bound follows from the exact same argument as above except that we now use the first case in (A.11).

The last part of the proof of Theorem A.3 aims at showing the second half of (A.13), that is, at proving a uniform bound for the primitive function of $\mathbf{G}$ in the case $|x| \leq \underline{\mathbf{c}}\, y$. Let us first assume $x \geq 0$ and thus $x \in [0, \underline{\mathbf{c}}\, y]$. We have already seen the relation:

$$\int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \;=\; 1 - \int_x^{\underline{\mathbf{c}}\, y} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \;-\; \int_{\underline{\mathbf{c}}\, y}^{+\infty} \mathbf{G}(\xi, y)\, \mathrm{d}\xi\,,$$

and we thus already have from the arguments above (namely, the case $x \geq \underline{\mathbf{c}}\, y$):

$$\left| \int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \right| \;\leq\; C \;+\; \int_x^{\underline{\mathbf{c}}\, y} |\mathbf{G}(\xi, y)|\, \mathrm{d}\xi\,.$$

The final integral on the right-hand side is estimated by using the second case in (A.11), which gives the uniform control (after an obvious change of variable in ξ):

$$\left| \int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \right| \;\leq\; C\,.$$

We can actually push a little further this argument and obtain from (A.11) a uniform bound for:

$$\int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi$$

as long as x belongs to the interval $[-y^{1/3}, \underline{\mathbf{c}}\, y]$ and not only when x belongs to $[0, \underline{\mathbf{c}}\, y]$ (use now the third case in (A.11)). It thus remains to examine the case $x \in [-\underline{\mathbf{c}}\, y, -y^{1/3}]$ for which we need to take into account the oscillating behavior of the Green's function. This is the only regime where applying the triangle inequality to estimate the primitive function does not (and can not !) work. Let therefore $x \in [-\underline{\mathbf{c}}\, y, -y^{1/3}]$. We decompose:

$$\int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \;=\; \int_{-\infty}^{-\underline{\mathbf{c}}\, y} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \;+\; \int_{-\underline{\mathbf{c}}\, y}^{x} \big(\mathbf{G}(\xi, y) - 2\, \mathrm{Re}\; \mathfrak{g}(\xi, y) \big)\, \mathrm{d}\xi$$
$$+\; 2\, \mathrm{Re} \int_{-\underline{\mathbf{c}}\, y}^{x} \mathfrak{g}(\xi, y)\, \mathrm{d}\xi\,.$$

The first term on the right-hand side has already been estimated, see (A.40), and the second term on the right-hand side is estimated by using the bound[5] (A.12). At this stage, we already have an estimate that reads:

$$\left| \int_{-\infty}^{x} \mathbf{G}(\xi, y)\, \mathrm{d}\xi \right| \le C + 2 \left| \int_{-\underline{\mathbf{c}}\, y}^{x} \mathfrak{g}(\xi, y)\, \mathrm{d}\xi \right| , \tag{A.41}$$

and the remaining point of the proof is to derive a uniform estimate for the primitive function of the *explicit* function $\mathfrak{g}$ whose expression is given in (A.10). The proof relies on integration by parts, as detailed below.

We first introduce the notation:

$$\beta_0 := \frac{c_4}{9\, c_3^2} > 0 \, , \quad \beta_1 := \frac{2}{3\sqrt{3\, c_3}} > 0 \, ,$$

so that, after performing a change of variable in the integral of (A.10), we obtain the expression:

$$\mathfrak{g}(x, y) := \frac{3\, \beta_1\, \mathrm{e}^{-\mathbf{i}\, \pi/4}}{2\sqrt{2}\, \pi} \frac{|x|^{1/2}}{y^{1/2}} \exp\left(-\beta_0 \frac{x^2}{y} \right) \exp\left(\mathbf{i}\, \beta_1 \frac{|x|^{3/2}}{y^{1/2}} \right) \int_{-1}^{1} \mathrm{e}^{-3\, \beta_1 \frac{|x|^{3/2}}{y^{1/2}} u^2}\, \mathrm{e}^{\beta_1 (1-\mathbf{i}) \frac{|x|^{3/2}}{y^{1/2}} u^3}\, \mathrm{d}u \, . \tag{A.42}$$

Let us define a function H on $\mathbb{R}^+$ as follows (the constant $\beta_1 > 0$ being fixed as above):

$$\forall\, w \ge 0 \, , \quad H(w) := \exp\left(\mathbf{i}\, \beta_1\, w \right) \int_{-1}^{1} \mathrm{e}^{-3\, \beta_1\, w\, u^2}\, \mathrm{e}^{\beta_1 (1-\mathbf{i})\, w\, u^3}\, \mathrm{d}u \, . \tag{A.43}$$

With the help of (A.43), we can rewrite (A.42) and obtain the relation:

$$\forall\, (x, y) \in \mathbb{R} \times \mathbb{R}^{+*} \, , \quad y^{1/2}\, \mathfrak{g}(y^{1/2}\, x, y) := \frac{\beta_1\, \mathrm{e}^{-\mathbf{i}\, \pi/4}}{\sqrt{2}\, \pi}\, \mathrm{e}^{-\beta_0\, x^2} \left(\frac{3}{2}\, y^{1/4}\, |x|^{1/2} \right) H(y^{1/4}\, |x|^{3/2}) \, . \tag{A.44}$$

where the factor $(3/2)\, y^{1/4}\, |x|^{1/2}$ equals, up to a sign, the derivative of the function $(x \mapsto y^{1/4}\, |x|^{3/2})$. We are thus in a very favorable position for applying integration by parts. Before going further, we prove the following Lemma.

Lemma A.1. *Let the function H be defined on $\mathbb{R}^+$ by (A.43). Then there exists a constant $C > 0$ that only depends on β_1 and such that:*

$$\forall\, w \ge 0 \, , \quad \left| \int_0^w H(w')\, \mathrm{d}w' \right| \le C \, .$$

Proof. We consider $w \ge 0$. By applying the Fubini Theorem, we get the relation:

$$\int_0^w H(w')\, \mathrm{d}w' = \int_{-1}^{1} \int_0^w \exp\Big(\beta_1\, w' \left(-3\, u^2 + u^3 + \mathbf{i}\, (1 - u^3) \right) \Big)\, \mathrm{d}w'\, \mathrm{d}u \, ,$$

and it turns out that the continuous function:

$$u \in [-1, 1] \longmapsto -3\, u^2 + u^3 + \mathbf{i}\, (1 - u^3) \, ,$$

[5] The whole point of (A.12) was precisely to extract from $\mathbf{G}$ its leading oscillating behavior so that the difference between $\mathbf{G}$ and this "leading order term" would actually become uniformly integrable.

does not vanish. Its modulus is thus uniformly bounded from below. We get:

$$\int_0^w H(w')\,\mathrm{d}w' \;=\; \int_{-1}^1 \frac{\exp\Big(\beta_1\,w\,\big(-3\,u^2+u^3\;+\,\mathbf{i}\,(1-u^3)\big)\Big)-1}{\beta_1\,\big(-3\,u^2+u^3\;+\,\mathbf{i}\,(1-u^3)\big)}\,\mathrm{d}u\,,$$

and the triangle inequality (as well as a lower bound for the modulus of the denominator) yields:

$$\left|\int_0^w H(w')\,\mathrm{d}w'\right| \;\le\; C\,\int_{-1}^1 \exp\big(\beta_1\,w\,(-3\,u^2+u^3)\big)+1\,\mathrm{d}u\,.$$

We have $-3\,u^2+u^3\le 0$ for $u\in[-1,1]$ and $w\ge 0$ so the uniform bound of Lemma A.1 follows. □

We now go back to our main problem, which is to prove a bound for the primitive function of $\mathfrak{g}$, see (A.41). We use the expression (A.44) and obtain:

$$\begin{aligned}
\int_{-\underline{\mathbf{c}}\,y}^x \mathfrak{g}(\xi,y)\,\mathrm{d}\xi \;&=\; \int_{-\underline{\mathbf{c}}\,y^{1/2}}^{x/y^{1/2}} y^{1/2}\,\mathfrak{g}(y^{1/2}\,w,y)\,\mathrm{d}w \\
&= \frac{\beta_1\,\mathrm{e}^{-\mathbf{i}\,\pi/4}}{\sqrt{2}\,\pi}\,\int_{|x|/y^{1/2}}^{\underline{\mathbf{c}}\,y^{1/2}} \mathrm{e}^{-\beta_0\,w^2}\,\left(\frac{3}{2}\,y^{1/4}\,w^{1/2}\,H(y^{1/4}\,w^{3/2})\right)\,\mathrm{d}w \\
&= \frac{\beta_1\,\mathrm{e}^{-\mathbf{i}\,\pi/4}}{\sqrt{2}\,\pi}\,\int_{|x|/y^{1/2}}^{\underline{\mathbf{c}}\,y^{1/2}} \mathrm{e}^{-\beta_0\,w^2}\,\frac{\mathrm{d}}{\mathrm{d}w}\left(\int_0^{y^{1/4}\,w^{3/2}} H(w')\,\mathrm{d}w'\right)\,\mathrm{d}w\,.
\end{aligned}$$

It then only remains to integrate by parts the final integral and to apply Lemma A.1 in order to derive the uniform bound:

$$\left|\int_{-\underline{\mathbf{c}}\,y}^x \mathfrak{g}(\xi,y)\,\mathrm{d}\xi\right| \;\le\; C\,.$$

This final argument is a prototype application of Abel's transform (in the continuous setting). This completes the proof of Theorem A.3.

A.6 Consequences

This final paragraph is devoted to applying Theorem A.3 in order to derive suitable bounds for the activation function $\mathfrak{A}$ and other quantities that arise in our decomposition of the Green's function of the operator $\mathscr{L}$ in (2.11). We therefore now go back to the framework of stationary discrete shock profiles and use the index r, resp. ℓ, to refer to the right, resp. left, state of the discrete shock (2.2). The analysis of Section 4.3 uses the following quantities defined for any $j_0\in\mathbb{N}^*$ and $n\in\mathbb{N}^*$:

$$\mathfrak{A}_r^n(j_0) \;:=\; \frac{1}{2\,\mathbf{i}\,\pi}\,\int_{\eta+\mathbf{i}\,\mathbb{R}} \mathrm{e}^{n\,\tau-j_0\,\varphi_r(\tau)}\,\frac{\mathrm{d}\tau}{\tau}\,,$$

where η is any positive number (the Cauchy formula shows that the definition is independent of η) and:

$$\mathfrak{B}_r^n(j_0) \;:=\; \frac{1}{2\,\mathbf{i}\,\pi}\,\int_{\mathbf{i}\,\mathbb{R}} \mathrm{e}^{n\,\tau-j_0\,\varphi_r(\tau)}\,\mathrm{d}\tau\,.$$

In both definitions of $\mathfrak{A}_r^n(j_0)$ and $\mathfrak{B}_r^n(j_0)$, the function φ_r is defined by:

$$\forall\,\tau\in\mathbb{C}\,,\quad \varphi_r(\tau) \;:=\; -\,\frac{1}{\alpha_r}\,\tau\;+\;\frac{1-\alpha_r^2}{6\,\alpha_r^3}\,\tau^3\;-\;\frac{1-\alpha_r^2}{8\,\alpha_r^3}\,\tau^4\,.$$

At last, we recall that α_r belongs to the interval $(-1,0)$, see (2.12). This is a consequence of Lax shock inequalities and the choice of the CFL parameter.

By using the parametrization $\tau = \mathbf{i}\,|\alpha_r|\,\theta$ in the definition of $\mathfrak{B}_r^n(j_0)$, we obtain the expression:

$$\mathfrak{B}_r^n(j_0) \;:=\; \frac{|\alpha_r|}{2\,\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathbf{i}\,(j_0+n\,\alpha_r)\,\theta} \, \mathrm{e}^{-\mathbf{i}\,j_0\,\frac{1-\alpha_r^2}{6}\,\theta^3} \, \mathrm{e}^{j_0\,\alpha_r\,\frac{1-\alpha_r^2}{8}\,\theta^4} \, \mathrm{d}\theta \,,$$

and the integral is convergent since j_0 is positive and α_r belongs to the interval $(-1,0)$. Going back to the definitions (A.4) (with the choice $\alpha = \alpha_r \in (-1,0)$) and (A.9), we have thus obtained the relation:

$$\forall\, j_0 \in \mathbb{N}^* \,, \quad \forall\, n \in \mathbb{N}^* \,, \quad \mathfrak{B}_r^n(j_0) \;:=\; |\alpha_r|\, \mathbf{G}_r \left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|} \right) \,, \tag{A.45}$$

which is the reason why Theorem A.3 will give us exactly what we need for proving the bounds we need in our analysis. The index r in $\mathbf{G}_r$ refers to the fact that we have made the choice $\alpha = \alpha_r$ when defining the constants c_3 and c_4 in (A.4).

Let us now turn to the activation function $\mathfrak{A}_r^n(j_0)$. Choosing the parametrization $\tau = |\alpha_r|\,(\eta + \mathbf{i}\,\theta)$ for any $\eta > 0$, we get:

$$\mathfrak{A}_r^n(j_0) \;:=\; \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathrm{e}^{-(j_0+n\,\alpha_r)\,(\eta+\mathbf{i}\,\theta)} \, \mathrm{e}^{j_0\,\frac{1-\alpha_r^2}{6}\,(\eta+\mathbf{i}\,\theta)^3} \, \mathrm{e}^{j_0\,\alpha_r\,\frac{1-\alpha_r^2}{8}\,(\eta+\mathbf{i}\,\theta)^4} \, \frac{\mathrm{d}\theta}{\eta+\mathbf{i}\,\theta} \,.$$

With the choice $\alpha = \alpha_r$ and the definition (A.4) for the coefficients c_3 and c_4, we can introduce the function $\mathbf{A}_r$ defined by:

$$\forall\,(x,y) \in \mathbb{R} \times \mathbb{R}^{+*} \,, \quad \mathbf{A}_r(x,y) \;:=\; \frac{1}{2\,\pi} \int_{\mathbb{R}} \mathrm{e}^{x\,(\eta+\mathbf{i}\,\theta)} \, \mathrm{e}^{-c_3\,y\,(\eta+\mathbf{i}\,\theta)^3} \, \mathrm{e}^{-c_4\,y\,(\eta+\mathbf{i}\,\theta)^4} \, \frac{\mathrm{d}\theta}{\eta+\mathbf{i}\,\theta} \,, \tag{A.46}$$

where the definition is independent of the choice of $\eta > 0$ because of Cauchy's formula. With this notation, we have expressed the activation function $\mathfrak{A}_r^n(j_0)$ as:

$$\mathfrak{A}_r^n(j_0) \;=\; \mathbf{A}_r \left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|} \right) \,. \tag{A.47}$$

It remains to connect the function $\mathbf{A}_r$ with the primitive function of $\mathbf{G}_r$ with respect to its first variable. From the dominated convergence theorem, we have that $\mathbf{A}_r$ is differentiable with respect to its first variable and (passing to the limit $\eta \to 0$ in the expression of the partial derivative):

$$\frac{\partial \mathbf{A}_r}{\partial x}(x,y) \;=\; \mathbf{G}_r(x,y) \,.$$

Moreover, the factor $\exp(x\,\eta)$ can be extracted from the integral and we thus have:

$$\lim_{x \to -\infty} \mathbf{A}_r(x,y) \;=\; 0 \,,$$

from which we get the general expression[6]:

$$\mathbf{A}_r(x,y) \;=\; \int_{-\infty}^{x} \mathbf{G}_r(\xi,y)\,\mathrm{d}\xi \,.$$

[6]The integral is convergent because of the bounds that we proved in Theorem A.3.

Going back to (A.47), this means that we have expressed the activation function $\mathfrak{A}_r^n(j_0)$ as follows:

$$\forall\, j_0 \in \mathbb{N}^*\,, \quad \forall\, n \in \mathbb{N}^*\,, \quad \mathfrak{A}_r^n(j_0) \;=\; \int_{-\infty}^{-j_0+n\,|\alpha_r|} \mathbf{G}_r\left(\xi, \frac{j_0}{|\alpha_r|}\right)\,\mathrm{d}\xi\,. \tag{A.48}$$

Theorem A.3 can be recast into the following compact form that will be helpful in the analysis of Chapter 4.

Corollary A.6. *Let the function $\mathbf{A}_r$ be defined in (A.46) with constants c_3 and c_4 as in (A.4) with the choice $\alpha = \alpha_r \in (-1,0)$. Then for any constant $\underline{\mathbf{c}} > 0$, there exist two positive constants C and c such that for any $n \in \mathbb{N}^*$ and any $j_0 \in \mathbb{N}^*$, there holds:*

$$|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)| \;\le\; \begin{cases} C\,\exp\left(-c\,\dfrac{|j_0-n\,|\alpha_r||^{4/3}}{n^{1/3}}\right)\,, & \text{if } -j_0+n\,|\alpha_r| \le -\underline{\mathbf{c}}\,n,\\ C\,, & \text{if } -\underline{\mathbf{c}}\,n \le -j_0+n\,|\alpha_r| \le \underline{\mathbf{c}}\,n. \end{cases}$$

and:

$$|1-\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)| \;\le\; C\,\exp\left(-c\,\frac{|j_0-n\,|\alpha_r||^{4/3}}{n^{1/3}}\right)\,, \quad \text{if } -j_0+n\,|\alpha_r| \ge \underline{\mathbf{c}}\,n.$$

In particular, there holds :

$$\sup_{j_0\in\mathbb{Z}\,,\,n\in\mathbb{N}^*} \;|\mathbf{A}_r(-j_0+n\,|\alpha_r|,n)| \;<\; +\infty\,.$$

A.7 Higher order estimates

Another crucial estimate that was needed in the analysis of Section 4.3 (see the proof of Lemma 4.2) aims at controlling the difference:

$$\mathbf{A}_r\left(-j_0+n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) \;-\; \mathbf{A}_r\left(-j_0+n\,|\alpha_r|,n\right)\,,$$

with the function $\mathbf{A}_r$ defined in (A.46) and $j_0, n \in \mathbb{N}^*$. Unsurprisingly, the most direct way to estimate this difference is to apply the mean value theorem, which gives rise to the partial derivative of $\mathbf{A}_r$ with respect to its second variable. This leads us to introduce the family of *correctors*:

$$\forall\, p \in \mathbb{N}^*\,, \quad \forall\,(x,y) \in \mathbb{R}\times\mathbb{R}^{+*}\,, \quad \mathbf{G}_p(x,y) \;:=\; \frac{1}{2\,\pi}\int_{\mathbb{R}} (\mathbf{i}\,\theta)^p\,\mathrm{e}^{\mathbf{i}\,x\,\theta}\,\mathrm{e}^{\mathbf{i}\,c_3\,y\,\theta^3}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,. \tag{A.49}$$

The case $p = 0$ corresponds to the definition (A.9) of $\mathbf{G}$ and the relevance of the functions $\mathbf{G}_p$ for controlling the above difference between the two values of $\mathbf{A}_r$ will be the purpose of Corollary A.7 below. We aim here at generalizing Theorem A.3 and at obtaining sharp bounds on $\mathbf{G}_p$ for any $p \in \mathbb{N}^*$. Our result is the following.

Theorem A.4. *Let us assume that the coefficient c_3 in (A.4) is positive, that is $\alpha \in (0,1)$. Let $p \in \mathbb{N}^*$. Let $y_{\min} > 0$ and let $\underline{\mathbf{c}} > 0$ be given. Then there exist some constants $C > 0$ and $c > 0$ such that, for any*

$(x,y) \in \mathbb{R} \times [y_{\min}, +\infty)$, *there holds:*

$$|\mathbf{G}_p(x,y)| \le \begin{cases} \dfrac{C}{y^{(p+1)/4}} \exp(-c\,x^{4/3}/y^{1/3})\,, & \text{if } x \ge \underline{\mathbf{c}}\,y, \\ \dfrac{C}{y^{1/3+p/4}} \exp(-c\,x^{3/2}/y^{1/2})\,, & \text{if } 0 \le x \le \underline{\mathbf{c}}\,y, \\ \dfrac{C}{y^{1/3+p/4}}\,, & \text{if } -y^{1/3} \le x \le 0, \\ \dfrac{C}{|x|^{1/4}\,y^{(p+1)/4}} \exp(-c\,x^2/y)\,, & \text{if } -\underline{\mathbf{c}}\,y \le x \le -y^{1/3}, \\ \dfrac{C}{y^{(p+1)/4}} \exp(-c\,|x|^{4/3}/y^{1/3})\,, & \text{if } x \le -\underline{\mathbf{c}}\,y. \end{cases} \tag{A.50}$$

If c_3 is negative, the same estimate holds for $\mathbf{G}_p$ with x being switched to $-x$.

Applying Theorem A.4 gives us the desired estimate for the difference between the two evaluations of $\mathbf{A}_r$, namely we have the following Corollary.

Corollary A.7. *Let the function $\mathbf{A}_r$ be defined in (A.46). There exist two positive constants C and c such that for any $n \in \mathbb{N}^*$ and any $j_0 \in \mathbb{N}^*$ that satisfies $j_0 \in [n\,|\alpha_r|/2, n]$, there holds:*

$$\left|\mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) - \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, n\right)\right|$$
$$\le C \begin{cases} \dfrac{1}{n^{1/3}} \exp\big(-c\,|j_0 - n\,|\alpha_r||^{3/2}/n^{1/2}\big)\,, & \text{if } j_0 \ge n\,|\alpha_r|, \\ \dfrac{1}{n^{1/3}}\,, & \text{if } 0 \le n\,|\alpha_r| - j_0 \le n^{1/3}, \\ \dfrac{1}{|j_0 - n\,|\alpha_r||^{1/4}\,n^{1/4}} \exp\big(-c\,|j_0 - n\,|\alpha_r||^2/n\big)\,, & \text{if } n\,|\alpha_r| - j_0 \ge n^{1/3}. \end{cases}$$

Proof of Theorem A.4. A very large part of the proof of Theorem A.4 follows that of Theorem A.3. We therefore feel free to refer to the various steps of the proof of Theorem A.3 (that corresponds to the case $p = 0$) and to shorten many details.

• Step 1: the uniform estimate. We follow the same argument as in the proof of Proposition A.1 but, keeping similar notation, we now use the choice $f(\theta) := x\,\theta + c_3\,y\,\theta^3$ and $g(\theta) := (\mathbf{i}\,\theta)^p \exp(-c_4\,y\,\theta^4)$, so that we have the estimates:

$$\|\,g\,\|_{L^\infty([a,b])} \le \frac{C}{y^{p/4}}\,, \quad \|\,g'\,\|_{L^1([a,b])} \le \frac{C}{y^{p/4}}\,,$$

for any $y > 0$, uniformly with respect to the interval $[a,b]$. By applying the same arguments as in the proof of Proposition A.1, we get the uniform estimate:

$$\forall\, y > 0\,, \quad \sup_{x \in \mathbb{R}} |\mathbf{G}_p(x,y)| \le \frac{C}{y^{1/3+p/4}}\,, \tag{A.51}$$

with a constant C that only depends on p.

• Step 2: the fast decaying side. Part I. Changing the integration line $\mathbb{R}$ to $\mathbf{i}\,\mu+\mathbb{R}$ for any $\mu\in\mathbb{R}$ thanks to the Cauchy formula, we then apply the triangle inequality and get a similar bound as the one we had obtained in (A.17), namely:

$$\forall\,\mu\in\mathbb{R}\,,\quad \forall\,(x,y)\in\mathbb{R}\times\mathbb{R}^{+*}\,,$$
$$|\mathbf{G}_p(x,y)|\,\le\,C_p\,\mathrm{e}^{-x\,\mu+c_3\,y\,\mu^3-c_4\,y\,\mu^4}\int_{\mathbb{R}}(|\mu|^p+|\theta|^p)\,\mathrm{e}^{-3\,y\,\mu\,(c_3-2\,c_4\,\mu)\,\theta^2}\,\mathrm{e}^{-c_4\,y\,\theta^4}\,\mathrm{d}\theta\,,\quad \text{(A.52)}$$

where the constant C_p only depends on $p\in\mathbb{N}^*$ that is a given fixed integer. The important property is that C_p does not depend on x,y nor μ.

Let us assume that c_3 is positive. We first consider the regime where x is positive and such that the positive parameter μ_0 defined by $\mu_0:=(x/(3\,c_3\,y))^{1/2}$ satisfies $2\,c_4\,\mu_0\le c_3/2$, which corresponds to the constraint $0<x\le\mathbf{c}_\sharp\,y$ for some well-defined positive constant $\mathbf{c}_\sharp$ (the same one as in the proof of Proposition A.2). Choosing the parameter μ_0 in (A.52), we follow the same arguments as in the proof of Proposition A.2 and get the bound (compare with (A.18)):

$$|\mathbf{G}_p(x,y)|\,\le\,C\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\int_{\mathbb{R}}\left(\frac{x^{p/2}}{y^{p/2}}+|\theta|^p\right)\mathrm{e}^{-c\,x^{1/2}\,y^{1/2}\,\theta^2}\,\mathrm{d}\theta\,,$$

for suitable constants C and c. Integrating with respect to θ, we thus get the bound:

$$|\mathbf{G}_p(x,y)|\,\le\,C\left\{\frac{1}{y^{1/4+p/3}\,\max(1,x^{1/4})}+\frac{1}{y^{(p+1)/4}\,\max(1,x^{(p+1)/4})}\right\}\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,,\quad \text{(A.53)}$$

which holds for any $y>0$ and $x\in(0,\mathbf{c}_\sharp\,y]$. We now argue similarly as in the proof of Corollary A.2. We consider $y\ge y_{\min}>0$ and $x\in[0,\mathbf{c}_\sharp\,y]$. For $x\le y^{1/(3(1+p))}$, we use the uniform bound (A.51) and for $x\in[y^{1/(3(1+p))},\mathbf{c}_\sharp\,y]$, we use (A.53). This combination of (A.51) and (A.53) gives the unified estimate:

$$\forall\,y\ge y_{\min}\,,\quad \forall\,x\in[0,\mathbf{c}_\sharp\,y]\quad |\mathbf{G}_p(x,y)|\,\le\,\frac{C}{y^{1/3+p/4}}\,\exp\left(-c\,\frac{x^{3/2}}{y^{1/2}}\right)\,,\quad \text{(A.54)}$$

where C and c are appropriate constants that do not depend on y and x.

• Step 3: the fast decaying side. Part II. The constant $\mathbf{c}_\sharp$ has been fixed and we consider the regime $y>0$, $x\ge\mathbf{c}_\sharp\,y$. We argue as in the proof of Proposition A.2 and first use Young's inequality to obtain:

$$\forall\,\mu>0\,,\quad |\mathbf{G}_p(x,y)|\,\le\,C\left\{\frac{\mu^{p-1/2}}{y^{1/2}}+\frac{1}{(\mu\,y)^{(p+1)/2}}\right\}\exp\left(f(\mu)\right)\,,\quad \text{(A.55)}$$

where f is the convex function defined in (A.20) whose minimum over $\mathbb{R}^+$ is attained at some $\underline{\mu}>0$. We have already shown the lower bound $\underline{\mu}\ge\mathbf{c}_\flat\,(x/y)^{1/3}$ in the proof of Proposition A.2 and the equality:

$$\underbrace{3\,c_3\,y\,\underline{\mu}^2}_{\ge 0}+32\,c_4\,y\,\underline{\mu}^3\,=\,x\,,$$

directly gives the upper bound $\underline{\mu} \le \mathbf{C}_\flat\,(x/y)^{1/3}$ for yet another constant $\mathbf{C}_\flat$. Choosing the optimal parameter $\underline{\mu}$ in (A.55) and following arguments as in the proof of Proposition A.2 for the upper estimate of $f(\underline{\mu})$, we end up with the estimate:

$$|\mathbf{G}_p(x,y)| \,\le\, C\,\left\{ \frac{1}{y^{p/4+9/24}} \,+\, \frac{1}{y^{(p+1)/2}} \right\} \exp\left(-c\,\frac{x^{4/3}}{y^{1/3}} \right) ,$$

for $x \ge \mathbf{c}_\sharp\, y$. Using now $y \ge y_{\min}$, we end up with the estimate:

$$\forall\, y \ge y_{\min}\,, \forall\, x \ge \mathbf{c}_\sharp\, y \quad |\mathbf{G}_p(x,y)| \,\le\, \frac{C}{y^{(p+1)/4}}\, \exp\left(-c\,\frac{x^{4/3}}{y^{1/3}} \right) , \tag{A.56}$$

for suitable constants C and c.

It remains to argue as in Corollaries A.2 and A.3 to pass from a given constant $\mathbf{c}_\sharp$ to an arbitrary given constant $\underline{\mathbf{c}} > 0$ given a priori. At his stage, we have already shown the validity of the first three estimates in (A.50).

• Step 4: the oscillating side. Part I. We still assume that c_3 is positive and now assume that x is negative. We follow the proof of Proposition A.3 and consider the same contour deformation as the one depicted in Figure A.1. This gives rise to a decomposition:

$$\mathbf{G}_p(x,y) \,=\, \varepsilon_1(x,y) \,+\, \varepsilon_2(x,y) \,+\, \mathscr{H}_\flat(x,y) \,+\, \mathscr{H}_\sharp(x,y)\,,$$

that is entirely similar to the one in (A.22) except that the four integrals now incorporate the contribution of the polynomial factor $(\mathbf{i}\,\theta)^p$. For instance, we have (keeping the notation $\omega := |x|/y$):

$$\varepsilon_2(x,y) \,=\, \frac{1}{2\,\pi}\, \exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}} - \frac{c_4}{9\,c_3^2}\,\omega^2\,y \right) \int_{\Xi(\omega)}^{+\infty} \left(\mathbf{i}\,\theta - \sqrt{\frac{\omega}{3\,c_3}} \right)^p \mathrm{e}^{\mathbf{i}\,\cdots}\, \mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,\theta^2 + \frac{2\,c_4}{c_3}\,\omega\,y\,\theta^2}\, \mathrm{e}^{-c_4\,y\,\theta^4}\, \mathrm{d}\theta\,,$$

where, again, the three dots within the integral stand for a *real* quantity whose precise expression is useless, and $\Xi(\omega)$ stands for the quantity defined in (A.23). Applying the triangle inequality yields the bound:

$$|\varepsilon_2(x,y)| \,\le\, C\, \exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}} \right) \int_{\Xi(\omega)}^{+\infty} \left(\theta^p \,+\, \omega^{p/2} \right) \mathrm{e}^{-\sqrt{3\,c_3\,\omega}\,y\,\theta^2 + \frac{2\,c_4}{c_3}\,\omega\,y\,\theta^2}\, \mathrm{e}^{-c_4\,y\,\theta^4}\, \mathrm{d}\theta\,,$$

where the constant C does not depend on ω and y. We restrict again $\omega = |x|/y$ by imposing the condition (A.27) so that we have:

$$\frac{2\,c_4}{c_3}\,\omega \,\le\, \frac{1}{2}\,\sqrt{3\,c_3\,\omega}\,.$$

This restriction corresponds to an inequality $\omega \le \omega_0$ for some well-chosen constant $\omega_0 > 0$. This yields the estimate:

$$|\varepsilon_2(x,y)| \,\le\, C\, \exp\left(\frac{4\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}} \right) \int_{2\,\sqrt{\frac{\omega}{3\,c_3}}}^{+\infty} \left(\theta^p \,+\, \omega^{p/2} \right) \mathrm{e}^{-\frac{\sqrt{3\,c_3\,\omega}}{2}\,y\,\theta^2}\, \mathrm{d}\theta\,. \tag{A.57}$$

We now use the following Lemma which follows from integration by parts and an induction argument (the proof is left to the reader).

Lemma A.2. *Let the sequence $(Q_k)_{k\in\mathbb{N}}$ of real polynomials be defined by:*

$$Q_0(Y) := \frac{1}{2}\,,$$
$$\forall\, k \in \mathbb{N}\,, \quad Q_{k+1}(Y) := \frac{1}{2}\,Y^{k+1} \,+\, (k+1)\,Q_k(Y)\,.$$

Then for any integer $\nu \in \mathbb{N}$ and for any real numbers $a > 0$ and $X > 0$, there holds:

$$\int_X^{+\infty} \theta^\nu\,\mathrm{e}^{-\,a\,\theta^2}\,\mathrm{d}\theta \;\leq\; \begin{cases} a^{-(\nu+1)/2}\,Q_{(\nu-1)/2}(a\,X^2)\,\mathrm{e}^{-\,a\,X^2}\,, & \text{if } \nu \text{ is odd,}\\ a^{-\nu/2-1}\,X^{-1}\,Q_{\nu/2}(a\,X^2)\,\mathrm{e}^{-\,a\,X^2}\,, & \text{if } \nu \text{ is even.}\end{cases}$$

Let us assume for a moment that p is odd. Applying Lemma A.2 in (A.57), we obtain the estimate:

$$|\varepsilon_2(x,y)| \;\leq\; C\,\exp\left(-\frac{2\,\omega^{3/2}\,y}{3\,\sqrt{3\,c_3}}\right)\left(\frac{Q(\omega^{3/2}\,y)}{\omega^{(p+1)/4}\,y^{(p+1)/2}} \,+\, \frac{\omega^{p/2}}{\omega\,y}\right)\,,$$

where Q is a real polynomial with nonnegative coefficients. Since the exponential term can absorb any polynomial expression of the same argument, we end up with the estimate:

$$|\varepsilon_2(x,y)| \;\leq\; C\,\exp\left(-c\,\omega^{3/2}\,y\right)\left(\frac{1}{\omega^{(p+1)/4}\,y^{(p+1)/2}} \,+\, \frac{\omega^{p/2}}{\omega\,y}\right)\,, \tag{A.58}$$

if p is odd. If p is even, applying Lemma A.2 in (A.57) yields the final estimate:

$$|\varepsilon_2(x,y)| \;\leq\; C\,\exp\left(-c\,\omega^{3/2}\,y\right)\left(\frac{1}{\omega^{p/4+1}\,y^{p/2+1}} \,+\, \frac{\omega^{p/2}}{\omega\,y}\right)\,. \tag{A.59}$$

Of course, there is a similar estimate for the contribution $\varepsilon_1(x,y)$ that is the complex conjugate of $\varepsilon_2(x,y)$.

Let us now turn to the contribution $\mathscr{H}_\flat(x,y)$ that corresponds to the inclined segment on the left in Figure A.1. Keeping the notation of the proof of Proposition A.3, we have:

$$\mathscr{H}_\flat(x,y) \;=\; \frac{\mathrm{e}^{p_0(\omega)\,y-\mathbf{i}\,\pi/4}}{2\,\pi}\,\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \big(\mathbf{i}\,\Theta(t)\big)^p\,\exp\left(y\,\sum_{k=1}^4 p_k(\omega)\,t^k\right)\mathrm{d}t\,,$$

where $(t \mapsto \Theta(t))$ parametrizes the segment so that we have a uniform estimate:

$$|\Theta(t)| \;\leq\; C\,\sqrt{\omega}\,,$$

for $\omega \leq \omega_0$.

Instead of trying to isolate the leading contribution in $\mathscr{H}_\flat(x,y)$ as we did in the proof of Proposition A.3, we rather apply the triangle inequality and use the behavior (A.29) of $p_0,\dots,p_4$ when ω is small. Starting from:

$$|\mathscr{H}_\flat(x,y)| \;\leq\; C\,\omega^{p/2}\,\mathrm{e}^{(\mathrm{Re}\;p_0(\omega))\,y}\,\int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\,\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\left(y\,\sum_{k=1}^4 (\mathrm{Re}\;p_k(\omega))\,t^k\right)\mathrm{d}t\,,$$

we use (A.29c) to first absorb the term $(\mathrm{Re}\, p_1(\omega))\, t = \mathcal{O}(\omega^{5/2})$ (uniformly with respect to t) by half of $\mathrm{Re}\, p_0(\omega)$. We can then absorb $(\mathrm{Re}\, p_3(\omega))\, t^3$ on the considered interval by part of $(\mathrm{Re}\, p_2(\omega))\, t^2$ (this argument was also used in the proof of Proposition A.3). The final term $(\mathrm{Re}\, p_4(\omega))\, t^4$ is $\mathcal{O}(\omega\, t^2)$ so it can also be absorbed by half of what is left from $(\mathrm{Re}\, p_2(\omega))\, t^2$ (up to choosing ω small enough). In the end we get an estimate (with uniform constants C and c):

$$|\mathscr{H}_\flat(x,y)| \le C\, \omega^{p/2}\, \mathrm{e}^{-c\,\omega^2\, y} \int_{-\sqrt{\frac{2\,\omega}{3\,c_3}}-\frac{2\sqrt{2}\,c_4}{9\,c_3^2}\,\omega}^{\sqrt{\frac{2\,\omega}{3\,c_3}}} \exp\Big(-c\,\omega^{1/2}\, y\, t^2\Big)\, \mathrm{d}t\,.$$

Estimating crudely the integral over the segment by the integral of the same function over $\mathbb{R}$, we end up with:

$$|\mathscr{H}_\flat(x,y)| \le C\, \frac{\omega^{p/2-1/4}}{y^{1/2}}\, \mathrm{e}^{-c\,\omega^2\, y}\,,$$

and the same estimate holds for $\mathscr{H}_\sharp(x,y)$ that is the complex conjugate of $\mathscr{H}_\flat(x,y)$.

Let us assume that p is odd. Combining the latter estimate with (A.58), we get:

$$|\mathbf{G}_p(x,y)| \le C\, \frac{\omega^{p/2-1/4}}{y^{1/2}}\, \mathrm{e}^{-c\,\omega^2\, y} + C\, \mathrm{e}^{-c\,\omega^{3/2}\, y} \left(\frac{1}{\omega^{(p+1)/4}\, y^{(p+1)/2}} + \frac{\omega^{p/2}}{\omega\, y} \right).$$

Since ω has been chosen smaller than some constant ω_0, there is no loss of generality in assuming $\omega_0 \le 1$ and we therefore have:

$$\mathrm{e}^{-c\,\omega^{3/2}\, y} \le \mathrm{e}^{-c\,\omega^2\, y}\,.$$

We recall the definition $\omega = |x|/y$ and rewrite the latter estimate in terms of x and y to obtain:

$$|\mathbf{G}_p(x,y)| \le C\, \exp\left(-c\,\frac{x^2}{y}\right) \left\{ \left(\frac{|x|}{\sqrt{y}}\right)^{p/2} \frac{1}{|x|^{1/4}\, y^{(p+1)/4}} + \frac{1}{|x|^{(p+1)/4}\, y^{(p+1)/4}} + \left(\frac{|x|}{\sqrt{y}}\right)^{p/2} \frac{1}{|x|\, y^{p/4}} \right\}.$$

We can absorb all polynomial expressions of $|x|/\sqrt{y}$ by the Gaussian function and we also use the inequalities:

$$|x| \ge |x|^{1/4}\, y^{1/4}\,, \quad |x|^{p/4} \ge y_{\min}^{p/12} > 0\,,$$

that hold for $|x| \ge y^{1/3}$ and $y \ge y_{\min}$. Eventually, we have shown that there exists some small constant $\mathbf{c}_\flat > 0$ and some constants C and c such that there holds:

$$\forall\, y \ge y_{\min}\,, \quad \forall\, x \in [-\mathbf{c}_\flat\, y, -y^{1/3}]\,, \quad |\mathbf{G}_p(x,y)| \le \frac{C}{|x|^{1/4}\, y^{(p+1)/4}}\, \exp\left(-c\,\frac{x^2}{y}\right). \tag{A.60}$$

The same kind of arguments lead to the estimate (A.60) in the case where p is even (starting now from (A.59)).

• Step 5: the oscillating side. Part II. There is no real difficulty in adapting the proof of Proposition A.4 to this slightly more general framework that incorporates the factor $(\mathbf{i}\,\theta)^p$ in the definition (A.49) of $\mathbf{G}_p$. By following the same arguments as in the proof of Proposition A.4 and absorbing polynomial terms by exponentially decaying ones, we can show that there exists a constant $\mathbf{C}_\flat > \mathbf{c}_\flat$ and some

$$\forall\, y \ge y_{\min}\,, \quad \forall\, x \le -\mathbf{C}_\flat\, y\,, \quad |\mathbf{G}_p(x,y)| \le \frac{C}{y^{(p+1)/4}}\, \exp\left(-c\,\frac{|x|^{4/3}}{y^{1/3}}\right). \tag{A.61}$$

• <u>Step 6: the oscillating side. Part III</u>. It remains to deals with the case $x \in [-\mathbf{C}_\flat\, y, -\mathbf{c}_\flat\, y]$ and this is done by merely adapting Proposition A.5. We choose again the contour depicted in Figure A.2 so that along the two inclined segments, the complex number θ is $\mathcal{O}(\delta)$ and the parameter δ is chosen such that it satisfies (A.37). This choice, that is uniform with respect to x and y in the considered regime, allows us to absorb the term $(\mathbf{i}\,\theta)^p$ into a constant along the two inclined segments. For the integrals along the two horizontal half-lines, the polynomial factor $(\mathbf{i}\,\theta)^p$ simply gives an algebraic factor that is harmless when compared with the exponentially decaying term $\exp(-c\,y)$. Overall, we leave as an exercise to the interested reader to prove that for suitable constants C and c, there holds:

$$\forall\, y \ge y_{\min}\,, \quad \forall\, x \in [-\mathbf{C}_\flat\, y, -\mathbf{c}_\flat\, y]\,, \quad |\mathbf{G}_p(x,y)| \,\le\, \frac{C}{y^{(p+1)/4}}\, \exp(-c\,y)\,. \tag{A.62}$$

We can then use the above estimates (A.60), (A.61), (A.62) and adapt the arguments of Corollaries A.2 and A.3 to show that for any given constant $\underline{\mathbf{c}} > 0$, there exist constants C and c such that the estimates corresponding to the last two cases of (A.50) are valid. This completes the proof of Theorem A.3. □

Proof of Corollary A.7. From the definition (A.46) of $\mathbf{A}_r$ and the definition (A.49) of the correctors $\mathbf{G}_p$, we have[7]:

$$\frac{\partial \mathbf{A}_r}{\partial y}(x,y) \,=\, -\,c_3\, \mathbf{G}_2(x,y) \,-\, c_4\, \mathbf{G}_3(x,y)\,.$$

We now consider $n \in \mathbb{N}^*$ and $j_0 \in \mathbb{N}^*$ such that j_0 belongs to the segment $[n\,|\alpha_r|/2, n]$. We observe that the segment $[j_0/|\alpha_r|, n]$ is included in $[n/2, n/|\alpha_r|]$ so the mean value theorem gives the bound:

$$\begin{aligned}&\left|\mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, \frac{j_0}{|\alpha_r|}\right) \,-\, \mathbf{A}_r\left(-j_0 + n\,|\alpha_r|, n\right)\right| \\ &\le\, C\,|n\,|\alpha_r| - j_0| \sup_{y \in [n/2, n/|\alpha_r|]} |\mathbf{G}_2(-j_0 + n\,|\alpha_r|, y)| \,+\, C\,|n\,|\alpha_r| - j_0| \sup_{y \in [n/2, n/|\alpha_r|]} |\mathbf{G}_3(-j_0 + n\,|\alpha_r|, y)|\,.\end{aligned} \tag{A.63}$$

Corollary A.7 then follows by applying Theorem A.4, keeping in mind that α_r is negative (so the relevant constant c_3 is negative) and that the relevant values of x and y satisfy here $|x|/y \le \underline{\mathbf{c}}$ for some constant $\underline{\mathbf{c}}$ that only depends on α_r (this is because of the bounds on j_0 in terms of n). Actually, the most critical case arises in the right-hand side of (A.63) with the term:

$$|n\,|\alpha_r| - j_0| \sup_{y \in [n/2, n/|\alpha_r|]} |\mathbf{G}_2(-j_0 + n\,|\alpha_r|, y)|\,,$$

in the regime $n\,|\alpha_r| - j_0 \ge n^{1/3}$. We then use Theorem A.4 for $p = 2$ and obtain a bound of the form:

$$C\,|n\,|\alpha_r| - j_0|\, \frac{1}{|n\,|\alpha_r| - j_0|^{1/4}\, n^{3/4}}\, \exp\left(-c\, \frac{|n\,|\alpha_r| - j_0|^2}{n}\right)\,,$$

so the Gaussian term can absorb the polynomial expression $|n\,|\alpha_r| - j_0|/\sqrt{n}$ and this gives a bound:

$$\frac{C}{|n\,|\alpha_r| - j_0|^{1/4}\, n^{1/4}}\, \exp\left(-c\, \frac{|n\,|\alpha_r| - j_0|^2}{n}\right)\,,$$

[7]We first differentiate the definition (A.46) with respect to y and then apply once again the Cauchy formula to let the abscissa η tend to zero.

just like we had in Corollary A.1 for the free Green's function. In all other regimes, the situation is more favorable and there are extra positive powers of n that can even be omitted in the end. $\square$

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